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IGOR BURBAN AND ALEXANDER ZHEGLOV

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and Commuting Differential Operators

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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FOURIER–MUKAI TRANSFORM ON WEIERSTRASS CUBICS AND COMMUTING DIFFERENTIAL OPERATORS

IGOR BURBAN AND ALEXANDER ZHEGLOV

ABSTRACT. In this article, we describe the spectral sheaves of algebras of commuting differential operators of genus one and rank two with singular spectral curve, solving a problem posed by Previato and Wilson. We also classify all indecomposable semi-stable sheaves of slope one and ranks two or three on a cuspidal Weierstraß cubic.

The purpose of this article is to study spectral sheaves of genus one commutative subalgebras in the algebra of ordinary differential operators $\mathfrak{D} = \mathbb{C}[[z]][\partial]$.

Let $\Lambda \subset \mathbb{C}$ be a lattice and $\wp(z)$ be the corresponding Weierstraß function. As it was observed by Wallenberg [51] in 1903, the ordinary differential operators

$$(0.1) \quad P = \partial^2 - 2\wp(z + \alpha) \quad \text{and} \quad Q = 2\partial^3 - 4\wp(z + \alpha)\partial - 3\wp'(z + \alpha),$$

commute for all $\alpha \in \mathbb{C}$ and obey the relation $Q^2 = 4P^3 + g_2P + g_3$, where g_2 and g_3 are the Weierstraß parameters of the lattice Λ , see [51].

In 1968 Dixmier discovered another interesting example [14]: for any $\kappa \in \mathbb{C}$, put $D := \partial^2 + z^3 + \kappa$ and consider

$$(0.2) \quad P = D^2 + 2z \quad \text{and} \quad Q = D^3 + \frac{3}{2}(zD + Dz).$$

Then P and Q commute and satisfy the relation $Q^2 = P^3 - \kappa$. Dixmier also shown that the subalgebra $\mathbb{C}[P, Q] \subset \mathfrak{D}$ is in fact *maximal*.

It turns out that any non-trivial commutative subalgebra \mathfrak{B} in \mathfrak{D} is finitely generated and has Krull dimension one. Moreover, the affine curve $X_0 = \text{Spec}(\mathfrak{B})$ admits a one-point compactification by a *smooth* point p to a projective curve X . The arithmetic genus of X is called *genus* of the algebra \mathfrak{B} . Additionally, the algebra \mathfrak{B} defines a coherent torsion free sheaf \mathcal{F} on the curve X characterized by the following properties:

- For any point $q \in X_0$ (smooth or singular) corresponding to an algebra homomorphism $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$, we have an isomorphism of vector spaces

$$\mathcal{F}|_q^* \longrightarrow \{f \in \mathbb{C}[[z]] \mid P \circ f = \chi(P)f \text{ for all } P \in \mathfrak{B}\}.$$

- The evaluation map $H^0(X, \mathcal{F}) \xrightarrow{\text{ev}_p} \mathcal{F}|_p$ is an isomorphism and $H^1(X, \mathcal{F}) = 0$.

The curve X (respectively, the sheaf \mathcal{F}) is called *spectral curve* (respectively, *spectral sheaf*) of the algebra \mathfrak{B} . The rank of the torsion free sheaf \mathcal{F} is called *rank* of \mathfrak{B} . Krichever correspondence [29] asserts that any non-trivial commutative subalgebra of \mathfrak{B} of *rank one* is essentially determined by its spectral data (X, p, \mathcal{F}) . The description of commutative subalgebras of \mathfrak{D} of higher rank is more complicated. It was first given by Krichever [27, 28, 29] and then elaborated by many authors, including Drinfeld [15], Mumford [40], Segal and Wilson [47], Verdier [50], Mulase [38] and others.

A first description of genus one and rank two commutative subalgebras of \mathfrak{D} was obtained by Krichever and Novikov [30], who also discovered a connection between this kind of problems and soliton solutions of certain non-linear PDE equations. In their Ansatz,

however, the spectral curve X was taken to be smooth. Since that time, the study of genus one commutative subalgebras of \mathfrak{D} attracted a considerable attention, see for example [19, 20, 13, 41, 36]. We refer to [41, Section 1] for an illuminative overview. It is not difficult to show that for any (normalized) genus one and rank two commutative subalgebra $\mathfrak{B} \subset \mathfrak{D}$ there exist two operators $L, M \in \mathfrak{B}$ such that $\mathfrak{B} = \mathbb{C}[L, M]$ and

$$(0.3) \quad L = \partial^4 + a_2\partial^2 + a_1\partial + a_0, \quad M = 2L_+^{\frac{3}{2}}, \quad M^2 = 4L^3 + g_2L + g_3$$

for some $g_2, g_3 \in \mathbb{C}$, see [21, 41] or Proposition 3.1 below. Here, $L^{\frac{3}{2}}$ is taken in the algebra of pseudo-differential operators $\mathbb{C}[[z]]((\partial^{-1}))$ and $L_+^{\frac{3}{2}}$ is the projection of $L^{\frac{3}{2}}$ onto \mathfrak{D} . A full description of all operators L as in (0.3) satisfying the constraint $[L, M] = 0$ for $M = 2L_+^{\frac{3}{2}}$ was obtained by Grünbaum [21], who also got convenient formulae for the coefficients a_0, a_1 and a_2 . In this article we deal with the following

Problem. What is the spectral sheaf \mathcal{F} of the algebra $\mathfrak{B} = \mathbb{C}[L, P]$ of genus one and rank two, expressed through the coefficients a_0, a_1 and a_2 from (0.3) in the case when the spectral curve X is *singular*? In particular, what is the spectral sheaf of Dixmier’s family (0.2) for $\kappa = 0$?

Previato and Wilson gave a complete solution of the above problem in the case the spectral curve X is smooth [41, Theorem 1.2]. Their answer was given in terms of Grünbaum’s classification [21] as well as of Atiyah’s classification of vector bundles on elliptic curves [2]. The description of the spectral sheaf in the case of a singular spectral curve was left as an open problem. Quoting [41, Page 109]: “We have not worked out the case when the curve X is singular, that is, when X is nodal or cuspidal cubic. It would probably be rather complicated (because of the need to consider torsion free sheaves)”.

It turns out that the problem of Previato and Wilson can be completely solved thanks to the technique of derived categories and Fourier–Mukai transforms on the Weierstraß cubics [8]. The main idea is that instead of dealing with the spectral sheaf \mathcal{F} directly, it is easier to describe the Fourier–Mukai transform \mathcal{T} of \mathcal{F} , which is a certain torsion sheaf on X . This approach brings a new light on the method of [41] and allows to treat the analogous problem for genus one commutative subalgebras of \mathfrak{D} of arbitrary rank.

The structure of this article is the following. In Section 1, we review the theory of commutative subalgebras in \mathfrak{D} . The major new result of this section is Theorem 1.25 explaining the appearance of Fourier–Mukai transforms in the study of spectral sheaves.

A classification of indecomposable vector bundles on a smooth elliptic curve was obtained by Atiyah in [2]. In this case, indecomposable vector bundles are automatically semi-stable. Semi-stable torsion free sheaves of integral slope on a nodal Weierstraß cubic were explicitly classified in [8], see also [17, 5]. However, the category of semi-stable torsion free sheaves of slope one on a cuspidal cubic curve turns out to be representation wild [16, 5]. Nevertheless, one can obtain a full classification of all rank two or three semi-stable coherent sheaves of slope one on a cuspidal cubic curve. This is done in Section 2, see in particular Theorem 2.8 (providing a self-contained classification in the nodal case as well) and Corollary 2.15.

In Section 3 we give a full answer on the question of Previato and Wilson [41], describing the spectral sheaf of a genus one and rank two commutative subalgebra of \mathfrak{D} with singular spectral curve, see Theorem 3.7, Theorem 3.11 and Theorem 3.16. In particular, we describe all such commutative subalgebras, whose spectral sheaf is indecomposable and not locally free, see Corollary 3.13. Finally, taking the Fourier transform of Dixmier’s example

(0.2), we illustrate how the spectral sheaf of a genus one and rank three commutative subalgebra of \mathfrak{D} can be explicitly determined, see Example 3.22.

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1. COMMUTATIVE SUBALGEBRAS IN THE ALGEBRA OF DIFFERENTIAL OPERATORS

Let $\mathfrak{D} = \mathbb{C}[[z]][\partial] = \left\{ \sum_{i=0}^n a_i(z)\partial^i \mid a_i(z) \in \mathbb{C}[[z]], 0 \leq i \leq n \right\}$ be the algebra of ordinary differential operators with coefficients in the algebra $\mathbb{C}[[z]]$ of formal power series. In this section we shall review the theory of commutative subalgebras of \mathfrak{D} . The first systematic study of this problem dates back to a work of Schur [46]. Burchnell and Chaundy [10, 11, 12] and Baker [3] obtained a full classification of pairs of commuting differential operators of coprime orders. The modern algebro-geometric treatment of arbitrary commutative subalgebras in \mathfrak{D} was initiated by Krichever [27, 28, 29]. This theory has been extensively applied by Novikov and his school in the study of soliton solutions of various non-linear partial differential equations, see for example the survey [30]. Krichever’s approach was formalized and further developed by Drinfeld [15], Mumford [40], Verdier [50], Segal and Wilson [47] and Mulase [38]. The literature dedicated to this area is vast and the described bibliography is definitely uncomplete. There are numerous survey articles on this subject, see for example [42, 52, 39]. Nonetheless, for our purposes we felt it was necessary to review this theory once again, setting the notation and introducing all the relevant notions. The main novelty of this section is Theorem 1.25 explaining the appearance of derived categories in the study of spectral sheaves of commutative subalgebras of \mathfrak{D} .

Let us begin with the following well-known result about automorphisms of \mathfrak{D} .

Lemma 1.1. *Let φ be an algebra endomorphism of \mathfrak{D} . Then there exist $u \in \mathbb{C}[[z]]$ satisfying $u(0) = 0$ and $u'(0) \neq 0$, and $v \in \mathbb{C}[[z]]$ such that*

$$(1.1) \quad \begin{cases} z & \xrightarrow{\varphi} & u \\ \partial & \xrightarrow{\varphi} & \frac{1}{u'}\partial + v. \end{cases}$$

In particular, φ is an automorphism of \mathfrak{D} , i.e. $\text{End}(\mathfrak{D}) = \text{Aut}(\mathfrak{D})$.

Proof. Let $u := \varphi(z) \in \mathfrak{D}$. It is not difficult to show that u belongs $\mathbb{C}[[z]]$ and satisfies the properties stated in the theorem. Let $P := \varphi(\partial) = a_n\partial^n + a_{n-1}\partial^{n-1} + \dots + a_0 \in \mathfrak{D}$ for some $n \in \mathbb{N}$, where $a_n \neq 0$. Clearly, $[P, u] = nu'a_n\partial^{n-1} + \text{l.o.t.}$, hence $[\partial, z] = 1 = [P, u]$ if and only if $n = 1$ and $a_1 = \frac{1}{u'}$. \square

Remark 1.2. Let $w \in \mathbb{C}[[z]]$ be a unit (i.e. $w(0) \neq 0$). Then for the inner automorphism $\text{Ad}_w : \mathfrak{D} \rightarrow \mathfrak{D}$, $P \mapsto w^{-1}Pw$, we have:

$$\begin{cases} z & \mapsto & z \\ \partial & \mapsto & \partial + \frac{w'}{w}. \end{cases}$$

Note that for any $\mathbb{C}[[z]] \ni v = \sum_{i=0}^{\infty} \beta_i z^i = \beta_0 + \tilde{v}$, the formal power series $w := \exp(v) = e^{\beta_0} \exp(\tilde{v})$ is a unit in $\mathbb{C}[[z]]$. Therefore, any automorphism $\varphi \in \text{Aut}(\mathfrak{D})$ satisfying $\varphi(z) = z$ is inner, see (1.1)

Proposition 1.3. *Let $P = a_n \partial^n + a_{n-1} \partial^{n-1} + \dots + a_0 \in \mathfrak{D}$, where $a_n(0) \neq 0$. Then there exists $\varphi \in \text{Aut}(\mathfrak{D})$ such that*

$$(1.2) \quad Q := \varphi(P) = \partial^n + b_{n-2} \partial^{n-2} + \dots + b_0$$

for some $b_0, \dots, b_{n-2} \in \mathbb{C}[[z]]$. Moreover, if $Q \in \mathfrak{D}$ is a normalized differential operator of positive order (i.e. a differential operator having the form (1.2)) and ψ an inner automorphism of \mathfrak{D} such that $\psi(Q) = Q$ then $\psi = \text{id}$.

Proof. By assumption, a_n is a unit in $\mathbb{C}[[z]]$. Therefore, there exists $a \in \mathbb{C}[[z]]$ such that $a^n = a_n$. It implies that $P = (a\partial)^n + \text{l.o.t.}$ Hence, there exists a change of variables transforming P into an operator of the form $\tilde{P} := \partial^n + c_{n-1} \partial^{n-1} + \dots + c_0$. Applying now to \tilde{P} an automorphism (1.1) with $u = z$ and $v = -\frac{c_{n-1}}{n}$, we get a normalized operator Q . This proves the first statement. The proof of the second statement is straightforward. \square

Definition 1.4. A differential operator $P = a_n \partial^n + a_{n-1} \partial^{n-1} + \dots + a_0 \in \mathfrak{D}$ of positive order n is called *formally elliptic* if $a_n \in \mathbb{C}^*$.

The following useful observation is due to Verdier [50, Lemme 1].

Lemma 1.5. *Let \mathfrak{B} be a commutative subalgebra of \mathfrak{D} containing a formally elliptic element P . Then all elements of \mathfrak{B} are formally elliptic.*

Remark 1.6. An algebra $\mathfrak{B} \subset \mathfrak{D}$ containing a formally elliptic element is called *elliptic*. There exists non-trivial non-elliptic commutative subalgebras in \mathfrak{D} , i.e. those which are not of the form $\mathbb{C}[P]$, where P is a non-elliptic operator. Nonetheless, the major interest concerns those commutative subalgebras of \mathfrak{D} which belong to the subalgebra $\mathbb{C}\{z\}[\partial]$ of ordinary differential operators, whose coefficients are *convergent* power series. If $P = a_n \partial^n + a_{n-1} \partial^{n-1} + \dots + a_0$ is such an operator then shifting the variable $z \mapsto z + \varepsilon$ with $\varepsilon \in \mathbb{C}$ such that $|\varepsilon|$ is sufficiently small, we may always achieve that $a_n(0) \neq 0$. Note that this operation can not be extended on the whole \mathfrak{D} . Nonetheless, one can show that all elements of \mathfrak{B} belong to $\mathbb{C}\{z\}[\partial]$ (this follows for example from Schur's theorem [39, Theorem 2.2], see for example [38, Lemma 5.3]) and one can choose a common radius of convergence for all coefficients of all elements of \mathfrak{B} . According to Proposition 1.3, we can transform P into a normalized formally elliptic differential operator. Therefore, in the sequel all commutative subalgebras of \mathfrak{D} are assumed

- to contain an elliptic operator of positive order (i.e. being elliptic)
- to be *normalized*, meaning that all elements of \mathfrak{B} of minimal positive order are normalized.

The last assumption eliminates redundant degrees of freedom in the problem of classification of commutative subalgebras of differential operators: if $\mathfrak{B} \subset \mathfrak{D}$ is a normalized elliptic subalgebra and φ an inner automorphism of \mathfrak{D} such that $\varphi(\mathfrak{B}) = \mathfrak{B}$ then $\varphi = \text{id}$.

Definition 1.7. Let \mathfrak{B} be a commutative subalgebra of \mathfrak{D} . We call the natural number

$$r = \text{rk}(\mathfrak{B}) = \text{gcd} \{ \text{ord}(P) \mid P \in \mathfrak{B} \}$$

the *rank* of \mathfrak{B} .

Theorem 1.8. *Let \mathfrak{B} be a commutative subalgebra of \mathfrak{D} .*

- (1) *Then \mathfrak{B} is finitely generated integral domain of Krull dimension one. In particular, \mathfrak{B} determines an integral affine algebraic curve $X_0 := \text{Spec}(\mathfrak{B})$.*
- (2) *Moreover, X_0 can be compactified to a projective algebraic curve X by adding a single smooth point p , which is determined by the valuation*

$$\text{val}_p : \mathfrak{Q} \longrightarrow \mathbb{Z}, \quad \frac{P}{Q} \mapsto \frac{\text{ord}(Q) - \text{ord}(P)}{r},$$

where \mathfrak{Q} is the quotient field of \mathfrak{B} and r is the rank of \mathfrak{B} .

Comment to the proof. Algebraic curves entered for the first time into the theory of commutative subalgebras of \mathfrak{D} in the works of Burchnell and Chaundy [10, 11] and in a greater generality in the works of Krichever [27, 28]. In the stated form, this result can be found in the articles of Mumford [40, Section 2] and Verdier [50, Proposition 1]. See also [38, Theorem 3.3].

Definition 1.9. The projective curve $X = X_0 \cup \{p\}$ is called *spectral curve* of a commutative subalgebra $\mathfrak{B} \subset \mathfrak{D}$. The arithmetic genus of X is called *genus* of \mathfrak{B} .

Example 1.10. In the example of Wallenberg (0.1), the algebra $\mathbb{C}[P, Q]$ has rank one and genus one. In the example of Dixmier (0.2), the algebra $\mathbb{C}[P, Q]$ has rank two and genus one for any $\kappa \in \mathbb{C}$.

Definition 1.11. Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra. Consider the right \mathfrak{D} -module $F := \mathfrak{D}/z\mathfrak{D} \xrightarrow{\cong} \mathbb{C}[\partial]$, $\overline{a(z)\partial^n} \mapsto a(0)\partial^n$. Clearly, the *right* action of \mathfrak{D} on $\mathbb{C}[\partial]$ is given by the following rules:

$$(1.3) \quad \begin{cases} p(\partial) \diamond \partial &= \partial \cdot p(\partial) \\ p(\partial) \diamond z &= p'(\partial). \end{cases}$$

Restricting the action (1.3) on the subalgebra \mathfrak{B} , we endow F with the structure of a \mathfrak{B} -module. Since the algebra \mathfrak{B} is commutative, we shall view F as a *left* \mathfrak{B} -module (although having the natural right action in mind).

Theorem 1.12. *Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra of rank r . Then F is finitely generated and torsion free over \mathfrak{B} . Moreover, $\mathfrak{Q} \otimes_{\mathfrak{B}} F \cong \mathfrak{Q}^{\oplus r}$, i.e. $\text{rk}_{\mathfrak{B}}(F) = \text{rk}(\mathfrak{B})$. In other words, the rank of the algebra \mathfrak{B} in the sense of Definition 1.7 coincides with the rank of F viewed as a \mathfrak{B} -module.*

Comment to the proof. In the stated form, this result can be found in [50, Proposition 3] and [40, Section 2]. See also [31, Theorem 2.1] for another treatment as well as for a generalization on the higher-dimensional cases.

Remind that according to the Nullstellensatz, the points of X_0 stand in bijection with the algebra homomorphisms $\mathfrak{B} \rightarrow \mathbb{C}$ (called in what follows *characters*).

Definition 1.13. Let $q \in X_0$ be any point and $\chi = \chi_q : \mathfrak{B} \rightarrow \mathbb{C}$ the corresponding character. We call the \mathbb{C} -vector space

$$(1.4) \quad \text{Sol}(\mathfrak{B}, \chi) := \{f \in \mathbb{C}[[z]] \mid P \circ f = \chi(P)f \text{ for all } P \in \mathfrak{B}\}$$

the *solution space* of the algebra \mathfrak{B} at the point q . Here, we take the usual left action \circ of \mathfrak{D} on $\mathbb{C}[[z]]$. Observe, that $\text{Sol}(\mathfrak{B}, \chi)$ has a natural \mathfrak{B} -module structure.

The geometric meaning of the \mathfrak{B} -module F is explained by the next result.

Theorem 1.14. *The following \mathbb{C} -linear map*

$$(1.5) \quad F \xrightarrow{\eta_\chi} \text{Sol}(\mathfrak{B}, \chi)^*, \quad \partial^i \mapsto (f \mapsto f^{(i)}(0))$$

is also \mathfrak{B} -linear, where $\text{Sol}(\mathfrak{B}, \chi)^ = \text{Hom}_{\mathbb{C}}(\text{Sol}(\mathfrak{B}, \chi), \mathbb{C})$ is the vector space dual of the solution space. Moreover, the induced map*

$$(1.6) \quad \mathfrak{B}/\text{Ker}(\chi) \otimes_{\mathfrak{B}} F \xrightarrow{\bar{\eta}_\chi} \text{Sol}(\mathfrak{B}, \chi)^*$$

is an isomorphism of \mathfrak{B} -modules.

Proof. These statements can be found in [40, Section 2] or [50, Proposition 5], where the proofs are briefly outlined. Since this result plays a central role in our work, we give a detailed proof here. First note that the following map

$$(1.7) \quad \text{Hom}_{\mathbb{C}}(F, \mathbb{C}) \xrightarrow{\Phi} \mathbb{C}[[z]], \quad \lambda \mapsto \sum_{p=0}^{\infty} \frac{1}{p!} \lambda(\partial^p) z^p$$

is an isomorphism of left \mathfrak{D} -modules. Let $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ be a character, then $\mathbb{C} = \mathbb{C}_\chi := \mathfrak{B}/\text{Ker}(\chi)$ is a left \mathfrak{B} -module. We obtain a \mathfrak{B} -linear map

$$(1.8) \quad \Psi : \text{Hom}_{\mathfrak{B}}(F, \mathbb{C}_\chi) \xrightarrow{I} \text{Hom}_{\mathbb{C}}(F, \mathbb{C}) \xrightarrow{\Phi} \mathbb{C}[[z]],$$

where I is the forgetful map. The image of I consists of those \mathbb{C} -linear functionals, which are also \mathfrak{B} -linear, i.e.

$$\text{Im}(I) = \{ \lambda \in \text{Hom}_{\mathbb{C}}(F, \mathbb{C}) \mid \lambda(P \diamond -) = \chi(P) \cdot \lambda(-) \text{ for all } P \in \mathfrak{B} \}.$$

This implies that $\text{Im}(\Psi) = \text{Sol}(\mathfrak{B}, \chi)$. Next, we have a canonical isomorphism of \mathfrak{B} -modules: $\text{Hom}_{\mathfrak{B}}(F, \mathbb{C}_\chi) \cong \text{Hom}_{\mathbb{C}}(\mathfrak{B}/\text{Ker}(\chi) \otimes_{\mathfrak{B}} F, \mathbb{C})$. Dualizing again, we get an isomorphism of vector spaces

$$\Psi^* : \text{Sol}(\mathfrak{B}, \chi)^* \longrightarrow (\mathfrak{B}/\text{Ker}(\chi) \otimes_{\mathfrak{B}} F)^{**} \cong \mathfrak{B}/\text{Ker}(\chi) \otimes_{\mathfrak{B}} F.$$

It remains to observe that Ψ^* is also \mathfrak{B} -linear and $(\Psi^*)^{-1} = \bar{\eta}_\chi$. \square

Remark 1.15. The isomorphism (1.6) has the following geometric meaning: if we view F as a coherent sheaf on $X_0 = \text{Spec}(A)$ then for any point $q \in X_0$ (smooth or singular) we have: $F|_q \cong \text{Sol}(\mathfrak{B}, \chi)^*$, where $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ is the character corresponding to the point q . Because of this fact, F is called *spectral module* of the algebra \mathfrak{B} .

Corollary 1.16. *Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra of rank r . Then for any character $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ we have: $r \leq \dim_{\mathbb{C}}(\text{Sol}(\mathfrak{B}, \chi)) < \infty$. Moreover, $\dim_{\mathbb{C}}(\text{Sol}(\mathfrak{B}, \chi)) \geq r+1$ if only if χ defines a singular point $q \in X_0$ and F is not locally free at q .*

According to Theorem 1.8, the affine curve $X_0 = \text{Spec}(\mathfrak{B})$ admits a canonical compactification X . It turns out that the spectral module F can also be *canonically* extended from X_0 on the whole projective curve X .

Theorem 1.17. *Let \mathfrak{B} be a commutative subalgebra in \mathfrak{D} of rank r . Then there exists a unique torsion free sheaf \mathcal{F} (called *spectral sheaf* of \mathfrak{B}) on the projective curve $X = X_0 \cup \{p\}$ such that*

- (1) $\Gamma(X_0, \mathcal{F})$ is isomorphic to F viewed as a \mathfrak{B} -module (remind that $\mathfrak{B} \cong \Gamma(X_0, \mathcal{O})$).
- (2) The image of the canonical restriction map $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X_0, \mathcal{F})$ is the vector space $\langle 1, \partial, \dots, \partial^{r-1} \rangle_{\mathbb{C}} \subset \mathbb{C}[\partial] = F \cong \Gamma(X_0, \mathcal{F})$. In particular, $\dim_{\mathbb{C}}(\Gamma(X, \mathcal{F})) = r$.

(3) The evaluation map $\Gamma(X, \mathcal{F}) \xrightarrow{\text{ev}_p} \mathcal{F}|_p$ is an isomorphism and $H^1(X, \mathcal{F}) = 0$.

Comment to the proof. Taking the long exact cohomology sequence, associated with the short exact sequence of coherent sheaves $0 \rightarrow \mathcal{F}(-[p]) \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_p \rightarrow 0$, we immediately see that the condition (3) of Theorem 1.17 on \mathcal{F} is equivalent to the vanishing

$$(1.9) \quad H^0(X, \mathcal{F}(-[p])) = 0 = H^1(X, \mathcal{F}(-[p])).$$

The statement follows from [38, Theorem 3.4], applied to the torsion free sheaf $\mathcal{F}(-[p])$.

Definition 1.18. The slope of a torsion free (but not necessarily locally free) coherent sheaf \mathcal{G} on X is the ratio $\mu(\mathcal{G}) := \frac{\chi(\mathcal{G})}{\text{rk}(\mathcal{G})}$, where $\chi(\mathcal{G}) := \dim_{\mathbb{C}}(H^0(X, \mathcal{G})) - \dim_{\mathbb{C}}(H^1(X, \mathcal{G}))$ is the Euler characteristic of \mathcal{G} and $\text{rk}(\mathcal{G})$ is the rank of \mathcal{G} . A coherent sheaf \mathcal{G} is semi-stable when for any subsheaf $\mathcal{G}' \subset \mathcal{G}$ we have: $\mu(\mathcal{G}') \leq \mu(\mathcal{G})$.

Corollary 1.19. Let \mathfrak{B} be a commutative subalgebra in \mathfrak{D} and \mathcal{F} be its spectral sheaf.

(1) The following sequence of coherent sheaves on X is exact:

$$(1.10) \quad 0 \longrightarrow \Gamma(X, \mathcal{F}) \otimes \mathcal{O} \xrightarrow{\text{ev}} \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0,$$

where \mathcal{T} is a torsion sheaf on X , whose support belongs to the affine part X_0 .

(2) \mathcal{F} is semi-stable of slope one.

Proof. (1) Let $\mathcal{T} := \text{Cok}(\Gamma(X, \mathcal{F}) \otimes \mathcal{O} \xrightarrow{\text{ev}} \mathcal{F})$. Because of the condition (3) in the axiomatic description of the spectral sheaf \mathcal{F} from Theorem 1.17, the infinite point $p \in X$ does not belong to the support of \mathcal{T} . It implies that \mathcal{T} is a torsion sheaf, whose support belongs to X_0 . Since the ranks of the torsion free sheaves $\Gamma(X, \mathcal{F}) \otimes \mathcal{O}$ and \mathcal{F} are both equal to r , the rank of $\text{Ker}(\text{ev})$ is equal to zero. This means that $\text{Ker}(\text{ev})$ is a torsion sheaf. On the other hand, $\text{Ker}(\text{ev})$ is a subsheaf of a torsion free sheaf $\Gamma(X, \mathcal{F}) \otimes \mathcal{O}$. Therefore, $\text{Ker}(\text{ev}) = 0$ and the sequence (1.10) is exact.

(2) The coherent sheaf $\tilde{\mathcal{F}} := \mathcal{F}(-[p])$ is semi-stable. Indeed, because of the vanishing (1.9), $\mu(\tilde{\mathcal{F}}) = 0$. If \mathcal{H} is a subsheaf of $\tilde{\mathcal{F}}$ then $H^0(X, \mathcal{H}) = 0$, thus $\mu(\mathcal{H}) \leq 0$. Hence, $\tilde{\mathcal{F}}$ is semi-stable, therefore \mathcal{F} is semi-stable as well. \square

Definition 1.20. Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra. Then the tuple (X, p, \mathcal{F}) is called *spectral data* of \mathfrak{B} . In particular, $\mathfrak{B} \cong \Gamma(X \setminus \{p\}, \mathcal{O})$ viewed as a \mathbb{C} -algebra.

Theorem 1.21 (Krichever correspondence). Consider the following two sets:

$$(1.11) \quad \text{DiffOp} = \{ \mathfrak{B} \subset \mathfrak{D} \mid \mathfrak{B} \text{ is commutative, elliptic and normalized} \}$$

and

$$(1.12) \quad \text{SpecData} = \left\{ (X, p, \mathcal{F}) \left| \begin{array}{l} X \text{ is an integral projective curve} \\ p \in X \text{ is a smooth point} \\ \mathcal{F} \text{ is torsion free, } H^1(X, \mathcal{F}) = 0 \\ \Gamma(X, \mathcal{F}) \xrightarrow{\text{ev}_p} \mathcal{F}|_p \text{ is an isomorphism} \end{array} \right. \right\}.$$

Then the Krichever map

$$(1.13) \quad \text{DiffOp} \xrightarrow{K} \text{SpecData}, \quad \mathfrak{B} \mapsto (X, p, \mathcal{F})$$

is surjective. Moreover, its restriction $\text{DiffOp}_1 \xrightarrow{K} \text{SpecData}_1$ on the set of commutative subalgebras $\mathfrak{B} \subset \mathfrak{D}$ of rank one, respectively the set of tuples (X, p, \mathcal{F}) with \mathcal{F} of rank one, is essentially a bijection (the word “essentially” means that the spectral data of \mathfrak{B} and \mathfrak{B}' are the same if and only if $\mathfrak{B}' = \varphi(\mathfrak{B})$ for $\varphi \in \text{Aut}(\mathfrak{D})$ induced by $z \mapsto \alpha z$ with $\alpha \in \mathbb{C}^*$).

Comment to the proof. In the case X is a smooth Riemann surface, this result has been proven by Krichever [28, Theorem 2.2]. Singular curves and torsion free sheaves which are not locally free were included into the picture by Mumford [40, Section 2] and Verdier [50, Proposition 4]. Their approach was further developed by Mulase [38, Theorem 5.6].

Example 1.22. It was already pointed out by Burchnell and Chaundy in 1923, that the Wallenberg’s family (0.1) exhausts the list of rank one commutative subalgebras of \mathfrak{D} , whose spectral curve X is elliptic [10, Section 8]. This perfectly matches with Theorem 1.21: in this case $X := \mathbb{C}/\Lambda \cong \text{Pic}^0(X)$. Next, if we wish the coefficients of the operators P and Q to be regular at 0, we have to demand that the parameter $\alpha \in \mathbb{C}$ from (0.1) does not belong to the lattice Λ . This corresponds to the exclusion of the structure sheaf \mathcal{O} from the set $\text{Pic}^0(X)$. For any $\alpha \in \mathbb{C}$, consider the following function

$$(1.14) \quad \psi_\alpha(z, t) = \frac{\sigma(t - \alpha - z)}{\sigma(t)\sigma(z + \alpha)} \exp(\zeta(t)(z + \alpha)),$$

where σ and ζ are the Weierstraß elliptic functions. Then we have:

$$(1.15) \quad \begin{cases} L_z \circ \psi_\alpha(z, t) &= \wp(t) \cdot \psi_\alpha(z, t) \\ P_z \circ \psi_\alpha(z, t) &= \wp'(t) \cdot \psi_\alpha(z, t). \end{cases}$$

Clearly, $q = (\wp(t), \wp'(t)) \in X_0 = V(y^2 - 4x^3 - g_2x - g_3)$ for all $t \in \mathbb{C} \setminus \Lambda$, where g_2 and g_3 are the Weierstraß parameters of the lattice Λ . The function $\psi_\alpha(z, t)$ is the genus one *Baker–Akhieser function*. An analogous expression for the Baker–Akhieser function exists for an arbitrary commutative subalgebra $\mathfrak{B} \subset \mathfrak{D}$ of rank one, such that the spectral curve X of \mathfrak{B} is smooth, see [28]. This provides another interpretation of the spectral sheaf \mathcal{F} .

Remark 1.23. The study of commutative subalgebras of \mathfrak{D} of arbitrary rank has been initiated by Krichever [27, 28, 29]. Although the Krichever map K is surjective, the algebra \mathfrak{B} can not be recovered from (X, p, \mathcal{F}) in the case $\text{rk}(\mathfrak{B}) \geq 2$. In order to study this “inverse scattering problem”, Krichever and Novikov introduced the formalism of vector-valued Baker–Akhieser functions. This method leads to explicit expressions for commutative subalgebras of genus one and rank two [30, Section 5] and three [36]. Using this approach, new commutative subalgebras of rank two and higher genus with polynomial coefficients were recently constructed in [34, 37].

Remark 1.24. Commutative subalgebras $\mathfrak{B} \subset \mathfrak{D}$ with *singular* spectral curve arise naturally in various applications in mathematical physics, see for instance [18, 53] and [49].

Main question. Assume we are given a commutative subalgebra $\mathfrak{B} \subset \mathfrak{D}$ of arbitrary rank. How to describe *explicitly* its spectral sheaf \mathcal{F} ?

The following observation plays a key role in our work, also explaining why the genus one case is so special.

Theorem 1.25. *The torsion sheaf \mathcal{T} from the short exact sequence (1.10) is isomorphic to the Seidel–Thomas twist of \mathcal{F} . If the arithmetic genus of X is one then the spectral sheaf \mathcal{F} can be recovered back from \mathcal{T} .*

Proof. For any projective variety X (smooth or singular) there exists an exact endofunctor $\mathbb{T} = \mathbb{T}_{\mathcal{O}} : D^b(\text{Coh}(X)) \rightarrow D^b(\text{Coh}(X))$ of the derived category of coherent sheaves $D^b(\text{Coh}(X))$ called *Seidel–Thomas twist functor* [48, Definition 2.5], assigning to a complex \mathcal{F}^\bullet another complex $\mathbb{T}(\mathcal{F}^\bullet)$ defined through the distinguished triangle

$$(1.16) \quad \text{RHom}^\bullet(\mathcal{O}, \mathcal{F}^\bullet) \otimes^{\mathbb{k}} \mathcal{O} \xrightarrow{\text{ev}} \mathcal{F}^\bullet \rightarrow \mathbb{T}(\mathcal{F}^\bullet) \rightarrow (\text{RHom}^\bullet(\mathcal{O}, \mathcal{F}^\bullet) \otimes^{\mathbb{k}} \mathcal{O})[1].$$

In our case, X is a curve, $\mathcal{F}^\bullet = \mathcal{F}[0]$ is a stalk complex, $\text{Ext}_X^1(\mathcal{O}, \mathcal{F}) = 0$ and the evaluation map $\text{Hom}_X(\mathcal{O}, \mathcal{F}) \otimes \mathcal{O} \xrightarrow{\text{ev}} \mathcal{F}$ is injective. Therefore, the distinguished triangle (1.16) is nothing but the short exact sequence (1.10). The key point is the following: $\mathbb{T}_{\mathcal{O}}$ is an auto-equivalence of $D^b(\text{Coh}(X))$ provided X is a Calabi–Yau variety, meaning that

$$\text{Ext}_X^i(\mathcal{O}, \mathcal{O}) = \begin{cases} \mathbb{C} & i = 0, \dim(X) \\ 0 & \text{otherwise.} \end{cases}$$

It remains to note that the irreducible Calabi–Yau curves are precisely the irreducible projective curves of arithmetic genus one (which are nothing but the Weierstraß cubics $X = X_{g_2, g_3} = \overline{V(y^2 - 4x^3 - g_2x - g_3)} \subset \mathbb{P}^2$, where $g_2, g_3 \in \mathbb{C}$). \square

The above Theorem 1.25 implies that the torsion sheaf \mathcal{T} is an important invariant of the algebra \mathfrak{B} , allowing to reconstruct the spectral sheaf \mathcal{F} in the genus one case. It turns out that at least the support of \mathcal{T} can be algorithmically determined.

Let $\mathbb{C}((z))$ be the field of formal Laurent series and $\tilde{\mathfrak{D}} = \mathbb{C}((z))[\partial]$ be the algebra of ordinary differential operators with coefficients in $\mathbb{C}((z))$. For any character $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ consider the \mathbb{C} -vector space

$$(1.17) \quad \text{Sol}'(\mathfrak{B}, \chi) := \{f \in \mathbb{C}((z)) \mid P \circ f = \chi(P)f \text{ for all } P \in \mathfrak{B}\}.$$

Obviously, $\text{Sol}(\mathfrak{B}, \chi) \subseteq \text{Sol}'(\mathfrak{B}, \chi)$. However, the following result is true.

Theorem 1.26. *Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra of rank r and $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ a character. Then we have: $\text{Sol}(\mathfrak{B}, \chi) = \text{Sol}'(\mathfrak{B}, \chi)$ and there exists a uniquely determined*

$$(1.18) \quad R_\chi = \partial^m + c_1\partial^{m-1} + \cdots + c_m \in \tilde{\mathfrak{D}}$$

such that $\text{Ker}(R_\chi) = \text{Sol}'(\mathfrak{B}, \chi)$. Moreover, $m \geq r$ and $m = r$ if and only if \mathcal{F} is locally free at the point $q \in X_0$ corresponding to χ . Finally, for any χ the operator R_χ is regular meaning that the order of the pole of $c_i(z)$ at $z = 0$ is at most i for all $1 \leq i \leq m$.

Proof. Let $P = \partial^n + a_1\partial^{n-1} + \cdots + a_n \in \mathfrak{D}$. Then the dimension of the \mathbb{C} -vector space $\text{Ker}(P) \subset \mathbb{C}((z))$ is n and $\text{Ker}(P) \subset \mathbb{C}[[z]]$. This implies that $\text{Sol}(\mathfrak{B}, \chi) = \text{Sol}'(\mathfrak{B}, \chi)$.

For any differential operators $Q_1, \dots, Q_l \in \tilde{\mathfrak{D}}$ we denote by $\langle Q_1, \dots, Q_l \rangle \subseteq \tilde{\mathfrak{D}}$ the left ideal generated by these elements. Recall that any left ideal $J \subseteq \tilde{\mathfrak{D}}$ is principal. Let $P_1, \dots, P_n \in \mathfrak{B}$ be the algebra generators of \mathfrak{B} (i.e. $\mathfrak{B} = \mathbb{C}[P_1, \dots, P_n]$) and $\alpha_i = \chi(P_i)$ for all $1 \leq i \leq n$. Then there exists a uniquely determined $R_\chi \in \tilde{\mathfrak{D}}$ as in (1.18) such that

$$(1.19) \quad \langle P - \chi(P)1 \mid P \in \mathfrak{B} \rangle = \langle P_1 - \alpha_1, \dots, P_n - \alpha_n \rangle = \langle R_\chi \rangle.$$

Let \mathfrak{K} be the universal Picard–Vessiot algebra of $\mathbb{C}((z))$, see [43, Section 3.2]. The algebra $\tilde{\mathfrak{D}}$ acts on \mathfrak{K} and any differential operator operator of order m from $\tilde{\mathfrak{D}}$ has exactly m linearly independent solutions with values in \mathfrak{K} . Obviously, $\text{Ker}(R_\chi) = \text{Sol}'(\mathfrak{B}, \chi) = \text{Sol}(\mathfrak{B}, \chi)$ viewed as subspaces of \mathfrak{K} . Moreover, $\dim_{\mathbb{C}}(\text{Ker}(R_\chi)) = \text{ord}(R_\chi)$. In virtue of Corollary 1.16, we get the statement about the order of R_χ . The regularity of R_χ follows from a classical theorem of Fuchs, see for example [23, Theorem 1.1.1]. \square

Definition 1.27. In what follows, the differential operator R_χ given by (1.19) will be called the *greatest common divisor* of $P_1 - \alpha_1, \dots, P_n - \alpha_n$.

Theorem 1.28. *Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra of rank r , $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ a character, $q \in X_0$ the corresponding point and R_χ the differential operator from Theorem 1.26. Then q belongs to the support of \mathcal{T} if and only if one of the following two cases occurs.*

- (1) $\text{ord}(R_\chi) \geq r + 1$. In this case, q is a singular point of X_0 and the spectral sheaf \mathcal{F} is not locally free at q .
- (2) $\text{ord}(R_\chi) = r$ and the coefficient c_1 of R_χ from the expansion (1.18) has a pole at $z = 0$. In this case, \mathcal{F} is locally free at q (which is allowed to be singular).

Proof. A point $q \in X_0$ belongs to the support of \mathcal{T} if and only if the evaluation map $\Gamma(X, \mathcal{F}) \xrightarrow{\text{ev}_q} \mathcal{F}|_q$ is not an isomorphism. If $\text{ord}(R_\chi) \geq r + 1$ then ev_q is not an isomorphism from the dimension reasons. Since $\dim_{\mathbb{C}}(\mathcal{F}|_q) > \text{rk}(\mathcal{F})$, the spectral sheaf \mathcal{F} is not locally free at q . From now on assume that $\text{ord}(R_\chi) = r$. Note that the following diagram

$$(1.20) \quad \begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xhookrightarrow{\iota} & \Gamma(X_0, \mathcal{F}) \\ \tilde{\eta}_\chi \downarrow & \searrow \text{ev}_q & \downarrow \text{ev}'_q \\ \text{Sol}(\mathfrak{B}, \chi)^* & \xleftarrow{\bar{\eta}_\chi} & \mathcal{F}|_q \end{array}$$

is commutative. Remind that $\Gamma(X_0, \mathcal{F}) \cong F = \mathbb{C}[\partial]$ as $\mathfrak{B} = \Gamma(X_0, \mathcal{O})$ -modules. The map ι is the canonical restriction map of a global section. By the construction of \mathcal{F} , the image of ι is the linear space $\langle 1, \partial, \dots, \partial^{r-1} \rangle_{\mathbb{C}}$, see Theorem 1.17. Next, $\bar{\eta}_\chi$ is the isomorphism (1.6) and $\tilde{\eta}_\chi$ assigns to the element $\partial^i \in \Gamma(X, \mathcal{F})$ the linear functional $(f \mapsto f^{(i)}(0)) \in \text{Sol}(\mathfrak{B}, \chi)^*$ for all $0 \leq i \leq r - 1$. Therefore, the map $\Gamma(X, \mathcal{F}) \xrightarrow{\text{ev}_q} \mathcal{F}|_q$ is an isomorphism if and only if $\tilde{\eta}_\chi$ is an isomorphism.

Now, assume that \mathcal{F} is locally free at the point q . Then the order of the differential operator R_χ is r , see Theorem 1.26. The map $\tilde{\eta}_\chi$ is an isomorphism if and only if the solution space $\text{Sol}(\mathfrak{B}, \chi)$ has a basis $(z^i w_i(z) \mid 0 \leq i \leq r - 1)$ with $w_i(0) \neq 0$ for all $0 \leq i \leq r - 1$. Since $\text{Sol}(\mathfrak{B}, \chi) \subset \mathbb{C}[[z]]$, the solution space has a basis of the form $(z^{\rho_i} \tilde{w}_i(z) \mid 1 \leq i \leq r)$, where $0 \leq \rho_1 < \rho_2 < \dots < \rho_r$ and $\tilde{w}_i(0) \neq 0$ for all $1 \leq i \leq r$. Therefore, $\tilde{\eta}_\chi$ is an isomorphism if and only if $(\rho_1, \dots, \rho_r) = (0, \dots, r - 1)$. Since the singularities of the differential operator R_χ are regular, the exponents ρ_1, \dots, ρ_r are the roots of the *indicial equation*

$$(1.21) \quad [x]_r + \gamma_1 [x]_{r-1} + \dots + \gamma_r = 0,$$

where $[x]_k = x(x-1)\dots(x-k+1)$ and γ_k is the residue of $z^{k-1}c_k(z)$ at the point $z = 0$ for all $1 \leq k \leq r$, see [26, Section 16.11]. Therefore, $(\rho_1, \dots, \rho_r) = (0, \dots, r - 1)$ if and only if $\gamma_1 = 0$. This implies the statement. \square

Theorem 1.28 provides a constructive approach to compute the support $Z \subset X_0$ of the torsion sheaf \mathcal{T} . If $q \in Z$ is a smooth point of X_0 then the knowledge of the roots of the indicial equation (1.21) permits to extract an additional information about the \mathcal{O}_q -module structure of \mathcal{T}_q , see [41]. To study the case when q is singular, we shall need a new ingredient: the spectral data for *families* of commuting differential operators.

Definition 1.29. Let R be an integral finitely generated \mathbb{C} -algebra and $\mathfrak{D}_R = R[[z]][\partial]$. A commutative R -subalgebra $\mathfrak{B} \subset \mathfrak{D}_R$ is called *elliptic* if it is flat over R and there exist two monic elements $P, Q \in \mathfrak{B}$ (i.e. elements whose coefficients at the highest power of ∂ is one) such that

$$(1.22) \quad \text{gcd}(\text{ord}(P), \text{ord}(Q)) = \text{gcd}(\text{ord}(L) \mid L \in \mathfrak{B}).$$

We call the number $r = \text{gcd}(\text{ord}(P), \text{ord}(Q))$ the rank of \mathfrak{B} .

Theorem 1.30. *Let R be an integral finitely generated \mathbb{C} -algebra, $B = \text{Spec}(R)$, $\mathfrak{B} \subset \mathfrak{D}_R$ an elliptic subalgebra and $X_0 = \text{Spec}(\mathfrak{B})$. Then the following statements are true.*

- (1) *The algebra \mathfrak{B} is finitely generated of Krull dimension $\text{kr.dim}(R) + 1$.*
- (2) *There exists an algebraic variety X_B , flat and projective morphism $X_B \xrightarrow{\pi} B$ and coherent sheaf \mathcal{F}_B on X_B such that*
 - (a) *π admits a section $B \xrightarrow{\sigma} X_B$, whose image belongs to the regular part of π ,*
 - (b) *if $\Sigma = \text{Im}(\sigma)$ then $X_B = \text{Spec}(\mathfrak{B}) \cup \Sigma$ and $\text{Spec}(\mathfrak{B}) \cap \Sigma = \emptyset$,*
 - (c) *\mathcal{F}_B is flat over B ,*
 - (d) *For any point $b \in B$, the tuple $(X_b, \sigma(b), \mathcal{F}_b)$ is the spectral data of the algebra $R/\mathfrak{m} \otimes_R \mathfrak{B} \subset \mathfrak{D}$, where \mathfrak{m} is the maximal ideal in R corresponding to b , $X_b = \pi^{-1}(b)$ and $\mathcal{F}_b = \mathcal{F}_B|_{X_b}$.*

Comment to the proof. To explain, how X, Σ and \mathcal{F} are defined, we follow the exposition of [31]. Let $F := \mathfrak{D}_R/z\mathfrak{D}_R \cong R[\partial]$. Then F is a right \mathfrak{D}_R -module with the action given by (1.3). For any $i \in \mathbb{N}_0$ we define:

$$F_i = \{Q \in F \mid \text{ord}(Q) \leq i\} \quad \text{and} \quad \mathfrak{B}_i = \{P \in \mathfrak{B} \mid \text{ord}(P) \leq i\}.$$

Consider the Rees algebra (respectively, the Rees module)

$$\tilde{\mathfrak{B}} := \bigoplus_{i=0}^{\infty} \mathfrak{B}_i t^i \subset \mathfrak{B}[t] \quad \text{respectively} \quad \tilde{F} := \bigoplus_{i=0}^{\infty} F_i t^i \subset F[t].$$

Then we pose $X_B := \text{Proj}_R(\tilde{\mathfrak{B}})$ and $\mathcal{F}_B := \text{Proj}_R(\tilde{F})$. The statements about $\text{kr.dim}(\mathfrak{B})$ and coherence of \mathcal{F}_B can be proven exactly in the same way as in [31].

Consider the short exact sequence of R -modules $0 \rightarrow \mathfrak{B}_i \rightarrow \mathfrak{B} \rightarrow \mathfrak{B}/\mathfrak{B}_i \rightarrow 0$. From the assumption (1.22) it follows that $\mathfrak{B}/\mathfrak{B}_i$ is a free R -module for all $i \in \mathbb{N}$ sufficiently large. Since \mathfrak{B} is flat, \mathfrak{B}_i is flat, too. Since \mathfrak{B}_i is finitely generated as R -module, it is projective for all i sufficiently large. The flatness of π follows from [24, Theorem 9.9]. Analogously, \mathcal{F}_B is flat over B , too. Consider $I = (t) \subset \tilde{\mathfrak{B}}$. Then $\Sigma := V(I) \subset X_B$. See also [44], in particular [44, Theorem 3.15 and Lemma 4.1], for a detailed study of the spectral data in the relative setting. \square

Remark 1.31. In this article arise commutative subalgebras $\mathfrak{B} \subset \mathfrak{D}_R$ with the following additional property: for any $i \in \mathbb{N}$ such that $\mathfrak{B}_i/\mathfrak{B}_{i-1} \neq 0$ there exists a monic element $L_i \in \mathfrak{B}_i$ with $\text{ord}(L_i) = i$. In this case, \mathfrak{B} is free (hence flat), viewed as an R -module.

2. SEMI-STABLE COHERENT SHEAVES ON THE WEIERSTRASS CUBIC CURVES

In this section, \mathbb{k} is an algebraically closed field of characteristic zero. We begin with a brief survey of various techniques which were used to study semi-stable coherent sheaves on irreducible curves of arithmetic genus one.

2.1. Fourier–Mukai transform on the Weierstraß cubic curves. Let

$$X = X_{g_2, g_3} = \overline{V(y^2 - 4x^3 - g_2x - g_3)} \subset \mathbb{P}_{\mathbb{k}}^2$$

be a Weierstraß cubic curve, where $g_2, g_3 \in \mathbb{k}$. Let $p = (0 : 1 : 0)$ be the infinite point of X (which is the neutral element with respect to the standard group law on the set of smooth points of X) and $X \xrightarrow{\iota} X, (x, y) \mapsto (x, -y)$ the standard involution of X . The following facts are well-known, see for example [25].

Theorem 2.1. *Any integral projective curve of arithmetic genus one is isomorphic to an appropriate Weierstraß cubic $X = X_{g_2, g_3}$. Moreover, if $\delta := g_2^3 + 27g_3^2$ then we have:*

- (1) *X is smooth if and only if $\delta \neq 0$. In this case, X is an elliptic curve.*
- (2) *Assume that $\delta = 0$, i.e. that X is singular. Then X has a unique singular point $s = (\xi, 0) = (\xi : 0 : 1)$ with*

$$\xi = \begin{cases} -\frac{3g_3}{2g_2} & g_2 \neq 0 \quad (s \text{ is a nodal singularity}), \\ 0 & g_2 = 0 \quad (s \text{ is a cuspidal singularity}). \end{cases}$$

Definition 2.2. For any coherent sheaf \mathcal{F} on the curve X , we define another coherent sheaf $\mathbb{F}(\mathcal{F}) := \text{Cok}(\Gamma(X, \mathcal{F}) \otimes \mathcal{O} \xrightarrow{\text{ev}} \mathcal{F})$, where ev is the evaluation morphism.

Theorem 2.3. *Let X be a Weierstraß cubic curve, $\text{Sem}(X)$ the category of semi-stable coherent sheaves on X of slope one and $\text{Tor}(X)$ the category of torsion coherent sheaves. Then the following results are true.*

- (1) *For any object \mathcal{F} of $\text{Sem}(X)$, the evaluation morphism ev is a monomorphism and the corresponding coherent sheaf $\mathbb{F}(\mathcal{F})$ is torsion, i.e. belongs to $\text{Tor}(X)$. In other words, the sequence (1.10) is exact for $\mathcal{T} = \mathbb{F}(\mathcal{F})$. Moreover,*

$$(2.1) \quad \text{Sem}(X) \xrightarrow{\mathbb{F}} \text{Tor}(X)$$

is an equivalence of categories.

- (2) *Similarly, for any object \mathcal{T} of $\text{Tor}(X)$, consider the coherent sheaf $\mathbb{G}(\mathcal{T})$ given by the universal extension sequence*

$$(2.2) \quad 0 \longrightarrow \text{Ext}^1(\mathcal{T}, \mathcal{O})^* \otimes \mathcal{O} \longrightarrow \mathbb{G}(\mathcal{T}) \longrightarrow \mathcal{T} \longrightarrow 0.$$

Then $\mathbb{G}(\mathcal{T})$ is semi-stable of slope one. Moreover, \mathbb{G} is an equivalence between the categories $\text{Tor}(X)$ and $\text{Sem}(X)$, which is quasi-inverse to \mathbb{F} .

- (3) *For any object \mathcal{F} of $\text{Sem}(X)$ and the corresponding object $\mathcal{T} = \mathbb{F}(\mathcal{F})$ of $\text{Tor}(X)$ the following results are true.*
 - (a) *The rank of \mathcal{F} is equal to the length of \mathcal{T} .*
 - (b) *\mathcal{F} is locally free if and only if \mathcal{T} has projective dimension one.*
 - (c) *Analogously, \mathcal{F} is not locally free if and only if the torsion sheaf \mathcal{T} has infinite projective dimension. In this case, the singular point of X belongs to the support of \mathcal{T} .*
- (4) *Moreover, the following diagram of categories and functors is commutative:*

$$(2.3) \quad \begin{array}{ccc} \text{Sem}(X) & \xrightarrow{\mathbb{D}} & \text{Sem}(X) \\ \mathbb{F} \downarrow & & \downarrow \mathbb{F} \\ \text{Tor}(X) & \xrightarrow{\mathbb{E}} & \text{Tor}(X), \end{array}$$

where

- (a) $\mathbb{D}(\mathcal{F}) := \iota^*(\mathcal{F}^\vee) \otimes \mathcal{O}(2[p])$ for \mathcal{F} from $\text{Sem}(X)$ with $\mathcal{F}^\vee := \text{Hom}_X(\mathcal{F}, \mathcal{O})$.
- (b) $\mathbb{E}(\mathcal{T}) := \text{Hom}_X(\mathcal{T}, \mathcal{K}/\mathcal{O})$ is the Matlis duality on $\text{Tor}(X)$, see e.g. [6, Section 3.2] or [8, Section 6]. Here, \mathcal{K} is the sheaf of rational functions on X .

Comment to the proof. The functorial correspondences \mathbb{F} and \mathbb{G} were essentially introduced by Atiyah [2, Part II], who used them to classify indecomposable vector bundles on elliptic curves [2, Theorem 7]. A translation of Atiyah's method into the formalism of derived

categories can be for instance found in [7]. The idea to use the functor \mathbb{F} to study semi-stable sheaves on singular Weierstraß curves and elliptic fibrations (the so-called *spectral cover construction*) is due to Friedman, Morgan and Witten, see [22, Section 1]. In [8, Section 2], the approach of [22] was elaborated and included into the framework of derived categories. We refer to [22, 8] for a proof of all statements of Theorem 2.3, see especially [8, Theorem 2.21 and Theorem 6.11]. \square

Remark 2.4. The described equivalence between the categories $\mathbf{Sem}(X)$ and $\mathbf{Tor}(X)$ can be best understood using the Seidel–Thomas twist functor \mathbb{T} , see Theorem 1.25. Namely, the following diagram of categories and functors is commutative:

$$(2.4) \quad \begin{array}{ccc} \mathbf{Sem}(X) & \xrightarrow{\mathbb{F}} & \mathbf{Tor}(X) \\ \mathbb{I} \downarrow & & \downarrow \mathbb{I} \\ D^b(\mathbf{Coh}(X)) & \xrightarrow{\mathbb{T}} & D^b(\mathbf{Coh}(X)), \end{array}$$

where \mathbb{I} assigns to a coherent sheaf the corresponding stalk complex, see [8, Theorem 2.21]. The twist functor \mathbb{T} is isomorphic to the *integral transform* \mathbb{M} with the kernel $\mathcal{P}^\bullet = \mathcal{I}_\Delta[1]$, where $\mathcal{I}_\Delta \subset \mathcal{O}_{X \times X}$ is the ideal sheaf of the diagonal $\Delta \in X \times X$, see [48, Lemma 3.2]. Remind, that the image of an object \mathcal{F}^\bullet from $D^b(\mathbf{Coh}(X))$ under \mathbb{M} is $\mathbb{M}(\mathcal{F}^\bullet) := \mathbb{R}\pi_{2*}(\pi_1^*(\mathcal{F}^\bullet) \otimes^{\mathbb{L}} \mathcal{P}^\bullet)$, where $\pi_i : X \times X \rightarrow X$ is the canonical projection for $i = 1, 2$, see for example [4]. In what follows, we shall call the functor \mathbb{F} *the Fourier–Mukai transform* of $D^b(\mathbf{Coh}(X))$. For a torsion free sheaf \mathcal{F} from $\mathbf{Sem}(X)$, the corresponding torsion sheaf \mathcal{T} will be called *Fourier–Mukai transform* of \mathcal{F} .

Remark 2.5. The formalism of integral transforms allows to extend the construction of functors \mathbb{F} and \mathbb{G} to the relative setting, where we start with a genus one fibration $X_B \xrightarrow{\pi} B$, see [22, 9, 4]. As in the absolute case, we can define the category $\mathbf{Sem}(X_B/B)$ consisting of those coherent sheaves \mathcal{F} on X_B , which are flat over B and such that for any $b \in B$ the restricted sheaf $\mathcal{F}|_{X_b}$ is semi-stable of slope one, where $X_b = \pi^{-1}(b)$. Analogously, we define the category $\mathbf{Tor}(X_B/B)$ of relative torsion coherent sheaves. Again, for any object \mathcal{F} of $\mathbf{Sem}(X_B/B)$, the canonical morphism $\pi^*(\pi_*(\mathcal{F})) \rightarrow \mathcal{F}$ is a monomorphism and we get an equivalence of categories $\mathbb{F}_B : \mathbf{Sem}(X_B/B) \rightarrow \mathbf{Tor}(X_B/B)$, given by the rule

$$0 \rightarrow \pi^*(\pi_*(\mathcal{F})) \rightarrow \mathcal{F} \rightarrow \mathbb{F}_B(\mathcal{F}) \rightarrow 0.$$

Clearly, for any $b \in B$ the following diagram of categories and functors is commutative:

$$\begin{array}{ccc} \mathbf{Sem}(X_B/B) & \xrightarrow{\mathbb{F}_B} & \mathbf{Tor}(X_B/B) \\ \iota_b^* \downarrow & & \downarrow \iota_b^* \\ \mathbf{Sem}(X_b) & \xrightarrow{\mathbb{F}} & \mathbf{Tor}(X_b), \end{array}$$

where $\iota_b : X_b \rightarrow X_B$ is the inclusion of the fiber over b . Let $\Delta_B \subset X_B \times_B X_B$ be the relative diagonal, $\mathcal{P}_B^\bullet = \mathcal{I}_{\Delta_B}[1]$ and \mathbb{M}_B the integral transform with the kernel \mathcal{P}_B^\bullet (the relative Fourier–Mukai transform). Then \mathbb{M}_B is an auto-equivalence of the derived category $D^b(\mathbf{Coh}(X_B))$ extending the equivalence \mathbb{F}_B similarly to the diagram (2.4).

Theorem 2.6. *Let X be a Weierstraß cubic curve. Then the following results are true.*

- (1) For any $r \in \mathbb{N}$ there exists a unique indecomposable vector bundle \mathcal{A}_r of rank r on X recursively defined through a short exact sequence

$$(2.5) \quad 0 \longrightarrow \mathcal{A}_r \longrightarrow \mathcal{A}_{r+1} \longrightarrow \mathcal{O} \longrightarrow 0,$$

where $\mathcal{A}_1 = \mathcal{O}$.

- (2) Let $q \in X$ be a smooth point, $r \in \mathbb{N}$ and $\mathcal{T}_{q,r} := \mathcal{O}_q/\mathfrak{m}_q^r$ the indecomposable torsion sheaf of length r supported at q . Then we have: $\mathbb{G}(\mathcal{T}_{q,r}) \cong \mathcal{O}([q]) \otimes \mathcal{A}_r$.

Comment to the proof. The first claim was established by Atiyah [2, Theorem 5]. The key point here is that the category of vector bundles on X admitting a filtration with quotients isomorphic to \mathcal{O} is equivalent to the category of finite dimensional modules over the discrete valuation ring $\mathbb{k}[[t]]$. It follows from the definition of the functor \mathbb{F} that $\mathbb{F}(\mathcal{O}([q]) \otimes \mathcal{A}_r) \cong \mathcal{T}_{q,r}$, implying the second part. \square

The following well-known result can be for instance found in [1, Example 8.9 (iii)].

Proposition 2.7. *Let $X_B \xrightarrow{\pi} B$ be a genus one fibration with irreducible fibers admitting a section $B \xrightarrow{\sigma} X_B$ such that $\sigma(B)$ belongs to the regular part of π . Let $\mathcal{L} \in \text{Pic}^d(X_B/B)$, i.e. \mathcal{L} is a line bundle on X_B such that $\deg(\mathcal{L}|_{X_b}) = d$ for all points $b \in T$, where $X_b = \pi^{-1}(b)$. Then there exists a unique section $B \xrightarrow{j} X_B$ with whose image also belongs to the regular part of π such that $\mathcal{L}|_{X_b} \cong \mathcal{O}_{X_b}((d-1)[\sigma(b)] + j(b))$ for all $b \in B$.*

2.2. Semi-stable sheaves of slope one and rank two on singular cubic curves.

In this subsection, let $\lambda \in \mathbb{k}$ and $X = X(\lambda) := \overline{V}(y^2 - x^3 - \lambda x^2) \subset \mathbb{P}_{\mathbb{k}}^2$ be the corresponding singular cubic curve. Let $p = (0 : 1 : 0)$ be the infinite point of X and $s = (0 : 0 : 1) = (0, 0)$ its singular point. Let $R = \mathbb{k}[x, y]/(y^2 - x^3 - \lambda x^2)$ be the coordinate ring of the affine curve $X_0 = X \setminus \{p\}$. Let $\mathcal{A} = \mathcal{A}_2$ be the Atiyah bundle or rank two on X , see (2.5). For any point $t = (\alpha : \beta) \in \mathbb{P}_{\mathbb{k}}^1$, let $I_t = \langle x^2, \alpha x + \beta y \rangle \subset R$ and $T_t = R/I_t$. Finally, let \mathcal{T}_t be the torsion coherent sheaf on X corresponding to the R -module T_t .

Theorem 2.8. *The following results are true.*

- (1) Let \mathcal{F} be an indecomposable semi-stable sheaf on X of rank two and slope one. Then either
 - (a) $\mathcal{F} \cong \mathcal{A} \otimes \mathcal{O}([q])$ for some smooth point $q \in X$, or
 - (b) $\mathcal{F} \cong \mathcal{B}_t := \mathbb{G}(\mathcal{T}_t)$ for some $t \in \mathbb{P}_{\mathbb{k}}^1$.
- (2) For any $\theta \in \mathbb{k}$, let $\mathcal{B}_\theta := \mathcal{B}_{(\theta:1)}$. Then \mathcal{B}_θ is locally free if and only if $\theta^2 - \lambda \neq 0$. In this case, $\det(\mathcal{B}_\theta) \cong \mathcal{O}([p] + [q_\theta])$, where $q_\theta = ((\theta^2 - \lambda) : \theta(\theta^2 - \lambda) : 1)$. In particular, $\mathcal{B}_\theta \cong \mathcal{B}_{\theta'}$ if and only if $\theta = \theta'$.
- (3) Similarly, $\mathcal{B}_\infty := \mathcal{B}_{(1:0)}$ is locally free with $\det(\mathcal{B}_\infty) \cong \mathcal{O}(2[p])$.
- (4) In the nodal case, the torsion free sheaves $\mathcal{U}_\pm := \mathcal{B}_{\pm\sqrt{\lambda}}$ are not isomorphic. In the cuspidal case, $\mathcal{U} := \mathcal{B}_0$ is the only indecomposable and not locally free object of $\text{Sem}(X)$ of rank two.

Proof. (1) If \mathcal{F} is indecomposable then the support of its Fourier–Mukai transform $\mathcal{T} := \mathbb{F}(\mathcal{F})$ is a single point $q \in X$. Since \mathcal{F} has rank two, the length of \mathcal{T} is two as well. If $q \neq s$ then $\mathcal{T} \cong \mathcal{T}_{q,2} = \mathcal{O}_q/\mathfrak{m}_q^2$ and hence $\mathcal{F} \cong \mathcal{A} \otimes \mathcal{O}([q])$.

From now on assume that \mathcal{T} is supported at the singular point s of X . Let $T = \Gamma(X \setminus \{p\}, \mathcal{T})$ be the R -module corresponding to \mathcal{T} . We claim that $T \cong R/J$, where J is an ideal in R with $\sqrt{J} = \mathfrak{m}_s$. Indeed, if $\mathfrak{m}_s T = 0$ then $T \cong R/\mathfrak{m}_s \oplus R/\mathfrak{m}_s$ is decomposable, contradiction. Hence, $\mathfrak{m}_s T \neq 0$. By Nakayama’s Lemma, $\mathfrak{m}_s T \neq T$. Therefore, there

exists elements $u \in T \setminus \mathfrak{m}_s T$ and $0 \neq v \in \mathfrak{m}_s T$. Since $\dim_{\mathbb{k}}(T) = 2$, the elements u and v form a basis of T . Moreover, $\langle v \rangle_{\mathbb{k}} = \mathfrak{m}_s u$, i.e. u is a cyclic vector of T . This shows that $T \cong R/J$ for some ideal J with $\sqrt{J} = \mathfrak{m}_s$. But all such ideals can be classified: it can be easily shown that $J = I_t$ for an appropriate $t \in \mathbb{P}_{\mathbb{k}}^1$.

(2) Let $\mathcal{F} = \mathcal{B}_t$ for some $t \in \mathbb{P}_{\mathbb{k}}^1$. Then \mathcal{F} is locally free if and only if the R -module $T = R/J$ has projective dimension one. The last property is true precisely when the localized ideal $J_s \subset R_s$ is principal. Clearly, $I_{\infty} = \langle x \rangle$ is already principal in R . Therefore, we assume that $t = (\theta : 1)$ for some $\theta \in \mathbb{k}$ and $I_t = \langle x^2, x + \theta y \rangle$. In the ring R , we have the equality $y^2 - \theta^2 x^2 = x^2(x + (\lambda - \theta^2))$. If $\lambda - \theta^2 \neq 0$, then x^2 belongs to the ideal generated by $y + \theta x$ in the local ring R_s . In particular, the localization of I_t is a principal ideal. On the other hand, if $\lambda + \theta^2 = 0$, the localization of I_t is not principal.

The determinant of the vector bundle \mathcal{B}_{θ} can be computed using the following trick. For any $\theta \in \mathbb{k}$, consider the line L_{θ} given by the equation $y + \theta x = 0$. Since $\lambda - \theta^2 \neq 0$, the line L_{θ} intersects the curve X at two points: the singular point s and another point $q'_{\theta} = (\theta^2 - \lambda : -\theta(\theta^2 - \lambda) : 1)$. Moreover, we have: $\mathcal{C}_{\theta} := \mathbb{G}(R/L_{\theta}) \cong \mathcal{B}_{\theta} \oplus \mathcal{O}([q'_{\theta}])$. Now we claim that $\det(\mathcal{C}_{\theta}) \cong \mathcal{O}(3[p])$. Indeed, consider the constant genus one fibration $X_B = X \times B$ over the base $B = \text{Spec}(\mathbb{k}[\tau])$ and the B -flat family of torsion sheaves given by the $\mathbb{k}[x, y, \tau]/(y^2 - x^3 - \lambda x^2)$ -module $\tilde{L} := \mathbb{k}[x, y, \tau]/(y^2 - x^3 - \lambda x^2, y - \tau + \theta x)$. Using the inverse relative Fourier–Mukai transform \mathbb{G}_B , we get a family $\tilde{\mathcal{C}} := \mathbb{G}_B(\tilde{L})$ of relatively semi-stable vector bundles on $X \times B$ with $\tilde{\mathcal{C}}|_{X \times \{0\}} \cong \mathcal{C}_{\theta}$. For $\zeta \neq 0$, we have: $\tilde{\mathcal{C}}|_{X \times \{\zeta\}} \cong \mathcal{O}([p_1]) \oplus \mathcal{O}([p_2]) \oplus \mathcal{O}([p_3])$, where p_1, p_2 and p_3 are the intersection points of X with the line $V(\varphi_{\theta, \zeta})$, where $\varphi_{\theta, \zeta}(x, y) := y - \zeta + \theta x$. However, the divisor of the function $\varphi_{\theta, \zeta}$ is $[p_1] + [p_2] + [p_3] - 3[p]$. It means that $\det(\tilde{\mathcal{C}}|_{X \times \{\zeta\}}) \cong \mathcal{O}(3[p])$ for all $\zeta \neq 0$. From Proposition 2.7 easily follows that $\det(\mathcal{C}_{\theta}) \cong \det(\tilde{\mathcal{C}}|_{X \times \{0\}}) \cong \mathcal{O}(3[p])$ as well. This fact implies that $\det(\mathcal{B}_{\theta}) \cong \mathcal{O}(3[p]) \otimes \mathcal{O}([q'_{\theta}])^{\vee} \cong \mathcal{O}(3[p]) \otimes \mathcal{O}([q_{\theta}] - 2[p]) \cong \mathcal{O}([p] + [q_{\theta}])$.

(3) Note that for $\theta \neq 0$ we have: $q_{\theta} = ((\theta^2 - \lambda) : \theta(\theta^2 - \lambda) : 1) = ((\xi - \lambda \xi^3) : (1 - \lambda \xi^2) : \xi^3)$ and $I_t = \langle x^2, x + \xi y \rangle$, where $\xi = \theta^{-1}$. From the continuity consideration similar to the previous paragraph, we deduce that $\det(\mathcal{B}_{\infty}) \cong \mathcal{O}(2[p])$.

(4) Let $\lambda \neq 0$, i.e. the curve X is nodal. We choose a square root $\rho = \sqrt{\lambda}$ and consider the ideal $I = \langle x^2, y + \rho x \rangle$ in the local ring R_s . Let \hat{R} be the completion of R_s and \mathfrak{m} the maximal ideal of \hat{R} . Consider the element $\hat{R} \ni w = x\sqrt{\lambda + x} := \rho x + \frac{1}{2\rho}x^2 + \dots$. Writing x as a power series in w we conclude that $x \equiv \frac{1}{\rho}w \pmod{\mathfrak{m}^2}$. Posing $u_{\pm} = y \pm x\sqrt{\lambda + x} := y \pm w \in \hat{R}$ we get: $\hat{R} = \mathbb{k}[[u_+, u_-]]/(u_+u_-)$. The next step is to determine the images of the generators of I under the completion map $R_s \rightarrow \hat{R}$. We see that $y + \rho x \equiv u_+ \pmod{\mathfrak{m}^2}$ and $x^2 = \frac{1}{4\rho^2}(u_+^2 + u_-^2) \pmod{\mathfrak{m}^4}$. Therefore, we conclude that

$$(2.6) \quad U_+ := R/(x^2, y + \rho x) \cong \hat{R}/(u_+, u_-^2) \quad \text{and} \quad U_- := \hat{R}/(x^2, y - \rho x) \cong \hat{R}/(u_-, u_+^2).$$

In particular, $\mathcal{U}_+ \not\cong \mathcal{U}_-$, where \mathcal{U}_{\pm} are torsion sheaves on X corresponding to U_{\pm} . \square

Remark 2.9. Let X be a singular Weierstraß cubic with the singular point s and the “infinite” point p . According to Theorem 2.8, for any smooth point $q \in X$ there exists a unique semi-stable vector bundle with determinant $\mathcal{O}([p] + [q])$, whose Fourier–Mukai transform is supported at the singular point s . Abusing the notation, we shall denote this vector bundle by \mathcal{B}_q in what follows. Such description of vector bundles from $\text{Sem}(X)$ is

advantageous since it eliminates unessential choices (for example, the dependence of λ in the nodal case, see part (2) of Theorem 2.8).

Corollary 2.10. *Let X be a singular Weierstraß cubic curve, $p \in X$ its point at infinity, $\mathbb{P}^1 \xrightarrow{\nu} X$ the normalization morphism, $\mathcal{S} = \nu_*(\mathcal{O}_{\mathbb{P}^1})$ and $\mathcal{A} = \mathcal{A}_2$ is the rank two Atiyah bundle on X . Let \mathcal{F} be a semi-stable torsion free sheaf on X of rank two and slope one, \mathcal{T} be the Fourier–Mukai transform of \mathcal{F} and $Z = \text{Supp}(\mathcal{T})$.*

- (1) *If \mathcal{F} is locally free and indecomposable, then it is either isomorphic to $\mathcal{A} \otimes \mathcal{O}([q])$ for some smooth point $q \in X$ or to $\mathcal{B}_{\bar{q}}$, where $\det(\mathcal{B}_{\bar{q}}) = \mathcal{O}([\bar{q}] + [p]) \in \text{Pic}^2(X)$, where \bar{q} is a smooth point of X (which can be arbitrary). In the first case $Z = \{q\}$, whereas in the second $Z = \{s\}$.*
- (2) *If \mathcal{F} is indecomposable but not locally free, then it is isomorphic to one of the sheaves \mathcal{U}_{\pm} (nodal case) or to \mathcal{U} (cuspidal case). In this case, $Z = \{s\}$.*
- (3) *If \mathcal{F} is decomposable, then it is isomorphic to $\mathcal{O}([q]) \oplus \mathcal{O}([q'])$, $\mathcal{O}([q]) \oplus \mathcal{S}$ or $\mathcal{S} \oplus \mathcal{S}$ for some smooth points $q, q' \in X$. We have: $Z = \{q, q'\}, \{q, s\}$ or $\{s\}$ respectively.*

For any object \mathcal{F} of $\text{Sem}(X)$ we have: $H^1(X, \mathcal{F}) = 0$. Moreover, $\Gamma(X, \mathcal{F}) \xrightarrow{\text{ev}_p} \mathcal{F}|_p$ is an isomorphism if and only if $p \notin Z$.

Remark 2.11. In fact, one can derive from [45] the following result. For $\lambda \in \mathbb{k}$, let $X = \overline{V}(y^2 - x^3 - \lambda x^2)$ and $s = (0, 0)$ be the singular point of X . Let $\text{Hilb}_s^2(X)$ be the Hilbert scheme of points of length two on X , supported at s . Then $\text{Hilb}_s^2(X) \cong \mathbb{P}^1$. Moreover, the corresponding universal ideal $\mathcal{J} \subset \mathcal{O}_{X \times \mathbb{P}^1}$ is $\langle x^2, z_0x - z_1y \rangle$, where $(z_0 : z_1)$ are homogeneous coordinates on \mathbb{P}^1 .

2.3. Regular semi-stable sheaves on a cuspidal cubic curve. In this subsection, $X = \overline{V}(y^2 - x^3) \subset \mathbb{P}_{\mathbb{k}}^2$ is a cuspidal cubic curve, $R = \mathbb{k}[t^2, t^3] = \mathbb{k}[x, y]/(y^2 - x^3)$ and $\widehat{R} := \mathbb{k}[[t^2, t^3]]$ is the completed local ring of X at the singular point $s = (0, 0)$. According to a result of Drozd [16], the category of finite dimensional \widehat{R} -modules is *representation wild*. This means that for any finitely generated \mathbb{k} -algebra Λ there exists an exact functor $\Lambda\text{-fdmod} \xrightarrow{\mathbb{J}} \widehat{R}\text{-fdmod}$ such that

- $\mathbb{J}(M) \cong \mathbb{J}(M')$ if and only if $M \cong M'$.
- $\mathbb{J}(M)$ is indecomposable if and only if M is indecomposable.

See also [5, Proposition 8] for a more detailed discussion of representation wildness and a simpler proof of Drozd's result. Therefore, the category $\text{Sem}(X)$ is representation-wild, too. Nevertheless, in this subsection we shall give a full classification of the indecomposable objects of $\text{Sem}(X)$ having rank three.

Definition 2.12. For any $n \in \mathbb{N}_0$ and $\theta \in \mathbb{k}$ consider the following ideals in \widehat{R} :

$$(2.7) \quad I_{n, \theta} = \langle t^n(t^2 + \theta t^3) \rangle \quad \text{and} \quad J_n = t^n \langle t^2, t^3 \rangle.$$

Lemma 2.13. *Let $I \subset \widehat{R}$ be a proper ideal. Then we have: $I = I_{n, \theta}$ or $I = J_n$ for some $n \in \mathbb{N}_0$ and $\theta \in \mathbb{k}$.*

Proof. Let $f = t^m + \theta t^{m+1} + \dots = t^m(1 + \theta t + \dots) = t^m \cdot w \in \mathbb{k}[[t]]$ be an element of I with the minimal multiplicity $m \in \mathbb{N}_{\geq 2}$, where $\theta \in \mathbb{k}$ is some scalar. Then for any $k \in \mathbb{N}_{\geq 2}$ the power series $t^k w^{-1}$ belongs to \widehat{R} . Therefore, t^{k+m} belongs to the principal ideal (f) in \widehat{R} , provided $k \geq 2$. Now, the following two cases can occur.

Case 1. The ideal I contains an element of multiplicity $m + 1$. Then $I = \langle t^m, t^{m+1} \rangle$.

Case 2. The ideal I does not contain any elements of multiplicity $m + 1$. Then $I = \langle f \rangle$. \square

Theorem 2.14. *Let M be an indecomposable \widehat{R} -module with $\dim_{\mathbb{k}}(M) = 3$. Then M is isomorphic to some module from the following list:*

- (1) $M_{\theta} := \widehat{R}/(t^3 + \theta t^4)$, where $\theta \in \mathbb{k}$.
- (2) $N := \widehat{R}/(t^4, t^5)$.
- (3) $N^{\#} := \mathbb{E}(N)$ (the Matlis dual of N).

Moreover, $\text{pr.dim}_{\widehat{R}}(M_{\theta}) = 1$ and $\text{pr.dim}_{\widehat{R}}(N) = \text{pr.dim}_{\widehat{R}}(N^{\#}) = \infty$.

Proof. For any $n \in \mathbb{N}$, an n -dimensional \widehat{R} -module is determined by an algebra homomorphism $\widehat{R} \rightarrow \text{Mat}_{n \times n}(\mathbb{k})$. Since $\widehat{R} \cong \mathbb{k}[[u, v]]/(v^2 - u^3)$, such a homomorphism is specified by a pair of *nilpotent* matrices $U, V \in \text{Mat}_{n \times n}(\mathbb{k})$ satisfying the conditions

$$UV = VU \quad \text{and} \quad V^2 = U^3.$$

Moreover, two such pairs (U, V) and (U', V') define isomorphic \widehat{R} -modules if and only if there exists a matrix $S \in \text{GL}_n(\mathbb{k})$ satisfying

$$U' = S U S^{-1} \quad \text{and} \quad V' = S V S^{-1}.$$

Case 1. Assume that $\text{rk}(U) = 2$. Then we may without loss of generality assume that

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{The equalities } UV = VU \text{ and } V^2 = 0 \text{ imply that } V = \begin{pmatrix} 0 & 0 & \theta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\theta \in \mathbb{k}$. It is easy to see that $(\mathbb{k}^3, U, V) \cong M_{-\theta}$.

Case 2. Assume that $\text{rk}(U) = 1$. Then we may without loss of generality assume that

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{From the equalities } UV = VU \text{ and } V^2 = 0 \text{ we conclude that } V = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \text{ for some } \alpha, \beta, \gamma \in \mathbb{k} \text{ such that } \alpha\beta = 0. \text{ In the case } \alpha = 0 = \beta, \text{ the module}$$

M contains the trivial module $\mathbb{k} = \widehat{R}/(t^2, t^3)$ as a direct summand. In particular, M is decomposable.

Assuming that $\alpha = 0$ and $\beta \neq 0$, we see that

$$\left(\mathbb{k}^3, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \right) \cong \left(\mathbb{k}^3, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right) \cong N.$$

Similarly, if $\beta = 0$ and $\alpha \neq 0$, we have:

$$\left(\mathbb{k}^3, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \cong \left(\mathbb{k}^3, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \cong N^{\#}.$$

The last isomorphism follows from the well-known fact that the Matlis duality in the category of finite dimensional \widehat{R} -modules is given by the rule $(U, V) \mapsto (U^t, V^t)$, where U^t is the transposed matrix of U , see for example [8, Remark 6.5].

Case 3. Finally, for $U = 0$ it is easy to show that M contains the trivial module $(\mathbb{k}, 0, 0)$ as a direct summand.

The statement about the projective dimension of M is obvious. \square

Corollary 2.15. *An indecomposable semi-stable coherent sheaf of rank three and slope one on a cuspidal cubic curve X is isomorphic to a one of the following sheaves:*

- (1) $\mathcal{O}([q]) \otimes \mathcal{A}_3$, where \mathcal{A}_3 is the Atiyah bundle of rank three from Theorem 2.6 and $q \in X$ is a smooth point.
- (2) $\mathcal{E}_{q(\theta)} := \mathbb{G}(M_\theta)$ for some $\theta \in \mathbb{k}$, where M_θ is the \widehat{R} -module from Theorem 2.14, viewed as a torsion sheaf on X . Moreover, $\mathcal{E}_{q(\theta)}$ is locally free and $\det(\mathcal{E}_{q(\theta)}) \cong \mathcal{O}([q_\theta] + 2[p])$, where $q_\theta = (\theta : 1 : \theta^3)$, see the next Lemma 2.16.
- (3) $\mathcal{V} := \mathbb{G}(N)$ and $\mathcal{V}^\dagger := \mathbb{G}(N^\sharp)$. They are not locally free and $\mathbb{D}(\mathcal{V}) \cong \mathcal{V}^\dagger$, where \mathbb{D} is the duality on $\mathbf{Sem}(X)$ from Theorem 2.3.

The following class of indecomposable semi-stable vector bundles on a cuspidal cubic curve X was introduced by Friedman, Morgan and Witten [22].

Lemma 2.16. *For any $n \in \mathbb{N}_{\geq 2}$ and $\theta \in \mathbb{k}$, consider the \widehat{R} -module $T_{n,\theta} = \widehat{R}/(t^n + \theta t^{n+1})$, which we view as a torsion sheaf on X . Let $\mathcal{E}(n, \theta) := \mathbb{G}(T(n, \theta))$. Then we have:*

- (1) $\mathcal{E}(n, \theta)$ is an indecomposable locally free sheaf of rank n on X with $\det(\mathcal{E}(n, \theta)) \cong \mathcal{O}([q_\theta] + (n-1)[p])$, where $q_\theta = (\theta : 1 : \theta^3)$.
- (2) $\mathbb{D}(\mathcal{E}(n, \theta)) \cong \mathcal{E}(n, \theta)$, where \mathbb{D} is the duality on $\mathbf{Sem}(X)$ from Theorem 2.3.

Proof. The fact that $\mathcal{E}(n, \theta)$ is an indecomposable locally free sheaf of rank n follows from the fact that $T_{n,\theta}$ is an indecomposable R -module with $\dim_{\mathbb{k}}(T_{n,\theta}) = n$ and $\text{pr.dim}_{\widehat{R}}(T_{n,\theta}) = 1$, and Theorem 2.3. Consider the vector bundle $\mathcal{C}(n, \theta) := \mathbb{G}(R/(t^n + \theta t^{n+1}))$. As in the proof of Theorem 2.8 we show that

- $\mathcal{C}(n, \theta) \cong \mathcal{E}(n, \theta) \oplus \mathcal{O}([q'_\theta])$, where $q'_\theta = (\theta : -1 : \theta^3)$.
- $\det(\mathcal{C}(n, \theta)) \cong \mathcal{O}((n+1)[p])$.

This implies that $\det(\mathcal{E}(n, \theta)) \cong \mathcal{O}((n+1)[p] - [q'_\theta]) \cong \mathcal{O}([q_\theta] + (n-1)[p])$.

Next, note that $T_{n,\theta}$ has a one-dimensional socle generated by the class of t^n . Therefore, its Matlis dual module $T_{n,\theta}^\sharp := \mathbb{E}(T_{n,\theta})$ has a simple top. Since $\dim_{\mathbb{k}}(T_{n,\theta}^\sharp) = n$, Lemma 2.13 implies that there exists some $\tilde{\theta} \in \mathbb{k}$ with $T_{n,\theta}^\sharp \cong T_{n,\tilde{\theta}}$. Therefore, $\mathbb{D}(\mathcal{E}(n, \theta)) \cong \mathcal{E}(n, \tilde{\theta})$. On the other hand, it is easy to see that $\det(\mathbb{D}(\mathcal{F})) \cong \det(\mathcal{F})$ for any locally free object of $\mathbf{Sem}(X)$. Therefore, $\tilde{\theta} = \theta$. \square

Remark 2.17. A full classification of all indecomposable semi-stable coherent sheaves of arbitrary integral slope on a nodal Weierstraß curve was given in [8, Theorem 5.1].

3. SPECTRAL SHEAVES OF RANK TWO AND GENUS ONE COMMUTATIVE SUBALGEBRAS

In this section, we classify the spectral sheaves of all rank two and genus one commutative subalgebras of \mathfrak{D} with singular spectral curve, completing the result of Previato and Wilson [41, Theorem 1.2].

3.1. Grünbaum's classification. We begin by recalling the classification of rank two and genus one commutative subalgebras of \mathfrak{D} , following Grünbaum's work [21]. In what follows, $\mathfrak{E} = \mathbb{C}[[z]]((\partial^{-1}))$ is the algebra of pseudo-differential operators and for any $Q \in \mathfrak{E}$ we denote by Q_+ the “differential part” of Q , i.e. the projection of Q onto \mathfrak{D} . We refer to [38] and [44, Appendix A] for a survey of properties of the algebra \mathfrak{E} .

The following result can be found in [21, Section 2], see also [41, Lemma 5.2]. For the readers's convenience, we give a detailed proof here.

Proposition 3.1. *Let $\mathfrak{B} \in \mathfrak{D}$ be a normalized commutative subalgebra of rank two and genus one. Then there exist two operators $L, M \in \mathfrak{B}$ such that $\mathfrak{B} = \mathbb{C}[L, M]$ and*

$$(3.1) \quad L = \partial^4 + a_2\partial^2 + a_1\partial + a_0, \quad M = 2L_+^{\frac{3}{2}}, \quad M^2 = 4L^3 + g_2L + g_3$$

for some $g_2, g_3 \in \mathbb{C}$.

Proof. If X is the spectral curve of \mathfrak{B} then $\mathfrak{B} \cong \Gamma(X \setminus \{p\}, \mathcal{O})$ as associative algebras. As \mathfrak{B} has genus one, there exist $g_2, g_3 \in \mathbb{C}$ such that $\Gamma(X \setminus \{p\}, \mathcal{O}) \cong \mathbb{C}[x, y]/(y^2 - 4x^3 - g_2x - g_3)$. In particular, we can find a pair of operators $L, M \in \mathfrak{B}$ such that $\mathfrak{B} = \mathbb{C}[L, M]$ and $M^2 = 4L^3 + g_2L + g_3$. As the rank of \mathfrak{B} is two, $\text{ord}(L) = 4$ and $\text{ord}(M) = 6$ is the only possibility. Since \mathfrak{B} is normalized, the operator L is normalized. Our next goal is to show that we can find a change of variables

$$\begin{cases} L & \mapsto \widehat{L} = L + \alpha \\ M & \mapsto \widehat{M} = M + \beta + \gamma L \end{cases}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\widehat{M} = 2\widehat{L}_+^{\frac{3}{2}}$. From the theory of pseudo-differential operators we know that there exists a uniquely determined operator $L^{\frac{1}{4}} = \partial + b_1\partial^{-1} + b_2\partial^{-2} + \dots$ in \mathfrak{E} . Then for any $i \in \mathbb{Z}$ we have: $L^{\frac{i}{4}} = \partial^i + b_1^{(i)}\partial^{i-2} + b_2^{(i)}\partial^{i-3} + \dots$. If $M \in \mathfrak{D}$ is such that $[L, M] = 0$ and $\text{ord}(M) = 6$ then there exist constants $\gamma_i \in \mathbb{C}$ for $i \in \mathbb{Z}_{\leq 6}$ such that

$$(3.2) \quad M = \sum_{i=6}^{-\infty} \gamma_i L^{\frac{i}{4}} = \sum_{i=6}^0 \gamma_i L_+^{\frac{i}{4}}.$$

Rescaling, assume that $\gamma_6 = 1$. Next, we have the following identity in the algebra $\mathfrak{B} \subset \mathfrak{E}$:

$$M^2 - L^3 = (2\gamma_5 L^{\frac{11}{4}} + \dots) = (2\gamma_5 L^{\frac{11}{4}} + \dots)_+ = 2\gamma_5 \partial^{11} + \text{l.o.t.}$$

Since \mathfrak{B} has rank two, it does not contain any differential operators of odd order. Therefore, $\gamma_5 = 0$ and

$$M = L^{\frac{3}{2}} + \gamma_4 L + \gamma_3 L^{\frac{3}{4}} + \gamma_2 L^{\frac{1}{2}} + \dots = L_+^{\frac{3}{2}} + \gamma_4 L + \gamma_3 L_+^{\frac{3}{4}} + \gamma_2 L_+^{\frac{1}{2}} + \gamma_1 L_+^{\frac{1}{4}} + \gamma_0.$$

Consider $N := M - \gamma_4 L \in \mathfrak{B}$. Again, we get the following equality in \mathfrak{B} :

$$N^2 - L^3 = (2\gamma_3 L^{\frac{9}{4}} + \dots) = (2\gamma_3 L^{\frac{9}{4}} + \dots)_+ = 2\gamma_3 \partial^9 + \text{l.o.t.}$$

This implies that $\gamma_3 = 0$, too.

Let $\alpha \in \mathbb{C}$ and $\widehat{L} = L + \alpha$. Obviously, we have: $\mathbb{C}[L, M] = \mathbb{C}[\widehat{L}, M]$. Moreover, $\widehat{L}_+^{\frac{3}{2}} = L_+^{\frac{3}{2}} + \frac{3}{2}\alpha L_+^{\frac{1}{2}}$ and $\widehat{L}_+^{\frac{i}{4}} = L_+^{\frac{i}{4}}$ for $i = 1, 2$. Therefore, we get a yet new identity in \mathfrak{B} :

$$\widehat{M} := M - \gamma_4 L - \gamma_0 = \widehat{L}_+^{\frac{3}{2}} + \gamma_1 \widehat{L}_+^{\frac{1}{4}} + \gamma_{-1} \widehat{L}_+^{-\frac{1}{4}} + \dots = \widehat{L}_+^{\frac{3}{2}} + \gamma_1 \widehat{L}_+^{\frac{1}{4}}.$$

As in the previous steps, we get an element $\widehat{M}^2 - \widehat{L} = 2\gamma_1 \partial^7 + \text{l.o.t.} \in \mathfrak{B}$ implying that $\gamma_1 = 0$, as $\text{rk}(\mathfrak{B}) = 2$. \square

The following result is due to Grünbaum [21].

Theorem 3.2. *Let $\mathfrak{B} \subset \mathfrak{D}$ be a genus one and rank two commutative subalgebra. Then*

$$\mathfrak{B} = \mathbb{C}[L, M] = \mathbb{C}[x, y]/(y^2 - 4x^3 - g_2x - g_3)$$

for some parameters $g_2, g_3 \in \mathbb{C}$. Here,

$$(3.3) \quad L = \left(\partial^2 + \frac{1}{2}c_2 \right)^2 + (c_1\partial + \partial c_1) + c_0$$

for certain $c_0, c_1, c_2 \in \mathbb{C}[[z]]$ obeying further constraints described below and $M = 2L_+^{\frac{3}{2}}$.

1. In the so-called formally self-adjoint case, $c_1 = 0$ and the following two subcases occur:

- (1) c_0 is a constant. Then the spectral curve is $y^2 = 4x^3 - 3c_0^2x - c_0^3$.
- (2) $c_0' \neq 0$. Then $c_0 = f$ and c_2 is given by the formula

$$(3.4) \quad c_2 = \frac{K_2 + 2K_3f + f^3 - f'''f' + \frac{1}{2}(f'')^2}{f'^2}.$$

for some $K_2, K_3 \in \mathbb{C}$. Other way around, if f, K_2, K_3 are such that c_2 is regular at $z = 0$ then $\mathfrak{B} = \mathbb{C}[L, M]$ has genus one and rank two. The spectral curve of \mathfrak{B} is given by the equation $y^2 = 4x^3 + 2K_3x - \frac{K_2}{2}$.

2. In the “generic” non-self-adjoint case, c_0, c_1 and c_2 are given by the formulae

$$(3.5) \quad \begin{cases} c_0 = -f^2 + K_{11}f + K_{12} \\ c_1 = f' \\ c_2 = \frac{K_{14} - 2K_{10}f + 6K_{12}f^2 + 2K_{11}f^3 - f^4 + f''^2 - 2f'f'''}{2f'^2} \end{cases}$$

where $f \in \mathbb{C}[[z]]$ satisfies $f(0) = 0$, and $K_{10}, K_{11}, K_{12}, K_{14} \in \mathbb{C}$. Other way around, if $f, K_{10}, K_{11}, K_{12}, K_{14}$ are such that c_2 is regular at $z = 0$ then $\mathfrak{B} = \mathbb{C}[L, M]$ has genus one and rank two. In this case, the Weierstraß parameters g_2 and g_3 of the spectral curve are given by the expressions

$$g_2 = K_{14} - 3K_{12}^2 - K_{10}K_{11} \text{ and } g_3 = \frac{1}{4}(K_{10}^2 - 2K_{10}K_{11}K_{12} - 4K_{12}^3 - K_{14}(K_{11}^2 + 4K_{12})).$$

Comment to the proof. Any normalized formally elliptic operator of order four can be written in the form (3.3), which turns out to be convenient for the computational purposes.

Then one takes the operator of order six $M := 2L_+^{\frac{3}{2}}$. The statement of the theorem follows from the analysis of the commutation relation $[L, M] = 0$, where one additionally has to rule out the rank one algebras $\mathbb{C}[L, M]$. \square

Remark 3.3. In the case $f'(0) = 0$, there are constraints between the coefficients of f and Grünbaum’s parameters K_{10}, K_{11}, K_{12} and K_{14} (respectively, K_2 and K_3) to insure that the Laurent series c_2 actually belongs to $\mathbb{C}[[z]]$. If that constraints are not satisfied, the resulting operators L and M still commute, but $\mathbb{C}[L, M]$ does not belong to \mathfrak{D} .

Remark 3.4. The different combinatorics of Grünbaum’s parameters c_0, c_1 and c_2 in the formally self-adjoint and non-self-adjoint cases looks like artificial. However, this separation turns out to be quite natural from the point of view of the computation of the greatest common divisor R_χ for a character $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$. See also Remark 3.19. For the reader’s convenience, and also following the work of Previato and Wilson [41], we decided to keep Grünbaum’s notations [21] in our article.

Although Grünbaum’s classification looks like quite massy on the first sight, it turns out to be perfectly suited to describe the spectral data (X, p, \mathcal{F}) of \mathfrak{B} in terms of Section 2. Krichever and Novikov derived their formulae [30] starting from the geometric side of Krichever’s correspondence and then obtained from it an explicit formula for the operator L . A comparison between the answers of [30] and [21] can be found in [21, Section 6]. At the present moment it is not clear to us, how to generalize the method of vector-valued Baker–Akhieser functions and deformations of Tyurin parameters of [30] on the case of singular Riemann surfaces.

Notation. In the sequel, the following notation will be used.

- $\mathfrak{B} = \mathbb{C}[L, M] \subset \mathfrak{D}$ is a genus one and rank two commutative subalgebra with L given by Grünbaum’s formulae from Theorem 3.2.
- Next, X is the compactified spectral curve of \mathfrak{B} , $p \in X$ is its point at infinity and $X_0 = X \setminus \{p\}$. If X_0 is singular then s denotes its unique singular point.
- Let \mathcal{F} be the spectral sheaf of \mathfrak{B} . See Corollary 2.10 for a list of possibilities.
- Finally, \mathcal{T} is the Fourier–Mukai transform of \mathcal{F} and $Z := \text{Supp}(\mathcal{T}) \subset X_0$.

Proposition 3.5. *Let $q = (\lambda, \mu) \in Z$ be such that \mathcal{F} is locally free at q . Let $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ be the character corresponding to q and*

$$R_\chi := \partial^2 + c_1\partial + c_2 = \gcd(L - \lambda, M - \mu) \in \tilde{\mathfrak{D}}.$$

Let $\nu := -\text{res}_0(c_1(z)) - 1$ and $(z^{\rho_1}w_1(z), z^{\rho_2}w_2(z))$ be a basis of the solution space $\text{Sol}(\mathfrak{B}, \chi) = \text{Ker}(R_\chi)$, where $0 \leq \rho_1 < \rho_2 \in \mathbb{N}_0$ and $w_i(0) \neq 0$ for $i = 1, 2$.

- (1) *We have: $0 \leq \nu \leq 3$ and $(\rho_1, \rho_2) \in \{(0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$.*
- (2) *Next, $(\rho_1, \rho_2) = (2, 3)$ if and only if q is a smooth point and $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$. This case occurs if and only if $\nu = 3$.*
- (3) *The case $\nu = 2$ is equivalent to $(\rho_1, \rho_2) = (1, 3)$. If q is a smooth point then $\mathcal{F} \cong \mathcal{A} \otimes \mathcal{O}([q])$. If q is singular then $\mathcal{F} \cong \mathcal{B}_{\bar{q}}$ for some smooth point $\bar{q} \in X$.*

Proof. All essential ideas are taken from [41].

(1) The indicial equation (1.21) implies that $\rho_1 + \rho_2 = \nu + 2$. By construction, $\text{Ker}(R_\chi) = \text{Sol}(\mathfrak{B}, \chi) \subset \text{Ker}(L - \lambda)$. Recall that $\text{ord}(L - \lambda) = 4$. If $z^\rho w(z) \in \text{Ker}(R_\chi)$ and $w(z) \neq 0$ then $\rho \leq 3$ (by the uniqueness of solution of a differential equation with regular coefficients). All together, this implies the first statement.

(2) Obviously, $\nu = 3$ if and only if $(\rho_1, \rho_2) = (2, 3)$. However, $\{0, 1\} \cap \{\rho_1, \rho_2\} = \emptyset$ if and only if the map $\tilde{\eta}_\chi$ from the commutative diagram (1.20) is zero. Going through the list of vector bundles from Corollary 2.10 we conclude that the map $\Gamma(X, \mathcal{F}) \xrightarrow{\text{ev}_q} \mathcal{F}|_q$ is zero if and only if q is a smooth point and $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$. See also [41, Proposition 3.1].

(3) If q is a smooth point then the stated result is [41, Theorem 1.2(ii)]. If q is singular, the result follows from Corollary 2.10. \square

3.2. Formally self-adjoint case. In this subsection, we describe the spectral sheaf of the algebra \mathfrak{B} from Grünbaum’s Theorem 3.2 in the formally self-adjoint case $c_1 = 0$.

Lemma 3.6. *Let $L = (\partial^2 + \frac{1}{2}c_2)^2 + \gamma$ for some $c_2 \in \mathbb{C}[[z]]$ and $c_0 = \gamma \in \mathbb{C}$ (degenerate self-adjoint case). Then X is singular and $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{S}$.*

Proof. According to Grünbaum [21, Section 2], we have: $M = 2(\partial^2 + \frac{1}{2}c_2)^3 + 3\gamma(\partial^2 + \frac{1}{2}c_2)$ and the equation of the spectral curve X_0 is $y^2 = 4x^3 - 3\gamma^2x - \gamma^3$. Clearly, X_0 is singular at the point $s = (-\frac{\gamma}{2}, 0)$. Let $P = (\partial^2 + \frac{1}{2}c_2)$. It is easy to see that

$$M = P \cdot \left(L + \frac{\gamma}{2} \right)$$

implying that the order of the greatest common divisor R_χ (1.18) for the character χ corresponding to the singular point s , is four. Therefore, we have: $\mathcal{F}|_s \cong \mathbb{C}^4$. It remains to observe that $\mathcal{S} \oplus \mathcal{S}$ is the only semi-stable sheaf of rank two and slope one on X , whose fiber over s is four dimensional, see Corollary 2.10. Note that $\mathbb{C}[L, M] \subset \mathbb{C}[P]$, hence $\mathbb{C}[L, M]$ is not maximal in this case. \square

Theorem 3.7. *Let L be given by (3.3) with $c_1 = 0$ and $f' \neq 0$ (non-degenerate formally self-adjoint case). Then \mathcal{F} is locally free. Let ν be the order of vanishing of $f'(z)$ at $z = 0$. Then Z is invariant under the involution $X_0 \xrightarrow{\iota} X_0$, $((\lambda, \mu) \mapsto (\lambda, -\mu))$ and the following results are true (we assume that $X_0 = V\left(y^2 - 4x^3 - 2K_3x + \frac{K_2}{2}\right)$ is singular):*

- (1) *If $\nu = 0$ then \mathcal{F} is isomorphic to*
 - (a) $\mathcal{O}([q]) \oplus \mathcal{O}([\iota(q)])$ if $Z = \{q, \iota(q)\}$ with $q \neq \iota(q)$.
 - (b) $\mathcal{A} \otimes \mathcal{O}([q])$ if $Z = \{q\} = \{\iota(q)\}$ and $q \neq s$.
 - (c) \mathcal{B}_p if $Z = \{s\}$.
- (2) *If $\nu = 1$ then $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([\iota(q)])$ with $q \neq \iota(q)$ and $Z = \{q, \iota(q)\}$.*
- (3) *If $\nu = 2$ then necessarily $Z = \{q\}$ with $q = \iota(q)$.*
 - (a) *If $q \neq s$ then $\mathcal{F} \cong \mathcal{A} \otimes \mathcal{O}([q])$.*
 - (b) *If $q = s$ then $\mathcal{F} \cong \mathcal{B}_p$.*
- (4) *If $\nu = 3$ then $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$, where $q = \iota(q)$ is a smooth point of X_0 . In this case, $Z = \{q\}$, what can occur only if X_0 is nodal.*

Proof. Let $q = (\lambda, \mu) \in X_0$ and $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ be the corresponding character. The key point is the following result [41, Section 5]: there exist $R, Q \in \mathfrak{D}$ both of order two such that

$$M - \mu = Q \cdot (L - \lambda) + R,$$

where $R = a_0\partial^2 + a_1\partial + a_2$ with $a_0 = (2\lambda + f)$ and $a_1 = -f'$. Since f is not a constant, the order of R_{X_q} is two for all $q \in X_0$ implying that the spectral sheaf \mathcal{F} is locally free. Note that ν coincides with the parameter introduced in Proposition 3.5.

By Theorem 1.28 we have: $Z = \{(\lambda_0, \pm\mu_0)\}$, where $\lambda_0 = -\frac{1}{2}f(0)$ and $\pm\mu_0$ are the roots of the equation $\mu^2 = h(\lambda_0)$ with $h(\lambda) = 4\lambda^3 + 2K_3\lambda - \frac{1}{2}K_2$. Unless $Z = \{s\}$, the description of \mathcal{F} can be obtained along the same lines as in [41, Theorem 1.2], see also Proposition 3.5. From now on we assume that $Z = \{s\}$. According to Corollary 2.10, $\mathcal{F} \cong \mathcal{B}_{\bar{q}}$ for some smooth point $\bar{q} \in X$ and we only have to show that $\bar{q} = p$. Note that

$$\text{res}_0\left(\frac{f'(z)}{f(z) - f(0)}\right) = \nu + 1.$$

Proposition 3.5 implies that $0 \leq \nu \leq 3$.

Case 1. Assume that Grünbaum's parameters K_2, K_3 and f are such that $f'(0) \neq 0$ (i.e. $\nu = 0$). Let \mathfrak{B} be the corresponding commutative subalgebra of \mathfrak{D} . Consider now the $\mathbb{C}[t]$ -flat family $\mathfrak{B}_B \subset (\mathbb{C}[t][z][\partial])$ defined by the Grünbaum's parameters $K_2, K_3(t) := K_3 + t$ and f . Let $X_B \xrightarrow{\pi} B$ be the corresponding spectral fibration (here, $B = \text{Spec}(\mathbb{C}[t])$) and \mathcal{F}_B be the corresponding spectral sheaf, see Theorem 1.30. For any $b \in B$ we denote by $X_b = \pi^{-1}(b)$ the fiber over b and $\mathcal{F}_b := \mathcal{F}_B|_{X_b}$. Clearly, $\mathcal{F}_0 \cong \mathcal{F}$ and $\mathcal{F}_b \cong \mathcal{O}_{X_b}([q_1(b)] + [q_2(b)])$ for $b \neq 0$ from some open neighbourhood $U \subset B$ of 0, where $\iota(q_1(b)) = q_2(b)$ in X_b . Therefore, $\det(\mathcal{F}_b) \cong \mathcal{O}_{X_b}(2[p])$ for all $b \in U \setminus \{0\}$. But then we also have: $\det(\mathcal{F}_0) \cong \mathcal{O}_{X_0}(2[p])$ and therefore $\mathcal{F} \cong \mathcal{B}_p$.

Case 2. Assume that Grünbaum's parameters K_2, K_3 and f are such that $f'(0) = 0$. Then f has an expansion of the form $f(z) = \alpha + \beta z^2 + \gamma z^3 + \delta z^4 + \dots$. Now we have to use the fact that Grünbaum's parameter $c_2(z)$ given by (3.4) is regular. This in particular implies that $\alpha^3 + 2K_3\alpha + K_2 + 2\beta^2 = 0$, i.e. $\left(-\frac{\alpha}{2}, \pm\beta\right) \in X_0$. Since we assumed that \mathcal{T} is supported at the singular point of X_0 , we get: $\beta = 0$. Hence, $\nu \geq 2$ and in virtue of Proposition 3.5 we have: $\nu = 3$, i.e. $\gamma \neq 0$. Requiring the regularity of $c_2(z)$, we get

the following constraint: $2K_3 + 3\alpha^2 - 24\delta = 0$. Observe that the point $(-\frac{\alpha}{2}, 0) \in X_0$ is singular if and only if $\delta = 0$. Summing up, we have in this case:

$$(3.6) \quad \begin{cases} f &= \alpha + \gamma z^3 + \tau z^5 + \dots, \text{ with } \gamma \neq 0, \\ K_2 &= 2\alpha^3, \\ K_3 &= -\frac{3}{2}\alpha^2. \end{cases}$$

Let $\mathfrak{B} = \mathbb{C}[L, M]$ be the corresponding commutative subalgebra of \mathfrak{D} . It admits the following *flat* deformation \mathfrak{B}_B over the base $B = \text{Spec}(\mathbb{C}[\delta])$:

$$(3.7) \quad \begin{cases} f(\delta) &= f + \delta z^4, \\ K_2(\delta) &= 2\alpha^3 - 24\alpha\delta, \\ K_3(\delta) &= 12\delta - \frac{3}{2}\alpha^2. \end{cases}$$

The total space of the corresponding genus one fibration $X_B \xrightarrow{\pi} B$ is given by the equation

$$X_B := \overline{V(y^2 - 4x^3 - (24\delta - 3\alpha^2)x - 2\alpha(\alpha^2 - 12\delta))} \subset \mathbb{P}_{(x,y)}^2 \times \mathbb{A}_\delta^1.$$

It is interesting to note that X_B is singular and $B \xrightarrow{\sigma} X_B, \delta \mapsto (-\frac{\alpha}{2}, 0, \delta)$ is a section of π . Let \mathcal{F}_B be the spectral sheaf of \mathfrak{B}_B , see Theorem 1.30. There exists an open subset $U \subset B$ with $0 \in U$ and such that for all $b \in U \setminus \{0\}$ we have: $\mathcal{F}_b := \mathcal{F}_B|_{Y_b} \cong \mathcal{A} \otimes \mathcal{O}([q])$ with $q = (-\frac{\alpha}{2}, 0)$ for $b \neq 0$. Therefore, $\det(\mathcal{F}_b) \cong \mathcal{O}(2[p])$ for $b \in U \setminus \{0\}$. This implies that $\det(\mathcal{F}_0) \cong \mathcal{O}(2[p])$ as well, see Proposition 2.7. Thus, $\mathcal{F} \cong \mathcal{B}_p$ as claimed. \square

Example 3.8. Let $\mathfrak{B} = \mathbb{C}[P, Q]$ be as in the example of Dixmier (0.2) for $\kappa = 0$. Then the spectral sheaf of \mathfrak{B} is \mathcal{B}_p .

3.3. Non-self-adjoint case. Let L be the fourth order differential operator given by Grünbaum's parameters $K_{10}, K_{11}, K_{12}, K_{14}$ and f as in (3.5). The equation of the affine spectral curve X_0 of the algebra $\mathbb{C}[L, M]$ is $y^2 = 4x^3 + g_2x + g_3$ with

$$(3.8) \quad \begin{cases} g_2 &= K_{14} - 3K_{12}^2 - K_{10}K_{11}, \\ g_3 &= \frac{1}{4}(K_{10}^2 - 2K_{10}K_{11}K_{12} - 4K_{12}^3 - K_{14}(K_{11}^2 + 4K_{12})). \end{cases}$$

For any $\lambda \in \mathbb{C}$ pose

$$(3.9) \quad \begin{cases} a(\lambda) &= \left(\lambda + \frac{1}{2}K_{12}\right)^2 + \frac{1}{4}K_{14} \\ b(\lambda) &= \left(\lambda + \frac{1}{2}K_{12}\right)K_{11} - \frac{1}{2}K_{10} \\ c(\lambda) &= -\lambda + K_{12} + \frac{1}{4}K_{11}^2. \end{cases}$$

Our analysis of the spectral sheaf \mathcal{F} is based in the following result from the article of Previato and Wilson [41, Section 5] attributed there to the PhD thesis of Latham [32].

Theorem 3.9. *Let (λ, μ) be any point of X_0 (smooth or singular) and $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ be the corresponding character. Let $\tilde{R}_\chi, \hat{R}_\chi \in \mathfrak{D}$ be the differential operators defined by the following conditions:*

$$(3.10) \quad \begin{cases} M - \mu &= \tilde{Q}_\chi \cdot (L - \lambda) + \tilde{R}_\chi, & \text{ord}(\tilde{R}_\chi) \leq 3 \\ L - \lambda &= \hat{Q}_\chi \cdot \tilde{R}_\chi + \hat{R}_\chi, & \text{ord}(\hat{R}_\chi) \leq 2. \end{cases}$$

Then we have: $\text{ord}(\tilde{R}_\chi) = 3$ and $\hat{R}_\chi = e_0(z; \lambda, \mu)\partial^2 - e_1(z; \lambda, \mu)\partial + e_2(z; \lambda, \mu)$ with

$$(3.11) \quad e_0 = a(\lambda) + b(\lambda)f + c(\lambda)f^2 \quad \text{and} \quad e_1 = \frac{1}{2}(b(\lambda) - \mu)f + c(\lambda)ff'.$$

Similarly to [41, Section 5], we have the following result.

Proposition 3.10. *A point $q = (\lambda, \mu) \in X_0$ belongs to Z if and only if $a(\lambda) = 0$ and $\mu = -b(\lambda)$.*

Proof. A lengthy but elementary computation shows the following

Fact. If a point $(\lambda, \mu) \in \mathbb{C}^2$ belongs to X_0 and $a(\lambda) = 0$ then necessarily $\mu = \pm b(\lambda)$.

Let $R_\chi := \gcd(L - \lambda, M - \mu)$ in the sense of Theorem 1.26.

Case 1. Assume that $a(\lambda) = b(\lambda) = c(\lambda) = 0$. Then $e_0(z; \lambda, \mu) = 0$. In virtue of the formulae (3.10) we see that $\text{ord}(\widehat{R}_\chi) \leq 1$ in this case. However, $\text{rk}(\mathbb{C}[L, M]) = 2$ and the only possibility for this to be true is that $\widehat{R}_\chi = 0$. Hence, $\text{ord}(R_\chi) = 3$. This case occurs if and only if X_0 is singular with the singular point $s = (\lambda, 0)$ and \mathcal{F} is not locally free at s . See Theorem 3.11 below. In this case, the singular point s belongs to the support of \mathcal{T} due to Theorem 1.28.

Case 2. Assume now that $(a(\lambda), b(\lambda), c(\lambda)) \neq (0, 0, 0)$. In this case, $e_0(z; \lambda, \mu) \neq 0$ and

$$R_\chi = \frac{1}{e_0(z; \lambda, \mu)} \widehat{R}_\chi = \partial^2 - \frac{e_1(z; \lambda, \mu)}{e_0(z; \lambda, \mu)} \partial + \frac{e_2(z; \lambda, \mu)}{e_0(z; \lambda, \mu)}.$$

According to Theorem 1.28, (λ, μ) belongs to the support of \mathcal{T} if and only if the Laurent power series $\frac{e_1(z; \lambda, \mu)}{e_0(z; \lambda, \mu)}$ has a pole at $z = 0$. Taking into account explicit expressions (3.11) for $e_i(z; \lambda, \mu)$ for $i = 0, 1$ as well as the assumption $f(0) = 0$, we see that $a(\lambda) = 0$. Therefore, $\mu = \pm b(\lambda)$. Note, that by assumption $(b(\lambda), c(\lambda)) \neq (0, 0)$.

If $\mu = -b(\lambda)$ then $\frac{e_1(z; \lambda, \mu)}{e_0(z; \lambda, \mu)} = \frac{f'(z)}{f(z)}$. This function has a pole at $z = 0$ as $f(0) = 0$. Therefore, the point $(\lambda, -b(\lambda))$ belongs to the support of \mathcal{T} due to Theorem 1.28. Moreover, the order of vanishing of f at 0 is at most four, see Proposition 3.5.

Now suppose that $\mu = b(\lambda)$. Then $\frac{e_1(z; \lambda, \mu)}{e_0(z; \lambda, \mu)} = \frac{c(\lambda)f'(z)}{b(\lambda) + c(\lambda)f(z)}$ has a pole at $z = 0$ if and only if $b(\lambda) = 0$ (and we are in the previous case). \square

Theorem 3.11. *Let $\mathcal{B} = \mathbb{C}[L, M]$ be a genus one and rank two commutative subalgebra, which is not formally self-adjoint and given by Grünbaum's parameters $K_{10}, K_{11}, K_{12}, K_{14}$ and f . Then we have:*

(1) *the spectral sheaf \mathcal{F} of \mathfrak{B} is not locally free if and only if*

$$(3.12) \quad \begin{cases} K_{10} &= (3K_{12} + \frac{1}{2}K_{11}^2)K_{11} \\ K_{14} &= -(3K_{12} + \frac{1}{2}K_{11}^2)^2. \end{cases}$$

(2) *Moreover, in this case \mathcal{F} is indecomposable (i.e. isomorphic to \mathcal{U}_\pm in the nodal case, respectively to \mathcal{U} in the cuspidal case) if and only if*

$$(3.13) \quad \Delta := 6K_{12} + K_{11}^2 = 0.$$

(3) *If $\Delta \neq 0$ then $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{O}([q])$, where $q = (-2K_{12} - \frac{1}{4}K_{11}^2, -\frac{1}{2}K_{11}(K_{11}^2 + 6K_{12}))$.*

Proof. (1) Assume that \mathcal{F} is not locally free. According to Theorem 1.28, this is equivalent to $\text{ord}(R_\chi) = 3$, where χ is the character, corresponding to some point $(\lambda_0, 0) \in X_0$. This can happen if and only if $\widehat{R}_\chi = 0$. In particular, $e_0(z; \lambda_0, 0) = 0$ implying that $a(\lambda_0) = b(\lambda_0) = c(\lambda_0) = 0$. From the equality $a(\lambda_0) = 0$ we get $\lambda_0 = K_{12} + \frac{1}{4}K_{11}^2$, whereas the vanishings $b(\lambda_0) = c(\lambda_0) = 0$ imply the constraints (3.12).

Other way around, assume that (3.12) are satisfied. A direct computation shows that the Weierstraß parameters g_2, g_3 given by the formulae (3.8), take the following form:

$$\begin{cases} g_2 &= -3\left(2K_{12} + \frac{1}{2}K_{11}^2\right)^2 \\ g_3 &= \left(2K_{12} + \frac{1}{2}K_{11}^2\right)^3. \end{cases}$$

By Theorem 2.1, the spectral curve X_0 is singular with the singular point $s = (\lambda_0, 0)$, where $\lambda_0 = K_{12} + \frac{1}{4}K_{11}^2$. Moreover, constraints (3.12) imply that $a(\lambda_0) = b(\lambda_0) = c(\lambda_0) = 0$, hence \mathcal{F} is indeed not locally free at s .

(2) The possibilities for the spectral sheaf \mathcal{F} are listed in Corollary 2.10. The case $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{S}$ is excluded since $\text{ord}(\tilde{R}_\chi) = 3$ by Theorem 3.9, implying that $\dim_{\mathbb{C}}(\mathcal{F}|_s) \leq 3$. Hence, \mathcal{F} is indecomposable if and only if \mathcal{T} is supported at the singular point of X_0 . According to Proposition 3.10, this occurs if and only if $K_{14} = 0$: otherwise, the equation $a(\lambda) = 0$ has two different solutions, both contributing to the support of \mathcal{T} due to Proposition 3.10. Since we already showed that the formulae (3.12) are true, the indecomposability of \mathcal{F} is equivalent to the vanishing $\Delta = 0$.

(3) Assume that the equations (3.12) are satisfied and $\Delta \neq 0$. Then the equation $a(\lambda) = 0$ has two different solutions: $\lambda_0 = K_{12} + \frac{1}{4}K_{11}^2$ and $\tilde{\lambda}_0 = -2K_{12} - \frac{1}{4}K_{11}^2$. The torsion sheaf \mathcal{T} is supported at $s = (\lambda_0, 0)$ and $q := (\tilde{\lambda}_0, -b(\tilde{\lambda}_0)) = (-2K_{12} - \frac{1}{4}K_{11}^2, -\frac{1}{2}K_{11}(K_{11}^2 + 6K_{12}))$. Taking into account Corollary 2.10, we get the statement. \square

Lemma 3.12. *Let $g \in \mathbb{C}[[z]]$. Then the Laurent series $h = \frac{2gg'' - g'^2}{g^2}$ is regular at $z = 0$ if and only if $g(0) \neq 0$ or $g(z) = z^2\tilde{g}(z)$ with $\tilde{g}(0) \neq 0$ and $\tilde{g}'(0) = 0$.*

Proof. Obviously, $h(z)$ is regular provided $g(0) \neq 0$. Assume that $g(z) = z^\rho\tilde{g}(z)$ with $\rho \in \mathbb{N}_0$ and $\tilde{g}(0) \neq 0$. Note that

$$(3.14) \quad h = \frac{g''}{g} + \left(\frac{g'}{g}\right)' = \left(\frac{\rho(\rho-1)}{z^2} + \frac{2\rho}{z}\frac{\tilde{g}'}{\tilde{g}} + \varphi\right) + \left(-\frac{\rho}{z^2} + \psi\right)$$

for appropriate $\varphi, \psi \in \mathbb{C}[[z]]$. If $\rho \geq 1$ then h is regular if and only if $\rho = 2$ and $\tilde{g}'(0) = 0$. Therefore, the series $g(z)$ has the form

$$(3.15) \quad g(z) = \zeta_2 z^2 + \sum_{i=4}^{\infty} \zeta_i z^i \quad \text{with } \zeta_2 \neq 0.$$

\square

Corollary 3.13. *Let $\mathfrak{B} = \mathbb{C}[L, M]$ be a genus one and rank two commutative subalgebra in \mathfrak{D} . Then the spectral sheaf of \mathfrak{B} is indecomposable and not locally free (i.e isomorphic to \mathcal{U}_\pm in the nodal case and to \mathcal{U} in the cuspidal case) if and only if L is formally non-self-adjoint and given by the formulae (3.3) with the parameters c_0, c_1 and c_2 :*

$$(3.16) \quad \begin{cases} c_0 &= -f^2 + \varrho f - \frac{\varrho^2}{6} \\ c_1 &= f' \\ c_2 &= \frac{2\varrho f^3 - \varrho^2 f^2 - f^4 + f''^2 - 2f'f'''}{2f'^2} \end{cases}$$

for an arbitrary $\varrho \in \mathbb{C}$ and any $f \in \mathbb{C}[[z]]$ satisfying $f(0) = 0$ and either of two conditions:

- $f'(0) \neq 0$ or
- $f'(0) = f''(0) = f^{(4)}(0) = 0, f'''(0) \neq 0$.

The equation of the spectral curve in this case is

$$(3.17) \quad y^2 = 4x^3 - \frac{1}{12}\varrho^4x + \frac{1}{216}\varrho^6.$$

Remark 3.14. In the notation of Theorem 3.2 we have $\varrho = K_{11}$. Note that the family (3.16) admits an obvious involution $\varrho \mapsto -\varrho$. It turns out that this involution corresponds to the flip $\mathcal{U}_\pm \mapsto \mathcal{U}_\mp$ on the level of spectral sheaves. The precise description of \mathcal{F} in the nodal case (i.e. \mathcal{U}_+ versus \mathcal{U}_-) is rather subtle, see the proof of Theorem 3.16.

Example 3.15. Let us set $\varrho = 0$ and $f = z$ in the equations (3.16). Then we get

$$L = \left(\partial^2 - \frac{z^4}{4}\right)^2 + 2\partial - z^2.$$

A straightforward computation shows that in this case $M := 2L_+^{\frac{3}{2}}$ is given by the formula $M = 2\partial^6 - \frac{3}{2}z^4\partial^4 + 6(1 - 2z^3)\partial^3 + z^2\left(\frac{3}{8}z^6 - 45\right)\partial^2 + z\left(3z^6 - \frac{3}{2}z^3 - 54\right)\partial + \left(-\frac{1}{32}z^{12} + \frac{37}{4}z^6 - 3z^3 - 14\right)$. Moreover, another straightforward computation yields:

$$R := \gcd(L, M) = \partial^3 - \frac{1}{2}z^2\partial^2 + z\left(-\frac{1}{4}z^3 + 1\right)\partial + \left(\frac{1}{8}z^6 - \frac{3}{2}z^3 + 1\right).$$

Since $\text{ord}(R) = 3$, the spectral sheaf of $\mathbb{C}[L, M]$ is the torsion free sheaf \mathcal{U} , as predicted. Notably, the coefficients of R are regular. We hope that a more detailed treatment of genus one commutative subalgebras in the Weyl algebra $\mathfrak{W} = \mathbb{C}[z][\partial]$ with a cuspidal spectral curve and the spectral sheaf which is not locally free will be helpful for various studies related to Dixmier's conjecture about $\text{Aut}(\mathfrak{W})$, see [35].

The following result characterizes those genus one and rank two commutative subalgebras of \mathfrak{D} , whose spectral curve X is singular and the associated torsion sheaf \mathcal{T} is indecomposable and supported at the singular point of X .

Theorem 3.16. *Let $\mathcal{B} = \mathbb{C}[L, M]$ be given by Grünbaum's parameters $K_{10}, K_{11}, K_{12}, K_{14}$ and f . Then the following results are true.*

- (1) *The (affine) spectral curve X_0 of \mathfrak{B} is singular and the torsion sheaf \mathcal{T} is supported at the singular point of X_0 if and only if $K_{10} = 0 = K_{14}$. In this case, $X_0 = \text{Spec}(R)$ with*

$$(3.18) \quad R = \mathbb{C}[x, y] / \left(y^2 - 4\left(x + \frac{K_{12}}{2}\right)^2(x - K_{12})\right).$$

- (2) *The spectral sheaf \mathcal{F} of \mathfrak{B} is locally free if and only if $\Delta := 6K_{12} + K_{11}^2 \neq 0$. In this case, $\mathcal{F} \cong \mathcal{B}_{\bar{q}}$ with*

$$(3.19) \quad \bar{q} = \left(\frac{1}{4}K_{11}^2 + K_{12}, \frac{K_{11}}{4}(6K_{12} + K_{11}^2)\right).$$

- (3) *Moreover, for the Fourier–Mukai transform \mathcal{T} of \mathcal{F} we have:*

$$(3.20) \quad \mathcal{T} \cong \widehat{R} / \left(\left(x + \frac{K_{12}}{2}\right)^2, y - K_{11}\left(x + \frac{K_{12}}{2}\right)\right).$$

Proof. (1) According to Proposition 3.10, the support of \mathcal{T} consists of a single point $q = (\lambda_0, \mu_0)$ if and only if $K_{14} = 0$. In this case, $\lambda_0 = -\frac{1}{2}K_{12}$ and $\mu_0 = -b(\lambda_0)$. If q is the singular point of X_0 then $b(\lambda_0) = 0$ implying that $K_{10} = 0$. Other way around, if $K_{10} = 0 = K_{14}$ then X_0 is given by the equation $y^2 = 4x^3 - 3K_{12}^2x - K_{12}^3$. According to Theorem 2.1, the curve X_0 is singular with the singular point $s = (\lambda_0, 0) = \left(-\frac{1}{2}K_{12}, 0\right)$. Moreover, $a(\lambda_0) = 0 = b(\lambda_0)$, hence \mathcal{T} is indeed supported at s .

(2) We already showed in Theorem 3.11 that the spectral sheaf \mathcal{F} is locally free if and only if $\Delta \neq 0$. Therefore, in this case $\mathcal{F} \cong \mathcal{B}_{\bar{q}}$ for some smooth point $\bar{q} \in X$ determined by the condition $\det(\mathcal{F}) \cong \mathcal{O}([p] + [\bar{q}])$. To compute the determinant of \mathcal{F} , we use again a deformation argument.

Case 1. Assume that $f'(0) \neq 0$. Then the power series c_2 given by (3.5) is automatically regular and the non-zero Grünbaum's parameters K_{11}, K_{12} do not correlate with the coefficients of the power series f . Keeping K_{11}, K_{12} and f unchanged and introducing new parameters $\alpha = K_{10}$ and $\beta = K_{14}$, we get a family \mathfrak{B}_B of commutative subalgebras in \mathfrak{D} given by (3.5), flat over the base $B = \text{Spec}(\mathbb{C}[\alpha, \beta])$ and such that $\mathfrak{B}_{(0,0)} \cong \mathfrak{B}$. Let \mathcal{F}_B be the corresponding spectral sheaf, see Theorem 1.30. Assume that $t = (\alpha, \beta) \in B$ is such that the support of the Fourier–Mukai transform \mathcal{T}_t of the corresponding spectral sheaf \mathcal{F}_t is locally free and supported at two different points of the spectral curve X_t . According to Proposition 3.10, the support of \mathcal{T}_t is $\{q_1, q_2\} = \{(\lambda_1, -b(\lambda_1)), (\lambda_2, -b(\lambda_2))\}$, where λ_1 and λ_2 are the roots of the equation $\lambda^2 + K_{12}\lambda + \frac{1}{4}(K_{12}^2 + \beta) = 0$. Moreover, $\mathcal{F}_t \cong \mathcal{O}([q_1]) \oplus \mathcal{O}([q_2])$, hence $\det(\mathcal{F}_t) \cong \mathcal{O}([q_1] + [q_2]) \cong \mathcal{O}([p] + [\bar{q}])$, where $\bar{q} = q_1 + q_2$ with “+” taken in the sense of the group law on the set of smooth points of X_t . Computing explicitly $q_1 + q_2 \in X_t$ and then setting $\alpha = \beta = 0$, we get: $\det(\mathcal{F}) \cong \mathcal{O}([p] + [\bar{q}])$ with \bar{q} given by (3.19).

Case 2. Suppose now that $f'(0) = 0$. The proof in this case is analogous to the previous one, but is technically more involved. First note that the Laurent series $\frac{f}{f'}$ is regular at $z = 0$. Since $K_{10} = K_{14} = 0$, the regularity of c_2 given by (3.5) is equivalent to the regularity of $\frac{f''^2 - 2f'f'''}{f'^2}$. Lemma 3.12 implies that the order of vanishing of f at $z = 0$

is precisely three. Moreover, f has the following form: $f = \xi_3 z^3 + \sum_{i=5}^{\infty} \xi_i z^i$ with $\xi_3 \neq 0$,

see (3.15). Setting $\begin{cases} f_{\xi} &= f + \xi z^4 \\ K_{10} &= -24\xi \end{cases}$ and keeping the parameters K_{11}, K_{12} untouched,

then we get a flat family of commutative subalgebras \mathfrak{B}_B over the base $B = \text{Spec}(\mathbb{C}[\xi])$ with $\mathcal{B}_0 \cong \mathfrak{B}$. As $K_{14} = 0$, the spectral sheaf \mathcal{F}_{ξ} is isomorphic to $\mathcal{A} \otimes \mathcal{O}([q_{\xi}])$ for $\xi \neq 0$, where q_{ξ} is a smooth point of the spectral curve (such behaviour is completely parallel to the self-adjoint case, see the proof of Theorem 3.7). Therefore, $\det(\mathcal{F}_{\xi}) \cong \mathcal{O}(2[q_{\xi}])$. In a similar manner we get again: $\mathcal{F} = \mathcal{B}_{\bar{q}}$ with \bar{q} given by (3.19).

(3) In the case $\Delta \neq 0$, the isomorphism (3.20) for the torsion sheaf \mathcal{T} follows from Theorem 2.8. It remains to describe \mathcal{T} in the case when $\Delta = 0$ and the spectral curve is nodal. Assume that $K_{10} = K_{14} = 0$, $K_{12} = \tau$ is fixed and $K_{11} = \theta$ can be varied. Furthermore, let $f \in z\mathbb{C}[[z]]$ be such that c_2 is regular at $z = 0$. Then we get a family of commutative subalgebras \mathfrak{B}_T flat over $T = \text{Spec}(\mathbb{C}[\theta])$, whose affine spectral surface $X_T \subset \mathbb{A}_{x,y}^2 \times T$ is given by the equation $y^2 = 4\left(x + \frac{\tau}{2}\right)^2(x - \tau)$. Let \mathcal{F}_T be the spectral sheaf of this family and \mathcal{T}_T its relative Fourier–Mukai transform. Let $b \in \mathbb{C} = T$ be such that $b^2 + 6\tau = 0$. Clearly, the torsion sheaf $(\mathcal{T}_T)|_{X_b}$ is a quotient of \mathcal{O}_{X_b} . Therefore, $\mathcal{T}_{T_0} := \mathcal{T}_T|_{T_0}$ is a quotient of $\mathcal{O}_{X_{T_0}}$ for some open neighbourhood $T_0 \subset T$ of b . Using Remark 2.11 as well as the universal property of the Hilbert scheme of points applied to (T_0, \mathcal{T}_{T_0}) , we get a uniquely determined morphism $T_0 \xrightarrow{\gamma} \mathbb{P}^1, \theta \mapsto (\gamma_0(\theta) : \gamma_1(\theta))$ such that

$$\mathcal{T}_{\theta} \cong \widehat{R}/\left(\left(x + \frac{\tau}{2}\right)^2, \gamma_0(\theta)\left(x + \frac{\tau}{2}\right) - \gamma_1(\theta)y\right).$$

From part (2) and Theorem 2.8(2) we already know that for $\theta \in T_0 \setminus \{b\}$ we have: $\gamma(\theta) = (1 : \theta)$. By continuity of γ we finally obtain: $\gamma(b) = (1 : b)$. Theorem is proven. \square

Remark 3.17. The description of the spectral sheaf \mathcal{F} of the algebra \mathfrak{B} in the case $s \notin Z$ is the same as in the work of Previato and Wilson [41]. In particular, $q = (\lambda, \mu)$ belongs to Z if and only if $a(\lambda) = 0$ and $\mu = -b(\lambda)$. There are namely the following possibilities:

- $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$ if $Z = \{q, q'\}$.
- $\mathcal{F} \cong \mathcal{O}([q]) \otimes \mathcal{A}$ or $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$ if $Z = \{q\}$. The last case occurs if and only if f has a zero of order four at $z = 0$.

3.4. Spectral sheaf of the Fourier transform of Dixmier's example. The methods developed in our article can be applied to determine the spectral sheaves of genus one and rank three commutative subalgebras of \mathfrak{D} .

Example 3.18. The Weyl algebra $\mathfrak{W} = \mathbb{C}[z][\partial]$ admits an algebra automorphism $z \mapsto \phi$ $\partial, \partial \mapsto -z$ called Fourier transform. Consider now the Fourier transform of Dixmier's example (0.2). Namely, for any $\kappa \in \mathbb{C}$, put $\widehat{D} := \phi(D) = \partial^3 + z^2 + \kappa$ and pose

$$(3.21) \quad \widehat{P} := \phi(P) = \widehat{D}^2 + 2\partial \quad \text{and} \quad \widehat{Q} := \phi(Q) = \widehat{D}^3 + \frac{3}{2}(\partial\widehat{D} + \widehat{D}\partial).$$

Then \widehat{P} and \widehat{Q} commute and satisfy the relation $\widehat{Q}^2 = \widehat{P}^3 - \kappa$. Moreover, the algebra $\widehat{\mathfrak{B}} := \mathbb{C}[\widehat{P}, \widehat{Q}]$ has genus one and rank three. Let $q = (\lambda, \mu) \in \text{Spec}(\widehat{\mathfrak{B}})$ and $\widehat{\mathfrak{B}} \xrightarrow{\chi} \mathbb{C}$ be the corresponding character. A straightforward computation gives the following formula for $R_\chi := \text{gcd}(\widehat{P} - \lambda, \widehat{Q} - \mu)$:

$$(3.22) \quad R_\chi = \partial^3 - \frac{1}{z + \mu}\partial^2 + \frac{\lambda}{z + \mu}\partial + \left(\kappa + z^2 - \frac{\lambda^2}{z + \mu} \right).$$

Let \mathcal{F} be the spectral sheaf of $\widehat{\mathfrak{B}}$. The formula (3.22) yields the following result.

- (1) If $\kappa \neq 0$ then $\mathcal{F} \cong \mathcal{O}([q_1]) \oplus \mathcal{O}([q_2]) \oplus \mathcal{O}([q_3])$, where $q_i = (\lambda_i, 0)$ with $\lambda_i^3 = \kappa$ for $i = 1, 2, 3$. In particular, $\det(\mathcal{F}) \cong \mathcal{O}(3[p])$.
- (2) If $\kappa = 0$ then $\mathcal{F} \cong \mathcal{E}_p$, where \mathcal{E}_p is the indecomposable rank three vector bundle on the cuspidal curve from Corollary 2.15. Indeed, \mathcal{F} is locally free and its Fourier–Mukai transform \mathcal{T} is supported only at the singular point of the spectral curve due to Proposition 3.10. Therefore, $\mathcal{F} \cong \mathcal{E}_q$ for some $q \in X$. From Proposition 2.7 we deduce that $\det(\mathcal{F}) \cong \mathcal{O}(3[p])$, hence $q = p$.

3.5. Summary. Combining the classification of Grünbaum [21], [41, Theorem 1.2] of Previato and Wilson with results of our article, we get the following picture. Let $\mathfrak{B} \subset \mathfrak{D}$ be a genus one and rank two commutative subalgebra. Then we have:

$$\mathfrak{B} = \mathbb{C}[L, M] = \mathbb{C}[x, y]/(y^2 - h(x)), \quad \text{where} \quad h(x) = 4x^3 + g_2x + g_3$$

for appropriate parameters $g_2, g_3 \in \mathbb{C}$. The operator L has the form

$$L = \left(\partial^2 + \frac{1}{2}c_2 \right)^2 + (c_1\partial + \partial c_1) + c_0 \quad \text{and} \quad M = 2L_+^{\frac{3}{2}}.$$

Let \mathcal{F} be the spectral sheaf of \mathfrak{B} , \mathcal{T} its Fourier–Mukai transform and Z the support of \mathcal{T} . If the spectral curve $X = \overline{V(y^2 - h(x))}$ is singular then s denotes its singular point. Finally, $X \xrightarrow{\iota} X, (\lambda, \mu) \mapsto (\lambda, -\mu)$ is the canonical involution of X and $p = (0 : 1 : 0)$ is the infinite point of X . We use the notation of Corollary 2.10 to describe \mathcal{F} .

1. The spectral curve X is singular and $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{S}$ if and only if $c_1 = 0$ and c_0 is a constant. See Lemma 3.6.
2. Let L be formally self-adjoint (i.e. $c_1 = 0$) with $c'_0 \neq 0$. Then c_0 and c_2 are given by

$$c_0 = f \quad \text{and} \quad c_2 = \frac{K_2 + 2K_3f + f^3 - f'''f' + \frac{1}{2}(f'')^2}{f'^2}$$

for some $f \in \mathbb{C}[[z]]$ and $K_2, K_3 \in \mathbb{C}$. We have in this case: $g_2 = 2K_3$ and $g_3 = -\frac{1}{2}K_3$. The spectral sheaf \mathcal{F} is automatically locally free and self-dual. Moreover, $Z = \{q_+, q_-\} = \{(\lambda, \mu_+), (\lambda, \mu_-)\}$, where $\lambda = -\frac{1}{2}f(0)$ and $\mu_{\pm}^2 = h(\lambda)$. According to Theorem 3.7, the following results are true.

- (1) If $q_+ \neq q_-$ then $\mathcal{F} \cong \mathcal{O}([q_+]) \oplus \mathcal{O}([q_-])$.
- (2) If $q_+ = q_- = q$ is a smooth point of X then $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$ in the case f' has zero of order three at $z = 0$ and $\mathcal{F} \cong \mathcal{A} \otimes \mathcal{O}([q])$ otherwise.
- (3) If X is singular and $Z = \{s\}$ then $\mathcal{F} \cong \mathcal{B}_p$.

3. Assume now that $c_1 \neq 0$, i.e. L is not self-adjoint case. Then c_0, c_1 and c_2 are given by

$$\begin{cases} c_0 &= -f^2 + K_{11}f + K_{12} \\ c_1 &= f' \\ c_2 &= \frac{K_{14} - 2K_{10}f + 6K_{12}f^2 + 2K_{11}f^3 - f^4 + f'^2 - 2f'f''}{2f'^2} \end{cases}$$

where $f \in z\mathbb{C}[[z]]$ and $K_{10}, K_{11}, K_{12}, K_{14} \in \mathbb{C}$. The Weierstraß parameters g_2 and g_3 of the spectral curve X are given by the formulae

$$g_2 = K_{14} - 3K_{12}^2 - K_{10}K_{11} \quad \text{and} \quad g_3 = \frac{1}{4}(K_{10}^2 - 2K_{10}K_{11}K_{12} - 4K_{12}^3 - K_{14}(K_{11}^2 + 4K_{12})).$$

Consider the following expressions:

$$\begin{cases} a(\lambda) &= \left(\lambda + \frac{1}{2}K_{12}\right)^2 + \frac{1}{4}K_{14} \\ b(\lambda) &= \left(\lambda + \frac{1}{2}K_{12}\right)K_{11} - \frac{1}{2}K_{10} \end{cases}$$

Let λ_1, λ_2 be the roots of $a(\lambda)$. Then $Z = \{q_1, q_2\} = \{(\lambda_1, -b(\lambda_1)), (\lambda_2, -b(\lambda_2))\}$.

- (1) If $q_1 \neq q_2$ are smooth then $\mathcal{F} \cong \mathcal{O}([q_1]) \oplus \mathcal{O}([q_2])$.
- (2) If $q_1 = q_2 = q$ is smooth then $\mathcal{F} \cong \mathcal{O}([q]) \oplus \mathcal{O}([q])$ in the case f' has zero of order three at $z = 0$ and $\mathcal{F} \cong \mathcal{A} \otimes \mathcal{O}([q])$ otherwise, see Proposition 3.5.
- (3) The spectral curve X is singular and $Z = \{s\}$ if and only if $K_{10} = K_{14} = 0$, see Theorem 3.16. In this case,

$$X = V\left(y^2 - 4\left(x + \frac{K_{12}}{2}\right)^2(x - K_{12})\right).$$

- (a) The spectral sheaf \mathcal{F} is locally free if and only if $\Delta := 6K_{12} + K_{11}^2 \neq 0$. Moreover, $\mathcal{F} \cong \mathcal{B}_q$ with $q = \left(\frac{1}{4}K_{11}^2 + K_{12}, \frac{1}{4}K_{11}(6K_{12} + K_{11}^2)\right)$.
- (b) If $\Delta = 0$ then \mathcal{F} is indecomposable but not locally free. If X is cuspidal (i.e. $K_{11} = K_{12} = 0$) then $\mathcal{F} \cong \mathcal{U}$. If X is nodal (i.e. $K_{12} \neq 0$) then then \mathcal{F} is isomorphic to one of the sheaves \mathcal{U}_{\pm} . More precisely, it is the inverse Fourier–Mukai transform of

$$\mathcal{T} := \widehat{R}/\left(\left(x + \frac{K_{12}}{2}\right)^2, y - K_{11}\left(x + \frac{K_{12}}{2}\right)\right),$$

where $\widehat{R} := \mathbb{C}[[x, y]]/\left(y^2 - 4\left(x + \frac{K_{12}}{2}\right)^2(x - K_{12})\right) \cong \widehat{\mathcal{O}}_s$.

- (4) The spectral curve X is singular and the spectral sheaf \mathcal{F} is decomposable and not locally free if and only if $K_{10} = (3K_{12} + \frac{1}{2}K_{11}^2)K_{11}$ and $K_{14} = -(3K_{12} + \frac{1}{2}K_{11}^2)^2 \neq 0$. In this case, $Z = \{s, q\}$ and $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{O}([q])$, where $q = (-2K_{12} - \frac{1}{4}K_{11}^2, -\frac{1}{2}K_{11}(K_{11}^2 + 6K_{12}))$, see Theorem 3.11.

Remark 3.19. We see from this description that L is non-degenerate formally self-adjoint (i.e. $c_1 = 0$ and $c'_0 \neq 0$) if and only if \mathcal{F} is locally free and $\det(\mathcal{F}) \cong \mathcal{O}(2[p])$. For such L we have: $\mathbb{D}(\mathcal{F}) \cong \mathcal{F}$, where \mathbb{D} is the duality from Theorem 2.3. The converse is however not true. Consider a non-self-adjoint operator L with $K_{10} = K_{11} = 0$ and $K_{14} \neq 0$. Then the spectral curve X is smooth (for generic K_{14}) and $\mathcal{F} \cong \mathcal{O}([q_1]) \oplus \mathcal{O}([q_2])$, where $\iota(q_i) = q_i$ for $i = 1, 2$. Therefore, $\mathbb{D}(\mathcal{F}) \cong \mathcal{F}$ in this case.

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UNIVERSITÄT ZU KÖLN, MATHEMATISCHES INSTITUT, WEYERTAL 86-90, D-50931 KÖLN, GERMANY
E-mail address: `burban@math.uni-koeln.de`

MOSCOW STATE UNIVERSITY, FACULTY OF MECHANICS AND MATHEMATICS, LENINSKIE GORY, GSP-1,
MOSCOW, 119899, RUSSIAN FEDERATION
E-mail address: `azheglov@math.msu.su`