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COMPOSITION OF IRREDUCIBLE MORPHISMS IN QUASI-TUBES

CLAUDIA CHAIO AND PIOTR MALICKI

Abstract. We study the composition of irreducible morphisms between indecomposable modules lying in quasi-tubes of the Auslander-Reiten quivers of artin algebras A in relation with the powers of the radical of their module category $\operatorname{mod} A$.

1. Introduction and the main results

Throughout this paper, by an algebra we mean an artin algebra over a fixed commutative artin ring R. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ consisting of one representative of each isomorphism class of indecomposable A-modules.

We denote the radical of the module category $\operatorname{mod} A$ by $\Re_A$. We recall that, for $X, Y \in \operatorname{ind} A$ the ideal $\Re_A(X,Y)$ is the set of all non-isomorphisms between X and Y. Inductively, the powers of $\Re_A(X,Y)$ are defined. By $\Re_A^\infty(X,Y)$ we denote the intersection of all powers $\Re_A^i(X,Y)$ of $\Re_A(X,Y)$ with $i \geq 1$.

There is a close relationship between irreducible morphisms and the powers of the radical of its module category. In [5], Bautista proved that a morphism $f : X \to Y$ between indecomposable modules X and Y in $\operatorname{mod} A$ is irreducible if and only if $f \in \Re_A(X,Y)\backslash\Re_A^2(X,Y)$. This was generalized by Igusa and Todorov in [20, Theorem 13.3] where they proved that, for a sectional path

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

of irreducible morphisms between indecomposable A-modules we have that their composition $f_n \dots f_2 f_1 \in \Re_A^n(X_0,X_n) \setminus \Re_A^{n+1}(X_0,X_n)$.

We denote by Γ_A the Auslander-Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. Recall that Γ_A is a valued translation quiver defined as follows: the vertices of Γ_A are the isomorphism classes $[X]$ of modules X in $\operatorname{ind} A$, we put an arrow from $[X] \to [Y]$ in Γ_A if there is an irreducible morphism from X to Y in $\operatorname{mod} A$. The valuation (d_{XY}, d'_{XY}) of an arrow $[X] \to [Y]$ in Γ_A is defined such that d_{XY} is the multiplicity of Y in the codomain of the minimal left almost split morphism for X and d'_{XY} is the multiplicity of X in the domain of the minimal right almost split morphism for Y. We shall not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. Moreover, the valuation $(1,1)$ of an arrow in Γ_A will be omitted and we will say that a component Γ of Γ_A has *trivial valuation* if all arrows in Γ have valuation $(1,1)$.

2010 *Mathematics Subject Classification.* 16G70, 16G20, 16E10.

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By a component of Γ_A we mean a connected component of the quiver Γ_A. In general, the Auslander-Reiten quiver Γ_A describes only the quotient category $\operatorname{mod} A/\Re_A^\infty$.

An important research direction towards understanding the structure of module categories is the study of compositions of irreducible morphisms between indecomposable modules.

In [22], S. Liu introduced the notion of degree of an irreducible morphism of modules (2.3) and using such a concept he described the shapes of the components of the Auslander-Reiten quivers of algebras of infinite representation type. Liu also, studied the composition of irreducible morphisms between indecomposable modules, generalizing Igusa and Todorov result concerning sectional paths. More precisely, Liu defined the notion of pre-sectional path (2.4) and proved that if

$$X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_{n-1} \longrightarrow X_n$$

is a pre-sectional path then there are irreducible morphisms $g_i : X_{i-1} \longrightarrow X_i$ for $i = 1, \dots, n$, such that their composition $g_n \dots g_2 g_1$ lies in $\Re_A^n(X_0, X_n) \backslash \Re_A^{n+1}(X_0, X_n)$.

Recently, there has been many new results related to the subject of the composition of irreducible morphisms and their relation with the power of the radical of their module category. Most of them involving the concept of degree. For instance, see [7, 8, 9, 11, 12, 13, 14, 15, 16].

In [11], the authors looked at the general situation of when the composite of two irreducible morphisms is a non-zero morphism and lies in $\Re_A^3$ for A an artin algebra. In particular, by [15] we are able to determine if a finite dimensional algebra over an algebraically closed field is of finite representation type by computing the degree of a finite number of irreducible morphisms. Moreover, in [9] whenever we deal with a representation-finite algebra, the minimal lower bound $m \geq 1$ such that $\Re_A^m$ vanishes, was given. This bound was determined in terms of the right and the left degree of irreducible morphisms, not depending on the maximal length of the indecomposable modules. This result was extended in [10] where the authors found the nilpotency of the radical of a module category for any artin algebra.

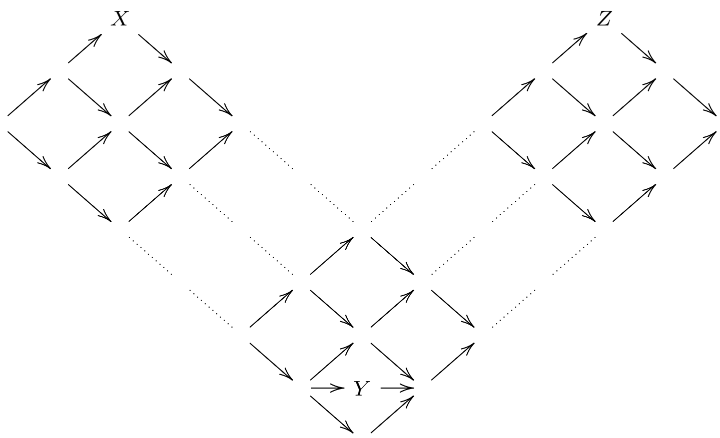
In [14], the authors studied the finiteness of degrees of irreducible morphisms between indecomposable modules lying in coherent almost cyclic components of Auslander-Reiten quivers of artin algebras.

In the representation theory of selfinjective algebras a prominent role played the components called *quasi-tubes*, whose stable parts are stable tubes. By general theory [22], [35], an infinite component Γ of the Auslander-Reiten quiver Γ_A of a selfinjective algebra A is a quasi-tube if and only if Γ contains an oriented cycle. The quasi-tubes occur in the Auslander-Reiten quivers of many selfinjective algebras, for example, for: the representation-infinite blocks of group algebras [18], [19], the representation-infinite tame algebras [34], the selfinjective algebras of wild canonical type [21], and the deformed preprojective algebras of generalized Dynkin type [6]. We also refer to the article [31] for the bound on the number of simple and projective modules in the quasi-tubes of the Auslander-Reiten quivers of finite dimensional selfinjective algebras over a field.

We would like to mention that the quasi-tubes occur also in the Auslander-Reiten quivers of the generalized multicoil algebras (see for instance [24, 25, 26, 27, 28, 29, 30, 32] for their structure and importance), which are obtained by sophisticated gluings of concealed canonical algebras using ten admissible algebra operations, generalizing the coil operations introduced in [2].

In this paper we are interested in the composition of irreducible morphisms between indecomposable modules lying in quasi-tubes of Auslander-Reiten quivers of artin algebras. In particular, we study the composition of irreducible morphisms between indecomposable modules in selfinjective algebras (where projective are also injective A-modules) and tubes in a general sense.

Let A be an artin algebra. In order to formulate one of our main results we define a special type of full translation subquiver of Γ_A. A full translation subquiver of Γ_A of the form



with X, Y and Z indecomposable projective-injective A-modules is said to be a *special configuration of modules.*

The main results proven in this work are the following theorems.

Theorem A. *Let A be a selfinjective artin algebra and Γ an infinite component of Γ_A without special configurations of modules and containing an oriented cycle. Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

be a path of irreducible morphisms with $X_i \in \Gamma$ for $i = 1, \dots, n+1$. Then, $f_n \dots f_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ if and only if $f_n \dots f_1 \in \Re_A^{\infty}(X_1, X_{n+1})$.

Theorem B. *Let A be an artin algebra and Γ a tube in Γ_A. Let $h_i : X_i \to X_{i+1}$ be n irreducible morphisms with $X_i \in \Gamma$ for $i = 1, \dots, n$. Then, $0 \neq h_n \dots h_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ if and only if $0 \neq h_n \dots h_1 \in \Re_A^{\infty}(X_1, X_{n+1})$.*

For basic background on the representation theory of algebras we refer to [1], [4] and [33].

2. Preliminaries

2.1. Let A be an algebra, X, Y be the modules in $\operatorname{ind} A$, and $f : X \to Y$ be an irreducible morphism in $\operatorname{mod} A$. If X is not injective, we shall denote by $\epsilon(X)$ the almost split sequence starting at X and by $\alpha(X)$ the number of indecomposable direct summands of the middle term of $\epsilon(X)$.

2.2. Let A be an algebra. Given $X, Y \in \operatorname{mod} A$, the ideal $\Re_A(X,Y)$ is the set of all the morphisms $f: X \to Y$ such that, for each $M \in \operatorname{ind} A$, each $h: M \to X$ and each $h': Y \to M$ the composition $h'fh$ is not an isomorphism. In particular, if $X, Y \in \operatorname{ind} A$ then $\Re_A(X,Y)$ is the set of all the morphisms $f: X \to Y$ which are not isomorphisms. Inductively, the powers of $\Re_A(X,Y)$ are defined. By $\Re_A^\infty(X,Y)$ we denote the intersection of all powers $\Re_A^i(X,Y)$ of $\Re_A(X,Y)$, with $i \geq 1$.

Next, we state the definition of degree of an irreducible morphism given by S. Liu in [22].

2.3. Let A be an algebra and let $f: X \to Y$ be an irreducible morphism in $\operatorname{mod} A$, with X or Y indecomposable. Following [22] the *left degree* $d_l(f)$ of f is infinite, if for each integer $n \geq 1$, each module $Z \in \operatorname{mod} A$ and each morphism $g \in \Re_A^n(Z,X) \setminus \Re_A^{n+1}(Z,X)$ we have that $fg \notin \Re_A^{n+2}(Z,Y)$. Otherwise, the left degree of f is the smallest positive integer m such that there is an A-module Z and a morphism $g \in \Re_A^m(Z,X) \setminus \Re_A^{m+1}(Z,X)$ such that $fg \in \Re_A^{m+2}(Z,Y)$.

The *right degree* $d_r(f)$ of an irreducible morphism f is dually defined.

2.4. Let A be an algebra. By a *path* in Γ_A we mean a sequence of irreducible morphisms between indecomposable modules $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$, and by a *non-zero path* (*zero-path*) we mean that the composition of the irreducible morphisms of the path does not vanish (vanishes).

In [5], Bautista defined the notion of sectional paths. A path $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$ in Γ_A is said to be *sectional* if for each $i = 2, \ldots, n-1$ we have that $Y_{i+1} \not\simeq {\tau_A}^{-1} Y_{i-1}$.

In [22], Liu generalized such a concept defining what he called a pre-sectional path. A path $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$ in Γ_A is said to be *pre-sectional* if, whenever $Y_{i-1} = \tau_A Y_{i+1}$ for $i = 2, \ldots, n-1$ then $Y_{i-1} \oplus \tau_A Y_{i+1}$ is a summand of the domain of a right almost split morphism for Y_i, or equivalently, whenever $\tau_A^{-1} Y_{i-1} = Y_{i+1}$ implies that $\tau_A^{-1} Y_{i-1} \oplus Y_{i+1}$ is a summand of the codomain of a left almost split morphism for Y_i. Observe that any sectional path is a pre-sectional path.

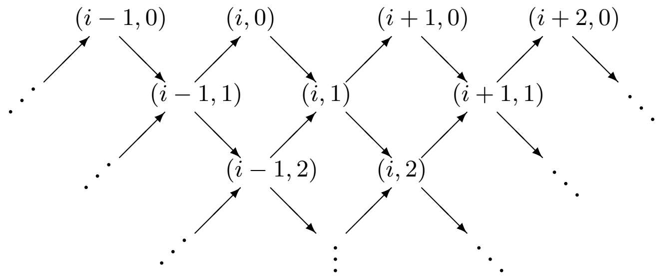
Furthermore, in [20] Igusa and Todorov proved that if

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

is a sectional path then the composition $f_n \ldots f_1 : X_0 \to X_n$ is such that $f_n \ldots f_1 \in \Re^n(X_0, X_n) \setminus \Re^{n+1}(X_0, X_n)$. In [22, Lemma 1.15], Liu extended the above result to pre-sectional paths and proved that if $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n$ is a pre-sectional path then there are irreducible morphisms $f_i : X_i \to X_{i+1}$ for $i = 0, \ldots, n-1$ such that $f_{n-1} \ldots f_0 \in \Re^n(X_0, X_n) \setminus \Re^{n+1}(X_0, X_n)$.

By a *cycle* in Γ_A we mean a sequence of irreducible morphisms between indecomposable modules of the form $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n \to Y_1$.

2.5. Recall that if $\mathbb{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$ (with trivial valuations (1,1)), then $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver of the form:



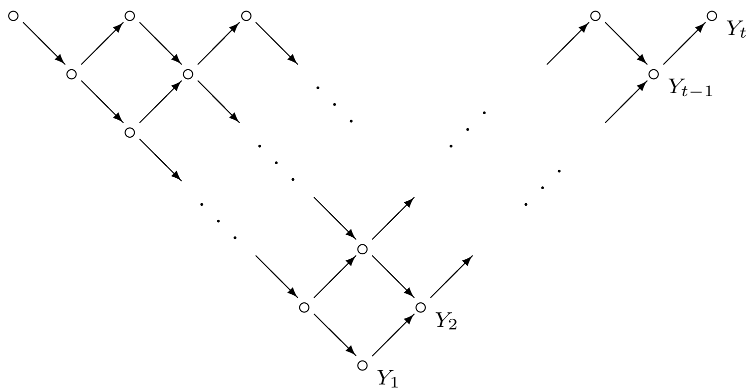
with $\tau(i,j) = (i-1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver Γ obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i,j) of $\mathbb{Z}\mathbb{A}_\infty$ with the vertex $\tau^r(i,j)$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$. The translation quiver of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is called *stable tube of rank* r. The *rank* of a stable tube Γ is the least positive integer r such that $\tau^r x = x$ for all x in Γ. The τ-orbit of a stable tube Γ formed by all vertices having exactly one direct predecessor is said to be the *mouth* of Γ.

Let (Γ, τ) be a translation quiver with trivial valuations. For a vertex X in Γ, called the *pivot*, we shall define two *admissible operations* [3] modifying (Γ, τ) to a new translation quiver (Γ', τ') depending on the shape of paths in Γ starting from X.

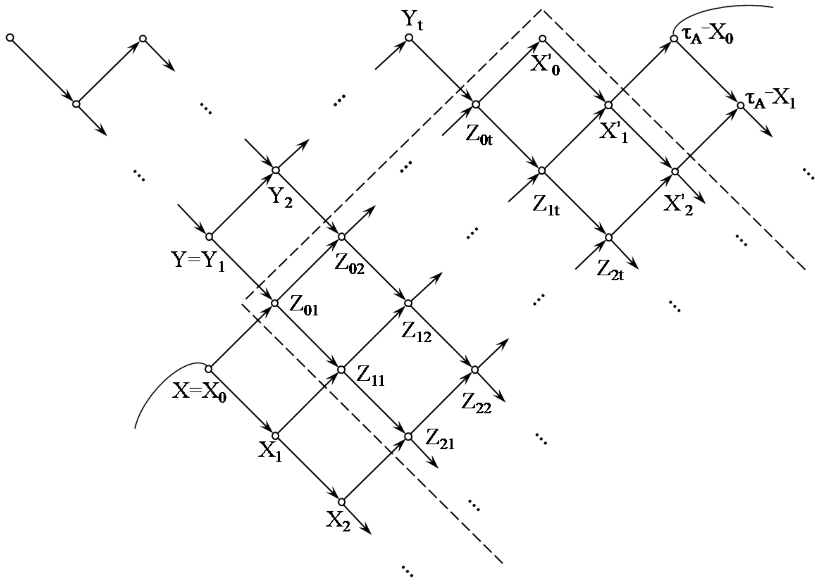
(ad 1) Suppose that Γ admits an infinite sectional path

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

starting at X, and assume that every sectional path in Γ starting at X is a subpath of the above path. For $t \geq 1$, let Γ_t be the following translation quiver, isomorphic to the Auslander-Reiten quiver of the full $t \times t$ lower triangular matrix algebra,



We then let Γ' be the translation quiver having as vertices those of Γ, those of Γ_t, additional vertices Z_{ij} and X_i' (where $i \geq 0$, $1 \leq j \leq t$) and having arrows as in the figure below

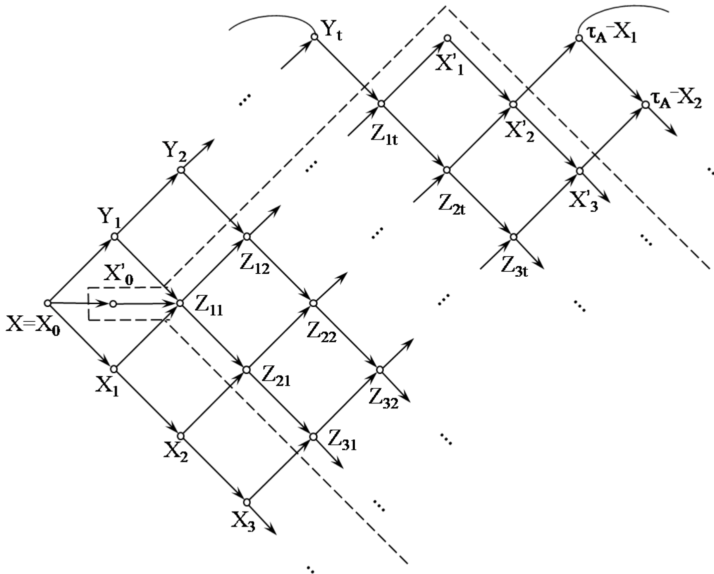


The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01}$ is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_t, respectively. If $t = 0$, the new translation quiver Γ' is obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

(ad 2) Suppose that Γ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

such that any sectional path starting at X is a subpath of one of these paths and X_0 is injective. Then Γ' is the translation quiver having as vertices those of Γ, additional vertices denoted by X'_0, Z_{ij}, X'_i (where $i \geq 1$, $1 \leq j \leq t$), and having arrows as in the figure below



The translation τ' of Γ' is defined as follows: X_0' is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X_i' = Z_{i-1,t}$ if $i \geq 2$, $\tau' X_1' = Y_t$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

We denote by (ad 1*) and (ad 2*) the admissible operations dual to the admissible operations (ad 1) and (ad 2), respectively.

A connected translation quiver Γ is said to be a *quasi-tube* if Γ can be obtained from a stable tube $\mathscr{T} = \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ by an iterated application of admissible operations (ad 1), (ad 2), (ad 1*) or (ad 2*). A *tube* (in the sense of [17]) is a quasi-tube having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*), that is, it contains a cyclical path and its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$, where S^1 is the unit circle and $\mathbb{R}^+$ is the nonnegative real line. Finally, if we apply only operations of type (ad 1) (respectively, of type (ad 1*)), then such a quasi-tube Γ is called a *ray tube* (respectively, a *coray tube*). Observe that a quasi-tube without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A quasi-tube Γ whose all non-stable vertices are projective-injective is said to be *smooth*.

The following proposition provides a characterization of quasi-tubes in the Auslander-Reiten quivers of selfinjective artin algebras ([26, Theorem A], [22],[35]).

Proposition 2.6. *Let A be a selfinjective artin algebra and Γ a component of Γ_A. The following statements are equivalent:*

(a) Γ *is a quasi-tube.*
(b) Γ^s *is a stable tube.*
(c) Γ *contains an oriented cycle.*

Here, Γ^s denotes the stable part of Γ, obtained from Γ by removing the projective-injective modules and the arrows attached to them.

Let A be an algebra, and let $\mathscr{T}$ be a stable tube of Γ_A. Then $\mathscr{T}$ has two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. Hence, for any module Z lying in $\mathscr{T}$, there is a unique sectional path $X_1 \to X_2 \to \cdots \to X_t = Z$ in $\mathscr{T}$ with X_1 lying on the mouth of $\mathscr{T}$ (consisting of arrows pointing to infinity) and there is a unique sectional path $Z = Y_1 \to Y_2 \to \cdots \to Y_t$ with Y_t lying on the mouth of $\mathscr{T}$ (consisting of arrows pointing to the mouth), and t is called the *quasi-length* of Z in $\mathscr{T}$, denoted by $\mathrm{ql}(Z)$. Now, let $\mathscr{C}$ be a smooth quasi-tube in Γ_A. Then the stable part $\mathscr{C}^s$ of $\mathscr{C}$ is a stable tube, and we may define the *smooth quasi-length* $\mathrm{sql}(X)$ of X from $\mathscr{C}$ as follows:

$$\mathrm{sql}(X) = \begin{cases} \mathrm{ql}(X) & \text{if } X \in \mathscr{C}^s, \\ \mathrm{ql}(X^+) & \text{otherwise,} \end{cases}$$

where for $X \in \mathscr{C} \setminus \mathscr{C}^s$, X^+ (respectively, X^-) denotes the immediate successor (respectively, immediate predecessor) of X in $\mathscr{C}$. Note that, if $X \in \mathscr{C} \setminus \mathscr{C}^s$ then $\mathrm{sql}(X) = \mathrm{ql}(X^+) = \mathrm{ql}(X^-)$.

2.7. We recall that a component Γ of Γ_A is called *almost cyclic* if all but finitely many modules of Γ lie on oriented cycles. Further, a component Γ of Γ_A is called *coherent* if conditions (C1) and (C2) are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path

$$P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$$

(C2) For each injective module I in Γ there is an infinite sectional path

$$\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I.$$

3. The results

We start this section recalling the definition of depth of a morphism given in [10].

Definition 3.1. *Let A be an artin algebra and $f : M \to N$ be a morphism in* $\mathrm{mod}\, A$. *We say that the depth of f, denoted by $\mathrm{dp}(f)$, is infinite in case $f \in \Re_A^\infty(M,N)$; otherwise, is the integer $n \geq 0$ for which $f \in \Re_A^n(M,N)$ but $f \notin \Re_A^{n+1}(M,N)$.*

For the convenience of the reader we state [8, Lemma 2.1] and [8, Proposition 2.2] which we will useful all through this paper. In fact, taking into account these results it is not hard to see that it is enough to study the irreducible morphisms satisfying the mesh relations of the components in consideration in order to have information on the irreducible morphisms of $\mathrm{mod}\, A$.

Lemma 3.2. *(*[8, Lemma 2.1]*) Let A be an artin algebra and Γ be a component of Γ_A with trivial valuation. Let $h_i : X_i \to X_{i+1}$ be an irreducible morphism with $X_i \in \Gamma$, for $i = 1, \ldots, n$. Then, for any choice of irreducible morphisms $f_i : X_i \to X_{i+1}$ we have that $h_n ... h_1 = \delta f_n \ldots f_1 + \mu$ with $\delta \in \mathrm{Aut}(X_{n+1})$ and $\mu \in \Re_A^{n+1}(X_1, X_{n+1})$.*

Let $f : X \to Y$ be an irreducible morphism between indecomposable modules in $\mathrm{mod}\, A$. We set

$$\mathrm{Irr}(X,Y) = \Re_A(X,Y)/\Re_A^2(X,Y).$$

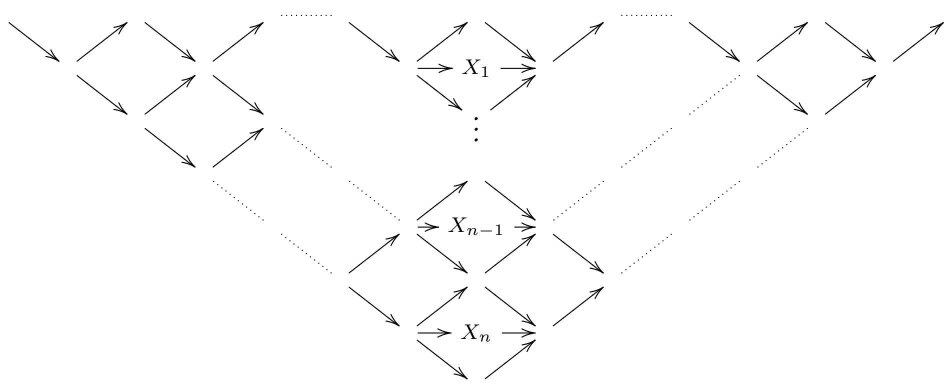
We recall that $\mathrm{Irr}(X,Y)$ is a $k_X - k_Y-$bimodule where $k_X = \mathrm{End}(X)/\Re_A(X,X)$ and $k_Y = \mathrm{End}(Y)/\Re_A(Y,Y)$. Moreover, k_Z is a division ring whenever Z is an indecomposable A-module.

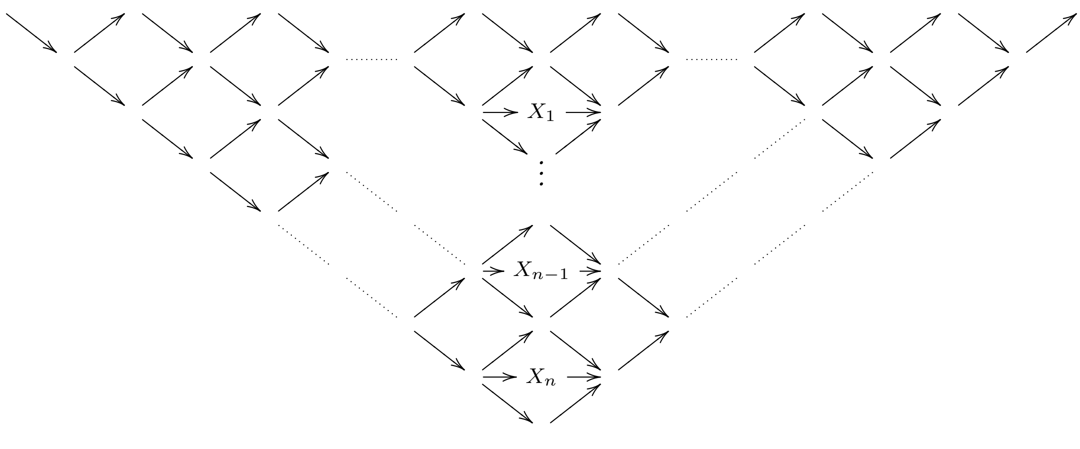
Proposition 3.3. [8, Proposition 2.2] *Let A be an artin algebra and $X_i \in \mathrm{ind}\, A$ for $1 \leq i \leq n+1$. Assume that $\dim_{k_{X_i}} \mathrm{Hom}_A(X_i, X_{i+1}) = \dim_{k_{X_{i+1}}} \mathrm{Hom}_A(X_i, X_{i+1}) = 1$, for $i = 1, \ldots, n$. Then, the following conditions are equivalent:*

(a) *There are irreducible morphisms $f_i : X_i \to X_{i+1}$ in $\mathrm{mod}\, A$, for $i = 1, \cdots, n$ with $f_n \cdots f_1 \notin \Re_A^{n+1}(X_1, X_{n+1})$.*
(b) *Given any irreducible morphisms $h_i : X_i \to X_{i+1}$ in $\mathrm{mod}\, A$, for $i = 1, \cdots, n$, then $h_n ... h_1 \notin \Re_A^{n+1}(X_1, X_{n+1})$.*

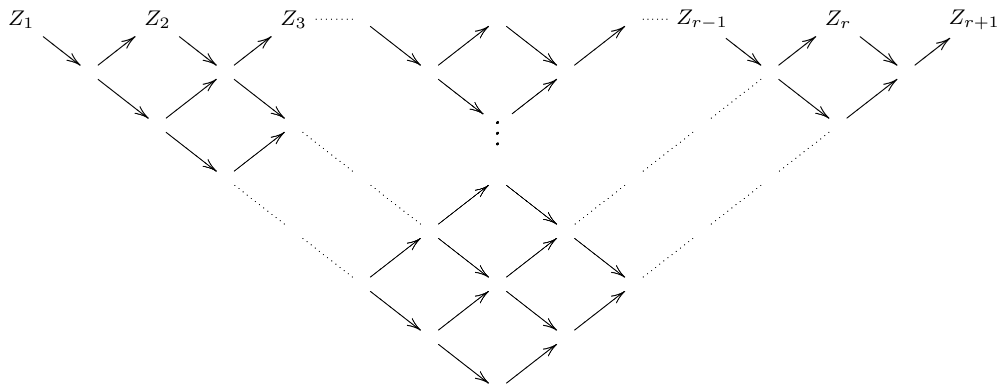
We shall dedicate the first part of this paper to study the composition of irreducible morphisms lying in an exceptional wing. We observe that these mentioned wings appear in coherent almost cyclic Auslander-Reiten components (see [26]). We start given the definition of exceptional wings.

Definition 3.4. *A full translation subquiver of Γ_A of one of the forms*





where $n \geq 1$, X_i, $1 \leq i \leq n$ are indecomposable projective-injective modules, and $X_n \neq 0$; or a wing in the sense of Ringel (see [33]*)*



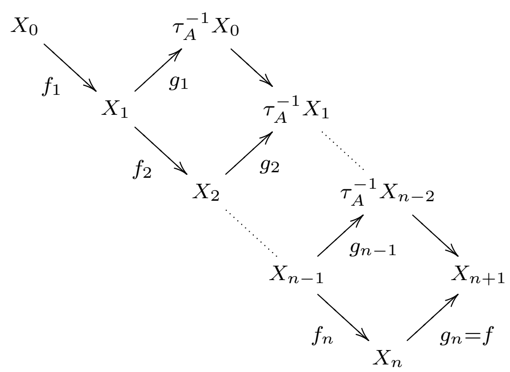
is said to be an exceptional wing. We denote it by $\mathcal{W}$. The two maximal sectional paths of $\mathcal{W}$ are called the borders of $\mathcal{W}$.

Definition 3.5. *We say that a composition $\varphi_{n_m} \dots \varphi_{n_1}$ of morphisms (irreducible morphisms, resp.) φ_{n_j}, for $j = 1, \dots, m$, in* $\operatorname{mod} A$ *(in a component Γ, resp.) behaves well whenever* $\operatorname{dp}(\varphi_{n_j}) = r_j$ *with $r_j \geq 0$ then we have that* $\operatorname{dp}(\varphi_{n_m} \dots \varphi_{n_1}) = r_m + \dots + r_1$.

We observe that for the proof of the converse of [7, Proposition 6.1] we do not need the hypothesis of Γ been a component of Γ_A satisfying $\alpha(\Gamma) \leq 2$ (the number of indecomposable direct summands of the middle term of all almost split sequence is less than or equal to 2). Such a hypothesis was only necessary for the other implication. In order to make this comment clear we shall include a proof of this fact in Lemma 3.6, Statement (a).

Next, we prove three technical lemmas which will allow us to study the composition of irreducible morphisms lying in an exceptional wing $\mathcal{W}$. More precisely, we shall prove that the composition of the irreducible morphisms in the borders of $\mathcal{W}$ behaves well.

Lemma 3.6. *Let A be an artin algebra, Γ a component of Γ_A, $X_i \in \Gamma$ for $i = 0, \dots, n$ and $n \geq 1$. Let $f : X_n \to X_{n+1}$ be an irreducible morphism and assume that there is a configuration of almost split sequences as follows*



with $\alpha(X_i) = 2$ *for* $i = 1, \dots, n-1$, $\alpha(X_0) = 1$ *and* $f_n \dots f_1$ *a sectional path. Moreover, assume there is a morphism* $\mu : X \to X_n$ *with* $X \in \Gamma$ *such that* $\mathrm{dp}(\mu) = m$ *for some positive integer* m *and* $f\mu \in \Re_A^{m+2}(X, X_{n+1})$. *Then,*

(a) *The left degree of* f *is* n *and* $m \geq n$.
(b) *There exists a morphisms* $\varphi_0 : X \to X_0$ *such that* $\mathrm{dp}(\varphi_0) = t$, *for some* $0 \leq t \leq m - n$, *and* $f_n \dots f_1\varphi_0 + \mu \in \Re_A^{m+1}(X, X_0)$.
(c) *If* φ_0 *is not an isomorphism then there exists a non-zero path of irreducible morphisms from* X *to* X_0 *in* $\mathrm{mod}\, A$ *of length at most* $m - n$.

Proof. (a). By hypothesis there exists a sectional path

$$\delta : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

with $\delta = f_n \dots f_1$. By [20] we know that $\delta \in \Re_A^n(X_0, X_n)\backslash\Re_A^{n+1}(X_0, X_n)$. We also have that $f\delta = 0$ then we get that $d_l(f) \leq n$.

On the other hand, since $\tau_A^{-1}X_{n-2} \oplus X_n$ is the middle term of $\epsilon(X_{n+1})$ by [22, Proposition 1.6] we get that $d_l(f) \geq n$. Hence, $d_l(f) = n$.

Now, since there is a morphism $\mu : X \to X_n$ with $X \in \Gamma$ such that $\mathrm{dp}(\mu) = m$ for some positive integer m and $f\mu \in \Re_A^{m+2}(X, X_{n+1})$, then, $d_l(f) \leq m$, that is, $n \leq m$.

(b). Since $\mathrm{dp}(\mu) = m$ and $f\mu \in \Re_A^{m+2}(X, X_{n+1})$ by [22, Lemma 1.2] there is a morphism $\varphi_{n-1} : X \to X_{n-1}$ such that $\varphi_{n-1} \notin \Re_A^m(X, X_{n-1})$, $g_{n-1}\varphi_{n-1} \in \Re_A^{m+1}(X, \tau_A^{-1}X_{n-2})$ and $f_n\varphi_{n-1}+\mu \in \Re_A^{m+1}(X, X_{n+1})$. Then, $f_n\varphi_{n-1} = -\mu+\mu_{m+1}$ with $\mu_{m+1} \in \Re_A^{m+1}(X, X_n)$. Therefore, $\mathrm{dp}(f_n\varphi_{n-1}) = m$. Then, we infer that $\mathrm{dp}(\varphi_{n-1}) = r$ for some $n-1 \leq r < m$. In fact, assume that $r < n-1$. Note that in such a case $n > 1$. If φ_{n-1} is an isomorphism and since $\mathrm{dp}(f_n\varphi_{n-1}) = m$ then $\mathrm{dp}(f_n) = m$ but $m > 1$, a contradiction to the fact that f_n is an irreducible morphism. Then, φ_{n-1} is not an isomorphism and $n \geq 2$.

With a similar argument as in the proof of Statement (a) we have that $d_l(g_{n-1}) = n-1$, getting a contradiction to the fact that since $\mathrm{dp}(\varphi_{n-1}) = r$ with $r < n-1$ and $g_{n-1}\varphi_{n-1} \in \Re_A^{m+1}(X, \tau_A^{-1}X_{n-2})$ then $d_l(g_{n-1}) < r < n-1$. Therefore, we prove that $\mathrm{dp}(\varphi_{n-1}) = r$ for some $n-1 \leq r \leq m-1$.

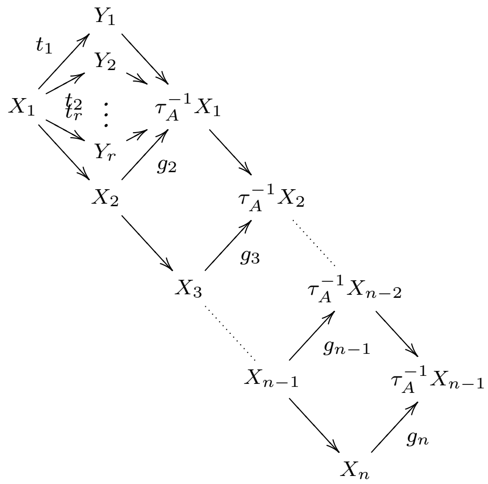
Now, since there is a morphism $\varphi_{n-1} : X \to X_{n-1}$ such that $\mathrm{dp}(\varphi_{n-1}) = r$ for some $n-1 \leq r \leq m-1$ and $g_{n-1}\varphi_{n-1} \in \Re_A^{m+1}(X, \tau_A^{-1}X_{n-2})$ then by [22, Lemma 1.2] we have that there is a morphism $\varphi_{n-2} : X \to X_{n-2}$ such that $\varphi_{n-2} \notin \Re_A^{m-1}(X, X_{n-2})$, $g_{n-2}\varphi_{n-2} \in \Re_A^m(X, \tau_A^{-1}X_{n-3})$ and $f_{n-1}\varphi_{n-2} + \varphi_{n-1} \in \Re_A^m(X, X_{n-1})$. With the same arguments as above we can show that $\mathrm{dp}(\varphi_{n-2}) = t$ for some $n-2 \leq t \leq m-2$. Moreover, $f_nf_{n-1}\varphi_{n-2} + \mu \in \Re_A^{m+1}(X, X_{n+1})$. In fact, since $f_{n-1}\varphi_{n-2} + \varphi_{n-1} \in \Re_A^m(X, X_n)$ then $f_nf_{n-1}\varphi_{n-2} + f_n\varphi_{n-1} \in \Re_A^{m+1}(X, X_{n+1})$ where $f_n\varphi_{n-1} = -\mu + \mu_{m+1}$ with $\mu_{m+1} \in \Re_A^{m+1}(X, X_{n+1})$ getting that $f_nf_{n-1}\varphi_{n-2} + \mu \in \Re_A^{m+1}(X, X_{n+1})$.

Iterating the same argument and applying successively [22, Lemma 1.2] to each possible morphism $\varphi_i : X \to X_i$ for $i = n-3, \dots, 0$ we get that there is a morphism $\varphi_0 : X \to X_0$ such that $\mathrm{dp}(\varphi_0) = t$, for some $0 \leq t \leq m-n$, and that $f_n \dots f_1\varphi_0 + \mu \in \Re_A^{m+1}(X, X_{n+1})$. Observe that φ_0 can be an isomorphism.

(c). Since φ_0 is not an isomorphism then $m > n$ and therefore $0 < t \leq m-n$. By [4, VI, Proposition 7.5] there exists a non-zero path of irreducible morphisms in $\mathrm{mod}\, A$

of length at most t. Hence, we infer that there is a path of irreducible morphisms of length at most $m-n$, getting the result. □

Lemma 3.7. *Let A be an artin algebra. Assume that there is a configuration of almost split sequences in* $\operatorname{mod} A$ *as follows*

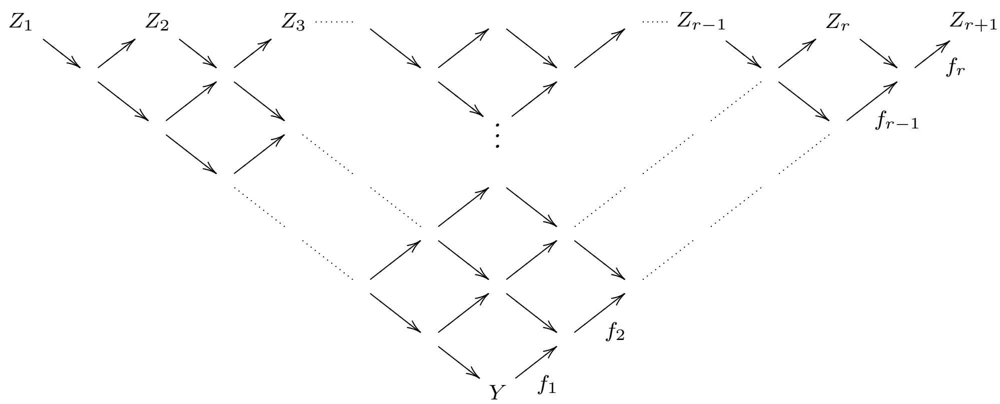


where $\alpha(X_i) = 2$ for $2 \le i \le n-1$. Suppose there exists $1 \le j \le r$ such that Y_j is projective and that the path $\delta : X_1 \to \cdots \to X_{n-1} \to X_n$ is sectional. Consider $\delta_i : X_1 \to \cdots \to X_i$ a subpath of δ and $g_i : X_i \to \tau_A^{-1}X_{i-1}$ irreducible morphisms. Then, $d_l(g_i) = \infty$ for $i \ge 2$. Moreover, the composition $g_i\delta_i$ behaves well for all $i \ge 2$.

Proof. Assume that $d_l(g_k) < \infty$, for some $2 \le k \le n-1$. By [22, Corollary 1.2] we know that $d_l(g_2) < \cdots < d_l(g_{n-1}) < d_l(g_n)$. Hence $d_l(g_2) < \infty$. Moreover, again by [22, Corollary 1.2] we get that $d_l((t_1, \ldots, t_r)^t) < \infty$, but by our assumption there is an integer $1 \le j \le r$ such that Y_j is projective getting a contradiction to [22, Lemma 1.2].

Finally, note that if $i \ge 2$ then the composition $g_i\delta_i$ behaves well, since δ_i is a sectional path and $d_l(g_i) = \infty$. □

Lemma 3.8. *Let A be an artin algebra and Γ be a component of Γ_A with trivial valuation. Assume we have an exceptional wing $\mathcal{W}$ in Γ as follows*



and that there exists $X \in \Gamma$ and a morphism $\mu : X \rightsquigarrow Y$ such that $\mathrm{dp}(\mu) = m$ *with* $m > r$*. Moreover, assume that any path of irreducible morphisms from $X \rightsquigarrow Z_i$ in Γ of length at most $m - r + 2(i-1)$ is zero. Then,* $\mathrm{dp}(f_r \dots f_1\mu) = m + r$.

Proof. Assume that $f_1\mu \in \Re_A^{m+2}$. Since $d_l(f_1) = r$ then by Lemma 3.6 (b) there exists a morphism $\varphi_0 : X \rightsquigarrow Z_1$ such that $\mathrm{dp}(\varphi_0) = t$ for some $0 < t \leq m - r$.

Observe that since $m > r$ then $m - r > 0$ and $X \not\simeq Z_1$. Hence φ_0 is not an isomorphism. By Lemma 3.6 (c), we know that there exists a non-zero path of irreducible morphisms $\varphi' : X \rightsquigarrow Z_1$ in $\mathrm{mod}\, A$ of length at most $m - r$ with $\varphi' \notin \Re_A^{m-r+1}(X, Z_1)$. We write the path φ' as follows

$$\varphi' : X \to Y_1 \to Y_2 \to \cdots \to Y_j \to Z_1.$$

On the other hand, if we consider $i = 1$ then by hypothesis any path of irreducible morphisms in Γ from $X \rightsquigarrow Z_1$ of length at most $m - r$ is zero. Therefore, any path γ in Γ going through the A-modules

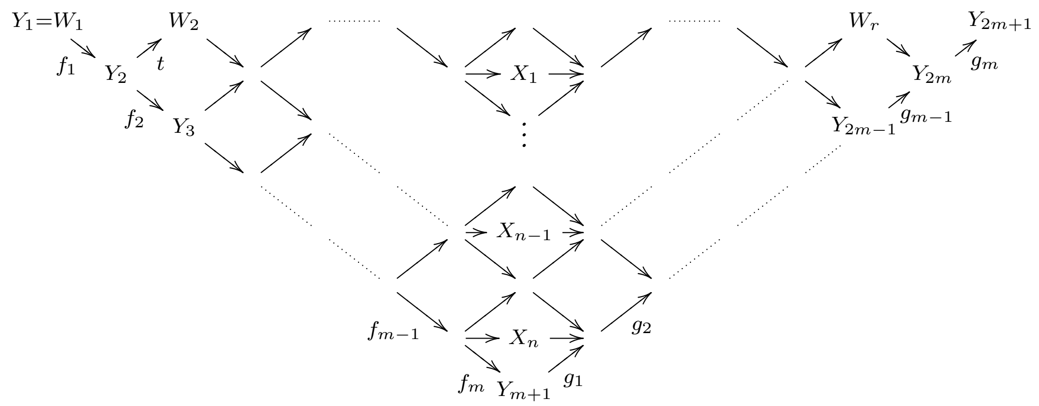
$$\gamma : X \to Y_1 \to Y_2 \to \cdots \to Y_j \to Z_1$$

vanishes. Since we are considering Γ a component of Γ_A with trivial valuation then, by Lemma 3.2 we have that $\varphi' = \delta\gamma + \mu$ with $\delta \in \mathrm{Aut}(Z_1)$ and $\mu \in \Re_A^{m-r+1}(X, Z_1)$. Hence, $\varphi' = \mu$ with $\mu \in \Re_A^{m-r+1}(X, Z_1)$ getting a contradiction to the fact that $\varphi' \notin \Re_A^{m-r+1}(X, Z_1)$. Therefore, $\mathrm{dp}(f_1\mu) = m + 1$.

Iterating this procedure over all the modules Z_i, for $2 \leq i \leq r$, we get that $\mathrm{dp}(f_r \dots f_1\mu) = m + r$. □

Now, applying Proposition 3.3 and the above lemmas we get the announced result concerning compositions of irreducible morphisms of the borders of an exceptional wing.

Proposition 3.9. *Let A be an artin algebra and Γ a component of Γ_A with trivial valuation. Assume we have an exceptional wing $\mathcal{W}$ in Γ containing a configuration of n almost split sequences with exactly three indecomposable middle terms as follows:*



Then,

(a) *The composition of irreducible morphisms in* $\mathrm{mod}\, A$ *between the indecomposable A-modules of the borders of $\mathcal{W}$ behaves well.*

(b) *Any composition of irreducible morphisms in $\mathcal{W}$ from W_1 to W_j, $2 \leq j \leq r$ is zero.*

Proof. Let $\mathcal{W}$ be an exceptional wing in Γ. Without loss of generality, it is enough to consider an exceptional wing as in the statement with $X_i \neq 0$ for $i = n$.

(a). First, if we consider a path involving the modules Y_i for $i = 1, \dots, m+1$ since any such a path is sectional we get the result by [20].

Now, by Lemma 3.7 we know that $\mathrm{dp}(g_1 f_m \dots f_1) = m+1$ since $f_m \dots f_1$ is a sectional path and $d_l(g_1) = \infty$.

Next, we proceed as follows. If X_{n-1} is projective module then $d_l(g_2) = \infty$. Hence $\mathrm{dp}(g_2 g_1 f_m \dots f_1) = m+2$ since $\mathrm{dp}(g_1 f_m \dots f_1) = m+1$. Otherwise, $X_{n-1} = 0$ and by Lemma 3.6 (a) we have that $d_l(g_2) = m-1$.

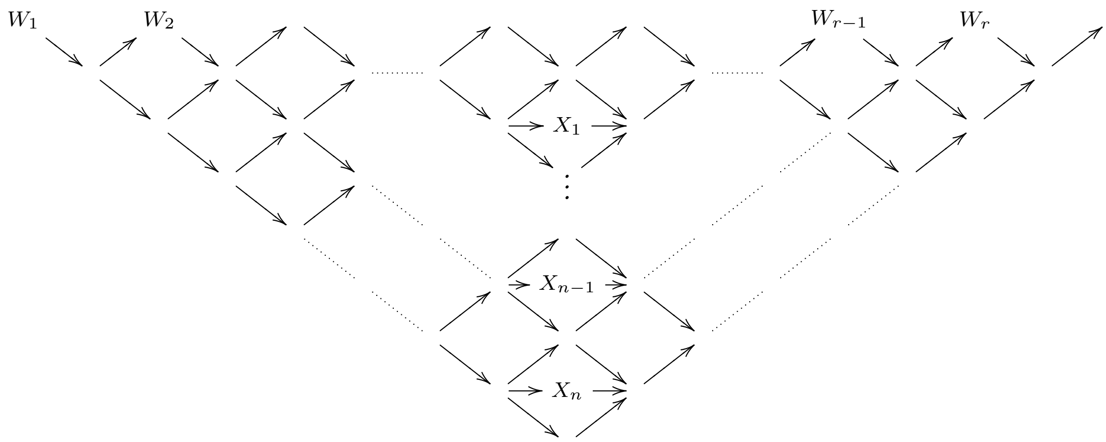
Assume that $g_2 g_1 f_m \dots f_1 \in \Re_A^{m+3}(W_1, Y_{m+2})$. Then, by Lemma 3.6 (c) there exists a non-zero path in $\mathrm{mod}\, A$ of length at most 2 from $W_1 \rightsquigarrow W_2$. Note that any path in Γ from W_1 to W_2 of length 2 is zero. In fact, observe that the only path of length two in Γ from W_1 to W_2 is the path $W_1 = Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{t} W_2$ whose irreducible morphisms belong to an almost split sequence with indecomposable middle term. Hence, $tf_1 = 0$. Since the arrows of Γ have trivial valuation then by Lemma 3.2 any other path of irreducible morphisms of length two between the same modules, let say, $W_1 = Y_1 \xrightarrow{h_1} Y_2 \xrightarrow{h_2} W_2$, is such that $h_2 h_1 = \delta t f_1 + \mu$ with $\delta \in \mathrm{Aut}(W_2)$ and $\mu \in \Re_A^3(W_1, W_2)$. Then $h_2 h_1 \in \Re_A^3(W_1, W_2)$. If $h_2 h_1 \neq 0$ then we get a contradiction to Lemma 3.6 (c). Therefore, we prove that we can not have a non-zero path of irreducible morphisms between W_1 and W_2 of length at most two. Then, by Lemma 3.8 we get that $\mathrm{dp}(g_2 g_1 f_m \dots f_1) = m+2$.

Analyzing the composition of each irreducible morphism g_i with $3 \leq i \leq n$ as above we get that $\mathrm{dp}(g_n \dots g_1 f_m \dots f_1) = m+n$. Finally, applying Lemma 3.8, we get the result.

Furthermore, any composition of the form $g_s \dots g_1 f_m \dots f_r$ with $1 \leq s \leq m$ and $1 \leq r \leq m$ also behaves well.

(b). It is an immediate consequence of the fact that all such paths may go through the almost split sequence starting at W_1 which has exactly one indecomposable middle term. $\square$

Proposition 3.10. *Let A be an artin algebra and Γ a component of Γ_A with trivial valuation. Assume we have an exceptional wing $\mathcal{W}$ in Γ containing a configuration of n almost split sequences with exactly three indecomposable middle terms as follows:*



Then,

(a) *The composition of irreducible morphisms in* $\operatorname{mod} A$ *between the indecomposable* A*-modules of the borders of* $\mathcal{W}$ *behaves well.*
(b) *Any composition of irreducible morphisms in* $\mathcal{W}$ *from* W_1 *to* W_j*,* $2 \leq j \leq r$ *is zero.*

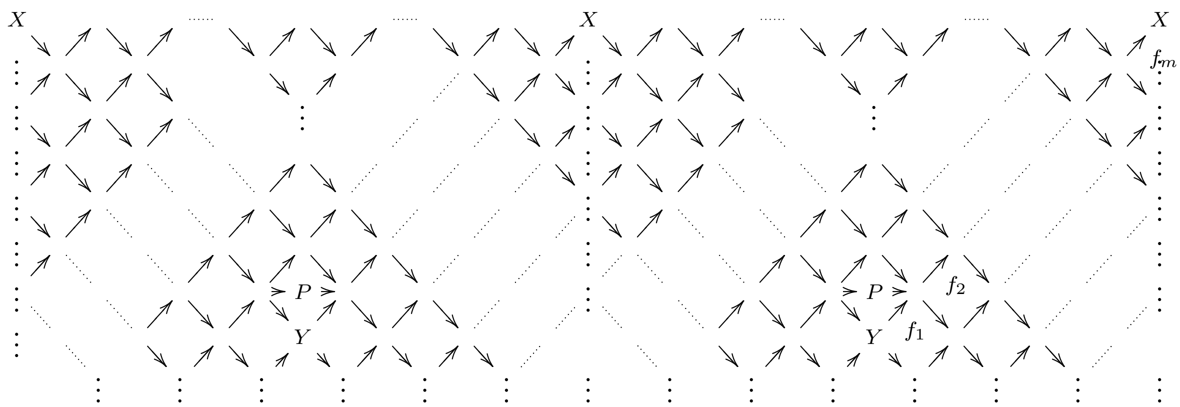
Proof. Similar to the proof of Proposition 3.9. □

Proposition 3.11. *Let* A *be an artin algebra and* $\Gamma \subset \Gamma_A$ *a smooth quasi-tube with only one almost split sequence with three indecomposable middle terms. Then the following conditions hold.*

(a) *The composition of* $r \geq 1$ *cycles in* Γ *(*$\operatorname{mod} A$*) from a projective-injective indecomposable module* P *with* $\operatorname{sql}(P) = rank\,(\Gamma^s)$ *(respectively, immediate predecessor or successor of* P*) behaves well.*
(b) *If* $\operatorname{sql}(P) < rank\,(\Gamma^s)$ *then the composition of* $r \geq 1$ *cycles in* Γ *(*$\operatorname{mod} A$*) from a projective-injective indecomposable module* P *(respectively, immediate predecessor or successor of* P*) is zero.*

Proof. Let A be an artin algebra, Γ be a smooth quasi-tube in Γ_A, and P be a projective-injective module in Γ (respectively, immediate predecessor or successor of P). Observe first that, it follows from [26, Lemma 4.9] (see also [23, Lemmas 2.5-2.8 and their duals]) that $\operatorname{sql}(P) \leq \operatorname{rank}(\Gamma^s)$.

(a). We illustrate the situation of this statement with the following diagram



By Proposition 3.9 we know that the composition of the morphisms in the borders δ_1 starting at X and ending at Y and δ_2 starting at Y and ending at X of the exceptional wing, behaves well. Moreover, $f_1\delta_1\delta_2\delta_1$ also behaves well since $d_l(f_1) = \infty$ and the irreducible morphisms of the border δ_1 have infinite left degree. By Lemma 3.8, we know that $f_m \dots f_1\delta_1\delta_2\delta_1 = \delta_2\delta_1\delta_2\delta_1$ behaves well.

Repeating this argument we get the result for Γ. Moreover, by Lemma 3.2 we get the result for $\operatorname{mod} A$.

(b). Let P be a projective-injective module, $\operatorname{sql}(P) < \operatorname{rank}(\Gamma^s)$ and

$$X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = P$$

be the unique sectional path in Γ with X_m lying on the mouth of Γ^s. Then $m = \operatorname{sql}(P)$. Observe that the smooth quasi-tube Γ is a coherent component of Γ_A (2.7). Since Γ is also a cyclic component of Γ_A, applying [26, Theorem A] (see also [23, Theorem 2.3]), we infer that Γ, considered as a translation quiver, can be obtained from a stable tube by an iterated application of admissible operations of type (ad 1), (ad 2), (ad 1^*) and (ad 2^*), described in Section 2. Moreover, by our assumption on the number of almost split sequences with three middle terms, we can apply only one admissible operation of type (ad 2) or (ad 2^*). Then it follows that $\operatorname{Hom}_A(P, X_i) = 0$ and $\operatorname{Hom}_A(P, Y_j) = 0$, where

$$X_m \to X_{m-1} \to \cdots \to X_1 \to Y_1 \to Y_2 \to \cdots$$

is the unique infinite sectional path in Γ consisting of arrows pointing to infinity and $1 \leq i \leq m$, $j \geq 1$. Therefore, $\operatorname{Hom}_A(P, X_0) = \operatorname{Hom}_A(P, P) = 0$ and then the composition of $r \geq 1$ cycles in Γ ($\operatorname{mod} A$) from P (respectively, from immediate predecessor or successor of P) is zero. $\square$

Proposition 3.12. *Let A be an artin algebra and Γ a smooth quasi-tube in Γ_A. Assume we have an exceptional wing $\mathcal{W}$ in Γ containing a configuration of n almost split sequences with exactly three indecomposable middle terms as on the figures in Definition 3.4. Then the following conditions hold.*

(a) *The composition of* $r \geq 1$ *cycles in* Γ *(*$\operatorname{mod} A$*) from a projective-injective indecomposable module* X_n *with* $\operatorname{sql}(X_n) = rank(\Gamma^s)$ *(respectively, immediate predecessor or successor of* X_n*) behaves well.*
(b) *If* $\operatorname{sql}(X_n) < rank(\Gamma^s)$ *then the composition of* $r \geq 1$ *cycles in* Γ *(*$\operatorname{mod} A$*) from a projective-injective indecomposable module* X_n *(respectively, immediate predecessor or successor of* X_n*) is zero.*

Proof. Similar to the proof of Proposition 3.11 using additionally induction on the number of projective-injective modules. □

The next result shall be useful for further purposes.

Lemma 3.13. *Let* A *be a selfinjective artin algebra and* $\Gamma \subset \Gamma_A$ *be a quasi-tube. Assume we have in* Γ *a zero path of irreducible morphisms* $X_1 \to X_2 \to \cdots \to X_n \to X_{n+1}$*. Then, any longest path in* Γ *from* $X_1 \rightsquigarrow X_{n+1}$ *vanishes.*

Proof. Let A be a selfinjective artin algebra and Γ be a quasi-tube in Γ_A. It follows from Proposition 2.6 that the stable part Γ^s of Γ is a stable tube. Moreover, observe that the quasi-tube Γ is a coherent component of Γ_A, that is, the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = U_1 \to U_2 \to U_3 \to \cdots$ starting at P;
(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to V_3 \to V_2 \to V_1 = I$ ending at I.

Since Γ is also a cyclic component of Γ_A, applying [26, Theorem A] (see also [23, Theorem 2.3]), we infer that Γ, considered as a translation quiver, can be obtained from a stable tube by an iterated application of admissible operations of type (ad 1), (ad 2), (ad 1*) and (ad 2*). Since the projectives and injectives vertices in Γ coincide, the projective-injective vertices in Γ are created as follows:

- for each operation (ad 1) with pivot X_0 and $t = 0$, the operation (ad 1*) with pivot at X_0' and $t = 0$ is applied;
- for each operation (ad 1*) with pivot X_0 and $t = 0$, the operation (ad 1) with pivot at X_0' and $t = 0$ is applied;
- for each operation (ad 1) with pivot X_0 and $t \geq 1$, the operation (ad 2*) with pivot at Z_{01} is applied;
- for each operation (ad 1*) with pivot X_0 and $t \geq 1$, the operation (ad 2) with pivot at Z_{01} is applied.

Now, let $\alpha : X_1 \to X_2 \to \cdots \to X_n \to X_{n+1}$ be a zero path of irreducible morphisms in Γ and i be the largest index such that a subpath $\beta : X_1 \to X_2 \to \cdots \to X_i$ of α is non-zero. Moreover, let $Z_p \to Z_{p-1} \to \cdots \to Z_1 \to X_i$ be the unique maximal sectional path in Γ starting at Z_p and formed by arrows pointing to the infinity. Then Z_p lies on the mouth of Γ^s. Then, it follows from the definition of admissible operations of types (ad 1), (ad 2), (ad 1*), (ad 2*) that, if $X_1 \to \cdots \to Y$ is a non-zero path of irreducible morphisms in Γ then Y lies in the infinite rectangle $\mathscr{S}(X_1, Z_p)$ consisting of the vertices bounded by:

- the infinite sectional path in Γ starting at X_1 and formed by arrows pointing to the infinity;
- the finite sectional path in Γ starting at X_1 and formed by arrows pointing to the mouth;
- the infinite sectional path $Z_p \to \cdots \to Z_1 \to X_i \to \cdots$ in Γ starting at Z_p and formed by arrows pointing to the infinity.

Therefore, any longest path in Γ from X_1 to X_{n+1} vanishes. □

Our next result shows that if A is a selfinjective artin algebra and Γ an infinite component of Γ_A without special configurations of modules and containing an oriented cycle then the composition of irreducible morphisms $f_n \dots f_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ if and only if $f_n \dots f_1 \in \Re_A^{\infty}(X_1, X_{n+1})$. To achive to such a result we start proving the following lemma.

Lemma 3.14. *Let A be a selfinjective artin algebra and Γ be a quasi-tube in Γ_A with at least two projective-injective modules and such that all projective-injectives belong to exactly two exceptional wings $\mathcal{W}$ and $\mathcal{W}'$ in Γ. Let $\alpha : X \to \cdots \to Y$, $\beta : Y \to \cdots \to Z$ be the borders of $\mathcal{W}$ and $\gamma : U \to \cdots \to V$, $\delta : V \to \cdots \to W$ be the borders of $\mathcal{W}'$. Then the following conditions hold.*

(a) *If $Z = U$ (respectively, $W = X$) then the composition of irreducible morphisms from X to W (respectively, from U to Z) behaves well.*
(b) *If $Z \neq U$ (respectively, $W \neq X$) then any composition of irreducible morphisms from X to W (respectively, from U to Z) is zero.*

Proof. Observe first that, it follows from [26, Lemma 4.9] (see also [23, Lemmas 2.5-2.8]) that there is in Γ the infinite rectangle $\mathscr{S}(Y, Z)$ consisting of the vertices bounded by:

- the infinite sectional path in Γ starting at Y and formed by arrows pointing to the infinity;
- the finite sectional path $\beta : Y \to \cdots \to Z$ in Γ;
- the infinite sectional path in Γ starting at Z and formed by arrows pointing to the infinity.

Moreover, all meshes in $\mathscr{S}(Y, Z)$ are with exactly two middle terms and for any T from $\mathscr{S}(Y, Z)$ we have $\mathrm{Hom}_A(X, T) \neq 0$.

Let $Z = U$. By Proposition 3.9 we know that the composition of irreducible morphisms of the borders of the exceptional wings behaves well. Let σ be the sectional path in Γ from infinity to V and ϱ be the sectional path in Γ from Y to infinity. Then σ intersects ϱ and denote by N their common module. Note that every composition of irreducible morphisms from Y to V in the rectangle $\mathscr{S}(Y, Z, V, N)$ is equal and non-zero. Therefore, the composition of irreducible morphisms from X to W behaves well.

If $Z \neq U$, then for the infinite sectional path $\tau_A^- Z = M_1 \to M_2 \to \cdots$ formed by arrows pointing to the infinity we have $\mathrm{Hom}_A(X, M_i) = 0$ where $i \geq 1$ and Z is the starting vertex of a mesh with exactly one middle term. Hence we get (b). □

Theorem 3.15. *Let A be a selfinjective artin algebra and Γ a quasi-tube of Γ_A without special configurations of modules. Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

be a path of irreducible morphisms with $X_i \in \Gamma$ for $i = 1, \dots, n+1$. Then, $f_n \dots f_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ if and only if $f_n \dots f_1 \in \Re_A^{\infty}(X_1, X_{n+1})$.

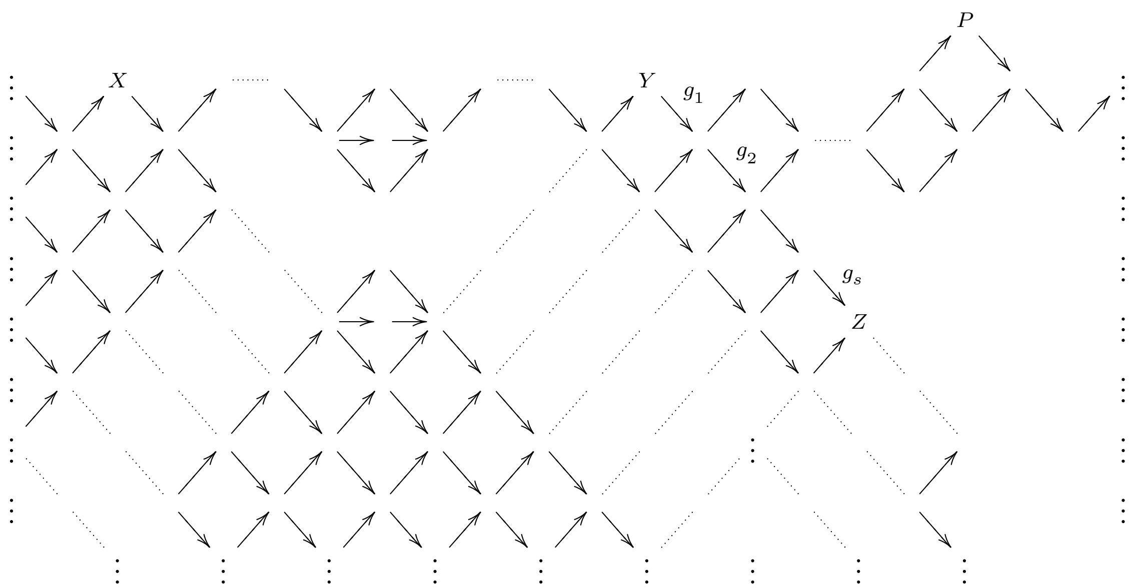
Proof. We only prove that, if $f_n \dots f_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ then $f_n \dots f_1 \in \Re_A^{\infty}(X_1, X_{n+1})$ since the other implication is clear.

To analyze the composition of irreducible morphisms in Γ we will start with the ones near the mouth of Γ. It is enough to study that all non-zero compositions behaves well.

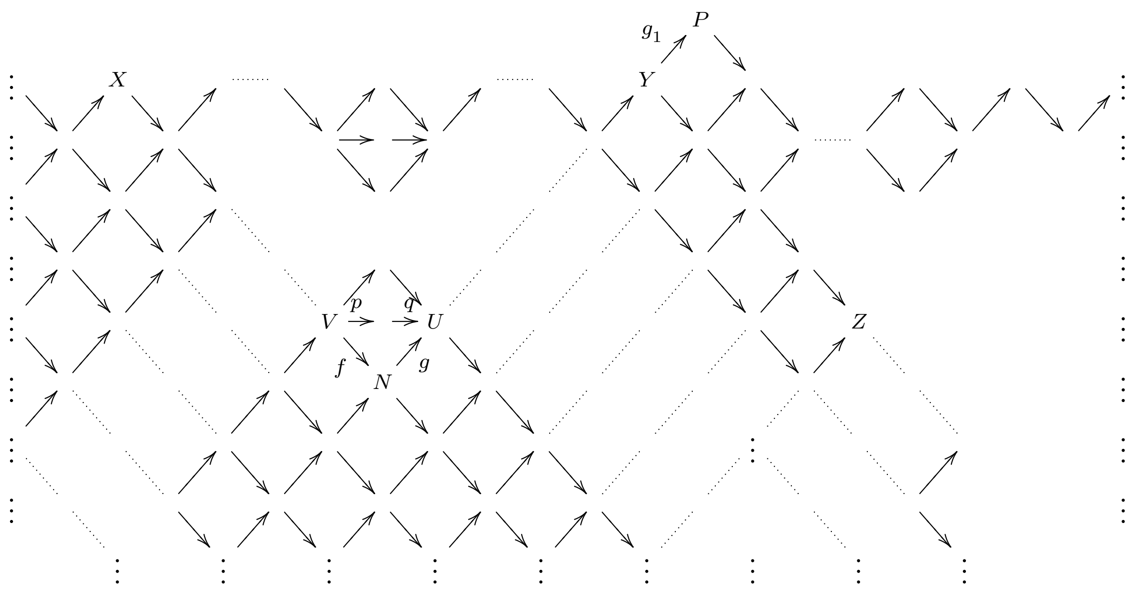
We will do induction on the number $n+m$ with $n, m \geq 1$, where n is the number of exceptional wings in Γ and m is the number of projective-injective vertices from almost split sequences with exactly two middle terms in Γ. We would like to note that on the figures below we present the first exceptional wing from Definition 3.4 but for the second exceptional wing containing the meshes with exactly three middle terms, the proof is the same. Let $n+m=2$, we have three cases.

(a). If $n=2$ then by Lemma 3.14 we get the result.

(b). Let $n=1$ and $m=1$. Let P be a projective-injective module in Γ belonging to a mesh with exactly two middle terms. Consider an exceptional wing $\mathcal{W}$ with the borders φ_1 starting at X and φ_2 ending in Y. Then, the only non-zero composition of irreducible morphisms from X to Z is $g_s ... g_1 \varphi_2 \varphi_1$, where $g_s ... g_1$ belong to the unique infinite sectional path in Γ starting at Y and passing through Z (formed by arrows pointing to the infinity), and $\varphi_2 \varphi_1$ is the composition of the borders of the wing $\mathcal{W}$. We illustrate the situation with the following diagram:



In fact, by Proposition 3.9 or Proposition 3.10 the composition $\varphi_2\varphi_1$ of the borders of the exceptional wing $\mathcal{W}$ behaves well. Now, since the left degree of the morphisms $g_1, \ldots, g_s$ are infinite then the composition $g_s \ldots g_1\varphi_2\varphi_1$ behaves well. Now, consider the situation illustrated with the following diagram:

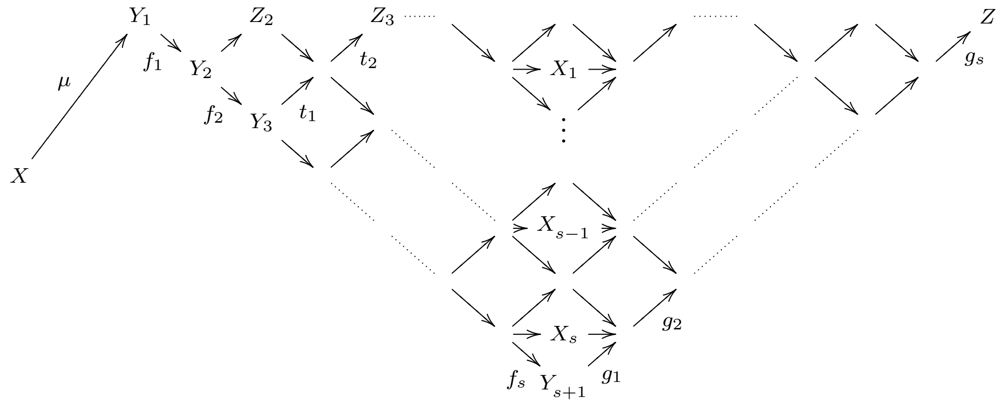


Denote by φ the unique sectional path from X to V, by ψ the unique sectional path from U to Y, and by η the unique sectional path from P to $\tau_A^- Z$. Note that the compositions $\eta g_1\psi qp\varphi$ and $\eta g_1\psi gf\varphi$ behaves well, since by [22] $d_l(g_1) = \infty$ and by [14] the irreducible morphisms in η have infinite left degree. Therefore, the only non-zero composition of irreducible morphisms from X to $\tau_A^- Z$ passing through P behaves well.

(c). Let $m = 2$ and X, Y be the projective-injective modules in Γ. In this case, if $\operatorname{Hom}_A(X, Z) \neq 0$ (respectively, $\operatorname{Hom}_A(Y, Z) \neq 0$) then Z belongs to the unique infinite sectional path starting at X (respectively, at Y). Moreover, it follows by Lemma 3.16 that any path from X to Y is zero. The non-zero paths are the ones which involves almost split sequences not going through (modulo mesh) the almost split sequences with only one indecomposable middle term. In fact, this follows because one can write such a composition as a chain of irreducible morphisms of a coray followed by a chain of irreducible morphisms in a ray. By [14] the right degree of the ones in the coray are infinite and the left degree of the ones in the ray are infinite.

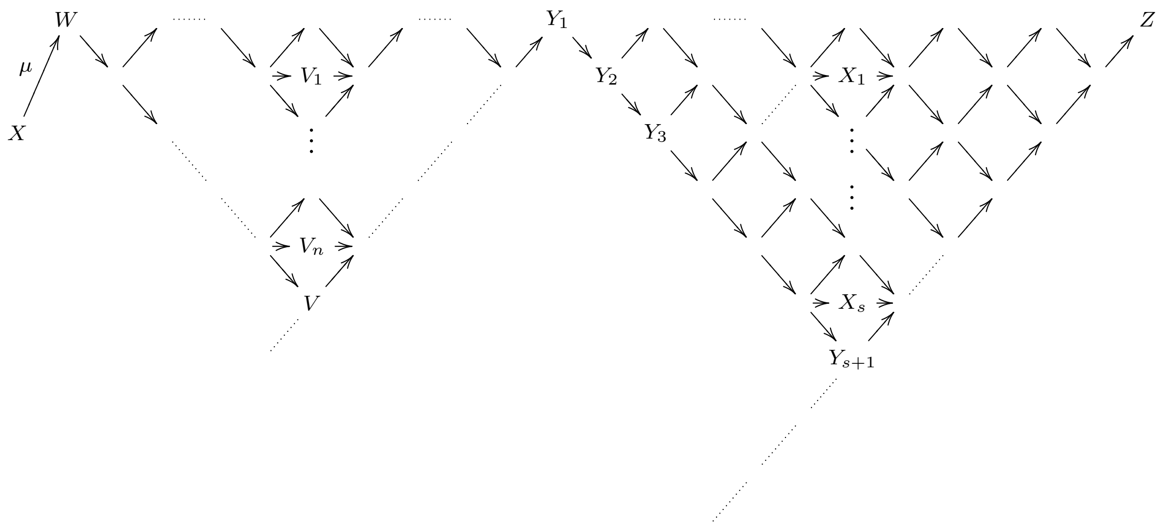
Assume that for $n + m - 1$ the result is true. We want to prove our theorem for $n + m$. We have two cases:

Case 1. We fix a configuration of almost split sequences of an exceptional wing $\mathcal{W}$ as follows:



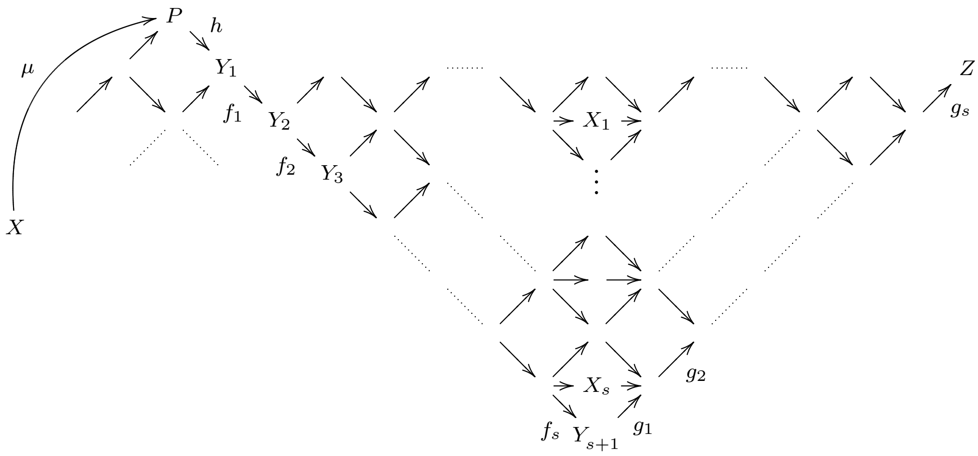
Since the left degrees of $f_1, \ldots, f_s, g_1$ are infinite then $g_1 f_s \ldots f_1 \mu$ behaves well. Now we proceed as in the proof of Proposition 3.9, that is, if X_{s-1} is projective-injective then $d_l(g_2) = \infty$ and hence $g_2 g_1 f_s \ldots f_1 \mu$ behaves well. Otherwise, if $X_{s-1} = 0$ then by Lemma 3.8 we know that any path in Γ from Y_1 to Z_3, say $\varphi_1 : Y_1 \rightsquigarrow Z_3$, is zero. Therefore, clearly, any longest path as $t_2 t_1 f_2 f_1 \mu$ is also zero. Iterating this procedure, we get that the composition $g_s \ldots g_2 g_1 f_s \ldots f_1 \mu$ behaves well. On the other hand, if $\mu : X \to Y_i$ for $2 \leq i \leq s$ then by inductive hypothesis we have that μ behaves well. Then, we have to consider the last configuration next to $\mathcal{W}$. We have two cases to consider.

Case 1-1. Consider the situation illustrated with the following diagram:



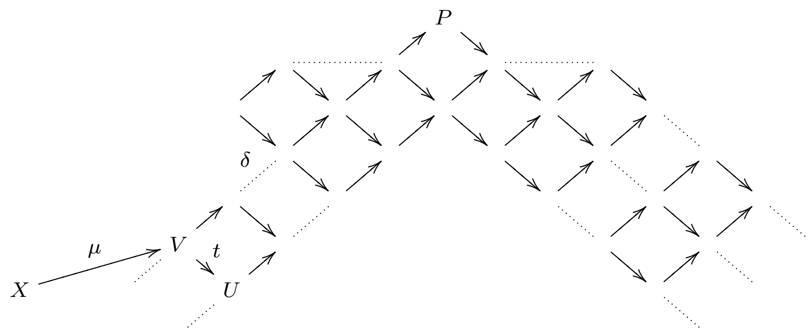
Then it follows from the inductive hypothesis, Lemma 3.14 and its proof that the composition $\lambda_2 \lambda_1 \sigma_2 \sigma_1 \mu$ behaves well, where $\sigma_1 : W \to \cdots \to V$, $\sigma_2 : V \to \cdots \to Y_1$, $\lambda_1 : Y_1 \to \cdots \to Y_{s+1}$, $\lambda_2 : Y_{s+1} \to \cdots \to Z$ are the borders of the above exceptional wings, and $\mathscr{S}(Y_1, Y_{s+1})$ is the infinite rectangle.

Case 1-2. Consider the situation illustrated with the following diagram:



with the exceptional wing $\mathcal{W}$ and projective-injective module P in Γ belonging to a mesh with exactly two middle terms. By the previous considerations it is enough to consider the composition $g_s \dots g_1 f_s \dots f_1 h\mu$. By the inductive hypothesis $\mu : X \rightsquigarrow P$ behaves well. Since the left degree of irreducible morphisms $h, f_1, \dots, f_s$ are infinite then the composition $f_s ... f_1 h\mu$ behaves well. Finally, also $g_s \dots g_1 f_s \dots f_1 h\mu$ behaves well.

Case 2. Assume we have the following situation:



where $\delta : V \to P$ is a sectional path in Γ and P is a projective-injective module belonging to a mesh with exactly two middle terms. By inductive hypothesis we know that μ behaves well. The irreducible morphisms in δ have infinite left degree by [14]. Hence, $\delta\mu$ behaves well. Moreover, again by [14] since t is an irreducible monomorphism then $d_l(t) = \infty$ and we get the result.

Then, it is enough to prove the result for zero paths in Γ, since if we have a non-zero path

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

in Γ then, as we see above, $f_n \dots f_1$ behaves well, getting a contradiction with our assumption. Therefore, $f_n \dots f_1 = 0$.

Now, any other composition of irreducible morphisms $h_i : X_i \to X_{i+1}$ for $i = 1, \dots, n$ is such that $h_n \dots h_1 = \delta f_n \dots f_1 + \mu$ with $\mu \in \Re_A^{n+1}(X_1, X_{n+1})$ and $\delta \in \mathrm{Aut}(X_{n+1})$. Hence, $h_n \dots h_1 \in \Re_A^{n+1}(X_1, X_{n+1})$.

Assume that $h_n \dots h_1 \notin \Re_A^{\infty}(X_1, X_{n+1})$, that is, the composition $h_n \dots h_1$ belongs to $\Re_A^{m}(X_1, X_{n+1}) \backslash \Re_A^{m+1}(X_1, X_{n+1})$ with $m > n$. Hence there is a non-zero path from $X_1 \rightsquigarrow X_{n+1}$ of length longest than n, contradicting Lemma 3.13. The proof is completed. □

Our next two results are fundamental for the study of the composition of irreducible morphisms lying in a tube.

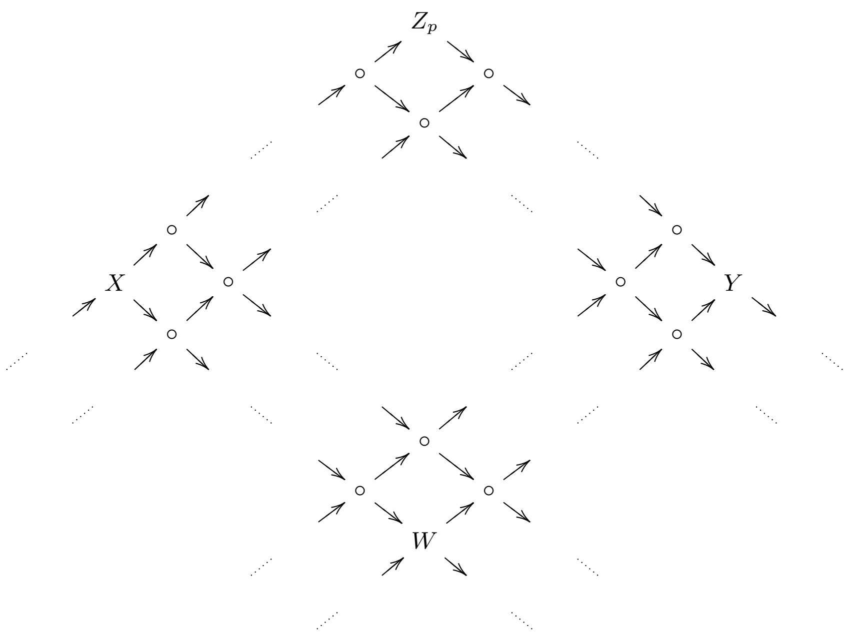
Lemma 3.16. *Let A be an artin algebra and Γ a tube in Γ_A. Then*

(a) *If there is a zero path in Γ from X to Y then any longest path in Γ from X to Y vanishes.*

(b) *If there is a non-zero path γ from X to Y in Γ of length m then $\mathrm{dp}(\gamma) = m$.*

Proof. (a). Let A be an artin algebra and Γ be a tube in Γ_A. From the definition of a tube we know that Γ considered as a translation quiver can be obtained from a stable tube by an iterated application of admissible operations of type (ad 1) and (ad 1*). Therefore, the statement follows from arguments similar to those applied in the proof of Lemma 3.13.

(b) Let $\gamma : X = X_1 \to X_2 \to \cdots \to X_m \to X_{m+1} = Y$ be a non-zero path in Γ of length m. Then we have in Γ the rectangle $\mathscr{S}(X, Z_p, Y, W)$ of the form



Observe that in this case any path in $\mathscr{S}(X, Z_p, Y, W)$ from X to Y is non-zero and has length m. Let $f : X \to Z_p$ be the composition of irreducible maps corresponding to the arrows of the sectional path $\alpha : X = X_1 \to \cdots \to Z_p$, and let $g : Z_p \to Y$ be the composition of irreducible maps corresponding to the arrows of the sectional path $\beta : Z_p \to \cdots \to X_{m+1} = Y$. Since by [22, Section 1] the arrows of the path α

(respectively, the path β) are of infinite right (respectively, left) degree, we infer that $gf \in \Re_A^m(X,Y) \setminus \Re_A^{m+1}(X,Y)$. Hence $\mathrm{dp}(\gamma) = m$. □

Lemma 3.17. *Let A be an artin algebra and Γ a tube in Γ_A. Let $h_i : X_i \to X_{i+1}$ be irreducible morphisms with $X_i \in \Gamma$ for $i = 1, \dots, n+1$. If $0 \neq h_n \dots h_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ then there exists $f_1, \dots, f_n$ such that $f_n \dots f_1 = 0$ for any choice of irreducible morphisms $f_i : X_i \to X_{i+1}$ satisfying the mesh relations of Γ.*

Proof. Consider irreducible morphisms $f_i : X_i \to X_{i+1}$ for $i = 1, \dots, n$ satisfying the mesh relations of Γ. By Lemma 3.2 we have that $h_n \dots h_1 = \delta f_n \dots f_1 + \mu$ with $\mu \in \Re_A^{n+1}(X_1, X_{n+1})$ and $\delta \in \mathrm{Aut}(X_{n+1})$. Hence, $f_n \dots f_1 \in \Re_A^{n+1}(X_1, X_{n+1})$.

Suppose that $f_n \dots f_1 \neq 0$. Then, by Lemma 3.16 (b), we get that $f_n \dots f_1$ behaves well getting a contradiction with the fact that $f_n \dots f_1 \in \Re_A^{n+1}(X_1, X_{n+1})$. Hence $f_n \dots f_1 = 0$. □

Next, we prove one of our main results. We observe that the proof is similar to [8, Theorem A]. For the convenience of the reader we state it here.

Theorem 3.18. *Let A be an artin algebra and Γ a tube in Γ_A. Let $h_i : X_i \to X_{i+1}$ be n irreducible morphisms with $X_i \in \Gamma$ for $i = 1, \dots, n$. Then, $0 \neq h_n \dots h_1 \in \Re_A^{n+1}(X_1, X_{n+1})$ if and only if $0 \neq h_n \dots h_1 \in \Re_A^{\infty}(X_1, X_{n+1})$.*

Proof. Assume that there are n irreducible morphisms $h_i : X_i \to X_{i+1}$ such that $0 \neq h_n \dots h_1 \in \Re_A^{n+1}(X_1, X_{n+1})$. By Lemma 3.17 there are n irreducible morphisms $f_i : X_i \to X_{i+1}$ in the mesh satisfying that $f_n ... f_1 = 0$.

Suppose that $h_n \dots h_1 \in \Re_A^{n+k}(X_1, X_{n+1}) \setminus \Re_A^{n+k+1}(X_1, X_{n+1})$, for some $k \geq 1$. By [4, V, Proposition 7.4] there is a non-zero path $\gamma : X_1 \to X_{n+1}$ of irreducible morphisms of length $n+k$, whose composition does not belong to $\Re_A^{n+k+1}(X_1, X_{n+1})$. Then, by Lemma 3.16 (a), we know that there is a zero path $\gamma' : X_1 \to X_{n+1}$ of length $n+k$ satisfying the mesh relations in Γ. By Lemma 3.2 we can write $\gamma' = \delta\gamma + \mu$ with $\delta \in \mathrm{Aut}(X_{n+1})$ and $\mu \in \Re_A^{n+k+1}(X_1, X_{n+1})$. Hence, we conclude that $\gamma' \in \Re_A^{n+k+1}(X_1, X_{n+1})$ a contradiction.

The converse is clear. □

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