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Near critical density irregular sampling in Bernstein spaces*

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Abstract

We obtain sharp estimates for the sampling constants in Bernstein spaces when the density of the sampling set is near the critical value

Keywords: Bernstein space; Beurling's sampling theorem; Sampling constant

1 Introduction

1.1 Beurling's sampling theorem

Definition 1 *Let σ be a positive number. The Bernstein space B_σ consists of all continuous bounded functions on $\mathbb{R}$ which are the Fourier transforms of distributions supported by $[-\sigma, \sigma]$.*

It is well–known that B_σ can be also characterized as the space of all bounded functions on $\mathbb{R}$ which can be extended to the complex plane as entire functions of exponential type σ.

Definition 2 *A set $\Lambda \subset \mathbb{R}$ is called a set of stable sampling (SS) for B_σ if*

$$\sup_{x\in\mathbb{R}} |f(x)| \leq C \sup_{\lambda\in\Lambda} |f(\lambda)|, \ \textit{for all } f \in B_\sigma, \tag{1}$$

where $C > 0$ is a constant.

We denote by $K(\Lambda, B_\sigma)$ the infimum over all C for which inequality (1) holds true, and call $K(\Lambda, B_\sigma)$ the *sampling constant.* We also set $K(\Lambda, B_\sigma) = \infty$ when Λ is not an SS for B_σ.

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Definition 3 *A set* Λ *is called uniformly discrete (u.d.) if*

$$\inf_{\lambda,\gamma\in\Lambda,\lambda\neq\gamma} |\lambda-\gamma| > 0.$$

If Λ is an SS for B_σ, then (see [1, Corollary of Theorem 2]) for every positive ϵ there is a u.d. subset Λ' of Λ satisfying $K(\Lambda', B_\sigma) < K(\Lambda, B_\sigma) + \epsilon$. Hence, in order to describe all sampling sets for B_σ, it suffices to describe the u.d. sampling sets. The classical theorem of Beurling [1] states that this description can be given in terms of the lower uniform density of Λ,

$$D^-(\Lambda) := \lim_{l\to\infty} \min_{a\in\mathbb{R}} \frac{|\Lambda\cap(a,a+l)|}{l}.$$

Here $|\Lambda\cap(a,a+l)|$ denotes the number of elements in $\Lambda\cap(a,a+l)$.

Theorem A (A. Beurling) *A u.d. set* Λ *is an SS for* $B_\sigma, \sigma > 0$, *if and only if*

$$D^-(\Lambda) > \frac{\sigma}{\pi}.$$

1.2 Sampling near critical density

Suppose $\Lambda\subset\mathbb{R}$ is a u.d. set satisfying $D^-(\Lambda) = 1$. By Theorem A, Λ is an SS for B_σ when $\sigma < \pi$, and is not an SS for B_σ when $\sigma \geq \pi$. One may check that when $\sigma < \pi$ and σ approaches the critical value π, then $K(\Lambda, B_\sigma)$ tends to infinity.

We ask how fast $K(\Lambda, B_\sigma)$ must grow when $\sigma \uparrow \pi$.

When $\Lambda = \mathbb{Z}$ is the set of integers, Bernstein [2] proved that $K(\mathbb{Z}, B_\sigma)$ has exactly the logarithmic growth:

Theorem B (S.N. Bernstein) Let $\Lambda = \mathbb{Z}$. Then

$$K(\mathbb{Z}, B_\sigma) = \frac{2}{\pi}\log\frac{\pi}{\pi-\sigma}(1+o(1)), \quad \sigma\uparrow\pi. \tag{2}$$

A slightly weaker result was proved by Boas and Schaeffer [3]. Some estimates of $K(\mathbb{Z}, B_\sigma)$ can be found in [11]. We mention also paper [4] which considers Gabor frames generated by the Gaussian window with respect to the lattice $a\mathbb{Z}\times a\mathbb{Z}$: An asymptotic behavior of the frame constants is obtained as constant a approaches the critical value $a = 1$.

The main result of this paper shows that the critical constants $K(\Lambda, B_\sigma)$ always have at least the logarithmic growth as $\sigma\uparrow\pi$:

Theorem 1 *For every $\Lambda, D^-(\Lambda) = 1$, and every $0 < \sigma < \pi$, we have*

$$K(\Lambda, B_\sigma) \geq C \log \frac{\pi}{\pi - \sigma}, \tag{3}$$

where $C > 0$ is an absolute constant.

Suppose a set Λ is an SS for B_σ. This means that the sampling constant $K(\Lambda, B_\sigma)$ is finite. Then, by Theorem A, $D^-(\Lambda) > \pi/\sigma$. Using Theorem 1, one can in a sense measure the stability of Theorem A by showing that $D^-(\Lambda)$ cannot be too close to π/σ unless the sampling constant $K(\Lambda, B_\sigma)$ is large:

Corollary 1 *Suppose a u.d. set Λ is an SS for $B_\sigma, \sigma > 0$. Then*

$$D^-(\Lambda) \geq \frac{\sigma}{\pi} \cdot \frac{1}{1 - \exp\{-CK(\Lambda, B_\sigma)\}}, \tag{4}$$

where $C > 0$ is an absolute constant.

To prove this corollary, one may observe that the relations

$$K(\Lambda, B_\sigma) = K(a\Lambda, B_{\sigma/a}), \tag{5}$$

$$D^-(a\Lambda) = D^-(\Lambda)/a$$

are true, where $a > 0$ and $a\Lambda = \{a\lambda, \lambda \in \Lambda\}$. Then, to get (4), one chooses $a = D^-(\Lambda)$ and applies (3).

We shall present two proofs of Theorem 1 based on two different approaches. The first approach is based on Faber's ideas in the interpolation theory, while the second one belongs to circle of Beurling's ideas.

Remark 1 *Since Beurling's Theorem A follows from Corollary 1, our first approach gives a new proof of this fundamental result.*

Remark 2 *By removing a single point from $\mathbb{Z}$, one gets a stronger estimate from below than (3):*

$$K(\mathbb{Z} \setminus \{0\}, B_\sigma) \geq \frac{\sigma}{\pi - \sigma}.$$

Indeed, set

$$f(x) = \frac{\sin \sigma x}{\sigma x}.$$

Then $\max_{x\in\mathbb{R}} |f(x)| = 1$ and

$$|f(n)| = \left|\frac{\sin \sigma n}{\sigma n}\right| = \left|\frac{\sin(\pi - \sigma)n}{\sigma n}\right| \le \frac{\pi - \sigma}{\sigma}, \quad n \in \mathbb{Z} \setminus \{0\},$$

from which the estimate above follows.

In fact, sampling constants $K(\Lambda, B_\sigma)$ may have arbitrarily fast growth:

Theorem 2 *For every function $\omega(\sigma) \uparrow \infty$ as $\sigma \uparrow \pi$, there exists $\Lambda, D^-(\Lambda) = 1$, such that*

$$K(\Lambda, B_\sigma) \ge \omega(\sigma), \quad \sigma < \pi.$$

2 Sampling constants for polynomials. Faber's approach

Let us denote by $\mathbb{T} := \{|z| = 1, z \in \mathbb{C}\}$ the unite circle in the complex plane, and by $C(\mathbb{T})$ the space of all continuous functions on $\mathbb{T}$ with the uniform norm $\|\cdot\|$. Let

$$P_n := \{\sum_{j=0}^{n} c_j z^j, |z| = 1\}$$

denote the subspace of $C(\mathbb{T})$ of the restrictions onto $\mathbb{T}$ of all complex polynomials of degree $\le n$.

Definition 4 *A set $\Lambda \subset \mathbb{T}$ is called a set of stable sampling (SS) for P_n if*

$$\|f\| \le C \sup_{\lambda\in\Lambda} |f(\lambda)|, \quad f \in P_n,$$

where C does not depend on f. The sampling constant $K(\Lambda, P_n)$ is defined to be the infimum over all such C.

Clearly, a set $\Lambda \subset \mathbb{T}$ is an SS for P_n if and only if $|\Lambda| > n$.

Our next result is an analogue of Theorem 1 for polynomials, and may have intrinsic interest.

Theorem 3 *There is an absolute constant $C > 0$ such that for every $\Lambda \subset \mathbb{T}, |\Lambda| > n$, we have*

$$K(\Lambda, P_n) \ge C \log \frac{n}{|\Lambda| - n}.$$

Remark 3 *Among all sets* $\Lambda \subset \mathbb{T}$ *satisfying* $|\Lambda| = n+1$, *the minimum of* $K(\Lambda, P_n)$ *is attained for the equally spaced nodes, i.e.*

$$K(\Lambda, P_n) \geq K(\mathbb{Z}_{n+1}, P_n),$$

see [7], [5] and [6]. Here $\mathbb{Z}_{n+1}$ *is the set of* $n+1-$*roots of unity. The inequality above was conjectured by Erdös in [8].*

In what follows, we will use a variant of Theorem 3 for trigonometric polynomials.

Denote by $C_{2\pi}$ the space of all continuous $2\pi-$periodic functions on $\mathbb{R}$ equipped with the uniform norm $\|\cdot\|$, and by

$$T_k := \{\sum_{j=-k}^{k} c_j e^{ijt},\ t \in \mathbb{R}\}$$

the $(2k+1)-$dimensional subspace of all trigonometrical polynomials of degree $\leq k$. Sampling sets $\Gamma \subset [0, 2\pi)$ for T_k and sampling constants $K(\Gamma, T_k)$ are defined as above. Clearly, $\Gamma \subset [0, 2\pi)$ is an SS for T_k if and only if $|\Gamma| > 2k$. Then we have

Theorem 3^* *There is an absolute constant* $C > 0$ *such that for every* $\Gamma \subset [0, 2\pi), |\Gamma| > 2k$, *we have*

$$K(\Gamma, T_k) \geq C \log \frac{2k}{|\Lambda| - 2k}.$$

It is easy to check that Theorems 3 and 3^* are equivalent. Indeed, since $K(\Lambda, P_n) \geq K(\Lambda, P_{n-1})$, one can check that Theorem 3 for odd n follows from the result for even n. Then take any even number n and set $k = n/2$. The relation $\Lambda = \{e^{i\gamma} : \gamma \in \Gamma\}$ establishes a one-to-one correspondence between sets $\Lambda \subset \mathbb{T}$ and $\Gamma \subset [0, 2\pi)$, and the relation $g(t) = e^{-ikt} f(e^{it})$ establishes a one-to-one norm preserving correspondence between functions $f \in P_n$ and $g \in T_k$. It follows that $K(\Lambda, P_n) = K(\Gamma, T_k), n = 2k$, which proves the equivalence between Theorems 3 and 3^*.

Our proof of Theorem 3^* involves some ideas going back to Faber.

Recall that a linear operator $U : C_{2\pi} \to T_k$ is called a projector if

$$Uf = f, \quad f \in T_k. \tag{6}$$

The following result is well known: Every projector $U : C_{2\pi} \to T_k$ satisfies the inequality

$$\|U\| > C \log k. \tag{7}$$

Here and below we denote by C some absolute positive constants, maybe different form line to line.

Inequality (7) follows directly from the fundamental observation due to Faber: By averaging of every projector with respect to translations, one gets a translation-invariant projector which is simply the $k-$th partial Fourier sum S_k. Precisely,

$$\frac{1}{2\pi}\int_0^{2\pi} H_h U H_{-h} dh = S_k, \quad U : C_{2\pi} \to T_k. \tag{8}$$

Here H_h is the translation operator and $S_k(f)$ means the $k-$th partial Fourier sum of f.

Remark 3 *Actually, Faber considered Lagrange interpolation projectors, which send f to the polynomial $q \in T_k$ interpolating f at given n nodes on the circle. Sometimes (see [9]) equality (8) for arbitrary projectors is called the Zygmund–Marzinkievich–Berman formula, while inequality (7) is called the Lozinski–Harshiladze theorem.*

The result above has a number of versions and applications. We shall use the following one due to Al.A. Privalov [13]:

Lemma 1 ([13]) *There is a constant $C > 0$ with the property: Given integers $1 \leq m \leq 2k$, a projector $U : C_{2\pi} \to T_k$, and linear functionals $\psi_j \in C^*_{2\pi}, j = 1, ..., m$, there is a function $f \in C_{2\pi}, \|f\| \leq 1$, such that $\psi_j(f) = 0, j = 1, ..., m$, and*

$$\|Uf\| \geq C \log \frac{2k}{m}.$$

The reader may find a list with additional references in [13].

For completeness of presentation, we prove this lemma in sec. 4.

Proof of Theorem 3^*. Denote by $m \geq 0$ the number such that $|\Gamma| = 2k + m + 1$. Since clearly, $K(\Gamma, T_k) \geq K(\Gamma^*, T_k)$ whenever $\Gamma \subset \Gamma^* \subset [0, 2\pi)$, we may assume that $m \geq 1$. Choose any subset $\Gamma_m \subset \Gamma$ such that $|\Gamma_m| = m$, and set $\Gamma' = \Gamma \setminus \Gamma_m$. Then $|\Gamma'| = 2k+1$.

Set

$$\varphi(t) := \prod_{\gamma\in\Gamma'} \sin\frac{t-\gamma}{2},$$

and define $U : C(\mathbb{T}) \to T_k$ to be the Lagrange interpolation operator

$$Uf(t) := \varphi(t)\sum_{\gamma\in\Gamma'} \frac{f(\gamma)}{2\varphi'(\gamma)\sin\frac{t-\gamma}{2}}.$$

It is easy to see that U is a projector onto T_k.

Now, for every $\gamma' \in \mathbb{T}\setminus\Gamma'$, the relation

$$\psi_{\gamma'}(f) := Uf(\gamma') = \varphi(\gamma')\sum_{\gamma\in\Gamma'} \frac{f(\gamma)}{2\varphi'(\gamma)\sin\frac{\gamma'-\gamma}{2}}$$

is a linear functional on $C_{2\pi}$. It follows from Lemma 1 that there exists $f \in C_{2\pi}, \|f\| \le 1$, such that $\psi_\gamma(f) = 0, \gamma \in \Gamma_m$, and $\|Uf\| \ge C\log(2k/m)$, from which Theorem 3* follows.

The following statement follows from Theorem 3* by an appropriate change of variable.

Corollary 2 *There is an absolute constant $C > 0$ with the property: Given an interval $(-N,N), N \in \mathbb{N}$, and a set $\Lambda \subset (-N,N), |\Lambda| > 2N$, there is a trigonometric polynomial*

$$P(t) = \sum_{j=-N}^{N} c_j e^{\frac{i\pi j}{N}t} \in B_\pi$$

such that

$$\max_{t\in\mathbb{R}} |P(t)| \ge C\log\frac{2N}{|\Lambda|-2N}\max_{\lambda\in\Lambda}|P(\lambda)|.$$

3 Sampling constants for Bernstein spaces

3.1 A sampling theorem for B_π

Let N be a positive integer and $\Lambda \subset \mathbb{R}$ be a set. Throughout this section we use the notation

$$\Lambda_N := \Lambda \cap (-N,N), \quad \Lambda(N) := \Lambda \cup (-\infty,-N] \cup [N,\infty).$$

Since $\Lambda(N)$ contains two infinite rays $|t| \geq N$, it is an SS for B_π. We show that for large N, the sampling constant $K(\Lambda(N), B_\pi)$ must be large unless the number of points of Λ in $(-N, N)$ is "much larger than" $2N$:

Theorem 4 *There is an absolute constant $C > 0$ such that for every set $\Lambda \subset \mathbb{R}, |\Lambda_N| > 2N$, we have*

$$K(\Lambda(N), B_\pi) \geq C \log \frac{2N}{|\Lambda_N| - 2N}. \tag{9}$$

Throughout the rest of this section we denote by C different positive absolute constants.

In order to prove this theorem we first need an auxiliary lemma.

Lemma 2 *Assume $M \in \mathbb{N}$ and $M^{-1/3}/2 < \delta < M^{-1/3}$. For every set $\Gamma \subset \mathbb{R}, |\Gamma_M| > 2M$, we have*

$$K(\Gamma(M), B_{\pi/(1-\delta)}) \geq C \log \frac{2M}{|\Gamma_M| - 2M}. \tag{10}$$

3.2 Proof of Lemma 2

We may assume that M is a sufficiently large number, so that the inequalities below hold true.

1. Let us show that it suffices to prove Lemma 2 for the case

$$2M + M^{2/3} \leq |\Gamma_M| \leq 3M. \tag{11}$$

Indeed, if $|\Gamma_M| > 3M$, then (10) is true for $C = 1/\log 2$.

Further, assume (10) holds for all sets Γ' satisfying (11). Let us show that it is then true for all sets Γ satisfying $2M < |\Gamma_M| < 2M + M^{2/3}$. For every such set Γ one may choose a set Γ' such that $\Gamma \subset \Gamma'$ and $0 \leq |\Gamma'_M| - (2M + M^{2/3}) \leq 1$. Then

$$K(\Gamma(M), B_{\pi/(1-\delta)}) \geq K(\Gamma'(M), B_{\pi/(1-\delta)}) \geq C \log \frac{2M}{M^{2/3} + 1} >$$

$$\frac{C}{3} \log M > \frac{C}{6} \log \frac{2M}{|\Gamma_M| - 2M},$$

which completes the proof.

2. Fix a number $m, M-2\sqrt{M} < m < M-\sqrt{M}$, such that there are no two distinct points $\gamma_1, \gamma_2 \in \Gamma_M$ satisfying $|\gamma_1 - \gamma_2| = m$. Set

$$\Gamma' := \Gamma_M + m\mathbb{Z} = \bigcup_{\gamma\in\Gamma_M} (\gamma + m\mathbb{Z}).$$

One may check that $\Gamma_M \subset \Gamma'$ and $|\Gamma'_m| = |\Gamma_M|$. By this and Corollary 2, there is a trigonometric polynomial

$$P(t) = \sum_{j=-m}^{m} c_j e^{\frac{i\pi j}{m} t} \in B_\pi$$

satisfying

$$\max_{t\in\mathbb{R}} |P(t)| \geq C \log \frac{2m}{|\Gamma_M| - 2m} \max_{\gamma\in\Gamma_M} |P(\gamma)|. \tag{12}$$

Denote by $|t_0| \leq m$ a maximum modulus point of P. Set

$$g(t) := \frac{P(t)}{P(t_0)} \frac{\sin(m^{-1/3}(t-t_0))}{m^{-1/3}(t-t_0)}$$

and $\delta := 1 - (1+m^{-1/3})^{-1} \in (M^{-1/3}/2, 2M^{-1/3})$. Then

$$g \in B_{\pi+m^{-1/3}} = B_{\pi/(1+\delta)}.$$

3. We now obtain some estimates of $|g|$ from above on the set $\Gamma(M)$. Firstly,

$$\max_{|t|\geq M} |g(t)| \leq \max_{|t|\geq M} \left| \frac{\sin(m^{-1/3}(t-t_0))}{m^{-1/3}(t-t_0)} \right| \leq \frac{1}{m^{-1/3}(M-m)} \leq \frac{2}{M^{1/6}}.$$

Further, by (11) and (12),

$$\max_{\gamma\in\Gamma_M} |g(\gamma)| \leq \frac{\max_{\gamma\in\Gamma_M} |P(\gamma)|}{\max_{t\in\mathbb{R}} |P(t)|} \leq$$

$$\left(C \log \frac{2m}{|\Gamma_M| - 2m}\right)^{-1} \leq \left(\frac{C}{2} \log \frac{2M}{|\Gamma_M| - 2M}\right)^{-1}.$$

Hence, since

$$\max_{t\in\mathbb{R}} |g(t)| = g(t_0) = 1,$$

we see that g satisfies

$$\max_{t\in\mathbb{R}} |g(t)| = 1 \geq \min\{\frac{C}{2} \log \frac{2M}{|\Gamma_M| - 2M}, \frac{M^{1/6}}{2}\} \max_{\gamma\in\Gamma(M)} |g(\gamma)|,$$

which proves (10).

3.3 Proof of Theorem 4

The argument in the first part of the previous proof shows that we may assume $2N + N^{2/3} \le |\Lambda_N| \le 3N$. Clearly, we may also assume that N is a large number.

Choose $N^{-1/3}/2 < \delta < 2N^{-1/3}/3$ such that $\delta N \in \mathbb{N}$, and set $M = (1-\delta)N$ and $\Gamma = (1-\delta)\Lambda$. It is clear that $|\Gamma_M| = |\Lambda_N| > 2M$, and one may check that $M^{-1/3}/2 < \delta < M^{-1/3}$. This means that we can apply Lemma 2, which gives

$$K(\Lambda(N), B_\pi) = K(\Gamma(M), B_{\pi/(1-\delta)}) \ge C \log \frac{2M}{|\Gamma_M| - 2M} =$$

$$C \log \frac{2(1-\delta)N}{|\Lambda_N| - 2(1-\delta)N} > \frac{C}{2} \log \frac{2N}{|\Lambda_N| - 2N},$$

which proves (9).

3.4 Proof of Theorem 1

Set $a = \sigma/\pi$ and $\Gamma = a\Lambda$. By (5), Theorem 1 is equivalent to the statement that for every set $\Gamma \subset \mathbb{R}$ satisfying $D^-(\Gamma) = \pi/\sigma > 1$, we have

$$K(\Gamma, B_\pi) \ge C \log \frac{\pi}{\pi - \sigma}. \tag{13}$$

Without loss of generality we may assume that $\pi - \sigma$ is a small number, and denote by N the integer satisfying

$$\frac{\sigma}{2\pi - 2\sigma} - 1 < N \le \frac{\sigma}{2\pi - 2\sigma}.$$

Then

$$D^-(\Gamma) = \frac{\pi}{\sigma} \le 1 + \frac{1}{2N}.$$

Therefore, there exists an interval of length $2N$ which contains at most $2N+2$ points of Γ. We may assume that $|\Gamma_N| \le 2N+2$. Since

$$K(\Gamma, B_\pi) \ge K(\Gamma(N), B_\pi),$$

estimate (13) follows from Theorem 4.

4 Proof of Lemma 1

Given $1 \le m < 2k$ linear functionals $\psi_j \in C^*_{2\pi}$, we have to show that there exists $g \in C_{2\pi}$ satisfying $\psi_j(g) = 0, j = 1, ..., m$, and

$$\max_{t\in[0,2\pi)} |Ug(t)| > C \log \frac{2k}{m}. \tag{14}$$

Here and throughout this proof we denote by C absolute constants.

1. Fix integer constants

$$\rho :\simeq \left(\frac{k}{m}\right)^{1/3}, \ m_1 :\simeq \frac{k}{\rho},$$

where $a \simeq b$ means $|a - b| < C$. We may assume that k/m, ρ and $m_1/m\rho$ are large numbers.

2. Set

$$Q_0(t) := \left(\frac{\sin 2m\rho t}{4m\rho \sin t/2}\right)^2, \ Q(t) := \sum_{l=1}^{4m\rho} \alpha_l Q_0\left(t - \frac{\pi l}{2m\rho}\right).$$

One can check that

$$\|Q\| \le \max\{|\alpha_l|; l = 1, ..., 4m\rho\}.$$

Observe that α_l can be chosen to satisfy the equalities

$$\psi_j(e^{im_1qt}Q(t)) = 0,$$

where

$$q = 0, \pm 1, ..., \pm(\rho - 1), \pm(\rho + 1), ..., \pm 2\rho, \ j = 1, ..., m.$$

This is so, since the number of equalities, $(4\rho - 1)m$, is less than the number of coefficients, $4m\rho$. Moreover, we may chose α_l so that

$$\max\{|\alpha_l|; l = 1, ..., 4m\rho\} = \alpha_{l_0} = 1.$$

Set $t_0 := \pi l_0/(2m\rho)$. Then $\|Q\| = \alpha_{l_0} = Q(t_0) = 1$.

3. Consider Fejér's polynomial

$$P(t) := \left(\frac{1}{\rho} + \frac{\cos t}{\rho - 1} + ... + \frac{\cos(\rho - 1)t}{1}\right) -$$

$$\left(\frac{\cos(\rho+1)t}{1} + ... + \frac{\cos 2\rho t}{\rho}\right) =: P_1(t) - P_2(t).$$

Clearly, $P_1(0) > \log\rho$ and it is well–known that $\|P_1 - P_2\| \le C$. Set $f_1(t) := CP_1(m_1 t)$, $f_2(t) := CP_2(m_1 t)$ and $f = f_1 - f_2$, where C is such that $\|f\| = 1$. Observe that

$$\|f_1\| = f_1(0) = CP_1(0) > C\log\rho \ge C\log\frac{k}{m}.$$

4. Consider the polynomials

$$g_\tau := (H_{-\tau} f) \cdot Q,$$

where $(H_\tau f)(t) := f(t-\tau)$. Clearly, $\|g_\tau\| \le 1$ and all our functionals vanish on g_τ, for every τ. To prove the lemma, we show that there exists τ such that g_τ satisfies (14).

Set

$$G(\tau) := (Ug_\tau)(t_0 - \tau) = G_1(\tau) + G_2(\tau),$$

where

$$G_j(\tau) := (U(H_{-\tau} f_j) \cdot Q)(t_0 - \tau).$$

In order to prove that g_τ satisfies (14) for some τ, it suffices to show that $\max_\tau |G(\tau)| > C\log k/m$. To prove the latter, it is convenient to use the de la Vallée Poussin means:

$$V_l(f)(x) := \frac{1}{l}\sum_{j=l}^{2l-1} S_j(f)(x),$$

where $V_l(f)$ denotes the $l-$th partial Fourier sum of f. It is well known that $\|V_l(G)\| < C\|G\|$.

It is easy to see that polynomial $G_2(t)$ contains only exponentials with exponents $j : |j| > 8m\rho$, so that $V_{4m\rho}(G_2) = 0$. Further, one may check that polynomial $(H_{-\tau} f_1) \cdot Q$ belongs to T_k, so that

$$G_1(\tau) = ((H_{-\tau} f_1) \cdot Q)(t_0 - \tau) = f_1(t_0)Q(t_0 - \tau).$$

Hence, $G_1 \in T_{4m\rho}$ which gives $V_{4m\rho}(G_1) = G_1$. We conclude that

$$\|G\| \ge C\|V_{4m\rho}(G)\| = C\|V_{4m\rho}(G_1)\| =$$

$$C\|G_1\| \ge C|G_1(0)| = C|f_1(0)||Q(t_o)| \ge C\log\frac{k}{m}.$$

5 Beurling's approach

5.1 Some results from Beurling's sampling theory

Beurling in [1] has built a general theory of balayage (or sweeping) of any finite measure from $\mathbb{R}^n$ to a given set Λ without changing the values on a compact set E of its Fourier transform. For a large class of sets E he established connection between balayage and sampling in the corresponding Bernstein space. We recall several facts from his theory which we use in the proof of Theorem 1 below.

Given two closed sets A and B on $\mathbb{R}$, the Fréchet distance $d(A, B)$ between A and B is the smallest number $d > 0$ such that $A \subset B + [-d, d]$ and $B \subset A + [-d, d]$. Let Λ_j be (not necessarily discrete) closed sets in $\mathbb{R}$. A set Λ is called a weak limit of Λ_j if for every closed interval I we have $d(\Lambda \cap I, \Lambda_j \cap I) \to 0, j \to \infty$. We shall use the fact that (see [1], Theorem 1)

$$K(\Lambda, B_\sigma) \leq \liminf_{j\to\infty} K(\Lambda_j, B_\sigma). \tag{15}$$

Given a u.d. set Λ, consider its translations $\Lambda - a, a \in \mathbb{R}$. For every sequence a_j, one can always choose a subsequence such that the corresponding translations of Λ converge weakly to some u.d. set Λ'. We denote by $W(\Lambda)$ the set of all possible weak limits of the translates of Λ. It follows from (15) that

$$K(\Lambda', B_\sigma) \leq K(\Lambda, B_\sigma), \quad \Lambda' \in W(\Lambda).$$

We say that a set Λ is a uniqueness set for B_σ if $f \in B_\sigma$ and $f|_\Lambda = 0$ imply $f = 0$. There is a beautiful connection between sampling and uniqueness properties for Bernstein spaces (see [1], Theorem 3):

Theorem C (Beurling) *Λ is an SS for B_σ if and only if every set $\Lambda' \in W(\Lambda)$ is a uniqueness set for B_σ.*

5.2 Some facts about entire functions

1. **A corollary of Jensen formula**.

Given an entire function f, denote by $n_f(r)$ the number of its zeros in the circle $|z| < r$. Then the inequality

$$n_f(r) \leq \max_{|z|\leq r} \log|f(ez)| - \log|f(0)| \tag{16}$$

holds provided $f(0) \neq 0$. This is an immediate corollary of Jensen formula (see [10], p.13).

2. **Bernstein's inequality**.
Set

$$\|f\| := \sup_{x\in\mathbb{R}} |f(x)|.$$

Bernstein's inequality ([10], p. 227, [14], p. 72) states

$$\|f'\| \leq \sigma \|f\|, \quad \text{for every} \quad f \in B_\sigma. \tag{17}$$

3. Observe that every $f \in B_\sigma$ satisfies ([14], Theorem 11, p. 70)

$$|f(x+iy)| \leq \|f\| e^{\sigma|y|}, \quad x+iy \in \mathbb{C}.$$

From this inequality and (16), for every $f \in B_\sigma, f(0) \neq 0$, it follows that

$$n_f(r) \leq \sigma e r + \log \frac{\|f\|}{|f(0)|}, \quad r > 0. \tag{18}$$

5.3 Proof of Theorem 1

Throughout the proof we denote by C some absolute constants.

To prove Theorem 1, we show that for every $\Lambda, D^-(\Lambda) = 1$, and every $\sigma < \pi$ there exists $f \in B_\sigma$ satisfying

$$\|f\| = 1, \quad |f(\lambda)| \leq \frac{C}{\log \frac{\pi}{\pi - \sigma}}, \quad \text{for every } \lambda \in \Lambda. \tag{19}$$

It is clear that it suffices to verify this only for $\sigma > \sigma_0$, with some $\sigma_0 < \pi$.

By Theorem C and (15), we may additionally assume that there exists $\varphi \in B_\pi$ such that $\varphi|_\Lambda = 0$. Without loss of generality, we may also assume that $0 \notin \Lambda$, $\|\varphi\| = 1$ and $|\varphi(0)| \geq 1/2$.

Fix any number $N \geq 64$, and set $\sigma := \pi - \pi N^{-3}$. Observe that

$$\log \frac{\pi}{\pi - \sigma} = 3 \log N. \tag{20}$$

To find a function f satisfying (19), we consider three cases.

1. Assume that the interval $[-N, N]$ contains a zero λ_0 of φ of multiplicity ≥ 2. Set

$$g(x) := \frac{\lambda_0 \varphi(x)}{x - \lambda_0}.$$

We have $g \in B_\pi, g|_\Lambda = 0$ and $|g(1/2)| = |\varphi(1/2)| \geq 1/2$. The latter shows that $\|g\| \geq 1/2$. Further,

$$|g(x)| \leq \frac{|\lambda_0|\|\varphi\|}{N^2 - |\lambda_0|} \leq \frac{2}{N}, \quad \text{for all } x, |x| \geq N^2.$$

Now set

$$f(x) := \frac{g(\frac{\sigma}{\pi}x)}{\|g\|}. \tag{21}$$

Then $f \in B_\sigma$ and $\|f\| = 1$. When $\lambda \in \Lambda, |\lambda| < N^2$, from (17) we have

$$|f(\lambda)| = \frac{|g(\lambda - \frac{\pi-\sigma}{\pi}\lambda)|}{\|g\|} \leq \frac{\pi - \sigma}{\pi}|\lambda|\frac{\|g'\|}{\|g\|} \leq \frac{\pi}{N}.$$

When $\lambda \in \Lambda, |\lambda| \geq N^2$, it follows from the estimate on $|g(x)|$ and $\|g\|$ above that $|f(\lambda)| < 4/N$. These estimate and (22) prove (a stronger inequality than) (19).

2. We may assume that φ has only simple zeros on $[-N, N]$. Assume additionally that every subinterval of $[-N, N]$ of length $\sqrt{N}$ contains at least $\sqrt{N}/8$ points of Λ. Then

$$\sum_{\lambda\in\Lambda, |\lambda|<N} \frac{1}{|\lambda|} \geq \frac{1}{8}\sqrt{N} \sum_{|j|<\sqrt{N}} \frac{1}{|j|\sqrt{N}} = C \log N. \tag{22}$$

Now, for every $\lambda \in \Lambda, |\lambda| < N$, we denote by c_λ the number such that $|c_\lambda| = 1$ and

$$-\frac{c_\lambda}{\lambda\varphi'(\lambda)} = \left|\frac{c_\lambda}{\lambda\varphi'(\lambda)}\right|.$$

Set

$$g(x) := \varphi(x) \sum_{\lambda\in\Lambda, |\lambda|<N} \frac{c_\lambda}{(x - \lambda)\varphi'(\lambda)}.$$

Then $g \in B_\pi, g(\lambda) = c_\lambda$ whenever $\lambda \in \Lambda, |\lambda| < N$, and $g(\lambda) = 0$ otherwise. Using Bernstein's inequality (17) for $\varphi'(\lambda)$ and (22), we get

$$|g(0)| = |\varphi(0)| \sum_{\lambda\in\Lambda, |\lambda|<N} \frac{1}{|\lambda\varphi'(\lambda)|} \geq C \log N.$$

Moreover, by (18), for every $|x| > 2N^2$ we have

$$|g(x)| \leq \sum_{\lambda\in\Lambda, |\lambda|<N} \frac{1}{|(2N^2 - \lambda)\varphi'(\lambda)|} \leq \frac{|g(0)|}{N}.$$

Hence, two estimates hold:

$$\frac{|g(\lambda)|}{\|g\|} \le \frac{|g(\lambda)|}{|g(0)|} \le \frac{C}{\log N}, \quad \lambda \in \Lambda, \tag{23}$$

and

$$\frac{|g(x)|}{\|g\|} \le \frac{C}{N}, \quad |x| > 2N^2. \tag{24}$$

Let f be defined by (21). Then $f \in B_\sigma$ and $\|f\| = 1$. When $\lambda \in \Lambda, |\lambda| < 2N^2$, we have $|\lambda|N^{-3} < 2/N$. Hence, by (17) and (23),

$$|f(\lambda)| = \frac{|g(\lambda - \lambda N^{-3})|}{\|g\|} \le \frac{|g(\lambda)| + |\lambda|\sigma N^{-3}\|g\|}{\|g\|} \le \frac{C}{\log N}.$$

From (24) it follows that the ratio above admits an even better estimate for $|\lambda| > 2N^2$. These estimates and (20) prove (19).

3. Assume that there is an interval I of length $\sqrt{N}$ which contains $< \sqrt{N}/8$ points of Λ. We may assume that $I = [-\sqrt{N}/2, \sqrt{N}/2]$.

We shall need the following lemma (see [12], Lemma 4.5):

Lemma 3 ([12]) *Given an integer n and a positive number $\omega < 1$, let P be an algebraic polynomial of degree $\le n$ which has a zero of multiplicity $\ge \omega n$ at the point 1. Then there exists a constant $0 < \eta < 1$ which depends only on ω such that*

$$\max_{|z|=1} |P(z)| = \max_{|z|=1; |z+1|>\eta} |P(z)|.$$

Set

$$\psi(x) := \left(\cos \frac{\pi}{\sqrt{N}} x\right)^{\sqrt{N}/4} \prod_{\lambda \in \Lambda \cap I} \sin \frac{2\pi}{\sqrt{N}}(x - \lambda).$$

One may check that ψ is $\sqrt{N}-$periodic, $\psi \in B_{\pi/2}$ and $\psi(\lambda) = 0, \lambda \in \Lambda \cap I$. Moreover, letting $z := \exp(2\pi i x/\sqrt{N})$, we see that

$$\psi(x) = z^{-\sqrt{N}/4}(z+1)^{\sqrt{N}/8} \prod_{\lambda \in \Lambda \cap I} (z^2 - \alpha_\lambda^2), \ \alpha_\lambda := e^{-2\pi i \lambda/\sqrt{N}}.$$

Hence, by Lemma 3 we conclude that there exists an absolute constant $0 < c < 1/2$ and a point $x_0, |x_0| \le c\sqrt{N}$ satisfying

$$\|\psi\| = |\psi(x_0)|.$$

Set

$$f(x) := \frac{\psi(x)}{\psi(x_0)} \frac{\sin \frac{\pi(x-x_0)}{4}}{\frac{\pi(x-x_0)}{4}}.$$

We have $f \in B_{3\pi/4}$, $\|f\| = f(x_0) = 1, f(\lambda) = 0$ whenever $\lambda \in \Lambda, |\lambda| < \sqrt{N}/2$. Moreover, since $|x - x_0| > (1/2 - c)\sqrt{N}, |x| > \sqrt{N}/2$, we get

$$|f(x)| \leq \left| \frac{\sin \frac{\pi(x-x_0)}{4}}{\frac{\pi(x-x_0)}{4}} \right| \leq \frac{4}{\pi(1/2 - c)\sqrt{N}}, \ |x| \geq \sqrt{N}/2,$$

from which (19) follows.

6 Proof of Theorem 2

Theorem 2 follows easily from our Theorem 4.

Take any function $\omega(\sigma) \uparrow \infty, \sigma \uparrow \pi$, and any sequence $0 < \sigma_1 < \sigma_2 < ..., \sigma_j \uparrow \pi$. To prove Theorem 2 it suffices to construct a u.d. set $\Lambda, D^-(\Lambda) = 1$, for which $K(\Lambda, B_{\sigma_j}) > \omega(\sigma_{j+1}), j \in \mathbb{N}$.

Set

$$\Lambda_1 := \{\frac{\pi}{\sigma_1} n : n \in \mathbb{Z}, |n| < N_1\} \cup \{x : x \in \mathbb{R}, |x| \geq N_1\}.$$

By Theorem 4 we may choose N_1 so large that

$$K(\Lambda_1, B_{\sigma_1}) > \omega(\sigma_2).$$

Next, we set

$$\Lambda_2 := \{\frac{\pi}{\sigma_1} n : n \in \mathbb{Z}, |n| < N_1\}$$

$$\cup \{\frac{\pi}{\sigma_2} n : n \in \mathbb{Z}, N_1 < |n| < N_2\} \cup \{x : x \in \mathbb{R}, |x| \geq N_2\}.$$

By Theorem 4 we may choose $N_2 > N_1$ so large that

$$K(\Lambda_2(N, \sigma), B_{\sigma_2}) > \omega(\sigma_3),$$

and so on. Proceeding like that we construct a sequence $N_j \to \infty$ and a sequence Λ_j which converge to some Λ. Since $\Lambda \subset \Lambda_j$ for each j, we have

$$K(\Lambda, B_{\sigma_j}) \geq K(\Lambda_j, B_{\sigma_j}) > \omega(\sigma_{j+1}).$$

Moreover, for every $j \in \mathbb{N}$ we have

$$\Lambda \cap \{x : N_j < |x| < N_{j+1}\} = \{\frac{\pi}{\sigma_j} n : N_j < |n| < N_{j+1}\}.$$

From this it easily follows that $D^-(\Lambda) = 1$, which completes the proof.

References

[1] Beurling, A. Balayage of Fourier–Stiltjes Transforms. In: The collected Works of Arne Beurling, Vol.2, Harmonic Analysis. Birkhauser, Boston, 1989.

[2] Bernstein, S. N. The extension of properties of trigonometric polynomials to entire functions of finite degree. (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, (1948). 421–444.

[3] Boas, R. P., Jr., Schaeffer, A. C. A theorem of Cartwright. Duke Math. J. 9, (1942). 879–883.

[4] Borichev, A, Gröchenig, K., Lyubarskii, Yu. Frame constants of Gabor frames near the critical density. (English, French summary) J. Math. Pures Appl. (9) 94 (2010), no. 2, 170–182.

[5] Brutman, L. On the polynomial and rational projections in the complex plane. SIAM J. Numer. Anal. 17 (1980), no. 3, 366–372.

[6] Brutman, L., Pinkus, A. On the Erdos conjecture concerning minimal norm interpolation on the unit circle. SIAM J. Numer. Anal. 17 (1980), no. 3, 373–375.

[7] De Boor, C., Pinkus, A. Proof of the conjectures of Bernstein, and Erdös concerning the optimal nodes for polynomial interpolation, J. Approximation Theory, 24 (1978), 289–303.

[8] Erdös, P. Problems and results on the convergence and divergence properties of the Lagrange interpolation polynomials and some extremal problems. Mathematica (Cluj) 10 (33) 1967, 65–73.

[9] Akilov, G.P., Kantorovich, L.V. Functional Analysis. Pergamon Press 1982.

[10] Levin, B. Ya. Lectures on Entire Functions. AMS. 1996.

[11] Liu, H. C., Macintyre, A. J. Cartwright's theorem on functions bounded at the integers. Proc. Amer. Math. Soc. 12 (1961), 460–462.

[12] Olevskii, A., Ulanovskii, A. Universal sampling and interpolation of band-limited signals. Geom. Funct. Anal. 18 (2008), no. 3, 1029–1052.

[13] Privalov, Al. A. The growth of the powers of polynomials, and the approximation of trigonometric projectors. (Russian) Mat. Zametki 42, no. 2 (1987), 207–214.

[14] Young, R.M. An introduction to Nonharmonic Fourier Series. Academic Press. 2001.