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Periodic Differential Delay Equation

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# ON PERIODIC SOLUTIONS AND GLOBAL DYNAMICS IN A PERIODIC DIFFERENTIAL DELAY EQUATION

ANATOLI F. IVANOV AND SERGEI I. TROFIMCHUK

ABSTRACT. Several aspects of global dynamics and the existence of periodic solutions are studied for the scalar differential delay equation  $x'(t) = a(t)f(x([t - K]))$ , where  $f(x)$  is a continuous negative feedback function,  $x \cdot f(x) < 0, x \neq 0$ ,  $0 \leq a(t)$  is continuous  $\omega$ -periodic,  $[\cdot]$  is the integer part function, and the integer  $K \geq 0$  is the delay. The case of integer period  $\omega$  allows for a reduction to finite-dimensional difference equations. The dynamics of the latter are studied in terms of corresponding discrete maps, including the partial case of interval maps ( $K = 0$ ).

## 1. INTRODUCTION

This paper deals with the global dynamics of solutions of differential delay equation

$$(1) \quad x'(t) = a(t)f(x([t - K])),$$

where the  $[\cdot]$  is the integer value function, and the non-negative integer  $K$  is the delay. We shall assume throughout the paper that  $f$  is a continuous real-valued function satisfying the negative feedback condition

$$(2) \quad x \cdot f(x) < 0 \quad \text{for all } x \neq 0,$$

and is bounded from one side

$$(3) \quad f(x) \leq M \text{ or } f(x) \geq -M \text{ for all } x \in \mathbb{R} \text{ and some } M > 0.$$

The coefficient  $a(t) > 0$  is a continuous periodic function with integer period  $\omega > 0$

$$(4) \quad a(t + \omega) = a(t) \quad \text{for all } t \in \mathbb{R}.$$

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Equation (1) is closely related to the more general differential delay equation

$$(5) \quad x'(t) = a(t)f(x(t - \tau)),$$

with the same  $f$  and  $a$ , and where  $\tau > 0$  is a constant delay. It can be viewed as a discrete version of equation (5) [5, 7, 11, 14, 15]. While the problem of global dynamics and existence of periodic solutions for general equation (5) is quite difficult to approach, equation (1) appears to be somewhat simpler to study in this regard. Equation (1) falls within the class of differential equations with piecewise constant argument, which have attracted a significant interest in recent years for their qualitative features and range of applications. Various aspects of their dynamics have been studied by many authors. Among those related to present work we would like to mention papers [1, 2, 3, 6, 10, 20].

When  $a(t) = a_0 > 0$  is a constant equation (5) is equivalent to the well studied autonomous equation

$$(6) \quad x'(t) = G(x(t - 1)).$$

It is well known that when  $G$  also satisfies the negative feedback condition (2), is one-sided bounded in the sense of (3), and  $G'(0) < -\pi/2$ , then the differential delay equation (6) has a slowly oscillating periodic solution [9, 12, 13, 16, 17]. The proof of this fact is rather non-trivial; it constitutes a part of an established theory for the existence of periodic solutions of functional differential equations, called the ejective fixed point techniques [9, 12, 17].

It is a natural next step to look for the existence of periodic solutions in similar but periodic functional differential equations of the form (5). As our initial approaches and analyses have indicated the use of the standard techniques of the ejective fixed point theory do not appear immediately applicable to this case. New approaches and techniques seem to be necessary. Our first step in this direction is to study periodic solutions and other dynamical properties of somewhat simpler differential delay equation (1).

The assumption that the period  $\omega$  is integer is crucial for all principal considerations of the paper. It simplifies the dynamics of solutions of equation (1) significantly: they are essentially reduced to the dynamics of finite-dimensional discrete maps (which can be quite complex by themselves). At present we don't have a clear workable idea how to approach the case when the period  $\omega$  is not commensurable with the delay in the equation ( $K$  for Eq. (1) and  $\tau$  for Eq. (5)). Even in the simplest case of  $K = 0$  and  $\omega$  being irrational, the corresponding

equation (1) seems to allow in some cases for the existence of quasi-periodic solutions.

## 2. PRELIMINARIES

We shall be using throughout the paper the standard notions and definitions related to functional differential and difference equations, as well as to interval maps, most of which can be found in monographs [4, 8, 9, 12, 18, 19].

For arbitrary initial function  $\varphi \in \mathcal{C} := C([-K, 0], \mathbb{R})$  the corresponding solution  $x = x(t, \varphi)$  of equation (1) is easily found by successive integration for  $t \geq 0$ . One has

$$x(t) = x(0) + f(\varphi(-K)) \int_0^t a(s) ds \quad \text{for all } t \in [0, 1)$$

with

$$x(1) = x(0) + \int_0^1 a(s) ds f(\varphi(-K)) := x(0) + a_1 \cdot f(\varphi(-K)).$$

Likewise

$$x(t) = x(1) + f(\varphi(-K + 1)) \int_1^t a(s) ds \quad \text{for all } t \in [1, 2)$$

with

$$x(2) = x(1) + \int_1^2 a(s) ds f(\varphi(-K + 1)) := x(1) + a_2 \cdot f(\varphi(-K + 1)),$$

and so on. Thus one can easily see that the solution  $x(t, \varphi), t \geq 0$  depends only on the values  $\varphi(-K), \varphi(-K + 1), \dots, \varphi(-1), \varphi(0)$  of the initial function  $\varphi \in \mathcal{C}$ . In Section 3, based on the above calculations, we derive the explicit form for the translation operator  $S^\omega$  along the solutions in the case of integer values  $\omega$  as a discrete finite-dimensional map on the set of initial values  $\mathbf{x}_0 = [x_0, x_{-1}, \dots, x_{-K}] = [x(0), x(-1), \dots, x(-K)]$ .

The oscillation of solutions of equation (1) is meant in a standard sense. A solution  $x(t)$  is called eventually positive (negative) if there exists  $T \geq 0$  such that  $x(t) > 0$  ( $x(t) < 0$ ) for all  $t > T$ . A nontrivial solution  $x(t)$  ( $x(t) \not\equiv 0$  for all  $t \geq T$ ) is called oscillatory if it is not eventually positive or negative. For every  $T \geq 0$  any oscillating solution  $x(t)$  of equation (1) changes sign on the interval  $[T, \infty)$ . This is easily seen from the positivity of  $a(t)$  and the negative feedback assumption (2) on  $f$ .

The oscillatory nature of solutions of differential delay equations is an important characteristic which can lead to certain implications such as

existence of nontrivial periodic solutions. It is a significant component of the ejective fixed point techniques used to prove the existence of slowly oscillating periodic solutions to equation (6).

**Proposition 2.1** (Eventual Uniform Boundedness). Suppose that nonlinearity  $f$  satisfies assumptions (2) and (3). There is a constant  $M_0$  such that for arbitrary initial function  $\varphi \in \mathcal{C}$  there exists time moment  $t_\varphi$  such that the corresponding solution  $x = x(t, \varphi)$  of equation (1) satisfies  $|x| \leq M_0$  for all  $t \geq t_\varphi$ .

*Proof.* The proof essentially follows from the fact of one-sided boundedness of the nonlinearity  $f$  and the periodicity of  $a(t)$  (therefore, the boundedness). Consider two potential possibilities for any solution  $x$ : (i) it has a finite number of zeros on the positive semiaxis  $\mathbb{R}^+ := \{t : t \geq 0\}$ , and (ii) it oscillates on  $\mathbb{R}^+$ .

In case (i), to be definite one can assume that  $x > 0$  in  $(t_0, t_0 + K + 1]$ , where  $t_0$  is the largest zero of  $x$  in  $\mathbb{R}^+$ . Then  $x(t)$  is decreasing in  $[t_0 + K + 1, +\infty)$  with  $\lim_{t \rightarrow \infty} x(t) = 0$ .

In case (ii), to be definite, assume that  $f$  is bounded from below,  $f(x) \geq -M$  for all  $x \in \mathbb{R}$  and some  $M > 0$ . Let  $a^* := \max\{a(t), t \in [0, \omega]\}$ . If  $t_0 \geq 0$  is a zero of the solution  $x(t)$ , then  $x(t) \geq -a^*M(t - t_0)$  for all  $t \in [t_0, t_0 + K + 1]$ , implying that  $x(t) \geq -a^*M(K + 1)$  holds there. We claim that the inequality  $x(t) \geq -a^*M(K + 1)$  is satisfied for all  $t \geq t_0$ .

Indeed, consider first the case when  $x(t_0 + K + 1) < 0$ . If  $x(t) < 0$  for all  $t \in (t_0, t_0 + K + 1]$  then  $x(t)$  is increasing in  $[t_0 + K + 1, t_1]$  where  $t_1$  is the first zero of  $x(t)$  following the point  $t_0 + K + 1$ . Therefore,  $x(t) \geq -a^*M(K + 1)$  holds in between the two consecutive zeros  $t_0$  and  $t_1$  for which also  $t_1 - t_0 \geq K + 1$  is satisfied. In case when  $x(t)$  has other zeros in the interval  $(t_0, t_0 + K + 1)$ , choose  $t^0$  as the rightmost zero there, and repeat the above reasoning to show that  $x(t) \geq -a^*M(K + 1)$  for all  $t \in [t^0, t^0 + K + 1]$ .

In the case of  $x(t_0 + K + 1) = 0$  one considers  $t_0 + K + 1$  as the new value of the zero  $t_0$  and repeats the reasoning to confirm the validity of the inequality for the values of  $t$  in  $[t_0 + K + 1, t_0 + 2K + 2]$ .

In the case when  $x(t_0 + K + 1) > 0$  there exists the next zero  $t_1 > t_0$  with  $x(t) > 0$  for all  $t \in (t_0, t_1)$ . This situation is reduced then to the previous case when  $x(t_0 + K + 1) = 0$ .

Thus, a sequence of zeros  $t_k, k = 0, 1, 2, \dots$  of the solution  $x(t)$  can be identified such that  $t_{k+1} - t_k \geq K + 1$  and  $x(t) \geq -a^*M(K + 1)$  for all  $t \in (t_{k+1}, t_k)$ .

Since the oscillating solution  $x(t)$  is now bounded from below for all  $t \geq t_0$ , the negative feedback assumption (2) implies that it is

also bounded from above for all  $t \geq t_1$ , where  $t_1$  is a zero satisfying  $t_1 \geq t_0 + K + 1$ . Indeed, this follows from the corresponding upper bound for its derivative and the reasoning similar to that in the first part of the proof. If one sets  $M^* := \max\{f(x), x \in [-M(K + 1), 0]\}$  then  $x'(t) \leq a^*M^*$  holds for all  $t \geq t_0 + K + 1$ .  $\square$

The following two propositions are closely related; they essentially reflect the fact that all solutions of equation (1) oscillate when either the coefficient  $a(t)$  or the derivative  $|f'(0)|$  is sufficiently large.

**Proposition 2.2** (Oscillation). Let  $f'(0) = f_0 < 0$  be fixed. There exists  $a_0 > 0$  such that for arbitrary  $\omega$ -periodic function  $a(t)$  with  $a(t) \geq a_0$  all solutions of equation (1) oscillate.

**Proposition 2.3** (Oscillation). Let the  $\omega$ -periodic function  $a(t) > 0$  be fixed. There exists  $f_0 < 0$  such that for arbitrary function  $f(x)$  with  $f'(0) < f_0$  all solutions of equation (1) oscillate.

*Proof.* The proofs of both Propositions 2.2 and 2.3 are straightforward and similar. They use a simple comparison argument which has been used multiple times in many other papers. We provide its outline here for the sake of completeness.

Indeed, assuming say  $x(t) > 0$  for all  $t \geq t_0$ , one sees that  $x(t)$  is decreasing for  $t \geq t_0 + K$  with  $\lim_{t \rightarrow +\infty} x(t) = 0$ . Comparing equation (1) with its linearization about  $x(t) \equiv 0$  one concludes that for arbitrary  $\varepsilon > 0$  there exists time moment  $t_*$  such that for all  $t \geq t_*$  the following holds

$$x'(t) < a^*[f'(0) + \varepsilon]x(t - K - 1) \text{ where } a^* := \max\{a(t), t \in [0, \omega]\}.$$

Integrating the last inequality on the interval  $[t - K - 1, t]$  and letting  $t \rightarrow +\infty$  one arrives at the estimate  $x(t) \leq [1 + a_0 f'(0)]x(t - K - 1)$ . The latter one contradicts the positiveness of  $x(t)$  when  $a_0 f'(0) < -1$ . The case of  $x(t)$  being eventually negative is treated in a completely analogous way.  $\square$

Note that Propositions 2.1, 2.2, and 2.3 are valid for the general equation (5). The proofs are similar to those above and are left to the reader.

### 3. MAIN RESULTS

**3.1. Shift by Time  $\omega$  Operator: Integer Period.** In this subsection we shall explicitly calculate the form of the shift-by-period operator along solutions of differential delay equation (1), in the case when the period  $\omega$  is a positive integer,  $\omega = N$ .

Define real numbers  $a_i, i = 1, 2, \dots, N$ , by the following integral values over the coefficient  $a(t)$

$$a_i = \int_{i-1}^i a(t) dt.$$

Introduce next the sequence of maps  $F_i$  of the Euclidean space  $\mathbb{R}^{K+1}$  into itself by

$$(7) \quad F_i := [u_0, u_1, \dots, u_{K-1}, u_K] \mapsto [u_0 + a_i f(u_K), u_0, u_1, \dots, u_{K-1}]$$

with the composite map  $F$  defined by

$$F := F_N \circ F_{N-1} \circ \dots \circ F_2 \circ F_1.$$

**Theorem 1** (Existence and Stability of Periodic Solutions).

Differential delay equation (1) has a periodic solution if and only if the map  $F$  has a non-trivial cycle. It has a periodic solution with period  $\omega$  if and only if the map  $F$  has a fixed point different from  $\mathbf{u}_* = [0, \dots, 0]$ . Moreover, the stability of any such periodic solution is the same as the stability of the corresponding cycle of  $F$ .

*Proof.* The proof is straightforward since the shift operator along solutions of equation (1) in the case  $\omega = N$  is equivalent to the map  $F$ . Indeed, for arbitrary  $t_0 \in [0, \omega]$  one finds the solution satisfying the initial condition  $x(t_0) = x_0$  by

$$x(t) = x_0 + \int_{t_0}^t a(s) f(x([s - K])) ds.$$

In particular, for any integer point  $t_0 = i \in [0, \omega]$  one has

$$(8) \quad x(t) = x(i) + \left( \int_i^t a(s) ds \right) f(x(i - K)), \quad \forall t \in [i, i + 1).$$

Let an initial function  $\varphi(s) \in \mathcal{C} = C([-K, 0], \mathbb{R})$  be given. Set  $\varphi(0) = x_0, \varphi(-1) = x_{-1}, \dots, \varphi(-K) = x_{-K}$ . As it is shown in the introduction, the corresponding solution  $x(t, \varphi)$  depends on the values  $\{x_0, x_{-1}, \dots, x_{-K}\}$  only, and does not depend on values of  $\varphi(t)$  at other non-integer times  $t \in [-K, 0]$ . By using (8) one easily finds for  $t \in [0, 1)$

$$x(t) = x_0 + \left( \int_0^t a(s) ds \right) f(x_{-K}).$$

At  $t = 1$ , by the continuity, one has

$$x(1) := x_1 = x_0 + \left( \int_0^1 a(t) dt \right) f(x_{-K}) := x_0 + a_1 f(x_{-K}).$$



The shift operator along the solution  $x(t, \varphi)$  by time  $T = 1$  is now defined as

$$\begin{aligned} x_0 &\mapsto x_0 + a_1 f(x_{-K}) := x'_0 \\ x_{-1} &\mapsto x_0 := x'_{-1} \\ x_{-2} &\mapsto x_{-1} := x'_{-2} \\ &\dots \quad \dots \\ x_{-K} &\mapsto x_{-K+1} := x'_{-K}, \end{aligned}$$

which is the map  $F_1$  applied to the point  $[x_0, x_{-1}, \dots, x_{-K}]$ .

Likewise, for  $t \in [1, 2)$  one has

$$x(t) = x_1 + \left( \int_1^t a(s) ds \right) f(x_{-K+1}),$$

with

$$x(2) := x_2 = x_1 + \left( \int_1^2 a(t) dt \right) f(x_{-K+1}) := x_1 + a_2 f(x_{-K+1}).$$

Therefore, the shift along the solution by time  $T = 2$  is given by the map

$$\begin{aligned} x_1 &\mapsto x_1 + a_2 f(x_{-K+1}) \\ x_0 &\mapsto x_1 \\ x_{-1} &\mapsto x_0 \\ &\dots \quad \dots \\ x_{-K+1} &\mapsto x_{-K+2}, \end{aligned}$$

which is the map  $F_2$  applied to the point  $F_1([x_1, x_0, x_{-1}, \dots, x_{-K+1}])$ , that is  $F_2 \circ F_1([x_1, x_0, x_{-1}, \dots, x_{-K+1}])$ .

By continuing this step-by-step integration procedure, one finds that the shift along the solution by period  $\omega = N$  is given by

$$F([x_1, x_0, x_{-1}, \dots, x_{-K+1}]) = F_N \circ \dots \circ F_2 \circ F_1([x_1, x_0, x_{-1}, \dots, x_{-K+1}]).$$

If there exists an initial vector  $\mathbf{u}_* = [u_0, u_1, \dots, u_K] \neq \mathbf{0}$  such that  $F(\mathbf{u}_*) = \mathbf{u}_*$  then any initial function  $\varphi \in \mathcal{C}$  with  $\varphi(-i) = u_i, i = 0, 1, \dots, K$  generates a non-trivial periodic solution of equation (1).

If an initial function  $\varphi \in \mathcal{C}$  results in a periodic solution  $x = p(t)$  of equation (1) then the vector  $\mathbf{u}^* = [u_0, \dots, u_K]$  with  $\varphi(-i) = u_i, i = 0, 1, \dots, K$  is a fixed point of the map  $F$ . Small perturbations  $\psi$  of the initial function  $\varphi$  in  $\mathcal{C}$ ,  $\|\varphi - \psi\|_{\mathcal{C}} < \delta$ , yield small perturbations of the vector  $\mathbf{u}^*$  in  $\mathbb{R}^{K+1}$ ,  $\|\mathbf{u}^* - \mathbf{u}\|_{\mathbb{R}^{K+1}} < \delta$ . And vice versa: small perturbations of the fixed point (vector)  $\mathbf{u}^*$  in  $\mathbb{R}^{K+1}$  can be translated into small perturbations of the corresponding initial function  $\varphi \in \mathcal{C}$  for

the periodic solution  $x = p(t)$ . Therefore, the stability of the periodic solution DDE (1) and of the corresponding cycle of the map  $F$  are the same. They are both either stable, or asymptotically stable, or unstable.

The same reasoning about the existence and stability applies to cycles of the map  $F$  and the corresponding periodic solutions of differential delay equation (1).  $\square$

**3.2. Case  $K = 0$ .** In this subsection we consider a special case of equation (1) when  $K = 0$  and  $\omega = N$  is a positive integer

$$(9) \quad \dot{x}(t) = a(t)f(x([t])).$$

We shall also assume throughout this subsection that  $f(x)$  is differentiable at  $x = 0$  with  $f'(0) < 0$ .

Given an initial value  $x(0) = x_0$  one easily solves equation (9) by the consecutive step-by-step integration for all  $t \geq 0$ , as described above.

Introduce the following auxiliary functions:

$$F_i(x) := x + \left( \int_{i-1}^i a(t) dt \right) f(x) := x + a_i f(x), \quad i = 1, 2, \dots, \omega$$

and set

$$(10) \quad F := F_\omega \circ F_{\omega-1} \circ \dots \circ F_1,$$

where the  $\circ$  stands for the composition of functions.

It is easy to see that  $x = 0$  is a fixed point of the map  $F$  which corresponds to the trivial solution  $x(t) \equiv 0$  of differential delay equation (9). Equation (9) has a non-trivial periodic solution if and only if map  $F$  has a nontrivial cycle of any period (including a fixed point). The stability of a cycle of map  $F$  and the stability of the corresponding periodic solution of differential delay equation (9) are the same.

We shall indicate and derive certain basic properties of the map  $F$  which are based on the properties of function  $f$  as a one-dimensional map.

**Theorem 2.** (Existence of Globally Attracting Interval). Suppose that nonlinearity  $f$  satisfies the assumptions (2) and (3). Then map  $F$  has a finite globally attracting interval  $I_0 = [\alpha_0, \beta_0]$ ,  $\alpha_0 \leq 0 \leq \beta_0$  such that

$$F(I_0) = I_0 \text{ and } \bigcap_{i \geq 0} F^i(\mathcal{U}) = I_0 \text{ for every open bounded set } \mathcal{U} \supset I_0.$$

Note that there is a possibility of interval  $I_0$  being a single point,  $\alpha_0 = \beta_0 = 0$ . In this case the only fixed point  $x_* = 0$  of the map  $F$  is globally attracting. The corresponding trivial solution  $x(t) \equiv 0$  of differential delay equation (9) is then globally asymptotically stable.

*Proof of Theorem 2.* We shall prove the theorem in several steps, starting with the simplest case  $N = 1$ . The principal idea is that any finite number of iterations of maps of the form  $F_a = x + af(x)$ , where  $a > 0$  is a parameter, possesses the same basic property described by the theorem as the single map  $F_a$  does.

**Case  $N = 1$ .** For arbitrary fixed  $a > 0$  map  $F_a$  has a closed globally attracting invariant interval  $I_a$  such that

$$F_a(I_a) = I_a \quad \text{and} \quad \bigcap_{n \geq 1} F_a^n(J) = I_a$$

for any interval  $J$  (open or closed) such that  $I_a \subseteq J$ .

To be definite, assume that  $f(x) \geq -M$  for some  $M > 0$  and all  $x \in \mathbb{R}$ . The case  $f(x) \leq M$  can be treated similarly (and therefore, it is left to the reader).

Due to the negative feedback assumption (2) one has that  $F_a(x) > x$  for all  $x < 0$  and  $F_a(x) < x$  for all  $x > 0$ . Besides,  $F_a(x) \geq x - Ma$ . Therefore, for every  $x \geq Ma$  one has that  $x > F_a(x) \geq 0$ .

Assume first that  $F_a(x) \geq 0$  also holds for all  $x \in [0, Ma]$ . Then for every  $x_0 \geq 0$  the sequence of its consecutive iterations  $x_n := F_a^n(x_0)$  is decreasing with  $x_{n+1} \leq x_n, n \neq 0$ . Therefore,  $\lim_{n \rightarrow \infty} x_n = 0$ .

The particular shape of  $F_a(x)$  in  $(-\infty, 0]$  is now of no importance: the fixed point  $x = 0$  is globally attracting. Indeed, for any  $x_0 < 0$  consider the sequence of its consecutive iterations,  $x_n := F_a^n(x_0)$ . If  $x_n < 0$  for all  $n \geq 0$  then  $\lim_{n \rightarrow \infty} x_n = 0$ . If  $x_{n_0} > 0$  for some  $n_0 \geq 1$  then the sequence  $y_n := F_a^n(x_{n_0})$  is nonnegative for all  $n \geq 0$  and decreasing with  $\lim_{n \rightarrow \infty} y_n = 0$ .

Suppose next that  $F_a(x)$  can assume negative values in the interval  $[0, Ma]$ . Set  $m = \min\{F_a(x), x \in [0, Ma]\}$ , and let  $L := \max\{F_a(x), x \in [m, 0]\} \geq 0$ .

If  $L = 0$  then  $F_a(x) \leq 0$  for all  $x \in [m, 0]$ . Therefore, for every  $x_0 \in [m, 0]$  the sequence of its consecutive iteration  $x_n := F_a^n(x_0)$  is increasing with  $\lim_{n \rightarrow \infty} x_n = 0$ . Thus, for every positive point  $x_0$  such that  $x_0 \geq 0$  or  $x_0 \in [m, 0]$  one has  $\lim_{n \rightarrow \infty} F_a^n(x_0) = 0$ . Choose next arbitrary  $x_0$  with  $x_0 < m$ . Consider the sequence of its consecutive iterations  $x_n := F_a^n(x_0)$ . If  $x_n \leq 0$  for all  $n \geq 0$  then  $\lim_{n \rightarrow \infty} x_n = 0$ . If  $x_{n_0} > 0$  for some  $n_0$  then for the sequence  $y_n := F_a^n(x_{n_0})$  one has  $\lim_{n \rightarrow \infty} y_n = 0$ , by the above reasoning. Thus,  $x = 0$  is globally attracting fixed point in the case  $L = 0$ .

Assume next that  $L > 0$ . Set  $\alpha_a := m$  and  $\beta_a := \max\{L, Ma\}$ . Then the interval  $[\alpha_a, \beta_a]$  is mapped into itself. This is evident from its construction. Set next  $I_a := \bigcap_{n \geq 0} F_a^n([\alpha_a, \beta_a])$ .  $I_a$  is a closed invariant interval (possibly degenerating into a single point  $\{0\}$ ) which attracts

all points from  $[\alpha_a, \beta_a]$ . We shall show that it is also a global attractor, i.e. it attracts all other points from  $\mathbb{R} \setminus [\alpha_a, \beta_a]$ . Indeed, for every point  $x_0 \geq Ma$  there exists  $n_0 \in \mathbb{N}$  such that  $F_a^{n_0}(x_0) \in [m, 0]$ . This is due to the fact that  $x > F_a(x) \geq x - Ma$  for  $x \geq Ma$ . For every  $x_0 < m$  the sequence of its iterations  $x_n := F_a^n(x_0)$  is either monotone increasing with  $x_n \leq 0$  for all  $n$ , or  $x_{n_0} > 0$  for some  $n_0 \in \mathbb{N}$ . In the first case,  $\lim_{n \rightarrow \infty} x_n = 0$ . In the second case, a subsequent finite iteration of  $y_0 = x_{n_0} > 0$  belong to the interval  $[m, 0]$ , due to the reasoning above. Therefore, every finite interval  $J \supset I_a$  (closed or open) has the property that  $F_a^{n_0}(J) \subset [\alpha_a, \beta_a]$  for some  $n_0 > 0$ . Thus  $\bigcap_{n \geq 0} F_a^n(J) = I_a$ .

**Case  $N = 2$ .** Consider two arbitrary maps  $F_a$  and  $F_b$ , where  $a > 0$  and  $b > 0$  are some fixed values. Let  $F(x) := F_b \circ F_a(x) = F_b(F_a(x))$ .

Note that in general  $F(x)$  does not satisfy the inequalities  $F(x) > x$  for all  $x < 0$  and  $F(x) < x$  for all  $x > 0$  used in case  $N = 1$ , even though both  $F_a$  and  $F_b$  do. However, it retains the following two basic properties that every  $F_a$  has:

(i) there exists  $x_+ \geq 0$  such that for all  $x \geq x_+$  function  $F(x)$  satisfies

$$x > F(x) \geq x - M_{ab}, \quad \text{where} \quad M_{ab} = M(a + b);$$

(ii) there exists  $x_- \leq 0$  such that for all  $x \leq x_-$  one has  $F(x) > x$ .

Indeed, since  $x - Ma \leq F_a(x) < x$  for all  $x \geq Ma$ , then for all sufficiently large  $x$  the following inequalities hold

$$F_a(x) - Mb \leq F_b(F_a(x)) < F_a(x) < x.$$

Which in turn implies that

$$x - M(a + b) \leq F_b(F_a(x)) < x$$

for all  $x > x_+$  with some  $x_+ \geq 0$ . This proves (i).

In order to prove (ii), note first that for arbitrary but fixed  $c > -\infty$  any map  $F_a$  has the property that  $\inf\{F_a(x), x \in [c, +\infty)\}$  exists and is finite. This is due to the fact that  $\lim_{x \rightarrow +\infty} F_a(x) = +\infty$ . Assume, to the contrary, that (ii) does not hold. Then there exists a sequence  $x_n \rightarrow -\infty$  such that  $F_b(F_a(x_n)) \leq x_n$ . Since  $F_a(x_n) > x_n$  then  $F_b(y_n) < y_n$  for the infinite sequence  $y_n := F_a(x_n)$ . This implies that the sequence  $y_n$  is bounded from below (it all belongs to some interval  $[c, \infty)$ ). But then the sequence  $x_n = F_b(y_n)$  must also be bounded from below, a contradiction with  $x_n \rightarrow -\infty$ .

The properties (i) and (ii) are sufficient to prove the existence of an invariant globally attracting interval for the map  $F$ . To show this, set  $\max\{F(x), x \in [x_-, x_+]\} := F^+$  and  $\min\{F(x), x \in [x_-, x_+]\} := F^-$ . Let  $\alpha := \min\{x_-, F^-\}$  and  $\beta := \max\{x_+, F^+\}$ . Then interval

$[\alpha, \beta]$  is mapped into itself,  $F([\alpha, \beta]) \subset [\alpha, \beta]$ . This is evident from its definition. Define  $I_0 = [\alpha_0, \beta_0] = \bigcap_{n \geq 0} F^n([\alpha, \beta])$ .  $I_0$  is a closed invariant interval (possibly coinciding with a single point  $\{0\}$ ). We claim that  $I_0$  is also globally attracting on  $\mathbb{R}$ , that is  $\bigcap_{n \geq 0} F^n(J) = I_0$  for every interval  $J \supset I_0$ . Indeed, if the initial value  $x_0$  is such that  $x_0 \geq \beta$  then the sequence  $x_n := F^n(x_0)$  is decreasing as long as  $x_n \geq \beta$ . Since there are no fixed points of  $F$  in  $[x_+, \infty)$ , there exists  $n_0$  such that  $x_{n_0} \in [\alpha, \beta]$ . Likewise, for any initial value  $x_0 \leq \alpha$  the sequence  $x_n := F^n(x_0)$  is increasing as long as  $x_n < \alpha$ . Then, for some  $n_0 \geq 1$ , either  $x_{n_0} \in [\alpha, \beta]$ , or  $x_{n_0} > \beta$ . The former means the invariance; for the latter case the first reasoning above should be applied again.

**Case  $N \geq 3$ .** The proof is done by induction, by repeating the reasoning of the case  $N = 2$ , as any finite composition of maps of the  $F_a$  type possesses the two properties (i) and (ii). This completes the proof.  $\square$

**Remark.** As it can be seen from the proof of the case  $N = 2$  of Theorem 2, the negative feedback condition (2) does not have to hold for all  $x \in \mathbb{R}$ . Therefore, a globally attracting interval  $I_0$  will always exist for the differential delay equation (9) if the nonlinearity  $f$  is bounded from one side and the negative feedback condition is satisfied for all sufficiently large  $x$ :  $x \cdot f(x) < 0$  for all  $|x| \geq x_0$  and some  $x_0 > 0$ .

**Corollary 3.1** (Uniform boundedness of solutions).

Suppose  $f$  satisfies (3). Then all solutions of equation (9) are bounded. Moreover, for arbitrary  $\varepsilon > 0$  and every initial function  $\varphi \in \mathcal{C}$  there exists time  $t_\varphi \geq 0$  such that the corresponding solution satisfies

$$\alpha_0 - \varepsilon \leq x(t) \leq \beta_0 + \varepsilon \quad \text{for all } t \geq t_\varphi.$$

The proof is straightforward from the fact of existence of the globally attracting interval  $I_0 = [\alpha_0, \beta_0]$  and its stability properties. The corollary is a more refined version of the general Proposition 2.1.

One can derive certain information about the global dynamics in differential delay equation (9) based on the size of the periodic function  $a(t)$ . Some of it is given by the following statements.

**Proposition 3.2** (Global asymptotic stability).

Given arbitrary  $f(x)$  satisfying assumptions (2) and (3), there exists  $a_0 > 0$  such that if  $a(t) \leq a_0 \forall t \in \mathbb{R}$  then the zero solution of differential delay equation (9) is globally asymptotically stable.

*Proof.* Indeed, for all sufficiently small  $a$ ,  $0 < a \leq a_0$ , any map  $F_a(x)$  has the property that  $x > F_a(x) > 0$  for all  $x > 0$ . But then any

composition  $F_{a_1} \circ F_{a_2}$ ,  $0 < a_1, a_2 \leq a_0$  also has the same property. This implies that  $x_* = 0$  is globally attracting fixed point. See the case  $N = 1$  of the proof of Theorem 2 for additional related details.  $\square$

The existence of nontrivial periodic solutions of period  $\omega$  is now given by the following

**Proposition 3.3** (Existence of periodic solutions of period  $\omega$ ).

Differential delay equation (9) has a periodic solution of period  $\omega$  if and only if map  $F$  given by (10) has a fixed point different from  $x = 0$ . The stability of such periodic solution is determined by the stability of the corresponding fixed point.

**Proposition 3.4** (Existence of periodic solutions of period  $\omega$ ).

Suppose that  $F'(0) > 1$ . Then differential delay equation (9) has at least two periodic solutions with period  $\omega$ .

*Proof.* Recall that  $I_0 = [\alpha_0, \beta_0] = \bigcap_{n \geq 0} F^n(I)$ , where  $I$  is an invariant interval of the map  $F$ . Note that in this case  $\alpha_0 < \beta_0$ . Since  $F(\alpha_0) \geq \alpha_0$  there exists a point  $x^\alpha \in [\alpha_0, 0)$  such that  $F(x^\alpha) = x^\alpha$ . Likewise, there exists  $x^\beta \in (0, \beta]$  such that  $F(x^\beta) = x^\beta$ .  $\square$

**Proposition 3.5** (Existence of periodic solutions of period  $2\omega$ ).

Suppose that  $F'(0) < -1$ . Then differential delay equation (9) has a periodic solution of period  $2\omega$ .

*Proof.* The differential delay equation (9) has a periodic solution of period  $2\omega$  if and only if the map  $F$  has a cycle of period 2. Since interval  $I$  is invariant under  $F$ , and  $x = 0$  is a repelling fixed point with the negative feedback condition satisfied locally

$$x \cdot F(x) < 0 \quad \text{for all } x \in [-\delta, \delta] \quad \text{for some } \delta > 0,$$

its instability implies the existence of a cycle of period two [4, 8, 19]. Note that the stability of such periodic solution is the same as the stability of the two-cycle.  $\square$

The value of  $F'(0)$  is easily calculated as

$$F'(0) = [1 + a_1 f'(0)] \cdot [1 + a_2 f'(0)] \cdot \dots \cdot [1 + a_N f'(0)] := \lambda.$$

Based on Propositions 3.4 and 3.5 we can state the following result on the existence of periodic solutions to equation (9).

**Theorem 3.** (Existence of Periodic Solutions)

- (i) Equation (9) has at least two periodic solutions with period  $\omega$  when  $\lambda > 1$ ;
- (ii) Equation (9) has a periodic solution with period  $2\omega$  when  $\lambda < -1$ .

The following corollary provides sufficient conditions for the existence of periodic solutions of differential delay equation (9) when either  $a(t)$  or  $|f'(0)|$  is sufficiently large. Set  $a_* := \min\{a(t), t \in [0, \omega]\} > 0$ .

**Corollary 3.6** (Existence of Periodic Solutions).

Suppose that  $a_* \cdot f'(0) < -2$ . Then

- (i) differential delay equation (9) has at least two periodic solutions with period  $\omega$  when  $N$  is even;
- (ii) differential delay equation (9) has a periodic solution with period  $2\omega$  when  $N$  is odd.

**Remark.** It is easy to see that, for respective values of  $N$ , the map  $F$  can be made such that  $\lambda > 1$  holds and it has exactly two additional non-zero fixed points. Likewise map  $F$  can be such that  $\lambda < -1$  and it possesses exactly one cycle of period two.

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