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LOCAL ASYMPTOTICS FOR THE AREA OF RANDOM WALK EXCURSIONS

DENIS DENISOV, MARTIN KOLB, AND VITALI WACHTEL

Abstract. We prove a local limit theorem for the area of the positive excursion of random walks with zero mean and finite variance. Our main result complements previous work of Caravenna and Chaumont, Sohier, as well as Kim and Pittel.

1. Introduction and statement of results

Let $\{S_n\}$ be an integer-valued centered random walk with finite second moments, and let τ denote the first time when the random walk is negative, i.e., $\tau := \min\{n \geq 1 : S_n \leq 0\}$. The path $\{S_1, S_2, \dots, S_{\tau-1}\}$ we shall call the positive excursion of $\{S_n\}$. It follows easily from recent results of Caravenna and Chaumont [3] and Sohier [13] that the rescaled excursion of the random walk conditioned on $\tau = n+1$ converges weakly to the standard Brownian excursion which we shall denote by $e(t), t \in [0,1]$. This implies that an appropriately rescaled area converges towards the corresponding functional of the Brownian excursion. More precisely,

$$\mathbf{P}\left(n^{-3/2}A_n \leq x \middle| \tau = n+1\right) \to \mathbf{P}\left(\int_0^1 e(t)dt \leq x\right),\ x > 0, \tag{1}$$

where

$$A_n := \sum_{k=1}^{n} S_k.$$

For simple random walks this convergence was proved by Takacs [15], who also identified the limiting distribution – the so-called Airy distribution. (We give below an exact expression for its density.) His motivation was partially rooted in combinatorics. More precisely, he was interested in the investigation of the asymptotic number of random trees on n vertices with given total height, see Takacs [15, 16, 17] and Spencer [14]. Using the well-known one-to-one correspondence between random trees and random walk excursions, this problem is equivalent to a problem concerning the area under random walk path. It is worth mentioning that areas of random walk excursions appear also in other combinatorial problems such as:

- analysis of linear probing hashing, Flajolet, Poblete and Viola [7];
- enumeration of paths below a line of rational slope, Banderier and Gittenberger [1];
- Winston-Kleitman problem on tournament scores, Winston and Kletman [21] and Takacs [15].

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Assertion (1) allows to find the asymptotic number of random trees on n vertices with the total height bounded by $xn^{3/2}$. But in order to find the number of trees with fixed total height one need a local version of (1). Moreover, such a result allows to confirm the Kleitman-Winston conjecture mentioned above, see Takacs [15, page 565]. This conjecture were proved by Kim and Pittel [10] by deriving a uniform upper bound for probabilities $\mathbf{P}(A_n = a|\tau = n+1)$ in the case of a simple random walk.

The main purpose of the present paper is to extend the result of Kim and Pittel to a local limit theorem for the excursion area of all random walks with finite variance.

We say that X is (d,ρ)-lattice if its distribution is lattice with span d and shift $\rho \in [0,d)$, i.e., h is the maximal number such that $\mathbf{P}(X \in \{\rho + dk,\ k \in \mathbb{Z}\}) = 1$. Furthermore, we define $N_x := \{n \geq 1 : \mathbf{P}(S_n = x) > 0\}$, $x \geq 0$.

Theorem 1. *Assume that* $\mathbf{E}X = 0$, $\mathbf{E}X^2 := \sigma^2 \in (0,\infty)$ *and* X *is* (d,ρ)*-lattice. Then*

$$\sup_{a \in n(n+1)\rho/2 + d\mathbb{Z}} \left| n^{3/2}\mathbf{P}\left(A_n = a \middle| \tau = n+1\right) - \frac{d}{\sigma} w_{ex}\left(\frac{a}{\sigma n^{3/2}}\right)\right| \to 0 \quad \text{as } n \to \infty \tag{2}$$

and, for every $x \geq 0$,

$$\sup_{a \in n(n+1)\rho/2 + d\mathbb{Z}} \left| n^{3/2}\mathbf{P}\left(A_n = a \middle| \tau = n+1, S_n = x\right) - \frac{d}{\sigma} w_{ex}\left(\frac{a}{\sigma n^{3/2}}\right)\right| \to 0 \tag{3}$$

as $N_x \ni n \to \infty$. *Here* w_{ex} *denotes the density of* $\int_0^1 e(t)dt$.

Takacs [15, Theorem 5] has obtained an exact expression for w_{ex}:

$$w_{ex}(x) = \frac{2^{13/6}}{3^{3/2}x^{10/3}} \sum_{k=1}^{\infty} a_k^2 \exp\left\{-\frac{2a_k^3}{27x^2}\right\} U\left(-\frac{5}{6}, \frac{4}{3}, \frac{2a_k^3}{27x^2}\right),$$

where $U(a,b,z)$ is a confluent hypergeometric function and $\{-a_k\}$ is a sequence of zeros of the Airy function

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + tx)dt$$

arranged so that $a_k < a_{k+1}$ for all k. (For further properties of the Airy function we refer to Janson [9, Section 12].)

Using the asymptotics $w_{ex}(x) \to 0$ as $x \to 0$ or $x \to \infty$ from [9, Section 15] we conclude $\sup_{x \geq 0} w_{ex}(x) < \infty$. From this fact and (2) we infer that

$$\sup_{a \geq 1} \mathbf{P}\left(A_n = a \middle| \tau = n+1\right) \leq \frac{C}{n^{3/2}}, \quad n \geq 1,$$

reproducing the main result of Kim and Pittel [10].

Example 2. In order to demonstrate the relevance of our theorem in a combinatorial context, we apply it to the following problem of enumeration of Dyck paths below a line of rational slope. Following Banderier and Gittenberger [1], we look at walks on $\mathbb{N}^2$ with steps $(1,0)$ and $(0,1)$ constrained to stay below a line $y = \frac{\alpha}{\beta}x$ with $\alpha, \beta \in \mathbb{N}$. We are interesed in the asymptotic number of such walks of length n which start at $(0,0)$, end on the line and have a fixed area between the line and the path. According to Theorem 8 in [1], this number is equal, up to the factor

$\alpha+\beta$, to the number of random walk excursions of length n with the endpoint 0 and the same area. The set of jumps of this random walk is $\{\alpha,-\beta\}$. Let $N(n,a)$ denote the number of excursions with area a. Then

$$N(n,a)=2^n\mathbf{P}(A_n=a,\tau=n,S_n=0),$$

where S_n is a random walk with $\mathbf{P}(X=\alpha)=\mathbf{P}(X=-\beta)=1/2$. This walk is obviously $(\alpha,\alpha+\beta)$-lattice. Since $\mathbf{E}X=\alpha-\beta$ is not necessarily zero, we can not apply Theorem 1 directly. In order to obtain a driftless random walk we perform an exponential change of measure. Set $h_0=(\alpha+\beta)^{-1}\log(\beta/\alpha)$ and define a new measure $\widehat{\mathbf{P}}$ by the equality

$$\widehat{\mathbf{P}}(X=x)=\frac{e^{h_0x}}{\varphi(h_0)}\mathbf{P}(X=x),\ x\in\{\alpha,-\beta\},$$

where $\varphi(h)=\mathbf{E}e^{hX}=(e^{\alpha h}-e^{-\beta h})/2$. Then

$$\mathbf{P}(A_n=a,\tau=n,S_n=0)=(\varphi(h_0))^n\,\widehat{\mathbf{P}}(A_n=a,\tau=n,S_n=0).$$

Combining now (3) with Theorem 6 in [20], we obtain

$$\widehat{\mathbf{P}}(A_n=a,\tau=n,S_n=0)=C(\alpha,\beta)w_{ex}\left(\frac{a}{\sigma n^{3/2}}\right)n^{-3}+o(n^{-3}),$$

where $\sigma^2=\sigma^2(\alpha,\beta)=\widehat{\mathbf{E}}X^2$. (We can not give an analytical expression for the constant $C(\alpha,\beta)$, due to the fact that we do not know exact form of the renewal function of ascending ladder epochs.) As a result we have

$$N(n,a)\approx C(\alpha,\beta)w_{ex}\left(\frac{a}{\sigma n^{3/2}}\right)n^{-3}\left(\left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}+\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)}\right)^n.$$

The proof of (2) is based on the following local limit theorem for the joint distribution of a discrete meander and its area.

Theorem 3. *Assume that the conditions of Theorem 1 are satisfied. Then, for every $z\ge 0$,*

$$\sup_{a\in n(n+1)\rho/2+d\mathbb{Z},x\in n\rho+d\mathbb{Z}}\left|n^2\mathbf{P}_z\left(A_n=a,S_n=x\middle|\tau>n\right)-\frac{d^2}{\sigma^2}h\left(\frac{a}{\sigma n^{3/2}},\frac{x}{\sigma n^{1/2}}\right)\right|\to 0,$$

where $h(u,v)$ is the density function of the vector $(\int_0^1 M_t dt, M_1)$ and P_z is the distribution of the walk starting at z.

Part of the proof of this theorem consists in showing that the distribution of the vector $(\int_0^1 M_t dt, M_1)$ has a continuous density.

Corollary 4. *As $n\to\infty$,*

$$\sup_{a\in n(n+1)\rho/2+d\mathbb{Z}}\left|n^2\mathbf{P}(A_n=a|\tau>n)-\frac{d}{\sigma}w_{me}\left(\frac{a}{\sigma n^{3/2}}\right)\right|\to 0,$$

where w_{me} is the density of $\int_0^1 M_t dt$.

This result is a local counerpart of Theorem 4 in Takacs [18].

2. Brownian meander and its area

As it was mentioned above one of the steps in the proof of Theorem 3 consists in the investigation of the distribution of $(\int_0^1 M_t dt, M_1)$. The next proposition contains all the properties of this distribution which are needed in the proof of Theorem 3.

Proposition 5. *The joint distribution of $\int_0^1 M_t dt$ and M_1 is absolutely continuous with a continuous density $h(u,v)$. There exists a measure ν such that*

$$h(u,v) = \sqrt{\frac{\pi}{2}}\left(6u - 2v + \sqrt{\frac{2}{\pi}}\right) p_1(0,0;u,v) \tag{4}$$
$$+ \sqrt{\frac{\pi}{2}}\int_0^1\int_0^\infty \nu(ds,dz)\left[p_1(0,0;u,v) - p_{1-s}(0,0;u-z,v)\right].$$

Proof. Set $I_t = \int_0^t B_s ds$. Let

$$p_t(x,y;u,v) = \frac{3}{\sqrt{\pi}t^2}\exp\left\{-\frac{6(u-x-ty)^2}{t^3} + \frac{6(u-x-ty)(v-y)}{t^2} - \frac{2(v-y)^2}{t}\right\}$$

be the transition function of the process $(I_t, B_t)_{t\ge 0}$ and define

$$\overline{p}_t(x,y;u,v) = \frac{\mathbf{P}_{(x,y)}(I_t\in du, B_t\in dv, \tau > t)}{dudv},$$

where $\tau := \inf\{t > 0 : B_t = 0\}$.

Using the strong Markov property it can be easily seen that

$$\begin{aligned}\overline{p}_1(0,\varepsilon;u,v) &= p_1(0,\varepsilon;u,v) - \int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds, I_s\in dz)p_{1-s}(z,0,u,v)\\ &= p_1(0,\varepsilon;u,v) - p_1(0,0;u,v)\mathbf{P}_{(0,\varepsilon)}(\tau\le 1) \qquad (5)\\ &\quad+ \int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds, I_s\in dz)\left[p_1(0,0;u,v) - p_{1-s}(z,0,u,v)\right].\end{aligned}$$

Since

$$\mathbf{P}_{(0,\varepsilon)}(\tau > 1) = 2\Phi(\varepsilon) - 1 \sim \sqrt{\frac{2}{\pi}}\varepsilon \quad \varepsilon\to 0,$$

we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}p_1(0,0;u,v)\mathbf{P}_{(0,\varepsilon)}(\tau > 1) = \sqrt{\frac{2}{\pi}}p_1(0,0;u,v).$$

Furthermore, by Taylor's formula,

$$\begin{aligned}&p_1(0,\varepsilon;u,v) - p_1(0,0;u,v)\\ &= \frac{3}{\sqrt{\pi}}\left(\exp\left\{-6(u-\varepsilon)^2 + 6(u-\varepsilon)(v-\varepsilon) - 2(v-\varepsilon)^2\right\} - \exp\left\{-6u^2 + 6uv - 2v^2\right\}\right)\\ &= \frac{3}{\sqrt{\pi}}\exp\left\{-6u^2 + 6uv - 2v^2\right\}(12u - 6u - 6v + 4v)\varepsilon + O(\varepsilon^2),\end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(p_1(0,\varepsilon;u,v) - p_1(0,0;u,v)\right) = p_1(0,0;u,v)(6u-2v).$$

As a result,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(p_1(0,\varepsilon;u,v)-p_1(0,0;u,v)\mathbf{P}_{(0,\varepsilon)}(\tau\le 1)\right)$$
$$=p_1(0,0;u,v)\left(6u-2v+\sqrt{\frac{2}{\pi}}\right). \qquad (6)$$

In order to deal with the integral term in (5) we write

$$\begin{aligned}
&\int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds, I_s\in dz)\left[p_1(0,0;u,v)-p_{1-s}(z,0;u,v)\right]\\
&=\int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds, I_s\in dz)\left[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)\right]\\
&\quad+\int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds, I_s\in dz)\left[p_{1-s}(0,0;u,v)-p_{1-s}(z,0;u,v)\right]\\
&=\int_0^{1/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\\
&\quad+\int_0^{1/\varepsilon^2}\int_0^\infty \mathbf{P}_{(0,1)}(\tau\in ds, I_s\in dz)\left[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\right],
\end{aligned}$$

where the last equality follows from the Brownian scaling. Fix some $r\in(0,1/2)$ and write

$$\begin{aligned}
&\int_0^{1/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\\
&=\int_0^{r/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\\
&\qquad+\int_{r/\varepsilon^2}^{1/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right].
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
\frac{\partial}{\partial t}p_t(0,0;u,v)=&-\frac{6}{\sqrt{\pi}t^3}\exp\left\{-\frac{6u^2}{t^3}+\frac{6uv}{t^2}-\frac{2v^2}{t}\right\}\\
&+\frac{2}{\sqrt{\pi}t^2}\exp\left\{-\frac{6u^2}{t^3}+\frac{6uv}{t^2}-\frac{2v^2}{t}\right\}\left(\frac{18u^2}{t^4}-\frac{12uv}{t^3}+\frac{2v^2}{t^2}\right).
\end{aligned}$$

Noting that this derivative is uniformly bounded, we infer from Taylor's formula that

$$\sup_{s\le 1/2} s^{-1}|p_1(0,0;u,v)-p_{1-s}(0,0;u,v)|\le C. \qquad (7)$$

Combining this with the estimate $\frac{\mathbf{P}_{(0,1)}(\tau\in ds)}{ds}\le(2\pi)^{-1/2}s^{-3/2}$, we obtain

$$\left|\int_0^{r/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\right|$$
$$\le C\int_0^{r/\varepsilon^2} s^{-3/2}\varepsilon^2 s\,ds=2C\sqrt{r}\varepsilon. \qquad (8)$$

Using the relation

$$\frac{\mathbf{P}_{(0,1)}(\tau\in ds)}{ds}\sim(2\pi)^{-1/2}s^{-3/2}\quad\text{as } s\to\infty,$$

we get, as $\varepsilon\to 0$,

$$\begin{aligned}&\int_{r/\varepsilon^2}^{1/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\\&=\frac{1+o(1)}{\sqrt{2\pi}}\int_{r/\varepsilon^2}^{1/\varepsilon^2}s^{-3/2}\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]ds\\&=\frac{1+o(1)}{\sqrt{2\pi}}\varepsilon\int_r^1 s^{-3/2}\left[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)\right]ds.\end{aligned}$$

Using (7) once again we conclude that

$$\begin{aligned}&\lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{r/\varepsilon^2}^{1/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\\&\qquad=\frac{1}{\sqrt{2\pi}}\int_0^1 s^{-3/2}\left[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)\right]ds.\end{aligned}$$

Combining this relation with (8), we finally get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{1/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)\left[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)\right]\\&\qquad=\frac{1}{\sqrt{2\pi}}\int_0^1 s^{-3/2}\left[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)\right]ds.\end{aligned}\tag{9}$$

We now turn to the integral

$$\int_0^{1/\varepsilon^2}\int_0^\infty\mathbf{P}_{(0,1)}(\tau\in ds,I_s\in dz)\left[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\right].$$

Since the derivative

$$\frac{\partial}{\partial u}p_t(0,0;u,v)=\frac{3}{\sqrt{\pi}t^2}\left\{-\frac{6u^2}{t^3}+\frac{6uv}{t^2}-\frac{2v^2}{t}\right\}\left(\frac{6v}{t^2}-\frac{12u}{t^3}\right)$$

is uniformly bounded in t,

$$|p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)|\le C(u,v)\varepsilon^3 z.$$

Therefore,

$$\begin{aligned}&\left|\int_0^{1/\varepsilon^2}\int_0^{r/\varepsilon^3}\mathbf{P}_{(0,1)}(\tau\in ds,I_s\in dz)\left[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\right]\right|\\&\qquad\qquad\le C\varepsilon^3\int_0^{r/\varepsilon^3}z\mathbf{P}_{(0,1)}(I_\tau\in dz).\end{aligned}$$

According to formula (2.10) in Isozaki and Watanabe [8]

$$\frac{\mathbf{P}_{(0,1)}(I_\tau\in dz)}{dz}=\frac{2^{1/3}}{3^{2/3}\Gamma(1/3)}z^{-4/3}\exp\{-2/9z\},\quad z>0.$$

This implies that

$$\int_0^{r/\varepsilon^3} z\mathbf{P}_{(0,1)}(I_\tau \in dz) \le \frac{2^{1/3}}{3^{2/3}\Gamma(1/3)} \int_0^{r/\varepsilon^3} z^{-1/3}dz = \frac{3^{1/3}}{2^{2/3}\Gamma(1/3)} r^{2/3}\varepsilon^{-2}$$

and, consequently,

$$\left| \int_0^{1/\varepsilon^2} \int_0^{r/\varepsilon^3} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz) \left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v) \right] \right| \\ \le Cr^{2/3}\varepsilon. \tag{10}$$

Since p_t is uniformly bounded in all variables,

$$\left| \int_0^{r/\varepsilon^2} \int_{r/\varepsilon^3}^{\infty} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz) \left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v) \right] \right| \\ \le C\mathbf{P}_{(0,1)} \left(\tau \le r\varepsilon^{-2}, I_\tau \ge r\varepsilon^{-3} \right) \le C\mathbf{P}_{(0,1)} \left(\tau \le r\varepsilon^{-2}, \max_{t \le \tau} B_t \ge \varepsilon^{-1} \right),$$

where in the last step we used the bound $I_\tau \le \tau \max_{t\le\tau} B_t$. Applying now a good-$\lambda$-inequality (see Durrett [6, p.153]) and Doob's inequality, we have

$$\mathbf{P}_{(0,1)} \left(\tau \le r\varepsilon^{-2}, \max_{t \le \tau} B_t \ge \varepsilon^{-1} \right) \le 4r\mathbf{P}_{(0,1)} \left(\max_{t \le \tau} B_t \ge \varepsilon^{-1} \right) \le 8r\varepsilon.$$

Therefore,

$$\left| \int_0^{r/\varepsilon^2} \int_{r/\varepsilon^3}^{\infty} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz) \left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v) \right] \right| \\ \le Cr\varepsilon. \tag{11}$$

It remains to consider the integral

$$\int_{r/\varepsilon^2}^{1/\varepsilon^2} \int_{r/\varepsilon^3}^{\infty} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz) \left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v) \right].$$

We start with the Laplace transform of the function $\mathbf{P}_{(0,1)}(\tau > t, I_\tau > z)$. It is easy to verify that

$$F(\lambda,\mu) := \lambda\mu \int_0^\infty \int_0^\infty e^{-\lambda t - \mu z} \mathbf{P}_{(0,1)}(\tau > t, I_\tau > z) dt dz \\ = 1 - \mathbf{E}_{(0,1)}[e^{-\lambda\tau}] - \mathbf{E}_{(0,1)}[e^{-\mu I_\tau}] + \mathbf{E}_{(0,1)}[e^{-\lambda\tau - \mu I_\tau}].$$

It is well-known that

$$\mathbf{E}_{(0,1)}[e^{-\lambda\tau}] = e^{-\sqrt{2\lambda}}.$$

Furthermore, for all positive μ one has, see [12, Theorem 1],

$$\mathbf{E}_{(0,1)}[e^{-\lambda\tau - \mu I_\tau}] = \frac{\mathrm{Ai}((2\mu)^{1/3} + 2\lambda/((2\mu)^{2/3}))}{\mathrm{Ai}(2\lambda/((2\mu)^{2/3}))}.$$

From these equalities we conclude that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} F(\varepsilon^2\lambda, \varepsilon^3\mu) = \sqrt{2\lambda} - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}(2\mu)^{1/3} + \frac{\mathrm{Ai}'(2\lambda/((2\mu)^{2/3})}{\mathrm{Ai}(2\lambda/((2\mu)^{2/3})}(2\mu)^{1/3}. \tag{12}$$

According to Theorem 2.1(i) in Omey and Willekens [11], the latter convergence implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbf{P}_{(0,1)}(\tau > t\varepsilon^{-2}, I_\tau > z\varepsilon^{-3}) = G(t,z), \tag{13}$$

where the function G is determined by the right hand side in (12).

By the fundamental theorem of calculus we have

$$\begin{aligned} &p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v) \\ &= \int_0^{\varepsilon^3 z}\int_0^{\varepsilon^2 s}\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)dqdw - \int_0^{\varepsilon^3 z}\frac{\partial}{\partial w}p_1(0,0;u-w,v). \end{aligned} \tag{14}$$

Using this representation and exchanging the integrals, we get

$$\begin{aligned} &\int_{r/\varepsilon^2}^{1/\varepsilon^2}\int_{r/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in ds, I_\tau\in dz)\left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\right] \\ &= \int_0^1\int_0^\infty \frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)\int_{(q\vee r)/\varepsilon^2}^{1/\varepsilon^2}\int_{(w\vee r)/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in ds, I_\tau\in dz)dwdq \\ &\qquad - \int_0^\infty\frac{\partial}{\partial w}p_1(0,0;u-w,v)\int_{(w\vee r)/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in(r/\varepsilon^2,1/\varepsilon^2), I_\tau\in dz)dw \\ &= \int_0^1\int_0^\infty\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)\mathbf{P}_{(0,1)}\left(\tau\in\left(\frac{q\vee r}{\varepsilon^2},\frac{1}{\varepsilon^2}\right), I_\tau > \frac{w\vee r}{\varepsilon^3}\right)dwdq \\ &\qquad - \int_0^\infty\frac{\partial}{\partial w}p_1(0,0;u-w,v)\mathbf{P}_{(0,1)}\left(\tau\in\left(\frac{r}{\varepsilon^2},\frac{1}{\varepsilon^2}\right), I_\tau > \frac{w\vee r}{\varepsilon^3}\right)dw \end{aligned}$$

Applying now (13), we obtain

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{r/\varepsilon^2}^{1/\varepsilon^2}\int_{r/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in ds, I_\tau\in dz)\left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\right] \\ &\to \int_0^1\int_0^\infty\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)(G(q\vee r, w\vee r) - G(1,w\vee r))dwdq \\ &\qquad - \int_0^\infty\frac{\partial}{\partial w}p_1(0,0;u-w,v)(G(r,w\vee r) - G(1,w\vee r))dw \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{15}$$

Let ν denote the measure which corresponds to G, that is,

$$G(t,z) = \int_t^\infty\int_z^\infty \nu(ds,dy).$$

Using this representation and (14) with $\varepsilon = 1$, we can rewrite the limit in (15) in the following way

$$\int_r^1\int_r^\infty \nu(ds,dz)\left[p_{1-s}(0,0;u,v) - p_{1-s}(0,0;u-z,v)\right].$$

Letting here $r\to 0$ and taking into account (10) and (11) we conclude that

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^{1/\varepsilon^2}\int_0^\infty\mathbf{P}_{(0,1)}(\tau\in ds, I_\tau\in dz)\left[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\right] \\ &\to \int_0^1\int_0^\infty\nu(ds,dz)\left[p_{1-s}(0,0;u,v) - p_{1-s}(0,0;u-z,v)\right] \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{16}$$

Substituting (6), (9) and (16) into (5), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\overline{p}_1(0,\varepsilon;u,v) = &\left(6u-2v+\sqrt{\frac{2}{\pi}}\right)p_1(0,0;u,v)\\ &+\frac{1}{\sqrt{2\pi}}\int_0^1 s^{-3/2}[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)]\\ &+\int_0^1\int_0^\infty \nu(ds,dz)\,[p_{1-s}(0,0;u,v)-p_{1-s}(0,0;u-z,v)]\end{aligned}$$

Noting that $(2\pi)^{-1/2}s^{-3/2}ds=\int_0^\infty \nu(ds,dz)$, we see that the limit is equal to

$$\left(6u-2v+\sqrt{\frac{2}{\pi}}\right)p_1(0,0;u,v)+\int_0^1\int_0^\infty \nu(ds,dz)\,[p_1(0,0;u,v)-p_{1-s}(0,0;u-z,v)]\,.$$

To complete the proof of (4) it remains to observe that

$$h(u,v)=\lim_{\varepsilon\to 0}\frac{\overline{p}_1(0,\varepsilon;u,v)}{\mathbf{P}_{(0,\varepsilon)}(\tau>1)}$$

and that $\mathbf{P}_{(0,\varepsilon)}(\tau>1)\sim\sqrt{\frac{2}{\pi}}\varepsilon$. The continuity of the density follows from the fact that all the limits are locally uniform in u and v. $\square$

The existence of the density $h(u,v)$ can be seen as follows. Denote

$$\psi^s_{me}(x,y,t)=\mathbf{E}_x[e^{-sI_t},B_t\in dy,\tau>t]/dy$$

By the Feynmann-Kac formula we conclude that the generator $\mathcal{A}_s$ of the semigroup corresponding to $\psi^s_{me}(x,y,t)$ is given by the differential operator

$$\mathcal{A}_s=\frac{1}{2}\frac{\partial^2}{\partial x^2}-sy$$

with the Dirichlet boundary condition. it is well-known that the spectrum of this operator is purely discrete, and its eigenvalues $-\lambda_n$ can be found by the solving the equation

$$f''(y)-2syf(y)=-\lambda_n f(y)$$

with the boundary condition $f(0)=0$. The general solution is given by $\mathrm{Ai}((2s)^{1/3}y-2^{1/3}\lambda/s^{2/3})$. In order to satisfy the boundary condition we need to require $\lambda_n=a_n s^{2/3}/2^{1/3}$, where $-a_n$ are zeros of the Airy function.

The sequence $(2s)^{1/6}\mathrm{Ai}(y(2s)^{1/3}-a_n)/\mathrm{Ai}'(-a_n)$ is orthonormal, see [19, Section 4.4] for more details. Therefore, by diagonalisation of the self-ajoint operator $\mathcal{A}_s$,

$$\psi^s_{me}(x,y,t)=-\sum_{n=1}^\infty e^{-2^{-1/3}s^{2/3}a_n t}(2s)^{1/3}\frac{\mathrm{Ai}(y(2s)^{1/3}-a_n)\mathrm{Ai}(x(2s)^{1/3}-a_n)}{(\mathrm{Ai}'(-a_n))^2}.$$

As $x\to 0$,

$$\frac{\psi^s_{me}(x,y,1)}{\mathbf{P}_x(\tau>1)}\to\sqrt{\frac{\pi}{2}}(2s)^{2/3}\sum_{n=1}^\infty e^{-2^{-1/3}s^{2/3}a_n}\frac{\mathrm{Ai}(y(2s)^{1/3}-a_n)}{\mathrm{Ai}'(-a_n)}.$$

Consequently,

$$\mathbf{E}[e^{-s\int_0^1 M_t dt}, M_1 \in dy]/dy = \sqrt{\frac{\pi}{2}}(2s)^{2/3}\sum_{n=1}^{\infty} e^{-2^{-1/3}s^{2/3}a_n}\frac{\mathrm{Ai}(y(2s)^{1/3}-a_n)}{\mathrm{Ai}'(-a_n)}. \tag{17}$$

Integrating over y we get the formula for the Laplace transform of the area of the standard meander, see also formula (209) in Janson [9],

$$\mathbf{E}[e^{-s\int_0^1 M_t dt}] = \sqrt{\frac{\pi}{2}}(2s)^{1/3}\sum_{n=1}^{\infty} r_n e^{-2^{-1/3}s^{2/3}a_n},$$

where

$$r_n := \frac{1}{\mathrm{Ai}'(-a_n)}\int_{-a_n}^{\infty}\mathrm{Ai}(z)dz,\ n\ge 1.$$

Setting $s = -ir = e^{-i\pi/2}r$ in (17) and noting that the real part of $(e^{-i\pi/2}r)^{2/3}$ is always positive, we conclude that

$$\mathbf{E}[e^{ir\int_0^1 M_t dt}, M_1 \in dy]/dy = \sqrt{\frac{\pi}{2}}(-ir)^{2/3}\sum_{n=1}^{\infty} e^{-2^{-1/3}(-ir)^{2/3}a_n}\frac{\mathrm{Ai}(y(-2ir)^{1/3}-a_n)}{\mathrm{Ai}'(-a_n)}$$

is decreasing exponentially. Therefore, the corresponding measure is absolutely continuous with respect to the Lebesgue measure, and the corresponding density is continuous.

3. Local asymptotics for discrete meanders: Proof of Theorem 3.

First we state some known limit theorems for random walks and discrete meanders.

Proposition 6. *If the variance of S_1 is one, then, for every $B \in \mathcal{B}(\mathbb{R}^2_+)$ and every starting point $z \ge 0$,*

$$\lim_{n\to\infty}\mathbf{P}_z\left(\left(\frac{A_n}{n^{3/2}},\frac{S_n}{n^{1/2}}\right)\in B\Big|\tau>n\right) = \mathbf{P}\left(\left(\int_0^1 M_t dt, M_1\right)\in B\right),$$

where M_t is the Brownian meander.

This convergence is immediate from the functional limit theorem for random walks conditioned to stay positive which was proved by Bolthausen [2].

Another crucial ingredient of the proof of Theorem 3 is the following result.

Proposition 7. *Under the conditions of Theorem 1,*

$$\sup_{a\in n(n+1)\rho/2+d\mathbb{Z}, x\in n\rho+d\mathbb{Z}}\left|n^2\mathbf{P}\left(A_n=a,S_n=x\right)-g\left(\frac{a}{n^{3/2}},\frac{x}{n^{1/2}}\right)\right|\to 0,$$

where $g(u,v)=p_1(0,0;u,v)$ is the density of the vector $(\int_0^1 B_t dt, B_1)$.

A version of this convergence for absolutely continuous distributions has been proved by Caravenna and Deuschel [4]. Since the case of discrete random walks needs only some obvious changes, we omit the proof of this result.

Proposition 7 and the boundednes of g imply the following result.

Corollary 8. *There exists a constant C such that*

$$\sup_{a,x\in\mathbb{Z}}\mathbf{P}\left(A_n=a,S_n=x\right)\le Cn^{-2},\quad n\ge 1. \tag{18}$$

To simplify notation we give a proof of Theorem 3 for $z=0$ only. Moreover, we assume, for the same reason, that $d=1$ and $\rho=0$.

We start by considering various 'boundary' values of a and x. Splitting the trajectory of S_n at $n-m$, we obtain

$$\begin{aligned}
&\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\\
&=\sum_{y,b\ge 1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau>m\right).
\end{aligned}\tag{19}$$

Applying now (18) to probabilities $\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau>m\right)$ and using the following well-known relation

$$\mathbf{P}(\tau>n)\sim\theta n^{-1/2},\tag{20}$$

we get

$$\sup_{a,x\in\mathbb{Z}}\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\le\frac{C}{m^2}\mathbf{P}(\tau>n-m)\le\frac{C}{n^{5/2}}.\tag{21}$$

If $a\le\delta n^{3/2}$ then we infer from (19) with $m=[n/2]$ that

$$\begin{aligned}
\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)&\le\mathbf{P}(A_{n-m}\le a,\tau>n-m)\frac{C}{m^2}\\
&\le\mathbf{P}(A_{n-m}\le\delta n^{3/2}|\tau>n-m)\frac{C}{n^{5/2}}.
\end{aligned}$$

In view of Proposition 6,

$$\mathbf{P}(A_{n-m}\le\delta n^{3/2}|\tau>n-m)\to\mathbf{P}\left(\int_0^1 M_t dt\le 2^{3/2}\delta\right).$$

According to formula (212) in Janson [9],

$$\mathbf{P}\left(\int_0^1 M_t dt\le u\right)\sim c_1e^{-c_2/u^2}\quad\text{as } u\to 0.$$

Consequently,

$$n^{5/2}\sup_{a\le\delta n^{3/2},\,x\ge 1}\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\le Ce^{-1/\delta}.\tag{22}$$

For $a\ge 2Rn^{3/2}$ we have

$$\begin{aligned}
&\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\\
&\qquad=\mathbf{P}\left(A_n=a,S_n=x,\max_{k\le n}S_k\ge 2R\sqrt{n},\tau>n\right)\\
&\qquad=\mathbf{P}\left(A_n=a,S_n=x,\max_{k\le m}S_k\ge R\sqrt{n},\tau>n\right)\\
&\qquad\quad+\mathbf{P}\left(A_n=a,S_n=x,\max_{k\le n-m}(S_{m+k}-S_m)\ge R\sqrt{n},\tau>n\right).
\end{aligned}$$

Using the Markov property and (18), we get

$$\begin{aligned}
&\mathbf{P}\left(A_n = a, S_n = x, \max_{k\le m} S_k \ge R\sqrt{n}, \tau > n\right)\\
&\qquad\le \mathbf{P}\left(\max_{k\le m} S_k \ge R\sqrt{n}, \tau > m\right)\sup_{y,b\in\mathbb{Z}} \mathbf{P}(A_{n-m} = b, S_{n-m} = y)\\
&\qquad\le \frac{C}{n^{5/2}}\mathbf{P}\left(\max_{k\le m} S_k \ge R\sqrt{n}\big|\tau > m\right) \le \frac{C}{n^{5/2}}\mathbf{P}\left(\sup_{t\le 1} M_t \ge R\sqrt{2}\right).
\end{aligned}$$

In the last step we used functional limit theorem for random walks conditioned to stay positive. Furthermore, using (21), we obtain

$$\begin{aligned}
&\mathbf{P}\left(A_n = a, S_n = x, \max_{k\le n-m}(S_{m+k} - S_m) \ge R\sqrt{n}, \tau > n\right)\\
&\qquad\le \sup_{y,b\in\mathbb{Z}} \mathbf{P}(A_m = b, S_m = y, \tau > m)\mathbf{P}\left(\max_{k\le n-m}(S_{m+k} - S_m) \ge R\sqrt{n}\right)\\
&\qquad\le \frac{C}{n^{5/2}}\mathbf{P}\left(\max_{t\le 1} B_t > R\sqrt{2}\right).
\end{aligned}$$

As a result we have

$$n^{5/2} \sup_{a\ge 2Rn^{3/2},\, x\ge 1} \mathbf{P}\left(A_n = a, S_n = x, \tau > n\right) \le \Delta(R), \tag{23}$$

where $\Delta(R) \to 0$ as $R \to \infty$. Since for $x \ge 2R\sqrt{n}$ the equality

$$\mathbf{P}\left(A_n = a, S_n = x, \tau > n\right) = \mathbf{P}\left(A_n = a, S_n = x, \max_{k\le n} S_k \ge 2R\sqrt{n}, \tau > n\right)$$

holds, we have

$$n^{5/2} \sup_{a\ge 1,\, x\ge 2R\sqrt{n}} \mathbf{P}\left(A_n = a, S_n = x, \tau > n\right) \le \Delta(R), \tag{24}$$

For $x \le 2\varepsilon\sqrt{n}$ we use an alternative representation for $\mathbf{P}\left(A_n = a, S_n = x, \tau > n\right)$. Set $X_i' := -X_{m+1-i}$, $i \in \{1, 2, \ldots, m\}$ and $S_k' = S_0' + \sum_{i=1}^k X_i'$, $A_k' = \sum_{i=1}^k S_i'$. Then it is easy to see that

$$\begin{aligned}
&\left\{ym + \sum_{i=1}^m S_i = a - b,\ y + S_m = x,\ \min_{i\le m}(y + S_i) > 0\right\}\\
&\quad = \left\{xm + \sum_{i=1}^m S_i' = a - b + y - x,\ x + S_m' = y,\ \min_{i\le m}(x + S_i') > 0\right\}.
\end{aligned}$$

Consequently,

$$\mathbf{P}_y\left(A_m = a - b, S_m = x, \tau > m\right) = \mathbf{P}_x\left(A_m' = a - b, S_m' = y, \tau' > m\right)$$

and

$$\begin{aligned}
\mathbf{P}\left(A_n = a, S_n = x, \tau > n\right) = \sum_{y,b\ge 1} &\mathbf{P}\left(A_{n-m} = b, S_{n-m} = y, \tau > n - m\right)\\
&\times \mathbf{P}_x\left(A_m' = a - b + y - x, S_m' = y, \tau' > m\right).
\end{aligned} \tag{25}$$

From this representation and (21) we conclude that

$$\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\le\frac{C}{(n-m)^{5/2}}\mathbf{P}_x(\tau'>m)\le\frac{C}{n^{5/2}}\mathbf{P}_{2\varepsilon\sqrt{n}}(\tau'>m).$$

It is immediate from the functional CLT that $\mathbf{P}_{2\varepsilon\sqrt{n}}(\tau'>[n/2])\le C\varepsilon$. Therefore,

$$n^{5/2}\sup_{a\ge1,\,x\le2\varepsilon\sqrt{n}}\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\le C\varepsilon. \tag{26}$$

We now turn to 'normal' values for the vector (A_n,S_n), that is,

$$\delta n^{3/2}\le a\le 2Rn^{3/2}\quad\text{and}\quad 2\varepsilon\sqrt{n}\le x\le 2R\sqrt{n}.$$

For every x define

$$B_1=B_1(x):=\{y\ge1:|y-x|\le\varepsilon\sqrt{n}\}\quad\text{and}\quad B_2=B_2(x):=\mathbb{Z}_+\setminus B_1(x).$$

For every $m\ge1$ we have

$$\begin{aligned}\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)&=\mathbf{P}\left(A_n=a,S_n=x,S_{n-m}\in B_1,\tau>n\right)\\&\quad+\mathbf{P}\left(A_n=a,S_n=x,S_{n-m}\in B_2,\tau>n\right).\end{aligned}\tag{27}$$

Set $m=[\varepsilon^3n]$. Then, applying (21), we obtain, uniformly in $a,x\ge1$,

$$\begin{aligned}&\mathbf{P}\left(A_n=a,S_n=x,S_{n-m}\in B_2,\tau>n\right)\\&=\sum_{y\in B_2,b\ge1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau>m\right)\\&\le\frac{C}{n^{5/2}}\sum_{y\in B_2}\mathbf{P}_y(S_m=x)\le\frac{C}{n^{5/2}}\mathbf{P}(|S_m|>\varepsilon\sqrt{n})\le\frac{C}{n^{5/2}}\overline{\Phi}(\varepsilon^{-1/2}),\end{aligned}\tag{28}$$

where $\overline{\Phi}(x)=\int_x^\infty\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du$.

Further,

$$\begin{aligned}&\mathbf{P}\left(A_n=a,S_n=x,S_{n-m}\in B_1,\tau>n\right)\\&=\sum_{y\in B_1,b\ge1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau>m\right)\\&=\sum_{y\in B_1,b\ge1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x\right)\\&\quad-\sum_{y\in B_1,b\ge1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau\le m\right)\end{aligned}$$

Applying (21) to the probabilities in the second sum, we obtain

$$\begin{aligned}&\sum_{y\in B_1,b\ge1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau\le m\right)\\&\qquad\le\frac{C}{n^{5/2}}\sum_{y\in B_1}\mathbf{P}_y\left(S_m=x,\tau\le m\right)=\frac{C}{n^{5/2}}\mathbf{P}_x\left(S'_m\in B_1,\tau'\le m\right).\end{aligned}$$

For $x\ge2\varepsilon\sqrt{n}$ we have

$$\mathbf{P}_x\left(S'_m\in B_1,\tau'\le m\right)\le\mathbf{P}\left(\max_{k\le m}S_k>2\varepsilon\sqrt{n}\right)\le C\overline{\Phi}(\varepsilon^{-1/2}).$$

Therefore,

$$\sum_{y\in B_1,b\ge 1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x,\tau\le m\right)$$
$$\le \frac{C}{n^{5/2}}\overline{\Phi}(\varepsilon^{-1/2}). \tag{29}$$

It follows from Proposition 7 that

$$\sum_{y\in B_1,b\ge 1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)\mathbf{P}_y\left(A_m=a-b,S_m=x\right)$$
$$=\sum_{y\in B_1,b\ge 1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)m^{-2}g\left(\frac{a-b-my}{m^{3/2}},\frac{x-y}{m^{1/2}}\right)$$
$$+o\left(m^{-2}\mathbf{P}(\tau>n-m)\right) \tag{30}$$

uniformly in $a,x\ge 1$. Recalling (20), we get

$$\sum_{y\in B_1,b\ge 1}\mathbf{P}\left(A_{n-m}=b,S_{n-m}=y,\tau>n-m\right)m^{-2}g\left(\frac{a-b-my}{m^{3/2}},\frac{x-y}{m^{1/2}}\right)$$
$$=\frac{1-\varepsilon^3}{\varepsilon^6}\frac{\theta}{n^{5/2}}\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 ny}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)1\{S_{n-m}\in B_1\}\middle|\tau>n-m\right].$$

Since $g(u,v)\to 0$ as $v\to\infty$ uniformly in u,

$$\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 ny}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)1\{S_{n-m}\in B_2\}\middle|\tau>n-m\right]\le r_1(\varepsilon), \tag{31}$$

where $r_1(\varepsilon)\to 0$ as $\varepsilon\to 0$. According to Proposition 6,

$$\lim_{n\to\infty}\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 ny}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)\middle|\tau>n-m\right]$$
$$=\int_{\mathbb{R}_+^2}g\left(\frac{an^{-3/2}-(1-\varepsilon^3)^{3/2}u}{\varepsilon^{9/2}},\frac{xn^{-1/2}-(1-\varepsilon^3)^{1/2}v}{\varepsilon^{3/2}}\right)h(u,v)dudv.$$

Furthermore, as $\varepsilon\to 0$,

$$\varepsilon^{-6}\int_{\mathbb{R}_+^2}g\left(\frac{b-(1-\varepsilon^3)^{3/2}u}{\varepsilon^{9/2}},\frac{y-(1-\varepsilon^3)^{1/2}v}{\varepsilon^{3/2}}\right)h(u,v)dudv\to h(b,y)$$

locally uniformly in b,y. From this convergence and (31) we infer

$$\limsup_{n\to\infty}\left|\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 ny}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)1\{S_{n-m}\in B_1\}\middle|\tau>n-m\right]\right.$$
$$\left.-h\left(\frac{a}{n^{3/2}},\frac{x}{n^{1/2}}\right)\right|\le r_2(\varepsilon). \tag{32}$$

Combining (27)–(30) and (32), we conclude that

$$\limsup_{n\to\infty}\left|n^{5/2}\mathbf{P}(A_n=a,S_n=x,\tau>n)-\theta h\left(\frac{a}{n^{3/2}},\frac{x}{n^{1/2}}\right)\right|\le r_3(\varepsilon)$$

uniformly in $\delta n^{3/2}\le a\le 2Rn^{3/2}$ and $2\varepsilon\sqrt{n}\le x\le 2R\sqrt{n}$. Taking into account (26)–(24), we arrive at the desired local asymptotic.

4. Proof of Theorem 1

We are going to split the path of the excursion and to inverse the time in the second half of the path. For this reason we need information on the position of our random walk immediately before τ occurs. Let $H(x)$ be the renewal function corresponding to strict ascending ladder epochs.

Lemma 9. *For every fixed $x \in \mathbb{Z}_+$,*

$$\mathbf{P}(S_n = x, \tau = n+1) \sim \frac{H(x)}{\sqrt{2\pi}} n^{-3/2} P(X < -x).$$

Furthermore,

$$\mathbf{P}(\tau = n+1) \sim \left(\sum_{x>0} H(x) P(X < -x) \right) (2\pi)^{-1/2} n^{-3/2}.$$

Proof. First we note that

$$\mathbf{P}(S_n = x, \tau = n+1) = \mathbf{P}(S_n = x, \tau > n)\mathbf{P}(X_{n+1} < -x).$$

Further, according to Theorem 6 in [20],

$$\mathbf{P}(S_n = x, \tau > n) \sim \frac{H(x)}{\sqrt{2\pi}} n^{-3/2}.$$

Thus, the first statement is proved.

Obviously,

$$\mathbf{P}(\tau = n+1) = \sum_{x>0} \mathbf{P}(S_n = x, \tau > n)\mathbf{P}(X_{n+1} < -x).$$

Since $\sup_x \mathbf{P}(S_n = x, \tau > n) \le Cn^{-1}$, see Lemma 19 in [20],

$$\begin{aligned}\sum_{x \ge N} \mathbf{P}(S_n = x, \tau > n)\mathbf{P}(X_{n+1} < -x) &\le \frac{C}{n} \sum_{x \ge N} \mathbf{P}(X_{n+1} < -x) \\ &\le \frac{C}{n} \mathbf{E}[|X|, |X| > N], \quad N \ge 0.\end{aligned}$$

From the finiteness of the second moment we infer that there exist δ_n such that $\delta_n \to 0$ and $\mathbf{E}[|X|, |X| > \delta_n n^{1/2}] = o(n^{-1/2})$. Consequently,

$$\sum_{x \ge \delta_n n^{1/2}} \mathbf{P}(S_n = x, \tau > n)\mathbf{P}(X_{n+1} < -x) = o(n^{-3/2}). \tag{33}$$

Using Theorem 6 in [20] once again, we obtain

$$\sum_{x < \delta_n n^{1/2}} \mathbf{P}(S_n = x, \tau > n) = \frac{1}{\sqrt{2\pi}} n^{-3/2}(1 + o(1)) \left(\sum_{x < \delta_n n^{1/2}} H(x)\mathbf{P}(X < -x) \right). \tag{34}$$

Combining (33), (34) and noting that $\sum_{x>0} H(x)\mathbf{P}(X < -x)$ is finite, we finish the proof. □

We are now in position to prove Theorem 1. We start with the representation

$$\begin{aligned}\mathbf{P}\left(A_n=a,\tau=n+1\right)&=\sum_{x=1}^{\infty}\mathbf{P}\left(A_n=a,S_n=x,\tau=n+1\right)\\&=\sum_{x=1}^{\infty}\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\mathbf{P}\left(X\le -x\right).\end{aligned}\tag{35}$$

Using (21) we conclude that there exists $\delta_n\to 0$ such that

$$\begin{aligned}\sum_{x\ge\delta_n n^{1/2}}&\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\mathbf{P}\left(X\le -x\right)\\&\le Cn^{-5/2}\mathbf{E}[-X,X\le-\delta_n n^{1/2}]=o(n^{-3}).\end{aligned}\tag{36}$$

Combining (19) and (21), we get

$$\begin{aligned}\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)&\le Cn^{-5/2}\sum_{y,b\ge1}\mathbf{P}_x\left(A'_{n/2}=a-b,S'_{n/2}=y,\tau'>n/2\right)\\&\le Cn^{-5/2}\mathbf{P}_x\left(\tau'>n/2\right).\end{aligned}$$

According to Corollary 3 from [5], $\mathbf{P}_x\left(\tau'>n/2\right)\le Cxn^{-1/2}$ uniformly in $x\le\delta_n n^{1/2}$. Therefore, uniformly in a,

$$\begin{aligned}\sum_{N\le x\le\delta_n n^{1/2}}&\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\mathbf{P}\left(X\le -x\right)\\&\le Cn^{-3}\mathbf{E}[X^2,X\le -N].\end{aligned}\tag{37}$$

It remains to consider fixed values of x. Applying Theorem 3 to both probabilities on the right hand side of (19), we get, uniformly in a,

$$\begin{aligned}&\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\\&\quad=\frac{H(x)+o(1)}{n^5}\sum_{y,b\ge1}h\left(\frac{2^{3/2}b}{n^{3/2}},\frac{2^{1/2}y}{n^{1/2}}\right)h\left(\frac{2^{3/2}(a-b+y-x)}{n^{3/2}},\frac{2^{1/2}y}{n^{1/2}}\right)\\&\quad=\frac{H(x)+o(1)}{n^3}\int_0^\infty\int_0^{a/n^{3/2}}h\left(2^{3/2}u,2^{1/2}v\right)h\left(2^{3/2}(a/n^{3/2}-u),2^{1/2}v\right)dudv\\&\quad=:\frac{H(x)+o(1)}{n^3}q(a/n^{3/2}).\end{aligned}$$

Summing over x from 1 to N, we get

$$\begin{aligned}n^3\sum_{x=1}^{N}&\mathbf{P}\left(A_n=a,S_n=x,\tau>n\right)\mathbf{P}(X\le -x)\\&=\left(\sum_{x=1}^{N}H(x)\mathbf{P}(X\le -x)\right)q(a/n^{3/2})+o(n^{-3}).\end{aligned}$$

Combining this with (35), (36) and (37), we conclude that

$$n^3\mathbf{P}(A_n=a,\tau=n+1)=q(a/n^{3/2})\sum_{x=1}^{\infty}H(x)\mathbf{P}(X\le -x)+o(n^{-3})$$

uniformly in a. Hence, in view of Lemma 9,

$$n^{3/2}\mathbf{P}(A_n = a|\tau = n+1) = \sqrt{2\pi}q(a/n^{3/2}) + o(n^{-3/2}). \tag{38}$$

Uniqueness of the limit implies that $\sqrt{2\pi}q(x) = w_{ex}(x)$. This completes the proof of the theorem.

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