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OWP 2007 - 02

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

A 3-LOCAL IDENTIFICATION OF THE ALTERNATING GROUP OF DEGREE 8, THE MCLAUGHLIN SIMPLE GROUP AND THEIR AUTOMORPHISM GROUPS

Abstract. In this article we give 3-local characterizations of the alternating and symmetric groups of degree 8 and use these characterizations to recognize the sporadic simple group discovered by McLaughlin from its 3-local subgroups.

1. Introduction

In [18] local characteristic p completions of weak BN-pairs are classified when p is an odd prime. The outcome of this classification is that such groups are either rank 2 Lie type groups in characteristic p, the weak BN-pair is of type $\mathrm{PSL}_3(p)$ or $p \in \{3, 5, 7\}$ and the weak BN-pairs have known structure. In these exceptional cases the techniques used in [18] break down. This is partly because the expected outcomes may not be Lie type groups of rank 2 and so cannot be identified from their action on a moufang polygon and partly because the p-rank is very low leading to difficulties in eradicating p'-cores in centralizers of involutions. The groups corresponding to weak BN-pairs of type $\mathrm{PSL}_3(p)$ are currently being investigated by Astill [2]. For the larger amalgams when $p \in \{5, 7\}$ the exceptional cases have been further analyzed in [17] and [19]. In the case $p = 3$ there are three different, but closely related, exceptional weak BN-pairs of characteristic 3. For more details on weak BN-pairs see [18]. The purpose of this paper is to handle one of these exceptional configurations. In fact we prove a much stronger result than the original motivating problem requires, anticipating that this result will also find application in the programme of Meierfrankenfeld on finite groups of local characteristic p [15]. Our main theorem characterizes McL, the simple group discovered by McLaughlin, and Aut(McL) in terms of certain 3-local data and is as follows (our notation will be explained below).

Date: July 31, 2007.

This research was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from April 30 to May 13, 2006.

Theorem 1.1. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z = Z(S)$ and J is an elementary abelian subgroup of S of order 3^4. Further assume that*

(i) $O^{3'}(N_G(J)) \sim 3^4.\mathrm{Alt}(6)$;
(ii) $O^{3'}(N_G(Z)) \sim 3_+^{1+4}.2{\cdot}\mathrm{Alt}(5)$; *and*
(iii) *for all non-trivial elements x of J, $C_G(O_3(C_G(x))) \le O_3(C_G(x))$.*

Then G is isomorphic to either McL *or* Aut(McL).

Let us consider one of the target groups in Theorem 1.1, namely $G = \mathrm{McL}$. Choosing an involution t in $L := O^{3'}(N_G(Z))$ we observe that $C_L(t) \cong 3 \times 2{\cdot}\mathrm{Alt}(5)$. From the ATLAS [5], we recall that $C_G(t) \cong 2{\cdot}\mathrm{Alt}(8)$. It is therefore tempting to try to prove that $C_G(t)$ must be isomorphic to $2{\cdot}\mathrm{Alt}(8)$ since we can then avail ourselves of identification results due to Janko and Wong [13]. This is the path we follow which brings us to

Theorem 1.2. *Let G be a finite group with $D = \langle y, z\rangle$ an elementary abelian Sylow 3-subgroup of G of order 9. Assume the following hold.*

(i) $C_G(D) = D$ *and* $N_G(D)/C_G(D) \cong \mathrm{Dih}(8)$.
(ii) $C_G(y)$ *is 3-closed,* $C_G(y)/D \cong 2$ *and* $N_G(\langle y\rangle)/D \cong 2^2$.
(iii) $C_G(z) \cong 3 \times \mathrm{Alt}(5)$ *and* $N_G(\langle z\rangle) \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(5)$ *(the diagonal subgroup of index 2 in* $\mathrm{Sym}(3) \times \mathrm{Sym}(5)$*).*

Then G is isomorphic to Alt(8).

The proof of Theorem 1.2 has many features in common with certain of the 3-local characterizations obtained by Higman ([8], [9]) and a number of his students. This line of development was eclipsed by the burgeoning work on the simple group classification in the 1970's, though 3-local characterizations continue to be of interest (see [12, 11]). The main aim in the proof of Theorem 1.2 is to pin down the structure of a Sylow 2-subgroup of G. The first step is to look at $C_G(F)/F$ where F is a Sylow 2-subgroup of $C_G(z)$ (so F is an elementary abelian group of order 4). Now the structure of $C_G(z)$ implies that $\langle z\rangle F/F$ is self-centralizing in $C_G(F)/F$ and so we may apply a result due to Feit and Thompson (see Theorem 2.1 below) to limit the structure of $C_G(F)/F$. Further work restricts the structure of $C_G(F)/F$ yet more until we see, in (3.7), that $C_G(F)$ has a normal Sylow 2-subgroup with $E = O_2(C_G(F))$ an elementary abelian group of order 2^4. After that we quickly get that $N_G(E)$ contains T, a Sylow 2-subgroup of G of order 2^6. At this point there are results we could quote to identify G as being Alt(8). However, wherever possible, we give elementary proofs rather than appealing to substantial results in the literature. So, using the Feit Thompson result two more times we next determine the structure of the centralizers of involutions in the two involution classes. This information, combined with the presentation for Alt(8) given in Lemma 2.6, enables us to show that G contains a subgroup isomorphic to Alt(8). An easy application of the Thompson order formula finally yields that $G \cong \mathrm{Alt}(8)$.

In seeking the second alternative of Theorem 1.1, just as for the McL case, we are led to establish characterizations for smaller groups. Our next result is the one we require. Note, however, the uninvited guest here - the Sym(8) possibility is the one occurring in Aut(McL).

Theorem 1.3. *Let G be a finite group with $D = \langle y, z\rangle$ an elementary abelian Sylow 3-subgroup of G of order 9. Assume the following hold.*

(i) $|C_G(D)/D| = 2$, $N_G(D)/D \cong \mathrm{Dih}(8)\times 2$ *and* $N_G(D)/C_G(D) \cong \mathrm{Dih}(8)$.
(ii) $C_G(y)$ *is 3-closed,* $C_G(y)/D \cong 2^2$ *and* $N_G(\langle y\rangle)/D \cong 2^3$.
(iii) $C_G(z) \cong 3 \times \mathrm{Sym}(5)$ *and* $N_G(\langle z\rangle) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(5)$.

Then G is isomorphic to either Sym(8) *or* $\mathrm{PGO}_4^+(5)$.

We remark that $\mathrm{PGO}_4^+(5)$ is isomorphic to $(\mathrm{Alt}(5)\wr 2).2$ by which we mean the the unique group X with $F^*(X) \cong \mathrm{Alt}(5)\times\mathrm{Alt}(5)$ a minimal normal subgroup of X and $X/F^*(X)$ elementary abelian of order 4. This group emerges in the proof of Theorem 1.3 as a certain subgroup of index 2 in the subgroup of Sym(10) which preserves a partition of the ten points into two sets of size five.

The hypotheses of Theorems 1.2 and 1.3 are very similar as indeed are the groups Alt(8) and Sym(8), yet some aspects of the proof of Theorem 1.3 are very different to that of Theorem 1.2. We begin, in the spirit of the proof of Theorem 1.2, by quoting a theorem of Prince's (see Theorem 2.3) to deduce that $C_G(t)$ is either $N_G(D)$ or is isomorphic to $2 \times \mathrm{Sym}(6)$. Here t is the involution in $Z(N_G(D))$. The former case, which gives rise to $G \cong (\mathrm{Alt}(5)\wr 2).2$, rapidly leads to considering a subgroup of G isomorphic to $(\mathrm{Alt}(5)\wr 2).2$ with the remainder of the proof directed to towards showing that it actually is G. The methods used are mostly 2-local in nature and culminate in a call to the classification of groups with an abelian Sylow 2-subgroup [22]. It is in the latter case that we take a very different tack. The basic idea is to start with the Coxeter presentations for the direct Sym(6) factor of $C_G(t)$ and the direct Sym(3) factor of $N_G(\langle z\rangle)$ and attempt to paste them together so as to obtain a Coxeter presentation for Sym(8). Let t_1 be an involution in this direct Sym(3) factor. The crucial step, carried out in (4.6), is to show that t and t_1 are G-conjugate. We suppose that this is not the case and examine whether or not certain involutions are in $\mathcal{T} = t^G$. Matters come to a head when this fusion information, seen within an elementary abelian 2-subgroup E of order 16, leads us to a predict the existence of a certain subgroup of $\mathrm{GL}_4(2)$ of order 42. This predicted subgroup is incompatible with the subgroup structure of $\mathrm{GL}_4(2)$ and so we have our contradiction.

As a consequence we obtain a subgroup X of G which is isomorphic to Sym(8). The rich 2-local structure of Sym(8) assists us to quickly establish that X contains a Sylow 2-subgroup of G and then, courtesy of the Thompson Transfer Lemma, we find that G contains a subgroup H of index 2. Now we may apply Theorem 1.2 to deduce that $H \cong \mathrm{Alt}(8)$, whence it follows that $G \cong \mathrm{Sym}(8)$. We mention that at the heart of the proofs of both Theorem 1.1 and 1.2 we apply results that crucially rely on character theory for their proofs. Namely for Theorem 1.1 we apply Theorem 2.1 and for Theorem 1.2 we apply Theorem 2.3.

Returning to the proof of Theorem 1.2 we mention that an amalgam consisting of the groups with shape $(A_1, A_2, A_3) = (\mathrm{Sym}(3) \wr \mathrm{Sym}(5), 3^2 : \mathrm{Dih}(8), 2^2 : \mathrm{Dih}(12))$, pairwise intersections $A_1 \cap A_2 \sim 3^2 : 2^2$, $A_2 \cap A_3 \cong \mathrm{Dih}(8)$, $A_1 \cap A_3 = \mathrm{Dih}(8)$ and triple intersection $A_1 \cap A_2 \cap A_3 \cong 2^2$ can be found in G. If it were possible to show that the universal completion of such an amalgam must be isomorphic to Alt(8), then we would have another proof of Theorem 1.2 which would have been much more akin to the generators and relations part of the proof of Theorem 1.3. However, calculations using MAGMA [4] and employing the small index and coset image routines reveals that the universal completion of this amalgam has quotients isomorphic to $\mathrm{Sym}(8) \times \mathrm{Alt}(8)$ and $3^{14}.(\mathrm{Alt}(8) \times 2)$, so this is a forlorn hope.

With Theorems 1.2 and 1.3 to hand we finally embark upon the proof of Theorem 1.1. Choosing t to be an involution in $O^{3'}(N_G(Z))$ we wish, as mentioned earlier, to use these two results to determine the structure of $C_G(t)/\langle t \rangle$. Of course we must verify that we have the hypotheses of these theorems and this is done in Lemmas 5.12 and 5.14.

All groups in this paper are assumed to be finite with our group theoretic notation being standard as given, for example, in [1] and [14]. For the description of group structures we follow the ATLAS [5] except that we shall use Sym(n) and Alt(n) to denote, respectively, the symmetric and alternating groups of degree n and Dih(n), Q(n) and SDih(n), respectively, to stand for the dihedral group, quaternion group and semidihedral group of order n. We also use Mat(n) to denote the Mathieu group of degree n. Finally $X \sim Y$ where X and Y are groups will indicate that X and Y have the same shape.

Acknowledgement. This work was carried out while the authors were enjoying (very much) the hospitality of the Mathematische Forschungsinstitut Oberwolfach, as part of the Research in Pairs Program, April 30th to May 13th, 2006. We would also like to express our thanks to the referees for their detailed comments on this paper which have lead to numerous improvements.

2. Background results

¿From here on we assume all groups are finite.

Theorem 2.1 (Feit Thompson Theorem). *Let G be a group which contains a self-centralizing subgroup of order 3. Then one of the following statements is true.*

(i) *G contains a nilpotent normal $3'$-subgroup N such that G/N is isomorphic to either* $\mathrm{Alt}(3)$ *or* $\mathrm{Sym}(3)$.
(ii) *G contains a normal 2-subgroup N such that G/N is isomorphic to* $\mathrm{Alt}(5)$.
(iii) *G is isomorphic to* $\mathrm{PSL}_2(7)$.

Proof. This is a theorem of Feit and Thompson [6]. □

The set of maximal abelian normal subgroups of a p-group P is denoted by $\mathrm{SCN}(P)$ and the subset of this set consisting of those groups with p-rank at least k is denoted by $\mathrm{SCN}_k(P)$.

Our next important result is a consequence of the soluble signalizer functor theorem [14].

Theorem 2.2. *Let G be a group of 2-rank at least 4 with $O_{2'}(G) = 1$ and $\mathrm{SCN}_3(S)$ non-empty. If the centralizer of every involution of G is soluble, then $O_{2'}(C_G(t)) = 1$ for every involution t of G.*

Proof. See for example [7, Theorem 2.2]. □

As mentioned in Section 1, the next result is used in the proof of Theorem 1.3.

Theorem 2.3 (Prince). *Let G be a group which has a self-centralizing Sylow 3-subgroup S of order 9. Suppose that $N_G(S)/S \cong \mathrm{Dih}(8)$ and that $C_G(x) \le N_G(S)$ for all $x \in S^\#$. Then either $S \trianglelefteq G$ or $G \cong \mathrm{Sym}(6)$.*

Proof. See [20, Lemma 3.2]. □

Theorem 2.4 (Janko, Wong). *Let G be a group which possesses an involution t such that $C_G(t) \cong 2{}^{\cdot}\mathrm{Alt}(8)$. Then either $G = C_G(t)O_{2'}(G)$ or $G \cong \mathrm{McL}$.*

Proof. See [13]. □

We shall also require the surprisingly effective Thompson Transfer Lemma.

Lemma 2.5 (Thompson Transfer Lemma)**.** *Let G be a group and $T \in \mathrm{Syl}_2(G)$. Suppose that there exists a maximal subgroup U of T and an involution t of T such that $t^G \cap U = \emptyset$. Then $t \notin O^2(G)$.*

Proof. See [14, 12.1.1]. □

Lemma 2.6. *The generators $x_1, \dots, x_6$ and the relations*

$$\begin{aligned} x_1^3 &= 1 \\ x_i^2 &= 1 \text{ for } 2 \le i \le 6 \\ (x_i x_{i+1})^3 &= 1 \text{ for } 1 \le i \le 5 \\ (x_i x_j)^2 &= 1 \text{ for } 1 \le i \le 4 \text{ and } i+1 < j \le 6 \end{aligned}$$

give a presentation for $\mathrm{Alt}(8)$.

Proof. This is well-known. See [10, page 138].

□

Lemma 2.7. *Suppose that G is group which contains a subgroup $M = \langle t \rangle \times M_0$ of odd index where t is an involution and $M_0 \cong \mathrm{Sym}(4)$. Then G contains a subgroup H of index 2 such that $t \notin H$ and $M \cap H \cong \mathrm{Sym}(4)$.*

Proof. Let $T \in \mathrm{Syl}_2(M)$ and $T_0 = T \cap \mathrm{M}_0 \cong \mathrm{Dih}(8)$. Since $T \cong 2 \times \mathrm{Dih}(8)$, $\mathrm{Out}(T)$ is a 2-group whence $N_G(T) = TC_G(T)$. Therefore no two distinct involutions in $Z(T)$ are G-conjugate by Burnside's Theorem [14, Theorem 7.1.5]. Let $T' = \langle u \rangle$. Then $Z(T) = \langle t, u \rangle$. Because M_0 has two conjugacy classes of involutions (and one of them contains u) we deduce that one of t and tu cannot be G-conjugate to any element in T_0. Using the Thompson Transfer Lemma we then see that there is an index 2 subgroup H of G with $t \notin H$ (and $tu \notin H$) and $M \cap H \cong \mathrm{Sym}(4)$. □

Lemma 2.8. *Suppose G is a 2-group containing an elementary abelian subgroup Q of index 2. Assume that t is an involution in $G \setminus Q$ and $C_Q(t) = [Q, t]$. Then tQ contains exactly $|C_Q(t)|$ involutions and they are all conjugate.*

Proof. If $g \in Q$, then tg is an involution precisely when t and g commute. Since $g \in C_Q(t) = [Q, t]$, we have that $tg = t[q, t] = t^{qt}$ for some $q \in Q$.

□

Lemma 2.9. *Suppose that $G \cong \mathrm{Sym}(8)$ and $H \le G$ with $H \cong \mathrm{Alt}(8)$. Let $T \in \mathrm{Syl}_2(G)$ and $T_0 = T \cap H$.*

(i) T_0 *contains a unique elementary abelian subgroup of order* 2^4.
(ii) *Let* $t \in Z(T)^\#$. *Then* $C_G(t)$ *contains exactly three subgroups of index* 2. *They are* $C_H(t)$, $\mathrm{Sym}(2) \wr \mathrm{Alt}(4)$ *and one other which we call* K. *The groups* $C_H(t)$ *and* $\mathrm{Sym}(2) \wr \mathrm{Alt}(4)$ *have Sylow* 2*-subgroups of exponent* 4 *and* K *contains elements of order* 8.

Proof. These results are consequences of calculations in Sym(8). □

3. A Characterization of Alt(8)

This section is devoted to proving Theorem 1.2. So we are assuming that G is a group with $D = \langle y, z\rangle$ an elementary abelian Sylow 3-subgroup of G of order 9. Setting $Z = \langle z\rangle$, $Y = \langle y\rangle$, $L = N_G(Z)$ and $M = N_G(D)$ we also have

(3.1)

(i) $C_G(D) = D$, $M/D \cong \mathrm{Dih}(8)$;
(ii) D is normal in $C_G(Y)$, $C_G(Y)/D \cong 2$ and $N_G(Y)/D \cong 2^2$; and
(iii) $C_G(Z) \cong 3 \times \mathrm{Alt}(5)$ and $L \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(5)$.

Since $M/D = N_G(D)/C_G(D) \cong \mathrm{Dih}(8)$, M has two orbits on the subgroups of D of order 3 and two orbits on $D^\#$. Note that $|M : M \cap L| = 2$. From (3.1) (ii) and (iii) it follows that

(3.2) G has two conjugacy classes of elements of order 3 namely, y^G and z^G.

Let $V \in \mathrm{Syl}_2(M \cap L)$. Then $V \cong 2^2$ and V contains an element inverting D and a non-trivial element which centralizes Z. Put $B = N_M(V) \cong \mathrm{Dih}(8)$. Letting a_0 and b_0 be involutions with $a_0 \in C_B(z)$ and $b_0 \in C_B(y)$ we observe that $a_0 \in V$, $B = \langle b_0, V\rangle$ and

(3.3) a_0 and b_0 are not conjugate in G. In particular, G has at least two conjugacy classes of involutions.

Note that $Z \le C_G(a_0)$ and $Y \le C_G(b_0)$. If a_0 and b_0 were G-conjugate, then by (3.2), 3^2 would divide $|C_G(a_0)|$ contrary to (3.1) (i). Thus (3.3) holds. ♠

Next we investigate the group generated by $N_L(V)$ and $B = N_M(V)$. Set $F = N_L(V) \cap C_G(Z)$ and choose $a \in F \setminus V$.

(3.4)

(i) $N_L(V) \cong \mathrm{Dih}(8)$;
(ii) $\langle N_L(V), B\rangle = KV$ where $K = \langle a, b_0\rangle \cong \mathrm{Dih}(12)$ is a complement to V in KV; and

(iii) $V^{\#} \subseteq a_0^G$.

The definition of V and the structure of L imply that $N_L(V) \cong \mathrm{Dih}(8)$, $F \in \mathrm{Syl}_2(O^3(C_G(Z)))$ and F is elementary abelian of order 4. In particular, we note that a is an involution.

Since a centralizes Z arguing as in (3.3) we see that a and b_0 are not conjugate. Set $K = \langle a, b_0 \rangle$ and $K_0 = \langle ab_0 \rangle$. Clearly we have $\langle N_L(V), B \rangle = KV$. Further, as a and b_0 are not G-conjugate, $Z(K)$ is non-trivial.

Since $Z(N_L(V))$ centralizes Z and the non-trivial element of $Z(B)$ inverts Z, $Z(N_L(V)) \neq Z(B)$. Using $N_L(V) = \langle a, V \rangle$, $B = \langle b_0, V \rangle$, $Z(N_L(V)) \neq Z(B)$ as well as $Z(B)Z(N_L(V)) \leq V$, we get that $Z(B)^a \neq Z(B)$ and $Z(N_L(V))^{b_0} \neq Z(N_L(V))$. Therefore K acts transitively on $V^{\#}$. If $K \cap V \neq 1$, then $Z(K) \cap V \neq 1$, contrary to K acting transitively on $V^{\#}$. So $K \cap V = 1$ and K is a complement to V in KV. Since 3 divides $|K|$, K_0 has order divisible by 6. Let x be an element of K_0 of order 3. By (3.2) x is G-conjugate to either y or z. If the former occurs (3.1) (ii) implies that $|K_0| = 6$ while if the latter occurs (3.1)(iii) gives $C_G(x) \cong 3 \times \mathrm{Alt}(5)$ and, as 2 divides $|K_0|$, we also get $|K_0| = 6$. Hence $K \cong \mathrm{Dih}(12)$ and we have (ii).

Finally, as $a_0 \in V$, the transitive action of K on $V^{\#}$ implies (iii) holds and so (3.4) is true. ♠

Observe again that F is elementary abelian of order 4, $a, a_0 \in F$ and $N_L(F) \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(4)$. Let $Z(K) = \langle b \rangle$ (where K is as in (3.4)(ii)). Then $b \in Z(VK)$ and, by (3.4)(ii), $VK/\langle b \rangle \cong \mathrm{Sym}(4)$.

(3.5)

(i) All the involutions in L are in $a^G = a_0^G$.
(ii) $b^G = b_0^G$, $b \in C_G(F) \setminus F$ and 8 divides $|C_G(F)|$.
(iii) The elements in $C_G(b)$ of order 3 are in y^G.

Because $N_L(F) \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(4)$, all the involutions in F are conjugate. Since all the involutions in V are in a_0^G by (3.4) (iii), we infer that all the involutions in L are in $a_0^G = a^G$ and (i) holds.

¿From $F \leq N_L(V) \leq VK$ and $b \in Z(VK)$, it follows that $b \in C_G(F)$ and so $b \in C_G(F) \setminus F$. Hence the 2-part of $|C_G(F)|$ is at least 8. Let x be an element of K of order 3 with x inverted by a and b_0 (which we recall are not G-conjugate). If $x \in z^G$, then, by (3.5)(i), all the involutions in $N_G(\langle x \rangle)$ are in a^G, which is not the case. Hence $x \in y^G$ and we have (ii) and (iii). ♠

Set $N = N_G(F)$, $C = C_G(F)$ and $\overline{C} = C_G(F)/F$. At the moment we know that $N/C \cong N_L(F)/C_L(F) \cong \mathrm{Sym}(3)$ and, by the definition of F, $Z \leq C$. We shall shortly obtain detailed information about N which will ultimately restrict the 2-structure of G. Since $L \cap C = ZF$, we have $C_{\overline{C}}(\overline{Z}) = \overline{Z}$. Hence Theorem 2.1 yields that

(3.6) C has a normal subgroup X with $F \leq X$ such that one of the following holds.

(i) $\overline{C} \cong \mathrm{PSL}_2(7)$ and $\overline{X} = 1$.
(ii) $\overline{C}/\overline{X} \cong \mathrm{Alt}(5)$ and $\overline{X}$ is a 2-group.
(iii) $\overline{C}/\overline{X} \cong \mathrm{Alt}(3)$ or $\mathrm{Sym}(3)$ and $\overline{X}$ is a nilpotent $3'$-group.

Let $D_0 \in \mathrm{Syl}_3(N)$ with $Z \leq D_0$. Notice that $D_0 \in \mathrm{Syl}_3(G)$ and so D_0 is G-conjugate to D. Thus D_0 contains a G-conjugate Z_1 of Z with $Z \neq Z_1$ and also two G-conjugates Y_1 and Y_2 of Y. Put $E = O_2(C)$.

(3.7)

(i) $C/E \cong \mathrm{Alt}(3)$ and $E = FC_E(Z_1) = C_G(E)$ is elementary abelian of order 2^4.
(ii) $|N| = 2^5.3^2$.
(iii) $E \cap a^G \neq \emptyset \neq E \cap b^G$.
(iv) $C_E(Y_1) = C_E(Y_2) = 1$.

We establish (3.7) by working through the list in (3.6). The first possibility cannot occur as $|\mathrm{Out}(\mathrm{PSL}_2(7))| = 2$ would force $C_G(x)$ to involve $\mathrm{PSL}_2(7)$ for some $x \in D_0^{\#}$. Next we consider the possibility that $C/X \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(3)$. Then $N_{C/X}(ZX/X) \cong \mathrm{Sym}(3)$. Since $F \leq N_C(Z)$ and $C_{X/F}(Z) = 1$, this gives $|N_C(Z)| = 2^3.3$ and then the Frattini Argument implies that $|N_N(Z)| = 2^4.3^2$. But $|N_G(Z)| = 2^3.3^2.5$, a contradiction. Therefore, by (3.6), $C/X \cong \mathrm{Alt}(3)$.

Now X is a $3'$-group upon which D_0 operates and so by [14, 8.3.4]

$$X = \langle C_X(Y_1), C_X(Y_2), C_X(Z_1), C_X(Z)\rangle.$$

Since $C_G(Y_1)$ is 3-closed by (3.1)(ii), $C_X(Y_1) \leq C_G(D_0) \cap X = D_0 \cap X = 1$. Similarly, $C_X(Y_2) = 1$. We have that $C_X(Z) = F$. ¿From $C_X(Z_1) \leq C_G(Z_1) \cong 3 \times \mathrm{Alt}(5)$ and the fact that D_0 normalizes $C_X(Z_1)$ we deduce that $C_X(Z_1)$ is elementary abelian of order 1 or 4. By (3.5)(ii), $|C|$ is divisible by 8 and so we conclude that $X = FC_X(Z_1)$ with $|X| = 2^4$. Therefore $E = X$ and $E = FC_X(Z_1)$ with E elementary abelian of order 2^4. Clearly $C_G(E) \leq C_C(E) = E$ and so we have (i) and (iv).

Since $N/C \cong \mathrm{Sym}(3)$ and $a, b \in C$, (ii) and (iii) follow. ♠

Put $P = N_G(E)$. Since Z and Z_1 are conjugate in $N_G(D_0) \cong 3^2 : \mathrm{Dih}(8)$ and any two fours subgroups of L which are normalized by D_0 are conjugate by an element of $N_L(D_0)$, we may find an element $f \in N_G(D_0)$ which conjugates $F(= C_E(Z))$ to $C_E(Z_1)$. Further, as $N_L(F) \cong$

$\mathrm{Sym}(3) \curlywedge \mathrm{Sym}(4)$, we see that elements of $(N_L(F) \cap N_G(D_0)) \setminus D_0$ invert D_0. Therefore, as $N_G(D_0)/D_0 \cong \mathrm{Dih}(8)$,

$$(N_L(F) \cap N_G(D_0))/D_0 = Z(N_G(D_0)/D_0)$$

and so $f^2 \in N_L(F)$. Therefore, by (3.7)(i), f normalizes $E = FC_E(Z_1)$ and hence $N\langle f \rangle \leq P$. In our next claim we pin down the structure of P.

(3.8)

(i) P has orbits of length 6 and 9 on $E^\#$ with representatives, respectively, a and b.

(ii) $P = N\langle f \rangle$ has order $2^6.3^2$ and P acts irreducibly on E. Moreover P is isomorphic to a subgroup of $\mathrm{Sym}(8)$.

Because a and b are not G-conjugate they are certainly not P-conjugate. Since, by (3.5) (i), b centralizes F, $b \in E$ by (3.7)(i). Also by (3.5)(iii), b is not centralized by any conjugate of Z. Thus (3.7)(iv) implies that b is not centralized by any nontrivial element of D_0. Therefore $|b^P| \geq 9$. As $F^f = C_E(Z_1)$ and $F \cap F^f = 1$, $|a^P| \geq 6$. This proves that (i) holds.

By (3.7)(i) and (ii), P/E is isomorphic to a subgroup of $\mathrm{GL}_4(2) \cong \mathrm{Alt}(8)$ of order divisible by 9. Considering the lengths of the orbits of $N\langle f \rangle$ on E implies that P acts irreducibly on E and also that P is a $\{2,3\}$-group with the same orbits on E as $N\langle f \rangle$. It follows that $|P| = 2^6.3^2$ or $2^7.3^2$ (with P/E being a subgroup of $\mathrm{O}_4^+(2) \cong \mathrm{Sym}(3) \wr 2$). In particular, D_0E is normalized by P. Now $N_P(Z) \leq N_L(F) \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(4)$ and D_0E/E contains exactly two conjugates of Z fused by f, so we infer that $P = N\langle f \rangle$ is of order $2|N| = 2^6.3^2$.

Recalling that $|N_L(F)| = 2^3.3^2$ we have that $|P : N_L(F)| = 8$. Thus, as E is the unique minimal normal subgroup of P, P is isomorphic to a subgroup of $\mathrm{Sym}(8)$. ♠

(3.9) If $P/ED_0 \cong 2^2$, then P is isomorphic to a subgroup of $\mathrm{Alt}(8)$.

If (3.9) is false, then, using (3.8)(ii), we have $P \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$ where, by virtue of $F = C_E(Z)$, we have that Z is contained in one of the direct factors of the decomposition of P. But then $C_P(Z) \cong 3 \times \mathrm{Sym}(4)$ and this is not isomorphic to a subgroup of $3 \times \mathrm{Alt}(5)$. Thus (3.9) holds. ♠

Let $T \in \mathrm{Syl}_2(P)$. Our grip on the 2-structure of G begins to tighten.

(3.10)

(i) E is the unique elementary abelian subgroup of T of order 2^4.

(ii) $T \in \mathrm{Syl}_2(G)$.

We first prove part (i). Suppose that $E_1 \leq T$ with $E \neq E_1$ and $E_1 \cong 2^4$. If P/ED_0 is cyclic, then $|E_1E/E| = 2$ and $|E \cap E_1| = 2^3$. So E_1 induces a transvection on E. However, E_1 inverts D_0E/E and so this is not possible. Thus P/D_0E is elementary abelian and hence, by (3.9), P is isomorphic to a subgroup of Alt(8) and there we readily verify our claim. Let $S \in \mathrm{Syl}_2(G)$ with $T \leq S$. Then, using (i), $N_S(T) \leq N_G(E) \cap S = P \cap S = T$, whence $T = S \in \mathrm{Syl}_2(G)$. ♠

We next investigate $C_G(b)$ further. Put $Q = O_2(C_G(b))$.

(3.11)

(i) $Q \cong 2_+^{1+4}$.
(ii) P is isomorphic to a subgroup of Alt(8).
(iii) $C_G(b)/O_{2'}(C_G(b))$ is isomorphic to the centralizer of an involution of cycle type 2^4 in Alt(8).

¿From (3.1) (ii) and (3.5) (iii) it follows that $C_G(b)/\langle b\rangle$ contains a self-centralizing subgroup of order 3. Further, by (3.4) (ii), $C_G(b)/\langle b\rangle$ contains a subgroup isomorphic to Sym(4) and, by (3.8) (i) and (3.10) (ii), has a Sylow 2-subgroup of order 2^5. Using Theorem 2.1 for a second time reveals that $C_G(b)/O_{2'}(C_G(b))$ has order $2^6.3$ with $|Q| = 2^5$. If $E \leq Q$, then E is a normal subgroup of $C_G(b)$ by (3.10) (i) and hence $C_G(b) \leq N_G(E) = P$, which is not the case. Therefore $E \not\leq Q$ and so $|E \cap Q| = 2^3$. Since $Q/\langle b\rangle$ has two Sym(3) non-central chief factors and $E \cap Q \cong 2^3$, we deduce that $Q/\langle b\rangle$ is elementary abelian. Hence $Q' = \langle b\rangle$ by (3.10) (i). Using the Sym(3) action we see that $Z(Q)$ is elementary abelian. If $Z(Q) \neq \langle b\rangle$, then $Z(Q) \cong 2^3$ and $E \cap Q \not\leq Z(Q)$. But then $Z(Q)(E \cap Q)$ is an elementary abelian subgroup of T which does not exist by (3.10) (i). Thus $Q \cong 2_+^{1+4}$. Since $EQ = T$, T/E must be elementary and so (3.9) gives (ii).

Now $KV \leq C_G(b)$ and KV contains a subgroup $\langle b_0^a, b_0\rangle V \cong \mathrm{Sym}(4)$. Since $b \notin \langle b_0^a, b_0\rangle V$, by (i) this group is core-free in $C_G(b)/O_{2'}(C_G(b))$. Therefore $X = C_G(b)/O_{2'}(C_G(b))$ is isomorphic to a subgroup of Sym(8). Moreover X has index 2 in the centralizer of an involution of cycle type 2^4 in Sym(8). Plainly X is not 2-closed and, as T is isomorphic to a Sylow 2-subgroup of Alt(8), X has no elements of order 8. Using Lemma 2.9(ii) yields (iii). ♠

(3.12) G has exactly two conjugacy classes of involutions.

Let g be an involution in G. By (3.10) (ii) we may suppose $g \in T$. If $g \in E$, then $g \in a^G \cup b^G$ by (3.8)(i). If $g \notin E$, then, by (3.11) (ii),

$C_E(g) = [E, g] \cong 2^2$ and hence all involutions in gE are conjugate by Lemma 2.8. So we may further suppose $g \in N_G(D_0)$ and then we see that $g \in N_G(Z) = L$ or $g \in C_G(Y_1) \cup C_G(Y_2)$. Again $g \in a^G \cup b^G$, so proving (3.12). ♠

(3.13) $C_G(a)/O_{2'}(C_G(a)) \cong C_P(a)$ is isomorphic to the centralizer of an involution of cycle type $1^4.2^2$ in $\mathrm{Alt}(8)$.

We have $C_G(a) \geq C_L(a)E$ which has order $2^5.3$. Since a is not conjugate to b, $C_L(a)E$ contains a Sylow 2-subgroup of $C_G(a)$. Thus we can calculate in a Sylow 2-subgroup of Alt(8) in which a corresponds to the involution $(1,2)(3,4)$ and has centralizer in P isomorphic to

$$\langle (1,2)(3,4), (1,3)(2,4), (5,6)(7,8), (5,7)(6,8), (5,6,7), (1,2)(5,6) \rangle.$$

Thus we see that $C_P(a)/\langle a \rangle \cong 2 \times \mathrm{Sym}(4)$. In particular, $C_G(a)/\langle a \rangle$ has Sylow 2-subgroups isomorphic to $\mathrm{Dih}(8) \times 2$ and $Z(C_P(a))/\langle a \rangle = F/\langle a \rangle$. By Lemma 2.7 $C_G(a)$ has a normal subgroup of index 2 not containing F. Suppose that (3.13) does not hold. Then applying Theorem 2.1 for the third and final time we get that the subgroup H of index 2 in $C_G(a)$ has $H/\langle a \rangle \cong \mathrm{PSL}_2(7)$. Since $\mathrm{Aut}(\mathrm{PSL}_2(7))$ has dihedral Sylow 2-subgroups, it follows that $C_G(a) \leq N_G(F) \leq P$ which is absurd as P is soluble. Thus (3.13) holds. ♠

(3.14) G is a simple group.

Let N be a non-trivial normal subgroup of G. By (3.1) and [14, 8.3.4] 3 divides $|N|$. Then, as $N_G(D)$ acts irreducibly on D, we get $D \leq N$. Hence $G = NN_G(D)$ by the Frattini Argument. Further $E = [E, D_0] \leq [E, N] \leq N$ and therefore N contains all the involutions in G by (3.7) (iii) and (3.12). Since a Sylow 2-subgroup of $N_G(D)$ is isomorphic to Dih(8), this implies that $G = NN_G(D) = N$, so proving (3.14). ♠

(3.15)

(i) $C_G(b)$ is isomorphic to the centralizer of an involution of cycle type 2^4 in Alt(8).

(ii) $C_G(a)$ is isomorphic to the centralizer of an involution of cycle type $1^4.2^2$ in Alt(8).

Combining (3.11) (iii), (3.12) and (3.13) yields that the centralizer of every involution in G is soluble. Since G has 2-rank at least 4 and, by (3.14), $O_{2'}(G) = 1$, Theorem 2.2 together with (3.11) (iii) and (3.13) gives (3.15). ♠

(3.16) G contains a subgroup isomorphic to Alt(8).

We show that G contains elements $x_1, \ldots, x_6$ which satisfy the relations detailed in Lemma 2.6. We start in the subgroup $L \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(5)$. We select $x_1 = z$ and then $x_3, \ldots, x_6 \in L$ are chosen to correspond to transpositions from Sym(5) satisfying the standard Coxeter relations and inverting x_1. We need to find an appropriate involution x_2. Set $z_1 = x_5x_6$. Then z_1 is an element of order 3 in $E(L) \cong \mathrm{Alt}(5)$. Since $|N_L(\langle z_1\rangle)| = 2^2.3^2$ and Z is normal in $N_L(\langle z_1\rangle)$, we deduce that z_1 is conjugate to z. Since x_1 and x_3 centralize z_1 and $\langle x_1, x_3\rangle \cong \mathrm{Sym}(3)$, we get $\langle x_1, x_3\rangle \le E(C_G(z_1)) \cong \mathrm{Alt}(5)$. Now select an involution $x_2 \in E(C_G(z_1))$ so that x_2 centralizes x_6, $\langle x_1, x_2\rangle \cong \mathrm{Alt}(4)$ and $\langle x_2, x_3\rangle \cong \mathrm{Sym}(3)$. There are two choices for such an element and they are conjugate by x_3. Notice at this stage we know that $x_1, \ldots, x_6$ satisfy all the relations listed in Lemma 2.6 apart from perhaps the relation between x_2 and x_4 which says that x_2x_4 has order 2. From (3.15) we have that $C_G(x_6) \cong \mathrm{Sym}(4) \curlywedge \mathrm{Dih}(8)$. Since $\langle x_2, x_3\rangle \cong \mathrm{Sym}(3)$, $\langle x_3, x_4\rangle \cong \mathrm{Sym}(3)$ and $C_G(x_6)/O_2(C_G(x_6)) \cong \mathrm{Sym}(3)$, we may finally choose x_2 so that $x_2O_2(C_G(x_6)) = x_4O_2(C_G(x_6))$. In particular we may and do choose x_2 so that x_2x_4 has order a power of 2. Now x_2x_3 and x_3x_4 are both elements of order 3. Hence $x_2x_3, x_3x_4 \in O^2(C_G(x_6)) \cong \mathrm{Alt}(4)$. Since $x_2x_4 = x_2x_3x_3x_4$ is an element of 2-power order, we infer that x_2x_4 has order 2 or 1. Since $x_2 \notin L$, $x_2 \neq x_4$ and so x_2x_4 has order 2. By Lemma 2.6 $\langle x_1, x_2, x_3, x_4, x_5, x_6\rangle \cong \mathrm{Alt}(8)$, which establishes (3.16). ♠

The Thompson order formula [1, 45.6] for the case of two involution conjugacy classes asserts that

$$|G| = |C_G(b)|n_a + |C_G(a)|n_b$$

where n_a is the number of ordered pairs $(\alpha, \beta) \in a^G \times b^G$ with $a \in \langle \alpha\beta\rangle$ and n_b is the number of ordered pairs $(\alpha, \beta) \in a^G \times b^G$ with $b \in \langle \alpha\beta\rangle$. Since the numbers n_a and n_b are determined by the structure of $C_G(a)$ and $C_G(b)$ and the fusion in these centralizers is exactly as it is in Alt(8) by (3.16), we deduce that $|G| = |\mathrm{Alt}(8)|$. Therefore (3.16) implies that $G \cong \mathrm{Alt}(8)$ and this completes the proof of Theorem 1.2.

4. A CHARACTERIZATION OF Sym(8) AND (Alt(5) ≀ 2).2

In this section we move on to establish Theorem 1.3 which characterizes Sym(8) and (Alt(5) ≀ 2).2. Thus G is a group with $D = \langle y, z\rangle$ an elementary abelian Sylow 3-subgroup of order 9. This time our further assumptions are

(4.1)

(i) $|C_G(D)/D| = 2$, $N_G(D)/D \cong \mathrm{Dih}(8) \times 2$ and $N_G(D)/C_G(D) \cong \mathrm{Dih}(8)$;
(ii) D is normal in $C_G(y)$, $C_G(y)/D \cong 2^2$ and $N_G(\langle y \rangle)/D \cong 2^3$; and
(iii) $C_G(z) \cong 3 \times \mathrm{Sym}(5)$ and $N_G(\langle z \rangle) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(5)$.

Our first task is to give what is essentially a presentation of $N_G(D)$. Let z_1, z_2 be two conjugates of z in D with $D = \langle z_1, z_2 \rangle$ and put $Z_i = \langle z_i \rangle$ for $i = 1, 2$. Let $\langle t \rangle = Z(N_G(D))$. So t has order 2. Let s and t_1 be involutions in $N_G(D)$ chosen so as $\langle s, t_1 \rangle \cong \mathrm{Dih}(8)$, $z_1^s = z_2$ and $[z_2, t_1] = 1$. Furthermore, assume that t_1 is chosen so that $C_G(\langle t_1, z_1 \rangle) \cong \mathrm{Sym}(5)$. Define $t_2 = t_1^s$, and set $B = \langle t, t_1, s \rangle$. Then $B \in \mathrm{Syl}_2(N_G(D))$, $Z(B) = \langle t, t_1 t_2 \rangle$ and $B' = \langle t_1 t_2 \rangle$. Set

$$L = N_G(Z_1) \text{ and } M = N_G(D).$$

Define $y_1 = z_1 z_2$ and put $Y = \langle y \rangle$. Now set

$$A = C_G(\langle t_1, z_1 \rangle) = C_L(\langle t_1, z_1 \rangle) \cong \mathrm{Sym}(5).$$

Note that $t \in A$ and t centralizes Z_2, and therefore we record that

(4.2) t is a transposition in A. ♠

(4.3) Either $M = C_G(t)$ or $C_G(t) \cong 2 \times \mathrm{Sym}(6)$.

Put $K = C_G(t)$ and $\widetilde{K} = K/\langle t \rangle$. Then $\widetilde{K} \geq \widetilde{D}$. From (4.1)(i), we have $N_{\widetilde{K}}(\widetilde{D})/\widetilde{D} \cong N_G(D)/C_G(D) \cong \mathrm{Dih}(8)$. Also, from (4.1)(ii) we have $C_{\widetilde{K}}(\widetilde{y}) \leq N_{\widetilde{K}}(\widetilde{D})$. Now as t is a transposition in A, $\widetilde{D}$ is normal in $\widetilde{C_L(t)}$. Thus $C_{\widetilde{K}}(\widetilde{z}) \leq N_{\widetilde{K}}(\widetilde{D})$. Hence Theorem 2.3 completes the proof of (4.3). ♠

(4.4) The following hold.

(i) Either t is G-conjugate to t_1 or t is A-conjugate to tt_2.
(ii) t is not G-conjugate to $t_1 t_2$.

We have $t_2 \in A \cong \mathrm{Sym}(5)$. If t_2 is a transposition in A, then t is A-conjugate to t_2 by (4.2) and as t_1 and t_2 are conjugate by s we have that t is conjugate to t_1. If t_2 is not a transposition in A (so $t_2 \in \mathrm{A}' \cong \mathrm{Alt}(5)$), then tt_2 is a transposition in A. Thus in this case tt_2 is A-conjugate to t by (4.2) again. This proves (i).

Since $t_1 t_2 \in B' \leq Z(B)$, $t_1 t_2$ is contained in the derived subgroup of $C_G(t_1 t_2)$. As, by (4.3), t is not in the derived subgroup of its centralizer, t and $t_1 t_2$ are not G-conjugate. Thus (ii) also holds. ♠

Set $t_* = t_1t_2t$ and note that t_* inverts every non-trivial element of D.

(4.5) If $C_G(t) = M$, then $G \cong (\mathrm{Alt}(5) \wr 2).2$.

Suppose that $C_G(t) = M$. Then, as M is soluble and $C_G(t_1)$ contains A which is not soluble, t is not conjugate to t_1 and thus t is A-conjugate to tt_2 by (4.4). Therefore, as t is a transposition in A, tt_2 is a transposition in A and so centralizes an A-conjugate Z_3 of Z_2. Observe, as Z_2 is inverted by tt_2 and Z_3 is centralized by tt_2, $\langle Z_2, Z_3\rangle \cong \mathrm{Alt}(5)$. Further $\{t_2, t, tt_2\}$ is the set of all involutions in a Sylow 2-subgroup of $C_M(Z_1) \cong 3 \times \mathrm{Sym}(3) \times 2$. Since t_1 centralizes A and Z_3 is A-conjugate to Z_2, $\{tt_2, t_1, t_*\}$ are the involutions in a Sylow 2-subgroup of the centralizer in $N_G(Z_1Z_3)$ of Z_3. Thus, as sets, $\{tt_2, t_1, t_*\}$ and $\{t_2, t, tt_2\}$ are G-conjugate. As t is not conjugate to t_1 and is conjugate to tt_2, we infer that t is conjugate to t_*. Therefore, as $M = C_G(t)$, $O_3(C_G(t_*)) = Z_3Z_4$ for some Z_4 conjugate to Z_1. Since $s \in C_G(t^*)$, s normalizes Z_3Z_4. Note that, as $\langle Z_2, Z_3\rangle \cong \mathrm{Alt}(5)$, $\langle Z_1, Z_2, Z_3\rangle \cong 3 \times \mathrm{Alt}(5)$. If $Z_3^s = Z_3$, then s would normalize $\langle Z_1, Z_2, Z_3\rangle \cong 3 \times \mathrm{Alt}(5)$. But then s would normalize $Z_1 = Z(\langle Z_1, Z_2, Z_3\rangle)$ which it does not. Hence $Z_3^s = Z_4$. Notice that $C_G(Z_3) \geq \langle Z_1, Z_4, t_1, t_*\rangle$. Hence

$$C_G(Z_4) = C_G(Z_3)^s \geq \langle Z_1, Z_4, t_1, t_*\rangle^s = \langle Z_2, Z_3, t_2, t_*\rangle.$$

It follows that $\langle Z_2, Z_3\rangle$ and $\langle Z_1, Z_4\rangle$ commute. Since

$$\langle Z_1, Z_4\rangle^s = \langle Z_2, Z_3\rangle,$$

we infer that

$$X = \langle Z_1, Z_4, Z_2, Z_3\rangle \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5).$$

Because t_* centralizes Z_3Z_4, and $C_X(Z_3Z_4) = Z_3Z_4$, $t_* \notin X$, but t_* does normalize X. Thus we have $N_G(X) \geq \langle X, s, t_*\rangle$, $t_* \notin X$, with t_* inverting all the non-trivial elements of D and s interchanging $\langle Z_1, Z_4\rangle$ and $\langle Z_2, Z_3\rangle$. Combining this with (4.1)(ii) and the fact that $\mathrm{Aut}(\mathrm{Alt}(5) \times \mathrm{Alt}(5)) \cong \mathrm{Sym}(5) \wr \mathrm{Sym}(2)$ yields that $H = N_G(X) \cong (\mathrm{Alt}(5) \wr 2)2$. Let $S \in \mathrm{Syl}_2(H)$. Since H is uniquely determined up to isomorphism, we may determine the conjugacy classes of involutions and how they correspond to the involutions in B. However, we note that there remains a small ambiguity with the identification of s and st (see Table 1). In Table 1 we have regarded H as a subgroup of index 2 in $\mathrm{Sym}(5) \wr 2$ viewed as a subgroup of $\mathrm{Sym}(10)$ and stabilizing the partition $\{\{1, \ldots, 5\}, \{6, \ldots, 10\}\}$ of $\{1, \ldots, 10\}$.

Since $C_G(t) = C_H(t) = M$, t is not G-conjugate to any of the other H-classes of involution. Furthermore, we calculate that in S, there are

Inv. i	Representative	$C_H(i)$ Structure
t_1	$(1,2)(3,4)$	$C_H(i) \sim (2^2 \times \mathrm{Alt}(5)).2$
t_1t_2	$(1,2)(3,4)(6,7)(8,9)$	$C_H(i) \cong 2^4.2^2$
t	$(1,2)(6,7)$	$C_G(i) = C_H(i) \cong 2 \times 3^2 : \mathrm{Dih}(8)$
s, st	$(1,6)(2,7)(3,8)(4,9)(5,10)$	$C_H(i) \cong 2 \times \mathrm{Sym}(5)$
s, st	$(1,7)(2,6)(3,8)(4,9)(5,10)$	$C_H(i) \cong 2 \times \mathrm{Sym}(5)$

TABLE 1

exactly 4 conjugates of t and they are all conjugate in S. In particular, $N_G(S) = C_{N_G(S)}(t)S \leq \langle C_G(t), S\rangle \leq H$. Hence $N_G(S) = S$ and so $S \in \mathrm{Syl}_2(G)$. Now we apply the Thompson Transfer Lemma to t with respect to the index 2 subgroup $(S \cap X)\langle s\rangle$ of S, to deduce that G has a subgroup G_2 with $|G : G_2| = 2$. Hence exactly one of s or $st \in G_2$ we denote which ever element it is by s_*. We know that s_* centralizes y, so as s_* is not conjugate to t, $Y \in \mathrm{Syl}_3(C_{G_2}(s_*))$ and, as $t \notin G_2$, we have that Y is self-centralizing in $C_{G_2}(s_*)/\langle s_*\rangle$. Because $C_X(s_*)$ contains a subgroup isomorphic to $\mathrm{Alt}(5)$, we infer from Theorem 2.1 that $C_X(s_*) \cong C_{G_2}(s_*)/O_{2'}(C_{G_2}(s_*))$. In particular, s_* is not G-conjugate to either t_1 or t_1t_2. Thompson's Transfer Lemma now delivers a subgroup G_4 of index 2 in G_2. Furthermore, G_4 has elementary abelian Sylow 2-subgroups of order 16. Therefore $N_{G_4}(S \cap X)$ controls the fusion of involutions in $S \cap X$ with respect to G_4. Since $|N_{G_4}(S \cap X)/C_{G_4}(S \cap X)|$ is divisible by $|N_X(S \cap X)/(S \cap X)| = 3^2$ and this number is odd, the subgroup structure of $\mathrm{GL}_4(2) \cong \mathrm{Alt}(8)$, shows that $|N_{G_4}(S \cap X)/C_{G_4}(S \cap X)| = 3^2$. It follows that there are three $N_{G_4}(S \cap X)$-conjugacy classes in $S \cap X$. Thus G_4 has three conjugacy classes of involutions.

Finally, we note that it easy to demonstrate that if $G > H$, then G_4 is a minimal normal subgroup of G which is a simple group with elementary abelian Sylow 2-subgroups and three conjugacy classes of involutions. This violates the classification of simple groups with abelian Sylow 2-subgroups given in [22] (see also [3]) as all such simple groups have one conjugacy class of involutions. Therefore we conclude that $G = H \cong (\mathrm{Alt}(5) \wr 2).2$. ♠

In view of (4.3) and (4.5), from now on we assume that $C_G(t) \cong 2 \times \mathrm{Sym}(6)$. The first configuration we study finally leads to a contradiction.

(4.6) t is G-conjugate to t_1.

Set $\mathcal{T} = t^G$ and, aiming for a contradiction, assume that $t_1 \notin \mathcal{T}$. Then $tt_2 \in \mathcal{T}$ by (4.4) (i). Furthermore, as t is a transposition in A, (4.4) (i) implies that tt_2 is also a transposition in A. Set $T = \langle t_1, t_2, t\rangle$. Then T is elementary abelian of order 8 and s normalizes T. Let $\mathcal{I}$ be the set of all involutions in G and set $\mathcal{N} = \mathcal{I}\backslash\mathcal{T}$. Since $(tt_1)^s = tt_1^s = tt_2$, we have

$$\mathcal{T} \supseteq \{t, tt_1, tt_2\}.$$

(4.6.1) Let $J = C_G(t) \cong 2 \times \mathrm{Sym}(6)$.

(i) $t_1t_2 \in J' \cong \mathrm{Alt}(6)$;
(ii) $S = C_J(t_1t_2) \cong 2 \times 2 \times \mathrm{Dih}(8) \in \mathrm{Syl}_2(J)$;
(iii) t and t_1t_2 are not G-conjugate; and
(iv) $\langle t_1\rangle J' \cong \mathrm{Sym}(6) \cong \langle tt_1\rangle J'$.

Part (i) and (ii) follows from $\langle t_1t_2\rangle = B' \le J'$. Part (iii) follows similarly from the fact that $t \notin C_G(t)'$, whereas $t_1t_2 \le B' \le C_G(t_1t_2)'$. Part (iv) now follows from (iii) as tt_1 and tt_2 are both conjugate to t.

We restate (4.6.1) (iii) by noting that

$$\mathcal{N} \supseteq \{t_1, t_2, t_1t_2\}.$$

Let $F = N_L(T)$. Then $F \cong \mathrm{Dih}(8) \times 2$. Select $f \in (F \cap A) \setminus T$. Observe that f centralizes t_1 and t_2, $(tt_2)^f = t$ and $(tt_1)^f = t_*$. Hence we have shown

(4.6.2) $\mathcal{T} \supseteq \mathcal{T} \cap T = \{t, tt_1, tt_2, t_*\}$.

Let $E = TZ(S)$. If $E = Z(S)$, then, as $B \le S$, $T \le Z(B)$ which is false. Therefore E is elementary abelian of order 2^4. It follows that $P = N_{C_G(t)}(E) \cong 2 \times 2 \times \mathrm{Sym}(4)$ and P has three orbits of length one and four orbits of length 3 on the involutions in E. Since t_1 and t_2 are conjugate by s and since $s \in P$, we see that there exists $t_3 \in E$ with $t_1^P = \{t_1, t_2, t_3\}$. In particular, $t_3 \in \mathcal{N}$. Since t and t_* are in $\mathcal{T}$ it follows that

(4.6.3) $\mathcal{T} \supseteq \{t, tt_1, tt_2, tt_3, tt_1t_2 (= t_*), tt_1t_3, tt_2t_3\}$.

Now $t_1t_2 \in \mathcal{N}$ and so $t_2t_3, t_1t_3 \in \mathcal{N}$ as well.

(4.6.4) $\mathcal{N} \supseteq \{t_1, t_2, t_3, t_1t_2, t_2t_3, t_1t_3\}$.

Let $P_1 \ge S$ be a maximal subgroup of $C_G(t)$ with $P_1 \ne P$. Then the structure of $\mathrm{Sym}(6)$ shows that $P_1 \cong \mathrm{Sym}(4) \times 2 \times 2$.

(4.6.5) Suppose that $R \in \{P, P_1\}$. If $|Z(R) \cap \mathcal{T}| \ge 2$, then the Sylow 3-subgroups of R are conjugate to Z.

Obviously $\mathcal{T} \cap R \supset \{t\}$. Suppose that $g \in Z(R) \cap \mathcal{T}$ with $t \neq g$. Then $R \leq C_G(g) \cong \mathrm{Sym}(6) \times 2$. Since R is a maximal subgroup of $C_G(t)$ and since $t \neq g$, we get that $R = C_G(t) \cap C_G(g)$. Let $U \in \mathrm{Syl}_3(R)$. Then $C_G(U) \geq \langle C_{C_G(t)}(U), C_{C_G(g)}(U)\rangle$. Now $|C_{C_G(t)}(U)| = 2^2.3^2 = |C_{C_G(g)}(U)|$. Since these two groups are not contained in R, we infer that $C_G(U) > C_{C_G(t)}(U)$ and so $|C_G(U)| > 2^2.3^2$. Hence (4.1) (ii) and (iii) imply that U is conjugate to Z. This proves (4.6.5).

Let $x = t_1t_2t_3$ and note that $x \in Z(P)$.

(4.6.6) $t_3 \in Z(P_1)$ and the Sylow 3-subgroups of P_1 are conjugate to Z.

Since $\{t_1, t_2, t_3\}$ is a P-orbit and $t_1^s = t_2$, we get have that t_3 is centralized by $\langle E, s\rangle = S$. Thus $t_3 \in Z(S)$. We calculate that the three subgroups of $Z(S)$ which contain t are $\langle t, t_1t_2\rangle$, $\langle t, x\rangle$ and $\langle t, t_3\rangle$. Since $t_1t_2 \in J$, it follows from the structure of $\mathrm{Sym}(6)$, that $P_1 = C_G(\langle t, t_3\rangle)$. Since $t_3 \in \mathcal{T}$, the result follows from (4.6.5).

Since the Sylow 3-subgroups of P_1 and of P are not conjugate in G, we have that $Z(P) \cap \mathcal{T} = \{t\}$. Hence

(4.6.7) $x, tx \in \mathcal{N}$.

Combining (4.6.3), (4.6.4) and (4.6.7), we have

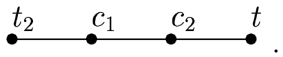
$$E \cap \mathcal{T} = \{t, tt_1, tt_2, tt_3, tt_1t_2(= t_*), tt_1t_3, tt_2t_3\}.$$

Finally we recall the element $f \in (A \cap N_L(T)) \setminus T$. As f normalizes T, f normalizes $C_G(T) = E$. Hence f induces an action on E and permutes the seven elements of $E \cap \mathcal{T}$. Now $t_*^f = tt_1$ and $tt_2^f = t$. Since the orbits of P on $E \cap \mathcal{T}$ are $\{t\}$, $\{t_*, tt_1t_3, tt_2t_3\}$ and $\{tt_1, tt_2, tt_3\}$, we infer that $H = N_G(E)$ acts transitively on $E \cap \mathcal{T}$. But then $|H : P| = 7$ and we get $|H/E| = 42$. Since $H/E \geq P/E \cong \mathrm{Sym}(3)$, we now have a contradiction to the structure of $\mathrm{GL}_4(2) \cong \mathrm{Alt}(8)$, so finishing the proof of (4.6). ♠

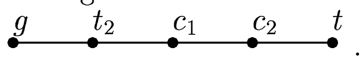
We now set $X = \langle C_G(t), M, L\rangle$.

(4.7) $X \cong \mathrm{Sym}(8)$ and $C_G(x) \leq X$ for all elements $x \in X$ of order 3.

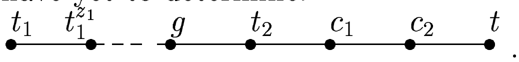
We begin by noting that $s \in C_G(t)$ and $M = (M \cap L)\langle s\rangle$, so $X = \langle C_G(t), L\rangle$. Since t and t_1 are conjugate by (4.6), we have $C_G(t_1) \cong 2 \times \mathrm{Sym}(6)$ and of course $C_G(t_1)$ contains $A \cong \mathrm{Sym}(5)$ with $t \in A$ a transposition in A. Let $J = \langle C_G(t_1)'\langle t\rangle$. Then $A \leq J$ and by considering the six-point action on the cosets of A, we identify t as a transposition in J. Recall that t_2 is conjugate to t and so is also a transposition in J. Since t and t_2 commute, there exist transpositions $c_1, c_2 \in A$ satisfying the Coxeter relations given by the Coxeter diagram



We have that $C_J(\langle c_1, c_2, t\rangle)$ is cyclic of order 2 containing the transposition g of J. Together with t_2, c_1, c_2 and t, g satisfies the Coxeter relations defined by the diagram



Therefore $\langle g, t_2, c_1, c_2, t\rangle \cong \mathrm{Sym}(6)$ and it commutes with t_1. Now we add elements t_1 and $t_1^{z_1}$. We know $\langle t_1, t_1^{z_1}\rangle = \langle t_1, z_1\rangle \cong \mathrm{Sym}(3)$ and that this subgroup commutes with $A = \langle t_2, c_1, c_2, t\rangle$. So we have the following Coxeter diagram where the dotted line denotes a product order which we have yet to determine.



Now $\langle tc_2\rangle$ is a cyclic group of order 3 which is contained in A. Since all subgroups of A of order 3 inverted by transpositions in A are A-conjugate to Z_2, we conclude that $\langle tc_2\rangle$ is conjugate to Z_1. Therefore $C_G(tc_2) \cong 3 \times \mathrm{Sym}(5)$. Since $\langle t_1, t_1^{z_1}, g, t_2\rangle \leq C_G(tc_2)$, we have that $\langle t_1, t_1^{z_1}, g, t_2\rangle$ is a subgroup of $A_1 = O^3(C_G(tc_2)) \cong \mathrm{Sym}(5)$. Furthermore, we have that

$$\langle t_1, t_1^{z_1}, t_2\rangle \leq N_{A_1}(Z_1) \cong 2 \times \mathrm{Sym}(3).$$

So $\langle t_1, t_1^{z_1}, t_2\rangle = N_{A_1}(Z_1)$. Since $J = \langle g, A\rangle$ does not normalize Z_1, we have $g \notin N_{A_1}(Z_1)$ and so $g \notin \langle t_1, t_1^{z_1}, t_2\rangle$. Hence, as $N_{A_1}(Z_1)$ is a maximal subgroup of A_1, $A_1 = \langle t_1, t_1^{z_1}, g, t_2\rangle \cong \mathrm{Sym}(5)$. We have that t_2 centralizes Z_1, therefore t_2 is a transposition in A_1 and since t_1 and $\langle t_2, g\rangle$ commute, t_1 is also a transposition in A_1. Hence g and $t_1^{z_1}$ are transpositions in A_1. Thus either $t_1^{z_1}$ and g commute or have product of order 3. If they commute, then $\langle t_1, t_2, g, t_1^{z_1}\rangle$ would have a subgroup of order 9 in $\mathrm{Sym}(5)$, which is absurd. Therefore $t_1^{z_1}g$ has order 3. It follows that $X = \langle t_1, t_1^{z_1}, g, t_2, c_1, c_2, t\rangle \cong \mathrm{Sym}(8)$. Finally, by considering the centralizers of elements of order 3 in $\mathrm{Sym}(8)$ and appealing to (4.1) we see that $C_G(x) \leq X$ for all $x \in X$ with x of order 3. ♠

Finally we prove that $X = G$. Let $S \in \mathrm{Syl}_2(X)$ be such that $t \in S$. Also let α, β, γ, t be the representatives for the four X-conjugacy classes of involutions in X where we assume that α has cycle type 2^4, β cycle type $1^2.2^3$ and γ has cycle type $1^4.2^2$. Of course t is a transposition. We have that $C_X(\alpha) \cong 2\wr\mathrm{Sym}(4)$, $C_X(\beta) \cong 2\times 2\wr\mathrm{Sym}(3)$ and $C_X(\gamma) \cong \mathrm{Dih}(8)\times\mathrm{Sym}(4)$. Plainly α and γ cannot be G-conjugate to t as their centralizers do not embed in $C_G(t) \cong 2 \times \mathrm{Sym}(6)$. Thus if t is G-conjugate to any of these involutions it must be conjugate to β. Suppose that t is G-conjugate to β. Then $C_G(\beta) \cong 2 \times \mathrm{Sym}(6)$

and a Sylow 3-subgroup $\langle d\rangle$ of $C_X(\beta)$ embeds into a Sylow 3-subgroup of D_0 of $C_G(\beta)$. Thus $D_0 \le C_G(d)$ and, since $|D_0| = 9$, $D_0 \not\le X$, which contradicts (4.7). Thus t and β are not G-conjugate. An easy calculation in a Sylow 2-subgroup of Sym(8) shows that S contains exactly four transpositions and that they are conjugate in S. The above calculation then shows that $t^G \cap S = t^S$. Therefore $N_G(S)$ acts on t^S and hence $N_G(S) = S(C_G(t) \cap N_G(S)) \le \langle S, C_G(t)\rangle \le X$. Thus $N_G(S) = N_X(S) = S$ and so $S \in \mathrm{Syl}_2(G)$. Because t is not G-conjugate to either α or γ, t is not G-conjugate to any element of $S \cap X'$. Therefore G has a normal subgroup H of index 2 by Thompson's Transfer Lemma. Because this subgroup contains $X' \cong \mathrm{Alt}(8)$ it satisfies the hypotheses of Theorem 1.2. Hence $H \cong \mathrm{Alt}(8)$ by Theorem 1.2. Consequently $G = X \cong \mathrm{Sym}(8)$, so proving Theorem 1.3. □

5. The McLaughlin Group

In this, our final section, we present the proof of Theorem 1.1. Much of our deliberations are concerned with getting into a position to use Theorems 1.2 and 1.3. We begin by recalling the hypotheses of Theorem 1.1.

Hypothesis 5.1. *G is a group, $S \in \mathrm{Syl}_3(G)$, $Z = Z(S)$ and J is an elementary abelian subgroup of S of order 3^4 such that the following hold*

(i) $O^{3'}(N_G(J)) \sim 3^4.\mathrm{Alt}(6)$;
(ii) $O^{3'}(N_G(Z)) \sim 3_+^{1+4}.2{\cdot}\mathrm{Alt}(5)$; *and*
(iii) *for all non-trivial elements x of J,* $C_G(O_3(C_G(x))) \le O_3(C_G(x))$.

The proof of Theorem 1.1 develops through a series of lemmas. Set $Q = O_3(N_G(Z))(= O_3(G_G(Z)))$. So Q is extraspecial of order 3^5. Further set

$$L = N_G(Z),\ L_* = O^{3'}(L),\ M = N_G(J) \text{ and } M_* = O^{3'}(M).$$

Thus we have

$$L_*/Q \cong 2{\cdot}\mathrm{Alt}(5) \cong \mathrm{SL}_2(5) \text{ and } M_*/J \cong \mathrm{Alt}(6) \cong \mathrm{PSL}_2(9).$$

Lemma 5.2. *The following hold:*

(i) $C_G(Q) = Z(Q)$;
(ii) $Z = Z(Q) \cong 3$; *and*
(iii) $C_G(J) = J$.

Proof. By Hypothesis 5.1 (ii), as $|S : Q| = 3$ and $Q \cong 3_+^{1+4}$, $|J \cap Q| = 3^3$ and $J \cap Q$ is a maximal elementary abelian subgroup of Q. Hence

$Z(Q) \leq J$. But then $C_G(O_3(C_G(Z(Q)))) \leq Q$ by Hypothesis 5.1 (iii). Hence (i) holds. Since $Z \leq C_G(Q)$, (ii) follows from (i).

Let $C = C_G(J)$. Then C is normalized by M. Since $C \geq J$, we have either $C \cap M_* = J$ or $C \cap M_* = M_*$ from Hypothesis 5.1 (i). However, $Z \leq J$ and so $C \leq N_G(Z)$. Thus Hypothesis 5.1 (ii) shows that it is impossible for $C \cap M_* = M_*$. So $C \cap M_* = J$ and hence $[M_*, C] \leq C \cap M_* = J$. If $x \in C$ has $3'$-order, then we see that $[S, x, x] \leq [J, x] = 1$. Thus $x \in C_G(S) \leq C_G(Q) = Z(Q)$, using (i), and so $x = 1$. Hence C is a 3-group. Since $N_G(J)/M_*$ is a $3'$-group, we conclude that $C = J$. □

Lemma 5.3. *Suppose that T is a 2-group of order 32 which contains a subgroup $R \cong \mathrm{Q}(16)$ and a normal cyclic subgroup F of order 4 with $C_R(F) \cong \mathrm{Q}(8)$. Then T is isomorphic to the group*

$$\langle a, b, c \mid a^4 = b^4 = c^8 = c^b c = c^a c^3 = b^2 c^4 = [a, c^2] = [a, b] = a^2 c^4 = 1 \rangle.$$

In particular, we note that $\langle ab, c\rangle \cong \mathrm{SDih}(16)$ and $\langle a, c^2, bc\rangle \cong 2 \times \mathrm{Q}(8)$.

Proof. Let $a \in F$ be of order 4 and $R = \langle b, c\rangle$ with $\langle c\rangle$ of order 8. Then, as R is a quaternion group, b has order 4, $c^b = c^{-1}$ and $b^2 = c^4$. Set $U = C_R(a) \cong \mathrm{Q}(8)$. Then we may assume notation is chosen so that $U = \langle b, c^2\rangle$. Since $a^2 \in F \cap U$, we deduce that $a^2 \in Z(R)$. Thus $a^2c^4 = 1$. Finally we note that a normalizes $\langle c\rangle$, centralizes c^2 and $c \notin C_R(a)$, so it follows that $c^a = c^5$ and this completes the presentation of T. □

Lemma 5.4. *As an M_*/J-module, J can be identified with the irreducible 4-dimensional section of the natural 6-point $\mathrm{GF}(3)$-permutation module for $\mathrm{Alt}(6)$.*

Proof. Since $C_G(J) = J$ by Lemma 5.2 (iii), J is a faithful M_*/J-module. Because 5 does not divide the order of $\mathrm{GL}_3(3)$, we infer that J is irreducible as a $\mathrm{GF}(3)\mathrm{Alt}(6)$-module. Using the fact that $\mathrm{Alt}(6)$ is isomorphic to $\mathrm{PSL}_2(9)$, we may apply the weight theory for $\mathrm{SL}_2(9)$ to determine the irreducible $\mathrm{Alt}(6)$-modules. We know that $\mathrm{SL}_2(9)$ has three basic modules in characteristic 3– they have dimensions 1, 2 and 3 and are all definable over $\mathrm{GF}(9)$. Steinberg's tensor product theorem then gives us all the irreducible modules for $\mathrm{SL}_2(9)$ as tensor products of basic modules and their algebraic conjugates by the automorphism of $\mathrm{GF}(9)$ of order 2. The only irreducible modules that, when defined over $\mathrm{GF}(3)$, have dimension 4 are the basic module of dimension 2 and its conjugate and the tensor product of those two modules. The latter one is then the unique 4-dimensional irreducible representation of $\mathrm{PSL}_2(9)$ of dimension 4 over $\mathrm{GF}(3)$. Since the module defined in the lemma is 4-dimensional, the result holds. □

Not surprisingly we shall need to know more about the module in Lemma 5.4.

Lemma 5.5. *Suppose that $X = \mathrm{Alt}(6)$ and let V be the $\mathrm{GF}(3)$-permutation module for X with standard basis $\{v_1, \dots, v_6\}$. Let $U_0 = \langle \sum_{i=1}^{6} v_i \rangle$ and $U = \langle v_i + 2v_j \mid 1 \le i, j \le 6 \rangle$. Set $W = U/U_0$. Then W is 4-dimensional and the following hold.*

(i) *X has three orbits on the one-dimensional subspaces of W, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, with representatives $\langle v_1 + v_2 + v_3 + U_0 \rangle$, $\langle v_1 + 2v_2 + v_3 + 2v_4 + U_0 \rangle$ and $\langle v_1 + 2v_2 + U_0 \rangle$ respectively. Furthermore, $|\mathcal{O}_1| = 10$ and $|\mathcal{O}_2| = |\mathcal{O}_3| = 15$. The stabilizers of a member of $\mathcal{O}_2$ and of a member of $\mathcal{O}_3$ are not conjugate in X.*

(ii) *If t is an involution in X, then $\dim C_W(t) = 2$ and $C_W(t)$ contains two subspaces from $\mathcal{O}_1$ and one each from $\mathcal{O}_2$ and $\mathcal{O}_3$. Furthermore, $C_X(t) \cong \mathrm{Dih}(8)$ interchanges the two members of $\mathcal{O}_1$ in $C_W(t)$ and $|C_X(t)/C_{C_X(t)}(C_W(t))| = 4$.*

(iii) *If $g \in X$ has order 4, then $C_W(g) = 0$.*

(iv) *If $D \in \mathrm{Syl}_3(X)$, then $\dim C_W(D) = \dim W/[W, D] = 1$ and $C_W(D) \in \mathcal{O}_1$;*

(v) *If $d \in X$ has order 3, then $\dim C_W(d) = 2$; and*

(vi) *If $D \in \mathrm{Syl}_3(X)$ and $t \in N_X(D)$ is an involution, then t centralizes $C_W(D)$ and $W/[W, D]$.*

Proof. This is an elementary calculation. □

Suppose that $K \le S$ is an abelian subgroup of order at least 3^4 and assume that $K \ne J$. Then either $JK = S$ and $|J \cap K| \ge 3^2$ or $|JK/J| = 3$ and $|J \cap K| \ge 3^3$. In the former case, $S = JK$ centralizes $J \cap K$ which is impossible as $Z = Z(S)$ has order 3 by Lemma 5.2 (ii). Thus $|JK/J| = 3$ and $C_J(K) = J \cap K$ has order 3^3. But this contradicts Lemma 5.5 (v). Thus $J = K$. In particular we have $J = J(S)$, the Thompson subgroup of S.

Lemma 5.6. *$N_G(S) = L \cap M$.*

Proof. Notice that $S = JQ$ and J and Q are characteristic in S. It follows that $N_G(S) \le L \cap M$. On the other hand $M \cap L$ normalizes $JQ = S$ and so $L \cap M \le N_G(S)$. □

We shall need some familiar facts, originally established by Schur, about the double covers of the symmetric group.

Lemma 5.7. *Suppose that $n \ge 4$ and $n \ne 6$. Then there are exactly two isomorphism types of group X such that $X/Z(X) \cong \mathrm{Sym}(n)$, $|Z(X)| = 2$ and $Z(X') = Z(X)$. These groups are denoted by $2^{-}\mathrm{Sym}(n)$ and*

$2^+\mathrm{Sym}(n)$ *and are distinguished by the fact that in the first case the preimage of a transposition has order* 4 *and in the second has order* 2. *Furthermore, in either case, if* $a, b \in X$ *project to disjoint transpositions in* $X/Z(X)$, *then* $[a, b] \neq 1$.

Proof. The first part of this result can be read from Schur's paper [21, page 166] and second part comes from [21, page 164]. Also see [1]. □

Lemma 5.8. *Suppose that* $X \cong 2^{\pm}\mathrm{Sym}(5)$ *and* $S \in \mathrm{Syl}_2(X)$. *If* $X \cong 2^+\mathrm{Sym}(5)$, *then* $S \cong \mathrm{SDih}(16)$ *whereas, if* $X \cong 2^-\mathrm{Sym}(5)$ *then* $S \cong \mathrm{Q}(16)$.

Proof. This uses the information given in [5, page 236]. □

Lemma 5.9. *Suppose that* $X \cong 2^-\mathrm{Sym}(8)$ *and* $T \in \mathrm{Syl}_2(X)$. *Then*

(i) X *has exactly* 3 *conjugacy classes of involutions. If* a, b, c *are representatives of these conjugacy classes then we may suppose that* $a \in Z(X)$, b *projects to an element of cycle type* 2^4 *and* c *projects to an element of cycle type* $1^2.2^3$.

(ii) $Z(T) = Z(X)$.

Proof. Let t_1, t_2, t_3 and t_4 be elements of X which project to pairwise disjoint transpositions. Then for each i, t_i has order 4 and by Lemma 5.7 $t_it_j = t_jt_iz$ where $z \in Z(X)^{\#}$. Using these relations it follows that elements which project to element of cycle type $1^2.2^3$ and 2^4 have order 2 and those which project to elements of cycle type $1^6.2$ and $1^4.2^2$ have order 4. Since $[t_1t_2t_3, t_4] = z$, $t_1t_2t_3$ is conjugate to t_1, t_2t_3z.

Now let $F_1 \leq X$ and $F_2 \leq X$ be such that $F_1/Z(X)$ and $F_2/Z(X)$ are disjoint four groups acting regularly on 4 points. Then $F_i \setminus X$ contains only elements of order 4. Hence $F_1 \cong F_2 \cong \mathrm{Q}(8)$. Furthermore F_1 and F_2 commute. If follows that F_1F_2 is extraspecial and so elements which project to elements of cycle type 2^4 are all conjugate in X (and indeed in X'). □

Lemma 5.10. *Suppose that* $X \cong 2{}^{\cdot}\mathrm{Alt}(8)$. *Then* X *contains exactly two conjugacy classes of involutions; the central one and ones which project to elements of cycle type* 2^4.

Proof. Since $X \cong Y'$ where $Y \cong 2^-\mathrm{Sym}(8)$, this follows from Lemma 5.9. □

In the next lemma we begin to close the net on our target groups by determining the structures of L and M. Set $L_0 = L_*N_{M_*}(S)$ and $M_0 = M_*N_{L_*}(S)$. Since $N_{M_*}(S) \leq L$ and $N_{L_*}(S) \leq M$, both L_0 and M_0 are subgroups of G.

Lemma 5.11. *The following hold.*

(i) $M_0/J \cong \mathrm{Mat}(10)$, $L_0/Q \cong 2^{-}\mathrm{Sym}(5)$ *and* $N_{L_*}(S)N_{M_*}(S)$ *has Sylow* 2*-subgroups which are isomorphic to* Q(8).

(ii) $|L : L_0| = |M : M_0| \leq 2$.

(iii) *If* $|L : L_0| = 2$, *then* $M/J \cong 2 \times \mathrm{Mat}(10)$ *and* $L/Q \sim (4 \circ \mathrm{SL}_2(5)).2$ *with* L *having Sylow* 2*-subgroups isomorphic to the group described in Lemma 5.3. Furthermore,* $N_G(S)$ *has Sylow* 2*-subgroups which are isomorphic to* $2 \times \mathrm{Q}(8)$.

Proof. Since L_* has no subgroup of index 2, L_* centralizes Z. Therefore $N_{L_*}(S)$ centralizes Z and so, as $N_{L_*}(S)$ has Sylow 2-subgroups which are cyclic of order 4 and $Z \leq J$, Lemma 5.5 (iii) implies that $N_{L_*}(S) \not\leq M_*$. Therefore, $|N_{L_*}(S) : N_{L_*}(S) \cap M_*| \geq 2$. Let t_1 be an involution in $N_{M^*}(S)$. Then t_1 inverts S/J and, by Lemma 5.5 (ii) and (vi) centralizes $J/[J,S]$ and Z and inverts $[J,S]/Z$. Now $[J,S] = J \cap Q$ and $Q/(J \cap Q) \cong QJ/J$ as a $\langle t_1 \rangle$ operator group. It follows that t_1 inverts Q/Z. On the other hand, $C_G(Q) = Z$ and so L_*/Q operates faithfully on Q/Z. Let t_2 be an involution in $N_{L_*}(S)$. Then $t_2Q \in Z(L_*/Q)$ and consequently t_2 also inverts Q/Z and centralizes Z. It follows that t_1t_2 centralizes $Q/Z(Q)$. But then t_1t_2 is a 3-element. Hence $t_1S = t_2S$ and $|N_{M_*}(S) \cap N_{L_*}(S)| = 3^6.2$.

Because $N_{L_*}(S)$ normalizes M_*, we have $N_{L_*}(S)N_{M_*}(S)$ is a subgroup of $N_G(S)$. Furthermore, $N_{L_*}(S)N_{M_*}(S)$ acts on $M_*/J \cong \mathrm{Alt}(6)$. Since $N_{L_*}(S)N_{M_*}(S)/S$ is generated by $N_{L_*}(S)/S$ and $N_{M_*}(S)/S$ both of which are cyclic groups of order 4 and $|N_{L_*}(S)/S \cap N_{M_*}(S)/S| = 2$, we infer that either $N_{L_*}(S)N_{M_*}(S)/S \cong \mathrm{Q}(8)$ or 4×2. As the normalizer of a Sylow 3-subgroup of $\mathrm{Aut}(\mathrm{Alt}(6))$ is isomorphic to $3^2 : \mathrm{SDih}(16)$ and $|M_0/M_*| = 2$, we deduce that

(a) if $N_{L_*}(S)N_{M_*}(S)/S \cong 4 \times 2$, then $M_0/J \cong 2 \times \mathrm{Alt}(6)$; and

(b) if $N_{L_*}(S)N_{M_*}(S)/S \cong \mathrm{Q}(8)$, then $M_0/J \cong \mathrm{Mat}(10)$.

Aiming for a contradiction, suppose that (a) occurs and let $X \in \mathrm{Syl}_2(N_{M_0}(S))$. Then $X \cong 2 \times 4$. Now $N_{L_*}(S)N_{M_*}(S) \leq N_G(Z)$ and so L_* has index 2 in $L_0 = XL_*$. If $|Z(L_0/Q)| = 2$, then from Lemma 5.8 we have that L_0/Q has either quaternion or semidihedral Sylow 2-subgroups and consequently has no subgroup isomorphic to $X \cong 2 \times 4$. Therefore $|Z(L_0/Q)| = 4$. Since L_0/Q acts faithfully on Q/Z and since the minimum dimension of a faithful $\mathrm{GF}(3)\mathrm{SL}_2(5)$-module is 4, we get that $Z(L_0/Q)$ is cyclic by Schur's Lemma. Thus $L_0/Q \cong 4 \circ \mathrm{SL}_2(5)$. However, as X has exactly two cyclic subgroups of order 4, in this case we see that $X \cap N_{M_*}(S)$ centralizes S/Q which as an X-space is isomorphic to $J/(J \cap Q) = J/[J,S]$ and this contradicts Lemma 5.5 (iii). Therefore (b) holds. In particular, if $X \in \mathrm{Syl}_2(N_{M_0}(S))$, then $X \cong \mathrm{Q}(8)$. It follows that the Sylow 2-subgroups of L_0 contains two quaternion subgroups of order 8 (X and a Sylow 2-subgroup of L_*). Since $X \in \mathrm{Syl}_2(N_{L_0}(S))$, $Z(L_0/Q) = Z(L_*/Q)$ and we have that $L_0/Q \cong 2^-\mathrm{Sym}(5)$ or $2^+\mathrm{Sym}(5)$ by Lemma 5.7. Since these groups have Sylow 2-subgroups which are respectively semidihedral and quaternion by Lemma 5.8, and semidihedral groups have a unique maximal subgroup which is quaternion, we deduce that $L_0/Q \cong 2^-\mathrm{Sym}(5)$. Thus (i) holds.

Suppose that $L > L_0$. Then, as $\mathrm{Aut}(\mathrm{Alt}(5)) \cong \mathrm{Sym}(5)$, we have that $C_{L/Q}(L_*/Q) > Z(L_*/Q)$. Since L_*/Q acts irreducibly on Q/Z, we use Schur's Lemma to deduce that $C_{L/Q}(L_*/Q)$ is cyclic of order 4. Thus $C_{L/Q}(L_*/Q)L_*/Q \cong 4 \circ \mathrm{SL}_2(5)$ (and has index 2 in L). Suppose that $C_{L/Q}(L_*/Q) = Z(L/Q)$. Then $Z(L/Q)$ is cyclic of order 4 and Schur's Lemma applied to the action of L/Q on Q/Z implies that there is a monomorphism from L/Q to $\mathrm{GL}_2(9)$. However, if F is a Sylow 5-subgroup of L/Q, then, as $L_0/Q \cong 2^-\mathrm{Sym}(5)$, we have that $N_{L/Q}(F)/C_{L/Q}(F)$ is cyclic of order 4 and this means that any faithful characteristic 3 representation of L/Q must have dimension at least 4. Thus we cannot have $C_{L/Q}(L_*/Q) = Z(L/Q)$. Therefore, $C_{L/Q}(L_*/Q)$ is not central in L/Q. It follows that $C_{L/Q}(L_*/Q)$ is not contained in the centre of any Sylow 2-subgroup of L/Q which contains it. Let $R_1 \in \mathrm{Syl}_2(L_0/Q)$. Then $R_1 \cong \mathrm{Q}(16)$ and since $C_{L/Q}(L_*/Q)$ commutes with $R_1 \cap L_*/Q \cong \mathrm{Q}(8)$, Lemma 5.3 implies that the Sylow 2-subgroups of L are isomorphic to the group given in Lemma 5.3. Let $C_{L/Q}(L_*/Q) = \langle a \rangle$ and $R_1 = \langle b, c \rangle$ be as in Lemma 5.3. By Lemma 5.8 we have that $N_{L_0/Q}(S/Q) \cong \mathrm{Q}(8)$. Thus $N_{L/Q}(S)$ has Sylow 2-subgroups of order 16 and contains a subgroup isomorphic to Q(8) which is contained in L_0/Q but not in L_*/Q. Since Q(16) contains exactly two subgroups isomorphic to Q(8), it follows that a Sylow 2-subgroup of $N_G(S)$ is conjugate to $\langle a, bc, c^2 \rangle \cong 2 \times \mathrm{Q}(8)$. Therefore $N_G(S)/S \cong 2 \times \mathrm{Q}(8)$ implies that M/J is not a subgroup of $\mathrm{Aut}(\mathrm{Alt}(6))$. Therefore $M/J \cong 2 \times \mathrm{Mat}(10)$. Hence (ii) and (iii) hold. □

Notice that the non-conjugate subgroups of Alt(6) of index 15 are conjugate in Mat(10). Therefore the orbits of M_* on J of length 15 as described in Lemma 5.5 (i) fuse into a single M-orbit of length 30. We reiterate this point in the proof of the next lemma.

Lemma 5.12. *The subgroup J contains exactly two G-conjugacy classes of subgroups of order* 3. *Furthermore, if $Y \leq J$ is a subgroup of order* 3 *and Y is not conjugate to Z, then $O_3(N_G(Y)) = J$ and $N_G(Y)$ is soluble.*

Proof. Since $J = J(S)$, J is weakly closed in S. Thus M controls fusion in J by [1, 37.6]. By Lemma 5.5(i), M_* has three orbits on the non-trivial cyclic subgroups of J. However, by Lemma 5.11 (i), $M_0/J \cong \mathrm{Mat}(10)$ and in $\mathrm{Mat}(10)$, there are no subgroups of index 15. Hence M_* has orbits of length 30 and 10 on the non-trivial cyclic subgroups of J. Furthermore, assuming that $Y \le J$ is a non-trivial cyclic subgroup of J and Y is not conjugate to Z, we get that $N_M(Y)/J \cong \mathrm{Sym}(4)$ or $2 \times \mathrm{Sym}(4)$ according to whether $|M : M_0| = 1$ or 2. Since Y is not conjugate to Z in G, $N_M(Y)$ contains a Sylow 3-subgroup of $N_G(Y)$. Therefore $O_3(N_G(Y)) \le J$ and then, as J is abelian, Hypothesis 5.1(iii) implies that $O_3(N_G(Y)) = J$ and $N_G(Y) = N_M(Y)$ is soluble. □

Let t be an element of order 2 in $L_* \cap M$. Then $tQ \in Z(L_*/Q)$, t centralizes Z and $C_{L_*}(t) \cong 3 \times \mathrm{SL}_2(5)$. Define

$$K = C_G(t) \text{ and } \overline{K} = K/\langle t \rangle.$$

Lemma 5.13. *One of the following holds:*

(i) $M/J \cong \mathrm{Mat}(10)$ *and* $K \cong 2{\cdot}\mathrm{Alt}(8)$*;*
(ii) $M/J \cong 2 \times \mathrm{Mat}(10)$ *and* $\overline{K} \cong (\mathrm{Alt}(5) \wr 2).2$*; or*
(iii) $M/J \cong 2 \times \mathrm{Mat}(10)$, $K \cong 2^{-}\mathrm{Sym}(8)$, *$G$ has a normal subgroup H of index* 2 *and* $K \cap H \cong 2{\cdot}\mathrm{Alt}(8)$.

Proof. Suppose first that $M/J \cong \mathrm{Mat}(10)$ (so $L = L_0$ and $M = M_0$). Since $C_{M/J}(tJ) \cong \mathrm{SDih}(16)$, Lemma 5.5 (ii) and the Frattini Argument imply $|M \cap K| = 3^2.2^4$ and $D = C_J(t) = O_3(M \cap K) \in \mathrm{Syl}_3(M \cap K)$ is elementary abelian of order 9. Let $T \in \mathrm{Syl}_2(M \cap K)$ with $t \in Z(T)$. Then $T \cong TJ/J \cong \mathrm{SDih}(16)$. Lemma 5.5 (ii) together with the structure of T imply $C_{M\cap K}(D) = D\langle t \rangle$ and $N_{M\cap K}(D)/C_{M\cap K}(D) \cong \overline{T} \cong \mathrm{Dih}(8)$. We recall from Lemma 5.12 that J has two conjugacy classes of non-trivial cyclic subgroups one represented by Z, the other by Y where $O_3(C_M(Y)) = J$. By Lemma 5.5 (ii) we may choose $Y \le D$, and we have $C_{\overline{K\cap M}}(\overline{Y}) = \overline{DC_T(Y)} \cong 3 \times \mathrm{Sym}(3)$.

Since t inverts Q/Z and centralizes Z, we have that $|K \cap L| = 2^4.3^2.5$ and $(K \cap L)/Z \cong L/Q \cong 2^{-}\mathrm{Sym}(5)$. Also, since Z is inverted in L, we have $\overline{K \cap L} \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(5)$.

We now calculate the centralizers and normalizers of Z and Y in K. Since Z and t have coprime orders, $C_{\overline{K}}(\overline{Z}) = \overline{C_K(Z)} = \overline{L_* \cap K} \cong 3 \times \mathrm{Alt}(5)$ and $N_{\overline{K}}(\overline{Z}) = \overline{N_K(Z)} = \overline{L \cap K} \cong \mathrm{Sym}(3) \curlywedge \mathrm{Sym}(5)$. Similarly, since $N_G(Y) \le M$, we have that $N_{\overline{K}}(\overline{Y})$ normalizes $\overline{D}$, $N_{\overline{K}}(\overline{Y})$ has order $2^2.3^2$ and has elementary abelian Sylow 2-subgroups and $|C_{\overline{K}}(\overline{Y})| = 3^2.2$. Since every non-trivial cyclic subgroup of $\overline{D}$ is conjugate to either $\overline{Z}$ or $\overline{Y}$ in $\overline{K}$ and since $\overline{D} \in \mathrm{Syl}_3(C_{\overline{K}}(\overline{Y}))$ and $\overline{D} \in \mathrm{Syl}_3(C_{\overline{K}}(\overline{Z}))$, we infer that $\overline{D} \in \mathrm{Syl}_3(\overline{K})$. Now $N_{\overline{K}}(\overline{D})$ cannot conjugate $\overline{Y}$ to $\overline{Z}$ and so we deduce that $|N_{\overline{K}}(\overline{D}) : N_{N_{\overline{K}}(\overline{D})}(\overline{Z})| = 2$ and therefore, as $|N_{N_{\overline{K}}(\overline{D})}(\overline{Z})/\overline{D}| = 4$ and $|N_{\overline{M\cap K}}(\overline{D})/\overline{D}| = 8$, $N_{\overline{K}}(\overline{D}) = N_{\overline{M\cap K}}(\overline{D})$. We have shown that $\overline{K}$ satisfies the hypothesis of Theorem 1.2 and thus $\overline{K} \cong \mathrm{Alt}(8)$. Since K contains $C_L(t)$ which in turn contains a subgroup isomorphic to $2{\cdot}\mathrm{Alt}(5)$ which itself contains t, we deduce that K is perfect. Hence $K \cong 2{\cdot}\mathrm{Alt}(8)$ as claimed.

Now suppose that $M/J \cong \mathrm{Mat}(10) \times 2$. Set $D = C_J(t)$. Then $D = O_3(K \cap M)$ and $K \cap M = DT$ where $T \in \mathrm{Syl}_2(L \cap M)$ and $T \cong C_{M/J}(tJ) \cong 2 \times \mathrm{SDih}(16)$. We choose notation so that $Z(T) = \langle e, t\rangle$ with $eJ \in Z(M/J)$ and $T = \langle e\rangle \times T_0$ with $T_0 \in \mathrm{Syl}_2(M_0)$. Since e inverts D, we see that $C_T(D) = \langle ef, t\rangle = \langle ef\rangle$ where $\overline{f} \in Z(\overline{T_0})$ (notice $(ef)^2 = t$ and f has order 4). It follows that $C_{L\cap M}(D) = D \times \langle ef\rangle$ and

$$N_{K\cap M}(D)/C_{K\cap M}(D) \cong T/\langle ef\rangle \cong \mathrm{Dih}(8).$$

As before we have two K-classes of non-trivial cyclic subgroups of order 3 in D with representatives Y and Z where $O_3(N_G(Y)) = J$. We have $N_{K\cap M}(Y)/D = \langle e, T_1\rangle D/D$ where $T_1 = T \cap M_* \cong \mathrm{Dih}(8)$ by Lemma 5.5 (i). It follows that $N_{\overline{K\cap M}}(\overline{Y})/\overline{D} \cong 2^3$ and $C_{\overline{K\cap M}}(\overline{Y})/\overline{Y} \cong 2^2$. Turning to $\overline{K\cap L}$, we see that $\overline{K\cap L}/\overline{Z} \cong 2 \times \mathrm{Sym}(5)$ from Lemma 5.11. Since $\overline{Z}$ is inverted in $\overline{K \cap L}$, we infer that $\overline{K\cap L} \cong \mathrm{Sym}(3) \times \mathrm{Sym}(5)$ and $C_{\overline{K\cap L}}(\overline{Z}) \cong 3 \times \mathrm{Sym}(5)$. We now argue exactly as in the previous case that the hypothesis of Theorem 1.3 holds. Therefore $\overline{K}$ is isomorphic either to $(\mathrm{Alt}(5) \wr 2).2$ or $\mathrm{Sym}(8)$.

To complete the proof of the lemma we need to establish the additional facts stated in (iii). So suppose that $\overline{K} \cong \mathrm{Sym}(8)$. As $O^3(C_{L_*}(t)) \cong 2{\cdot}\mathrm{Alt}(5)$, we have $K \cong 2^+\mathrm{Sym}(8)$ or $2^-\mathrm{Sym}(8)$. Now we have seen that $ef \in C_K(D)$. Therefore, $\overline{ef}$ is a transposition in $\overline{K}$. Hence $K \cong 2^-\mathrm{Sym}(8)$ as ef has order 4. Let $R \in \mathrm{Syl}_2(K)$. Then, as $K \cong 2^-\mathrm{Sym}(8)$, $Z(R) = \langle t\rangle$ by Lemma 5.9 (ii). It follows that $N_G(R) \le K$ and so $R \in \mathrm{Syl}_2(G)$. Let $K_* = O^2(K) \cong 2{\cdot}\mathrm{Alt}(8)$. Then K_* has two conjugacy classes of involutions by Lemma 5.10, the central one and the ones which project to involutions of cycle type 2^4. We know that both $\overline{e}$ and $\overline{f}$ invert $\overline{D}$ and so since e is an involution and f is an element of order 4, we deduce that $\overline{e}$ has cycle type $1^2.2^3$ and $\overline{f}$ has cycle type $1^4.2^2$ from Lemma 5.9 (i). In particular, $e \notin K_*$ and $et \notin K_*$. Now $C_G(e) \ge C_M(e) \cong 2 \times \mathrm{Mat}(10)$ and $2 \times \mathrm{Mat}(10)$ is not isomorphic to a subgroup of K. Therefore, t is not G-conjugate to e. Let $T_1 \in \mathrm{Syl}_2(C_{M_*}(t))$. Then $T_1 \cong \mathrm{Dih}(8)$ and, as all the involutions in T_1J/J are conjugate in M_*/J, we infer that all the involutions in T_1 are conjugate to t. Since $T_1 \le K$ and the involutions in K_*e are all conjugate to e, we deduce that $T_1 \leq K_*$ and the involutions in K_* are all conjugate to t. Finally, as R is a Sylow 2-subgroup of G and, as e is not conjugate to any element of $R \cap K_*$, Thompson's Transfer Lemma implies that G has a subgroup H of index 2. Since K_* is perfect, we have $K \cap H = K_*$ and we are done. $\square$

Lemma 5.14. $\overline{K}$ *is not isomorphic to* $(\mathrm{Alt}(5) \wr 2).2$

Proof. Assume that $\overline{K}$ is isomorphic to $(\mathrm{Alt}(5) \wr 2).2$ and set $K_* = O^{3'}(K)$. Then $\overline{K_*} \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$. By Lemma 5.13, $M/J \cong 2 \times \mathrm{Mat}(10)$. Let R be a Sylow 2-subgroup of $C_{M_*}(t)$. Then, as $M_*/J \cong \mathrm{Alt}(6)$, $R \cong \mathrm{Dih}(8)$. Because $N_{M*}(S) = (R \cap N_{M_*}(S))S = (R \cap L)S$, Lemma 5.11(i) tells us $L_0 = (R \cap L)L_*$, $R \cap L$ is cyclic of order 4 as well as $L_0/Q \cong 2^{-}\mathrm{Sym}(5)$. Let $A = O^3(C_{L_*}(t))$. Then $A \cong \mathrm{SL}_2(5)$ and, as $R \cap L$ normalizes A, $A(R \cap L) \cong L_0/Q \cong 2^{-}\mathrm{Sym}(5)$. In particular, as $\overline{K_*}$ is perfect, K_* does not contain $R \cap L$. On the other hand, as A is perfect, A is contained in K_* and we infer that $K_* \geq \langle t \rangle$ and K/K_* is elementary abelian of order 4. Now let T be a Sylow 2-subgroup of M containing R. Then, as $M/J \cong 2 \times \mathrm{Mat}(10)$, $T \cong 2 \times \mathrm{SDih}(16)$. Significantly, if we let F_1 and F_2 be the two fours groups of R, then there is an element $f \in T \leq K$ such that $F_1^f = F_2$. Suppose for a moment that $RK_* = K$. Then, as $|R| = 8$ and $|RK_*/K_*| = 4$, $R \cap K_* = \langle t \rangle$ and $F_1K_* \neq F_2K_*$. But, as K/K_* is abelian, this means that

$$F_1K_* = (F_1K_*)^f = F_2K_* = F_1F_2K_* = RK_*,$$

which is a contradiction. Therefore, RK_*/K_* has order 2 and thus, as $R \cap L \not\leq K_*$, $RK_* = (R \cap L)K_*$. Now $R \not\leq L$ implies there exists $s \in R$ such that $Z^s \neq Z$. Since s normalizes $C_J(t)$, we have $ZZ^s = C_J(t) \in \mathrm{Syl}_3(K)$ and, as $A = O^3(C_K(Z))$, we have $A^s \neq A$. In particular, as $RK_* = (R \cap L)K_*$ and $R \cap L$ normalizes A, A is not normal in K_*. Let K_1 and K_2 be the two distinct subgroups of K such that, for $i = 1, 2$, $K_i \geq \langle t \rangle$, $\overline{K_i} \cong \mathrm{Alt}(5)$ and $K_i \trianglelefteq K_*$. Let $W = ZZ^s = C_J(t)$. Then K_1 centralizes $W \cap K_2$. As $W \leq J$, it follows from Lemma 5.12 that $W \cap K_1$ is a conjugate of Z and then using Lemma 5.5(ii) we may assume $W \cap K_1 = Z$. But then $K_1 = A$. Hence A is normal in K_*, which as we remarked above is impossible. This contradiction shows $\overline{K}$ is not isomorphic to $(\mathrm{Alt}(5) \wr 2).2$. $\square$

Proof of Theorem 1.1. Suppose that $M/J \cong \mathrm{Mat}(10)$. Then Lemma 5.13(i) together with Theorem 2.4 implies that either $G = KO_{2'}(G)$ or $G \cong \mathrm{McL}$. The former possibility contradicts the fact that M acts irreducibly on J. Therefore $G \cong \mathrm{McL}$. If $M/J \cong \mathrm{Mat}(10) \times 2$, then, by Lemmas 5.13 (ii), (iii) and 5.14, G has normal subgroup H of index 2 such that $C_H(t) \cong 2{}^{\cdot}\mathrm{Alt}(8)$. Employing Theorem 2.4 again gives

$H \cong \mathrm{McL}$. Since $K \cong 2^{-}\mathrm{Sym}(8)$, $C_G(H) \leq Z(K) \leq H$, whence, as $Z(H) = 1$, we get G is contained in the automorphism group of H. Hence $G \cong \mathrm{Aut}(\mathrm{McL})$. This completes the proof of Theorem 1.1. □

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