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ANALYTIC STRUCTURE IN FIBERS

RICHARD M. ARON, JAVIER FALCÓ, DOMINGO GARCÍA, AND MANUEL MAESTRE

Abstract. Let B_X be the open unit ball of a complex Banach space X, and let $\mathcal{H}^\infty(B_X)$ and $\mathcal{A}_u(B_X)$ be, respectively, the algebra of bounded holomorphic functions on B_X and the subalgebra of uniformly continuous holomorphic functions on B_X. In this paper we study the analytic structure of fibers in the spectrum of these two algebras. For the case of $\mathcal{H}^\infty(B_X)$, we prove that the fiber in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ over any point of the distinguished boundary of the closed unit ball $\overline{B}_{\ell_\infty}$ of ℓ_∞ contains an analytic copy of B_{ℓ_∞}. In the case of $\mathcal{A}_u(B_X)$ we prove that if there exists a polynomial whose restriction to the open unit ball of X is not weakly continuous at some point, then the fiber over every point of the open unit ball of the bidual contains an analytic copy of $\mathbb{D}$.

1. Introduction

This paper continues the study of the analytic structure of fibers in the spectrum of algebras of holomorphic functions. In the next paragraph, we will give a very brief description of the focus of this manuscript. Subsequently, we will review all the relevant terms that will be used.

We concentrate here on the two most important algebras of holomorphic functions, $\mathcal{H}^\infty(B_X)$ (bounded holomorphic functions $f : B_X \to \mathbb{C}$ on the open unit ball of the Banach space X) and the subalgebra $\mathcal{A}_u(B_X)$ (uniformly continuous holomorphic functions). When endowed with the supremum norm, each is a commutative Banach algebra with identity. Denoting either of these algebras by $\mathcal{A}$, we let $\mathcal{M}(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathbb{C} \mid \varphi \text{ is a (non-zero) homomorphism }\}$. There is a natural surjective mapping $\pi : \mathcal{M}(\mathcal{A}) \to \overline{B}_{X^{**}}$. Our interest will be in the *fibers* in $\mathcal{M}(\mathcal{A})$, i.e. $\pi^{-1}(z)$, for $z \in \overline{B}_{X^{**}}$, which we denote by $\mathcal{M}_z(\mathcal{A})$. Depending

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on the geometric structure of X and on whether $\mathcal{A} = \mathcal{H}^\infty(B_X)$ or $\mathcal{A}_u(B_X)$, our fibers will be very large, at times, and very small at other times.

We now expand on the material that has been outlined above. First, there is a natural inclusion, $\delta : b \in B_X \rightsquigarrow \delta_b \in \mathcal{M}(\mathcal{H}^\infty(B_X))$, where $\delta_b(f) \equiv f(b)$ for $f \in \mathcal{H}^\infty(B_X)$. It is evident that this inclusion extends to $\overline{B}_X$ in the case of the subalgebra $\mathcal{A}_u(B_X)$. There is a canonical extension mapping $f \in \mathcal{H}^\infty(B_X) \rightsquigarrow \tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$ (see, e.g., [2] and [12]), which is (i) norm-preserving, (ii) multiplicative, and (iii) takes $\mathcal{A}_u(B_X)$ to $\mathcal{A}_u(B_{X^{**}})$. Because of this, the mapping δ extends to $\tilde{\delta} : B_{X^{**}} \to \mathcal{M}(\mathcal{H}^\infty(B_X))$ and also from $\overline{B}_{X^{**}} \to \mathcal{M}(\mathcal{A}_u(B_X))$, $\tilde{\delta}(z)(f) \equiv \tilde{\delta}_z(f) = \tilde{f}(z)$. Next, both $\mathcal{M}(\mathcal{H}^\infty(B_X))$ and $\mathcal{M}(\mathcal{A}_u(B_X))$ are compact Hausdorff spaces when endowed with the weak-star (Gelfand) topology. Calling $\mathcal{A}$ either of these Banach algebras, and noting that X^* is a subspace of $\mathcal{A}$, one defines the mapping $\pi : \mathcal{M}(\mathcal{A}) \to \overline{B}_{X^{**}}$ by $\pi(\varphi) := \varphi|_{X^*}$. (In the classical case $X = \mathbb{C}$ with $\mathcal{A} = \mathcal{H}^\infty(\mathbb{D})$ or $\mathcal{A}(\mathbb{D})$, the mapping π reduces to the usual function $\pi(\varphi) = \varphi(z \rightsquigarrow z)$). It is routine that π is continuous when $\overline{B}_{X^{**}}$ has the weak-star topology, and that $\pi \circ \delta = id_{B_X}$. Hence, by Goldstine's theorem, the range of $\pi(\mathcal{M}(\mathcal{A}))$ is all of $\overline{B}_{X^{**}}$.

Our central interest is in the contents and structure of fibers $\pi^{-1}(z)$, over various points $z \in \overline{B}_{X^{**}}$. We concentrate on $\mathcal{A} = \mathcal{H}^\infty(B_X)$ in Section 2, where we continue work of Schark [19] and B. Cole, T. W. Gamelin, and W. B. Johnson [10] (see also [11]). We prove that in any Banach space X the fiber over any point x_0 in the unit sphere S_X of X contains a copy of the unit disk, and also that for $X = c_0$, the fiber in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ over any point in the unit sphere of c_0^{**} contains a copy of the entire maximal ideal space $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Section 3 is devoted to the generalized disk algebra, $\mathcal{A}_u(B_X)$. Our main goal will be to prove that for a fixed complex Banach space X, if there exist a polynomial $P : B_X \to \mathbb{C}$ and a point $x_0 \in B_X$ such that $P|_{B_X}$ is not weakly continuous at x_0, then the fiber over every $z \in B_{X^{**}}$ contains a complex disk. In relation to a result of J. Farmer [15], it is shown that in the case of $\mathcal{A}_u(B_{\ell_p})$, the entire closed ball $\overline{B}_{\ell_p}$ can be continuous injected into the fiber over any point of $B_{\ell_p}, 1 < p < \infty$.

We emphasize that the apparent dichotomy in our results, whereby the (mere) complex disk is contained in the fiber in one instance and the (entire) ball of a Banach space is contained in the fiber in another, is-at least in part-the "fault" of the theory itself. For instance, recall that in the case of $\mathcal{M}(\mathcal{A}_u(B_{\ell_p}))$, $1 \leq p < \infty$, the fiber over a boundary point z_0 is precisely δ_{z_0}. See [3, Corollary 2.5 and Theorem 4.1] and [4, Proposition 3.1].

2. Disks in fibers of $\mathcal{H}^\infty(B_X)$

Before stating the main result of this section, we state and prove the following straightforward result. Recall that it is classical that the disk can be analytically injected in the fiber over the point $z_0 = 1$ of the maximal ideal of $\mathcal{H}^\infty(\mathbb{D})$ (see, e.g, [19]). We now show that this result holds for the fiber over any point in the unit sphere of a complex Banach space.

Proposition 2.1. *Let X be a Banach space and $x_0 \in S_X$. Then the complex disk $\mathbb{D}$ can be analytically injected into $\mathcal{M}_{x_0}(\mathcal{H}^\infty(B_X))$.*

Proof. We first show that we can embed $\mathcal{M}_1(\mathcal{H}^\infty(\mathbb{D}))$ into $\mathcal{M}_{x_0}(\mathcal{H}^\infty(B_X))$. Take $i : \mathbb{C} \longrightarrow X$, $i(\lambda) = \lambda x_0$, $\lambda \in \mathbb{C}$, and define $\Phi : \mathcal{H}^\infty(B_X) \longrightarrow \mathcal{H}^\infty(\mathbb{D})$ by $\Phi(f)(\lambda) = f \circ i(\lambda)$, for every $f \in \mathcal{H}^\infty(B_X)$ and every $\lambda \in \mathbb{D}$.

Obviously, Φ is linear, multiplicative and continuous and hence its transpose

$$\Phi^* : (\mathcal{H}^\infty(\mathbb{D}))^* \longrightarrow (\mathcal{H}^\infty(B_X))^*$$

is linear and w^*-w^*-continuous. Moreover its restriction to the corresponding spectra, which we also denote by Φ^*,

$$\Phi^* : \mathcal{M}(\mathcal{H}^\infty(\mathbb{D})) \longrightarrow \mathcal{M}(\mathcal{H}^\infty(B_X)),$$

is w^*-w^*-continuous.

We claim that Φ^* is injective. Indeed, if $\varphi \in (\mathcal{H}^\infty(\mathbb{D}))^* \backslash \{0\}$ then there exists $g \in \mathcal{H}^\infty(\mathbb{D})$ such that $\varphi(g) \neq 0$. Choose $x_0^* \in X^*$ such that $x_0^*(x_0) = \|x_0^*\| = 1$. We have $g \circ x_0^* \in \mathcal{H}^\infty(B_X)$ and

$$\Phi^*(\varphi)(g \circ x_0^*) = \varphi(\Phi(g \circ x_0^*)) = \varphi(\lambda \rightsquigarrow g \circ x_0^*(\lambda x_0)) = \varphi(\lambda \rightsquigarrow g(\lambda)) = \varphi(g) \neq 0 \ .$$

On the other hand, an easy computation shows that for all $a \in \overline{\mathbb{D}}$ the inclusion

$$\Phi^*(\mathcal{M}_a(\mathcal{H}^\infty(\mathbb{D}))) \subset \mathcal{M}_{ax_0}(\mathcal{H}^\infty(B_X))$$

holds.

Since

$$\Phi^* : \mathcal{M}_a(\mathcal{H}^\infty(\mathbb{D})) \longrightarrow \mathcal{M}_{ax_0}(\mathcal{H}^\infty(B_X))$$

is injective and continuous and both sets are compact, $\Phi^*(\mathcal{M}_a(\mathcal{H}^\infty(\mathbb{D}))$ is homeomorphic to $\mathcal{M}_a(\mathcal{H}^\infty(\mathbb{D}))$. In particular for $a = 1$ we have that

$$\Phi^* : \mathcal{M}_1(\mathcal{H}^\infty(\mathbb{D})) \longrightarrow \mathcal{M}_{x_0}(\mathcal{H}^\infty(B_X))$$

is a homeomorphism onto its image.

Now, by [19] there exists an injective and analytic map $F : \mathbb{D} \longrightarrow \mathcal{M}_1(\mathcal{H}^\infty(\mathbb{D}))$, and therefore

$$\Phi^* \circ F : \mathbb{D} \longrightarrow \mathcal{M}_{x_0}(\mathcal{H}^\infty(B_X))$$

is also injective and analytic. The proof is complete. □

Note that the above argument also works for points $x_0 \in S_{X^{**}}$ provided that there is an element $x_0^* \in S_{X^*}$ at which x_0 attains its norm. We do not know if Proposition 2.1 holds for all points in $S_{X^{**}}$.

Reference [10] contains many deep results about when it is possible to inject huge sets into $\mathcal{M}_0(\mathcal{H}^\infty(B_X))$, for an infinite dimensional Banach space X. In particular it is proved in [10, 6.7 Theorem] that there exists a holomorphic injective function $\phi : B_{\ell_\infty} \times B_{\ell_\infty} \to \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ such that $\Phi(z, w) \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ for all $z, w \in B_{\ell_\infty}$. Our aim here is to study the size of fibers over points of the sphere of $c_0^{**} = \ell_\infty$. We begin by examining what happens for the fibers $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ when z belongs to the distinguished boundary of the unit ball of ℓ_∞. This will be our main result in this section.

Theorem 2.2. *Let $z_0 = (z_1, z_2, ..., z_n, ...)$ be a point of the distinguished boundary $\mathbb{T}^{\aleph_0}$ of $\overline{B}_{\ell_\infty}$ (i.e. $|z_j| = 1$ for all $j \in \mathbb{N}$). Then there exists an injection $\Psi : B_{\ell_\infty} \to \mathcal{M}_{z_0}(\mathcal{H}^\infty(B_{c_0}))$ which is biholomorphic onto its image.*

In fact, to somewhat simplify our notation, we will prove this only for the case $z_0 = (1, 1, 1, \ldots)$. The general case, for any $z \in \mathbb{T}^{\aleph_0}$, follows by applying Möbius transformations coordinate-wise.

The proof of Theorem 2.2 will follow directly from the following results. Our argument is modelled on that of Schark [19]

Proposition 2.3. *Fix any point* $\lambda = (\lambda_1, \lambda_2, ...) \in \overline{B}_{\ell_\infty}$. *Define* $L(\lambda)$ *by*

$$L(\lambda_1, ..., \lambda_n, ...) = (\frac{\lambda_1 + i(\lambda_1 - 1)}{1 + i(\lambda_1 - 1)}, ..., \frac{\lambda_n + i(\lambda_n - 1)}{1 + i(\lambda_n - 1)}, ...).$$

Then $L(\lambda)$ *is a bijective holomorphic map on* B_{ℓ_∞} *which, when considered as acting from* $\overline{B}_{\ell_\infty}$ *onto* $\overline{B}_{\ell_\infty}$, *has* $\lambda = (1, 1, 1, \dots)$ *as its only fixed point.*

Proof. The function $\lambda_n \mapsto \frac{\lambda_n + i(\lambda_n - 1)}{1 + i(\lambda_n - 1)} = \frac{(1+i)\lambda_n - i}{i\lambda_n + (1-i)}$ is a Möbius transformation from $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$ with inverse function $\beta_n \mapsto \frac{(1-i)\beta_n + i}{-i\beta_n + (1+i)} = \frac{\beta_n + i(1-\beta_n)}{1 + i(1-\beta_n)}$. Hence $L(\lambda)$ is a bijective holomorphic map from $\overline{B}_{\ell_\infty}$ onto $\overline{B}_{\ell_\infty}$. Since the only fixed point of the function $\lambda_n \mapsto \frac{\lambda_n + i(\lambda_n - 1)}{1 + i(\lambda_n - 1)}$ is the point 1, the only fixed point of the map L is the point $(1, 1, 1, \dots)$.

By a result of Dunford, in order to prove that L is holomorphic it suffices to show that L is $w(\ell_\infty, \ell_1)-$holomorphic (see [16, Theorem 3.10.1, p. 93]). But given $(a_n) \in \ell_1$

$$((a_n) \circ L)(\lambda) = \sum_{n=1}^{\infty} a_n \frac{\lambda_n + i(\lambda_n - 1)}{1 + i(\lambda_n - 1)}$$

is the uniform limit of holomorphic functions, and hence it is holomorphic. $\square$

Remark 2.4. It is routine to check that the j^{th} composition of L with itself, $L^{(j)}$, has the form

$$L^{(j)}(\lambda_1, ..., \lambda_n, ...) = (\frac{\lambda_1 + ji(\lambda_1 - 1)}{1 + ji(\lambda_1 - 1)}, ..., \frac{\lambda_n + ji(\lambda_n - 1)}{1 + ji(\lambda_n - 1)}, ...)$$

where $j \in \mathbb{Z}$ is allowed. Therefore $L^{(j)}$ is a bijective holomorphic map from $\overline{B}_{\ell_\infty}$ onto $\overline{B}_{\ell_\infty}$ for all $j \in \mathbb{Z}$. Also for $j \neq 0$ the unique fixed point of $L^{(j)}$ is $(1, 1, 1, 1, \dots)$.

For every integer j and $\lambda \in B_{\ell_\infty}$, we have that $\widetilde{\delta_{L^{(j)}(\lambda)}} \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ and that the n^{th} coordinate of $L^{(j)}(\lambda)$ is $\widetilde{\delta_{L^{(j)}(\lambda)}}(p_n) = \widetilde{p_n}(L^{(j)}(\lambda)) = (L^{(j)}(\lambda))(n)$, where $\{p_n\}_{n=1}^\infty$ stands for the projection mappings on c_0. So

$$(\widetilde{\delta_{L^{(j)}(\lambda)}}(p_1), \widetilde{\delta_{L^{(j)}(\lambda)}}(p_2), \dots) = ((L^{(j)}(\lambda))(1), (L^{(j)}(\lambda))(2), \dots) = L^{(j)}(\lambda),$$

and hence the homomorphism $\widetilde{\delta_{L^{(j)}(\lambda)}}$ lies in the fiber of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ over $L^{(j)}(\lambda)$.

Proposition 2.5. *The mappings*

$$\begin{array}{rccc} \Psi_j : & B_{\ell_\infty} & \longrightarrow & \mathcal{M}(\mathcal{H}^\infty(B_{c_0})) \\ & \lambda & \rightsquigarrow & \widetilde{\delta_{L^{(2^j)}(\lambda)}} \end{array}$$

$j = 1, 2, \ldots$ *are holomorphic of norm one, when considering* $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ *as a subset of* $\left(\mathcal{H}^\infty(B_{c_0})\right)^*$.

Proof. It is enough to check that for each $f \in \mathcal{H}^\infty(B_{c_0})$ the function $\Psi_j(\cdot)(f) \in \mathcal{H}^\infty(B_{\ell_\infty})$. For this, notice that for each $f \in \mathcal{H}^\infty(B_{c_0}))$ the function

$$\begin{array}{rccc} \tilde{f}(L^{(2^j)}): & B_{\ell_\infty} & \longrightarrow & \mathbb{C} \\ & \lambda & \rightsquigarrow & \tilde{f}(L^{(2^j)}(\lambda)) \end{array}$$

is the composition of two holomorphic functions, since $\tilde{f} \in \mathcal{H}^\infty(B_{\ell_\infty})$ and by Proposition 2.3 $L^{(2^j)}$, is holomorphic. Therefore $\tilde{f}(L^{(2^j)})$ is holomorphic.

Also Ψ_j has norm one since for every $\lambda \in B_{\ell_\infty}$, $\widetilde{\delta_{L^{(2^j)}(\lambda)}} \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Hence $\|\widetilde{\delta_{L^{(2^j)}(\lambda)}}\| = 1$.

□

An application of the Ascoli theorem yields that the set of holomorphic mappings from B_{ℓ_∞} to $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ is compact with the compact open topology. In particular the sequence $(\Psi_j)_{j\in\mathbb{N}}$ has at least one accumulation point $\Psi : B_{\ell_\infty} \to \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ that is also holomorphic.

Proposition 2.6. *Any accumulation point* Ψ *of the sequence* $(\Psi_j)_{j\in\mathbb{N}}$ *is such that* $\Psi(\lambda)$ *is in the fiber over the point* $\vec{1} = (1, 1, ..., 1, ...) \in \ell_\infty$ *for every* $\lambda \in B_{\ell_\infty}$. *In particular,* $\Psi \neq \Psi_j$ *for* $j \in \mathbb{N}$.

Proof. We want to check that $\Psi(\lambda) \in \pi^{-1}(\vec{1})$, i. e. $\Psi(\lambda)(p_n) = 1$ for all projection mappings p_n. We will check that for every point $y = (y_1, y_2, ..., y_n, ...) \in \ell_1$, $\Psi(\lambda)(y) = \sum_n y_n = \vec{1}(y)$. Therefore, $\pi(\Psi(\lambda)) = \vec{1}$.

Fix $y = (y_1, y_2, ..., y_n, ...) \in \ell_1$. Since Ψ is an accumulation point of (Ψ_j), there exists a subnet (Ψ_α) convergent to Ψ. Since, for each fixed λ, the sequence $\left\{\frac{\lambda_n + 2^j i(\lambda_n - 1)}{1 + 2^j i(\lambda_n - 1)}\right\}_{j=1}^\infty$ converges to 1, then the subnet $\left\{\frac{\lambda_n + 2^\alpha i(\lambda_n - 1)}{1 + 2^\alpha i(\lambda_n - 1)}\right\}_\alpha$ also converges to 1. Hence, since $y \in \ell_1$, we have

$$\Psi(\lambda)(y) = \lim_\alpha \Psi_\alpha(\lambda)(y) = \lim_\alpha \sum_{n=1}^\infty y_n \frac{\lambda_n + 2^\alpha i(\lambda_n - 1)}{1 + 2^\alpha i(\lambda_n - 1)} = \sum_{n=1}^\infty y_n.$$

Indeed, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{n\geq n_0} |y_n| < \frac{\varepsilon}{2}$. Now the convergence of $\left\{\frac{\lambda_n + 2^\alpha i(\lambda_n - 1)}{1 + 2^\alpha i(\lambda_n - 1)}\right\}_\alpha$ to 1 for $n = 1, \ldots, n_0$ gives the existence of α_0 such that

$$\sum_{n=1}^\infty |y_n||1 - \frac{\lambda_n + 2^\alpha i(\lambda_n - 1)}{1 + 2^\alpha i(\lambda_n - 1)}| < \varepsilon,$$

for every $\alpha \geq \alpha_0$.

Thus $\Psi(\lambda)(y) = \sum_{n=1}^{\infty} y_n$. Therefore $\Psi(\lambda) \in \pi^{-1}(\bar{1})$.

To finish let us show that $\Psi \neq \Psi_j$ for all $j \in \mathbb{N}$. Observe that $|\Psi_j(\lambda)(p_n)| = |\widetilde{\delta_{L^{(2^j)}(\lambda)}}(p_n)| = |\big(L^{(2^j)}(\lambda)\big)(n)| < 1$ for all $n, j \in \mathbb{N}$ so we have that $\Psi_j(\lambda) \notin \pi^{-1}(\vec{1})$ and the result follows. $\square$

Proposition 2.7. *Given an accumulation point Ψ of the sequence $(\Psi_j)_{j\in\mathbb{N}}$, Ψ takes B_{ℓ_∞} homeomorphically into $\mathcal{M}_{\vec{1}}(\mathcal{H}^\infty(B_{c_0})) = \pi^{-1}(\vec{1})$.*

Proof. Define $h : B_{\ell_\infty} \to B_{\ell_\infty}$ as follows:

$$\begin{aligned}
&h(\lambda_1, \lambda_2, ..., \lambda_n, ...) = \\
&= (\lambda_1 \Pi_{k=0}^{\infty} \frac{\lambda_1 - 2^k i(\lambda_1 - 1)}{1 - 2^k i(\lambda_1 - 1)}, ..., \lambda_n \Pi_{k=0}^{\infty} \frac{\lambda_n - 2^k i(\lambda_n - 1)}{1 - 2^k i(\lambda_n - 1)}, ...) \\
&= (\lambda_1 \Pi_{k=0}^{\infty} L^{(-2^k)}(\lambda)(1), ..., \lambda_n \Pi_{k=0}^{\infty} L^{(-2^k)}(\lambda)(n), ...).
\end{aligned}$$

Observe that $h : B_{c_0} \to B_{c_0}$.

Notice that for every r, $0 < r < 1$, and every natural number n, the product $\Pi_{k=0}^{\infty} \frac{\lambda_n - 2^k i(\lambda_n - 1)}{1 - 2^k i(\lambda_n - 1)}$ converges uniformly on the complex disk $r\overline{\mathbb{D}}$, since

$$\big|L^{(-2^k)}(\lambda)(n) - 1\big| \leq K_r \frac{1}{2^k}$$

for some constant K_r. Therefore the product converges and since $|\frac{\lambda_n - 2^k i(\lambda_n - 1)}{1 - 2^k i(\lambda_n - 1)}| \leq 1$ we have

$$\big|\lambda_n \Pi_{k=0}^{\infty} \frac{\lambda_n - 2^k i(\lambda_n - 1)}{1 - 2^k i(\lambda_n - 1)}\big| \leq |\lambda_n|.$$

Hence h is well defined.

Now, for every coordinate and for every complex disk of radius $0 < r < 1$ we have

$$\begin{aligned}
|h(L^{(2^m)}(\lambda))(n) - \lambda_n| &= |L^{(2^m)}(\lambda)(n) \prod_{k=0}^{\infty} L^{(-2^k)}(L^{(2^m)}(\lambda))(n) - \lambda_n| \\
&= |L^{(2^m)}(\lambda)(n) \prod_{k=0}^{\infty} L^{(2^m-2^k)}(\lambda)(n) - \lambda_n| \\
&= |\lambda_n| \Big| L^{(2^m)}(\lambda)(n) \prod_{k=0}^{m-1} L^{(2^m-2^k)}(\lambda)(n) \prod_{k=m+1}^{\infty} L^{(2^m-2^k)}(\lambda)(n) - 1 \Big| \\
&\leq |\lambda_n| \Big[|L^{(2^m)}(\lambda)(n) - 1| + \sum_{\substack{k=0 \\ k\neq m}}^{\infty} |L^{(2^m-2^k)}(\lambda)(n) - 1| \Big] \\
&\leq |\lambda_n| K_r \Big[\frac{1}{2^m} + \sum_{k=0}^{m-1} \frac{1}{2^m - 2^k} + \sum_{k=m+1}^{\infty} \frac{1}{2^k - 2^m} \Big] \\
&\leq |\lambda_n| K_r \frac{m+2}{2^{m-1}},
\end{aligned}$$

and this converges to zero as m goes to infinity.

Let

$$\begin{array}{rccl} \hat{h}: & \mathcal{M}(\mathcal{H}^\infty(B_{c_0})) & \longrightarrow & B_{\ell_\infty} \\ & \phi & \rightsquigarrow & \big(\phi(p_1 \circ h), \phi(p_2 \circ h), \phi(p_3 \circ h), \dots\big). \end{array}$$

Since h is continuous, $\hat{h}$ is also continuous. Therefore, as Ψ is a cluster point of the sequence $(\Psi_j)_{j\in\mathbb{N}}$,

$$\begin{aligned}
(\hat{h} \circ \Psi)(\lambda) &= \hat{h}(\Psi(\lambda)) \\
&= \lim_{\alpha} \hat{h}(\Psi_\alpha(\lambda)) \\
&= \lim_{\alpha} \big(h(L^{(2^\alpha)}(\lambda))(1), h(L^{(2^\alpha)}(\lambda))(2), \dots\big) \\
&= (\lambda_1, \lambda_2, \dots) = \lambda.
\end{aligned}$$

Then $\hat{h} \circ \Psi = Id$ on B_{ℓ_∞}.

Hence, Ψ is bijective onto $\Psi(B_{\ell_\infty})$ with inverse being the restriction of $\hat{h}$ to the range of Ψ. Since $\hat{h}$ is continuous, the restriction of $\hat{h}$ to $\Psi(B_{\ell_\infty})$ is also continuous. Therefore, Ψ is a continuous bijective map from B_{ℓ_∞} onto $\Psi(B_{\ell_\infty})$ with continuous inverse $\hat{h}|_{\Psi(B_{\ell_\infty})}$. Thus, Ψ is a homeomorphism. □

Corollary 2.8. *Let $z_0 = (z_1, z_2, ..., z_n, ...)$ be a point of the boundary of B_{ℓ_∞} such that there exists n_0 with $|z_n| = 1$ for all $n > n_0$. Then there exists an injection $\Psi : B_{\ell_\infty} \to \mathcal{M}_{z_0}(\mathcal{H}^\infty(B_{c_0}))$ which is homeomorphic onto its image.*

For the proof of this corollary, we can assume, without loss of generality, that $\max\{|z_1|, \ldots, |z_{n_0}|\} < 1$. Now the lemma below and Theorem 2.2 give the conclusion.

Lemma 2.9. *Given $(b_1, w_0) := (b_1, z_2, \ldots, z_n, \ldots) \in \overline{B}_{\ell_\infty}$ with $|b_1| < 1$, we have that $\mathcal{M}_{(b_1,w_0)}(\mathcal{H}^\infty(B_{c_0}))$ is homeomorphic to $\mathcal{M}_{w_0}(\mathcal{H}^\infty(B_{c_0}))$.*

Proof. Define $R : \mathcal{M}_{w_0}(H^\infty(B_{c_0})) \to \mathcal{M}_{(b_1,w_0)}(H^\infty(B_{c_0}))$ by

$$R(\varphi)(f) = \varphi(w \rightsquigarrow f(b_1, w)).$$

We claim that R has the following properties: R is $w^* - w^*$ continuous; R is one-to-one; and R is onto. Once we have shown these three properties of R, we will be able to conclude that the fiber over w_0 in $\mathcal{M}(H^\infty(B_{c_0}))$ is topologically the same as the fiber of (b_1, w_0) in $\mathcal{M}(H^\infty(B_{c_0}))$.

For the first, suppose that $\varphi_\alpha \to \varphi$ weak-star in $\mathcal{M}(H^\infty(B_{c_0}))$. So, for all $h \in H^\infty(B_{c_0}), \varphi_\alpha(h) \to \varphi(h)$. Fixing $f \in H^\infty(B_{c_0})$, we see that $R(\varphi_\alpha)(f) = \varphi_\alpha(w \rightsquigarrow f(b_1, w))$ converges to $\varphi(w \rightsquigarrow f(b_1, w)) = R(\varphi)(f)$. Thus, R is $w^* - w^*$ continuous.

Next, we show that R is one-to-one. Suppose that $\varphi_1 \neq \varphi_2$ in $\mathcal{M}_{w_0}(H^\infty(B_{c_0}))$. So, for some $h \in H^\infty(B_{c_0}), \varphi_1(h) \neq \varphi_2(h)$. Define $f \in H^\infty(B_{c_0})$ by $f(z_1, w) = h(w)$, for all $(z_1, w) \in B_{c_0}$. Then

$$R(\varphi_1)(f) = \varphi_1(w \rightsquigarrow f(b_1, w)) = \varphi_1(h)$$

$$\neq \varphi_2(h) = \varphi_2(w \rightsquigarrow f(b_1, w)) = R(\varphi_2)(f).$$

Finally, let's show that R is onto. So, fix $\psi \in \mathcal{M}_{(b_1,w_0)}(H^\infty(B_{c_0}))$. Let $\varphi : H^\infty(B_{c_0}) \to \mathbb{C}$ be given by $\varphi(h) = \psi((z_1, w) \rightsquigarrow h(w))$. We claim that $R(\varphi) = \psi$. First, $\pi(\varphi) = (x_n)_{n=1}^\infty$ where $x_n = \varphi(e_n)$ and e_n is the n-th element of the canonical Schauder basis of $c_0^* = \ell_1$. But, by definition of φ

$$(x_n)_{n=1}^\infty = (\psi((z_1, w) \rightsquigarrow w_n))_{n=1}^\infty = w_0.$$

This follows since $\psi \in \mathcal{M}_{(b_1,w_0)}(H^\infty(B_{c_0}))$ implies that $\pi(\psi) = ((\psi((z_1, w) \rightsquigarrow z_1, (\psi((z_1, w) \rightsquigarrow w_n))_{n=1}^\infty) = (b_1, w_0)$. It then follows that

$$R(\varphi)(f) = \varphi(h : w \rightsquigarrow f(b_1, w)) = \psi((z_1, w) \rightsquigarrow h(w)) = \psi((z_1, w) \rightsquigarrow f(b_1, w)).$$

Next, for a fixed $f \in H^\infty(B_{c_0})$, define $g : B_{c_0} \to \mathbb{C}$ by

$$g(z_1, w) = \begin{cases} \frac{f(z_1,w)-f(b_1,w)}{z_1-b_1} & \text{if } z_1 \neq b_1 \\ \frac{\partial f}{\partial z_1}(b_1, w) & \text{if } z_1 = b_1. \end{cases}$$

The function g is locally bounded and separately holomorphic on the open unit ball of the Banach space c_0 and hence it is holomorphic. Moreover, g is bounded on B_{c_0}. To see this, consider z_1 in $\mathbb{D}$ so that $|z_1 - b_1| \geq \frac{|1-b_1|}{2}$. Then $|g(z_1, w)| \leq \frac{2}{|1-b_1|}(|f(z_1, w)| + |f(b_1, w)|) \leq \frac{4}{|1-b_1|}\|f\|_{B_{c_0}}$. If $0 < |z_1 - b_1| \leq \frac{|1-b_1|}{2}$, since for each $w \in B_{c_0}$ the function $z_1 \to g(z_1, w)$ is holomorphic on $\mathbb{D}$, the Maximum Modulus Theorem implies that g attains its maximum at a point of the circle centered at b_1 and of radius $\frac{|1-b_1|}{2}$. Thus, $\|g\|_{B_{c_0}} \leq \frac{4}{|1-b_1|}\|f\|_{B_{c_0}}$, and we have that

$$f(z_1, w) = f(b_1, w) + (z_1 - b_1)g(z_1, w),$$

with $g \in H^\infty(B_{c_0})$. Using the fact that $\psi(g)$ is well-defined, we see that

$$\psi(f) = \psi((z_1, w) \rightsquigarrow f(b_1, w) + (z_1 - b_1)g(z_1, w)) =$$

$$= \psi((z_1, w) \rightsquigarrow f(b_1, w)) + \psi((z_1, w) \rightsquigarrow (z_1 - b_1)g(z_1, w)) = \psi((z_1, w)) \rightsquigarrow f(b_1, w)),$$

since $\psi((z_1, w) \rightsquigarrow z_1 - b_1) = b_1 - b_1 = 0$. □

Remark 2.10. An argument analogous to that in the proof of Theorem 2.2 implies that there exists an injection $\Psi : B_{\ell_\infty} \to \mathcal{M}_{(1,0,1,0,\ldots,1,0,\ldots)}(\mathcal{H}^\infty(B_{c_0}))$ which is biholomorphic onto its image. However, we do not know if the fibers of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ over the points $(1, 0, 1, 0, \ldots, 1, 0, \ldots)$ and $(1, 1, \ldots, 1, \ldots)$ are homeomorphic or not. Also we do not know if B_{ℓ_∞} can be embedded in the fiber $\mathcal{M}_{(z_n)}(\mathcal{H}^\infty(B_{c_0}))$, when (z_n) is in the unit sphere of ℓ_∞ but $|z_n| < 1$ for all n, as for example $(\frac{n-1}{n})$.

Remark 2.11. Let us observe that the arguments of the proofs of Corollary 2.8 and Lemma 2.9, can easily be adapted to produce counterpart results for fibers of the maximal ideal of $\mathcal{H}^\infty(\mathbb{D}^2)$, the space of bounded holomorphic function on the bidisk. This finite dimensional approach is part of a work in progress.

Proposition 2.12. *For all $f \in \mathcal{H}^\infty(B_{c_0})$ there exists $g \in \mathcal{H}^\infty(B_{c_0})$ such that $\Psi(\lambda)(g) = f(\lambda)$ for all $\lambda \in B_{c_0}$.*

Proof. Let $g = f \circ h \in \mathcal{H}^\infty(B_{c_0})$, where $h : B_{c_0} \to B_{c_0}$ is the function introduced in Proposition 2.7. Consider

$$\begin{array}{rccc} \hat{g}: & \mathcal{M}(\mathcal{H}^\infty(B_{c_0})) & \longrightarrow & \mathbb{C} \\ & \phi & \rightsquigarrow & \phi(g). \end{array}$$

Therefore, for all $\lambda \in B_{c_0}$,

$$\begin{aligned} \Psi(\lambda)(g) &\equiv \hat{g}(\Psi(\lambda)) = \hat{g}(\lim_\alpha \Psi_\alpha(\lambda)) \\ &= \lim_\alpha \hat{g}(\Psi_\alpha(\lambda)) = \lim_\alpha (\Psi_\alpha(\lambda))(g) = \lim_\alpha \widetilde{\delta_{L^{(2^\alpha)}(\lambda)}}(g) \\ &= \lim_\alpha \widetilde{\delta_{L^{(2^\alpha)}(\lambda)}}(f \circ h) = \lim_\alpha \widetilde{f \circ h}(L^{(2^\alpha)}(\lambda)). \end{aligned}$$

By [9, Corollary 2.2] and using the fact that the space c_0 is symmetrically regular (see, e.g., [1]) we have that $\widetilde{f \circ h}(L^{(2^\alpha)}(\lambda)) = (\tilde{f} \circ \tilde{h})(L^{(2^\alpha)}(\lambda))$. Hence

$$\begin{aligned} \Psi(\lambda)(g) &= \lim_\alpha \widetilde{f \circ h}(L^{(2^\alpha)}(\lambda)) = \lim_\alpha (\tilde{f} \circ \tilde{h})(L^{(2^\alpha)}(\lambda)) \\ &= \lim_\alpha (\tilde{f} \circ h)(L^{(2^\alpha)}(\lambda)) = \tilde{f}(\lim_\alpha (h(L^{(2^\alpha)}(\lambda)))) \\ &= \tilde{f}(\lambda) = f(\lambda). \end{aligned}$$

□

Now, proceeding analogously to [19], we can use Ψ to transfer the analytic structure of B_{ℓ_∞} into $\Psi(B_{\ell_\infty})$. In this way, for every function $f \in \mathcal{H}^\infty(B_{c_0})$ we can consider the extension $\tilde{f} \in \mathcal{H}^\infty(B_{\ell_\infty})$. Since $\mathcal{M}(\mathcal{H}^\infty(B_{c_0})) \subseteq S_{(\mathcal{H}^\infty(B_{c_0}))^*}$, the restriction of $\tilde{f}$ to $\Psi(B_{\ell_\infty})$ is a bounded analytic function on $\Psi(B_{\ell_\infty})$.

In other words, we can say that if $B :\equiv \mathcal{H}^\infty(\Psi(B_{\ell_\infty}))$ is endowed with the supremum norm, then B is a uniformly closed Banach algebra of continuous function on $\Psi(B_{\ell_\infty})$, which is isomorphic to $\mathcal{H}^\infty(B_{c_0})$. Then, the maximal ideal space of B is

$$\mathcal{M}(B) = \{\phi \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0})); \phi(f) = 0 \text{ whenever } \tilde{f} = 0 \text{ on } \Psi(B_{\ell_\infty})\}.$$

Therefore the maximal ideal space can be identified with $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Hence the fiber $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$, for any z in $\mathbb{T}^{\aleph_0}$, contains a homeomorphic copy of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$.

Roughly speaking, the maximal ideal space $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ reproduces itself ad infinitum inside each of the fibers $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ for any z in $\mathbb{T}^{\aleph_0}$.

Recently, Johnson and Ortega Castillo [18] studied some properties of the fibers of the maximal ideal space of the Banach algebra $\mathcal{H}^\infty(B_{C(K)})$, when K is a scattered Hausdorff compact set and $C(K)$ is the space of continuous complex valued function on K. We are going to show that these results can be translated to some of the fibers of $\mathcal{M}(\mathcal{H}^\infty(B_{C(K)}))$.

Recall that a Hausdorff compact set K is called *scattered* (or *dispersed*) if every point has a basis of clopen neighborhoods. Hence if K is infinite it is always possible to find a sequence (U_n) of pairwise disjoint, non-empty clopen subsets. Observe that χ_{U_n} belongs to $C(K)$ for every n.

Lemma 2.13. *Let K be an infinite scattered compact Hausdorff set and let (U_n) be a sequence of non empty clopen sets pairwise disjoint subsets of K. Let $\Phi : c_0 \to C(K)$ be defined by*

$$\Phi((a_n)) = \sum_{n=1}^{\infty} a_n \chi_{U_n}.$$

(i) The mapping $R : \mathcal{M}(\mathcal{H}^\infty(B_{c_0})) \to \mathcal{M}(\mathcal{H}^\infty(B_{C(K)}))$,

$$R(\varphi)(F) = \varphi(F \circ \Phi)$$

is an injective and holomorphic mapping, for every φ in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ and every F in $\mathcal{H}^\infty(B_{C(K)})$.

(ii) If $z = (b_n)$ belongs to the closed unit ball of ℓ_∞ and we consider $\sum_{n=1}^\infty b_n \chi_{U_n}$ as an element of the closed unit ball of $C(K)^{**}$, then

$$R(\mathcal{M}_{(b_n)}(\mathcal{H}^\infty(B_{c_0}))) \subset \mathcal{M}_{\sum_{n=1}^\infty b_n \chi_{U_n}}(\mathcal{H}^\infty(B_{C(K)})).$$

Proof. Clearly $\sum_{n=1}^\infty a_n \chi_{U_n}$ is continuous on K for any null sequence (a_n) and $\|\sum_{n=1}^\infty a_n \chi_{U_n}\| = \|(a_n)\|$. Hence Φ is well-defined and is a linear isometry. As a consequence $T : \mathcal{H}^\infty(B_{C(K)}) \to \mathcal{H}^\infty(B_{c_0})$, $T(F) = F \circ \Phi$, is also well-defined and continuous. Thus $T^* : \mathcal{H}^\infty(B_{c_0})^* \to \mathcal{H}^\infty(B_{C(K)})^*$ is linear and continuous too (therefore an entire function). Moreover, since T is multiplicative, $T^*(\mathcal{M}(\mathcal{H}^\infty(B_{c_0})) \subset \mathcal{M}(\mathcal{H}^\infty(B_{C(K)}))$. Since R is the restriction of the mapping T^* to $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ to finish the proof of (i), we only have to check that R is injective. For that consider φ in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ and let $g \in \mathcal{H}^\infty(B_{c_0})$ with $\varphi(g) \neq 0$. As before, denote by $\tilde{g}$ in $\mathcal{H}^\infty(B_{\ell_\infty})$ the canonical extension of g and choose t_n in U_n for every n. Given h in $B_{C(K)}$, as $(h(t_n))$ is in B_{ℓ_∞}, we can define $F_0(h) = \tilde{g}(h(t_n))$. Clearly F_0 belongs to $\mathcal{H}^\infty(B_{C(K)})$ and

$$R(\varphi)(F_0) = \varphi(g) \neq 0.$$

(ii) If L is in $C(K)^*$, and $z = (a_n)$ is any element of B_{c_0}, then

$$L \circ \Phi((a_n)) = L(\sum_{n=1}^{\infty} a_n \chi_{U_n}) = \sum_{n=1}^{\infty} a_n L(\chi_{U_n}).$$

Hence, the sequence $(L(\chi_{U_n}))$ is in ℓ_1 and $\langle L, \sum_{n=1}^{\infty} b_n \chi_{U_n} \rangle := \sum_{n=1}^{\infty} b_n L(\chi_{U_n})$ defines a continuous linear form on $C(K)^*$ for each $z = (b_n)$ in $\overline{B}_{\ell_\infty}$. Now, for fixed $(b_n) \in \overline{B}_{\ell_\infty}$, if φ is in $\mathcal{M}_{(b_n)}(\mathcal{H}^\infty(B_{c_0}))$, then

$$\pi(R(\varphi))(L) = \varphi(L \circ \Phi) = \varphi(L(\chi_{U_n})) = \sum_{n=1}^{\infty} b_n L(\chi_{U_n}),$$

for every L in $C(K)^*$ and (ii) is proved. □

Corollary 2.14. *Let K be an infinite scattered compact Hausdorff set. The following hold:*

(i) B_{ℓ_∞} is continuously injected in the fiber $\mathcal{M}_0(\mathcal{H}^\infty(B_{C(K)}))$.

(ii) There exists an injection from B_{ℓ_∞} into $\mathcal{M}_{\sum_{n=1}^{\infty} \chi_{U_n}}(\mathcal{H}^\infty(B_{C(K)}))$, which is biholomorphic onto its image.

Proof. (i) is a consequence of [10, 6.7 Theorem] and of Lemma 2.13.(ii).

(ii) in turn, is a consequence of Theorem 2.2 and of Lemma 2.13.(ii). □

3. Disks in fibers of the algebra of the ball of a Banach space

The main goal of this section is to prove the following theorem.

Theorem 3.1. *Let X be a Banach space such that there exists a polynomial P and $x_0 \in B_X$ such that $P|_{B_X}$ is not weakly continuous at x_0. Then the complex disk $\mathbb{D}$ can be analytically injected in $\mathcal{M}_z(\mathcal{A}_u(B_X))$ for every $z \in B_{X^{**}}$.*

Remark 3.2. It might well seem that this unprepossessing result treats a very special situation and is therefore of little interest. In fact, the condition in Theorem 3.1 is close to being necessary and sufficient, in the following sense. If it happens that every polynomial is weakly continuous at each point of B_X and X^* has the approximation property, by using a similar argument to [5, Theorem 7.2] we have $\mathcal{M}_z(A_u(B_X)) = \{\delta_z\}$ for every $z \in \overline{B}_{X^{**}}$. Hence the fibers contains no disk at any point in the open unit ball of the bidual of X. It is unknown if the same holds if X^* does not have the approximation property.

Also it is worth recalling that for many Banach spaces, the fiber $\mathcal{M}_x(A_u(B_X))$ at any point x in the unit sphere of X is reduced to the evaluation at x. This can occur even in the case that the space has a polynomial whose restriction to the corresponding unit ball is not weakly continuous at any point. Examples of such spaces include the uniformly convex spaces, (consequence of [15, Proposition 4.1] and [3, Lemma 2.4], see also the remark preceding [4, Corollary 2.3]). Another example is ℓ_1 [4, Proposition 3.1].

In order to prove Theorem 3.1 we first need to study the set of points in which a continuous polynomial P defined on a complex Banach space is discontinuous when restricted to bounded sets that are endowed with a weaker topology than the norm. Our discussion is related to [6] and [7].

Given $x \in X$ and $r > 0$, $B_X(x,r)$ will stand for the open ball centered at x with radius r.

Lemma 3.3. *Let U be an open subset of a Banach space X and $f \in \mathcal{H}(U)$. Let E be a subset of X^* separating points of X. If for some $x_0 \in U$ there exists $r > 0$ such that $B_X(x_0,r) \subset U$ and $f|_{B_X(x_0,r)}$ is $w(X,E)$-continuous at x_0 then, for each $n \in \mathbb{N}$ and $R > 0$, the mapping $x \rightsquigarrow \frac{\hat{d}^n f}{n!}(x_0)(x)$ is $w(X,E)$-continuous at 0 when restricted to $B_X(0,R)$.*

Proof. Let us write $P_n(x) := \frac{\hat{d}^n f}{n!}(x_0)(x)$, $x \in X$, so P_n is a continuous n-homogeneous polynomial that is the n^{th} Taylor coefficient of f at x_0. It is routine that

$$P_n(x) = \frac{1}{2\pi i}\int_{|\lambda|=1} \frac{f(x_0+\lambda x)}{\lambda^{n+1}} d\lambda$$

if $\|x\| < r$.

By hypothesis, given $\varepsilon > 0$ there exist $x_1^*, \dots, x_k^* \in E$ and $0 < \delta < r$ such that $\sup_{j=1,\dots,k} |x_j^*(x)| < \delta$ implies that $|f(x_0+x) - f(x_0)| < \varepsilon$. It follows that

$$\begin{aligned} |P_n(x) - P_n(0)| &= \Big|\frac{1}{2\pi i}\int_{|\lambda|=1} \frac{f(x_0+\lambda x) - f(x_0)}{\lambda^{n+1}} d\lambda\Big| \\ &\le \max_{|\lambda|=1} |f(x_0+\lambda x) - f(x_0)| < \varepsilon \ , \end{aligned}$$

for every $x \in B(0,r)$ such that $\sup_{j=1,\dots,k} |x_j^*(x)| < \delta$.

Now, if $(x_\alpha)_{\alpha\in\Lambda}$ is a net that is $w(X,E)$-convergent to 0 in $RB_X = B_X(0,R)$, then $(\frac{r}{R}x_\alpha)_{\alpha\in\Lambda}$ is in $B_X(0,r)$ and is also $w(X,E)$-convergent to 0. Thus $P_n(x_\alpha) =$

$(\frac{R}{r})^n P_n(\frac{r}{R}x_\alpha)$ is $w(X,E)$-convergent to $(\frac{R}{r})^n P_n(0) = 0 = P_n(0)$. Therefore, $P_n|_{RB_X}$ is $w(X,E)$-continuous at 0 for every $R > 0$ and every $n \in \mathbb{N}$. $\square$

Proposition 3.4. *Let $R > 0$ and let $E \subset X^*$ separate points of X. If $P : X \longrightarrow \mathbb{C}$ is a continuous polynomial that is not $w(X,E)$-continuous at $x_0 \in B_X(0,R)$ when restricted to the open ball $B_X(0,R)$ then, for every $r > 0$ such that $B_X(x_0,r) \subset B_X(0,R)$, $P|_{B_X(x_0,r)}$ is not $w(X,E)$-continuous at x_0.*

Proof. We can write $P(x_0 + x) = P(x_0) + \sum_{j=1}^m P_j(x)$, for every $x \in X$, where m is the degree of P and each $P_j : X \to \mathbb{C}$ is a j-homogeneous polynomial. If $r_0 > 0$ were such that $B_X(x_0,r_0) \subset B_X(0,R)$ and $P|_{B_X(x_0,r_0)}$ is $w(X,E)$-continuous at x_0 then, by Lemma 3.3, each P_j $(j = 1,\dots,m)$ would be $w(X,E)$-continuous at 0 when restricted to $B_X(0,R)$. Hence $P|_{B_X(0,R)}$ would be $w(X,E)$-continuous at x_0, a contradiction. $\square$

Proposition 3.5. *If $P : X \longrightarrow \mathbb{C}$ is an m-homogeneous continuous polynomial that is not $w(X,E)$-continuous at 0 when restricted to the unit ball B_X then, for every $x \in B_X$ and every $r > 0$ such that $B_X(x,r) \subset B_X$, $P|_{B_X(x,r)}$ is not $w(X,E)$-continuous at x. Here, as above, $E \subset X^*$ separates points of X.*

Proof. By Proposition 3.4, it is enough to prove that for each x in B_X, $P|_{B_X}$ is not $w(X,E)$-continuous at x.

Here we follow the idea of the proofs of Boyd-Ryan [7, Proposition 1] and [7, Corollary 2]. First we will show that the set of points where $P|_{B_X}$ is $w(X,E)$-continuous is closed in norm. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points convergent in norm to x_0, where for each n, $P|_{B_X}$ is $w(X,E)$-continuous at x_n. Let $\epsilon > 0$ and $(y_\alpha)_{\alpha\in A}$ be a bounded net that is $w(X,E)$-convergent to 0 with $x_0 + y_\alpha, x_n + y_\alpha \in B_X$ for all $n \in \mathbb{N}$. Since P is an m-homogeneous continuous polynomial, P is uniformly continuous when restricted to B_X. Then there exists $n_\epsilon \in \mathbb{N}$ such that $|P(x_0) - P(x_n)| \le \epsilon/3$ and $|P(x_0 + y_\alpha) - P(x_n + y_\alpha)| \le \epsilon/3$ for all $n \ge n_\epsilon$ and for all $\alpha \in A$. Also since P is $w(X,E)$-continuous at x_{n_ϵ} there exists α_ϵ such that $|P(x_{n_\epsilon} + y_\alpha) - P(x_{n_\epsilon})| \le \epsilon/3$ for all $\alpha \ge \alpha_\epsilon$. It follows that

$$|P(x_0 + y_\alpha) - P(x_0)| \le \epsilon,$$

for all $\alpha \ge \alpha_\epsilon$, and so the set of points where $P|_{B_X}$ is $w(X,E)$-continuous is closed.

Let us now show that $P|_{B_X}$ is not $w(X,E)$-continuous at any point $x \in B_X$. So, suppose that for some $x_0 \in B_X$, $P|_{B_X}$ is $w(X,E)-$continuous at x_0. Hence, for any $r>0$ satisfying $B_X(x_0,r) \subset B_X$, $P|_{B_X(x_0,r)}$ is $w(X,E)-$continuous at x_0. Fix $n \in \mathbb{N}$ and apply the contrapositive of Proposition 3.4 to P and the case $R=n$. It follows that $P|_{nB_X}$ is $w(X,E)-$continuous at x_0. Hence, by homogeneity, $P|_{B_X}$ is $w(X,E)-$continuous at $\frac{1}{n}x_0$. However, since the set of points where $P|_{B_X}$ is $w(X,E)-$continuous is closed, it follows that $P|_{B_X}$ is $w(X,E)-$continuous at 0, which contradicts the hypothesis. Therefore $P|_{B_X}$ is not $w(X,E)$-continuous at any $x \in B_X$. □

Typical examples of subsets E satisfying the hypotheses of the preceding results are $E = X^*$ or $E = Y$ whenever $X = Y^*$.

Now, we translate the above results to the two weak topologies that are of greatest interest to us, the weak $w(X,X^*)$ and the weak-star $w(X^{**},X^*)$. It must be pointed out that the following Corollary complements results of V. Dimant, such as Corollary 1.7 and Remark 1.8 in [13].

Corollary 3.6. *If $P : X \longrightarrow \mathbb{C}$ is an m-homogeneous continuous polynomial that is not weakly continuous at 0 when restricted to the unit ball B_X then the following hold,*

(1) *For every $x \in B_X$ and every $r>0$ such that $B_X(x,r) \subset B_X$, $P|_{B_X(x,r)}$ is not weakly continuous at x.*
(2) *For every $z \in B_{X^{**}}$ and every $r>0$ such that $B_{X^{**}}(z,r) \subset B_{X^{**}}$, the canonical extension $\tilde{P}|_{B_{X^{**}}(z,r)}$ of P is not weak-star continuous at z.*

Proof. (1) Apply Proposition 3.5 for the space X and $E = X^*$.

(2) Apply Proposition 3.5 for the space X^{**} and $E = X^* \subset X^{***}$ and use the fact that $\tilde{P}$ is not $w(X^{**},X^*)$-continuous at 0 whenever P is not weakly continuous at 0 (restricted to the corresponding unit balls). □

Corollary 3.6 above is a refinement of Proposition 1, Corollary 2 and Proposition 3 of [7]. Also the results from Lemma 3.3 to Example 3.9 could be stated in terms of weakly sequential continuity of polynomials, in which case Corollary 3.6 is related to Proposition 1.(1) and 1.(2) of [6].

Corollary 3.7. *Let X be a Banach space, $x_0 \in B_X$, $0<r<1-\|x_0\|$, and $f \in \mathcal{A}_u(B_X)$ such that $f|_{B_X(x_0,r)}$ is not weakly continuous at x_0. Then there* *exists an m-homogeneous continuous polynomial P such that $P|_{B_X}$ is not weakly continuous at any point of B_X, and the canonical extension $\tilde{P}|_{B_{X^{**}}}$ is not weak-star continuous at any point of $B_{X^{**}}$.*

Proof. Let $P_n := \frac{\hat{d}^n f}{n!}(x_0)$, $n \in \mathbb{N}$. We have that $\sum_{m=0}^{\infty} P_m(x - x_0)$ converges absolutely and uniformly to f on $B_X(x_0, r)$. If every P_m were weakly continuous at 0 (restricted to any ball centered at 0) then by uniform convergence $f|_{B_X(x_0,r)}$ would be weakly continuous at x_0. Now the conclusion follows from Corollary 3.6. □

Proposition 3.8. *Let X be a complex Banach space and $P : X \mapsto \mathbb{C}$ be a continuous polynomial of degree m. If $G := \{x \in B_X : P|_{B_X}$ is weakly continuous at $x\}$ has nonempty interior then P restricted to bounded sets is weakly continuous.*

Proof. Let $x_0 \in G$ and $r > 0$ such that $B_X(x_0, r) \subset G \subset B_X$. Then

$$P(x_0 + u) = P(x_0) + \sum_{n=1}^{m} \frac{\hat{d}^n P}{n!}(x_0)(u),$$

for all $u \in B_X(0, r)$.

Applying the Cauchy integral formula for $u, u_0 \in B_X(0, r)$, we get

$$P_n(u) - P_n(u_0) = \frac{1}{2\pi i} \int_{\lambda=1} \frac{P(x_0 + \lambda u) - P(x_0 + \lambda u_0)}{\lambda^{n+1}} d\lambda.$$

Fix $u_0 \in B_X(0, r)$. By the weak continuity of $P|_{B_X}$ at any point of $B_X(x_0, r)$, given $\epsilon > 0$, for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ there exist a finite set $A_\lambda \subset X^*$ and a positive number δ_λ such that if $x - (x_0 + \lambda u_0)$ is in the weakly open set $V_\lambda := \{y \in B_X : \sup_{x^* \in A_\lambda} |x^*(y)| < \delta_\lambda\}$ and $x \in B_X$ then

$$|P(x) - P(x_0 + \lambda u_0)| < \epsilon/2. \tag{3.1}$$

By the compactness of the set $\{\lambda u_0 : |\lambda| = 1\}$ there exist complex numbers $\lambda_1, \ldots, \lambda_p$ of modulus one such that $\{\lambda u_0 : |\lambda| = 1\} \subset \cup_{l=1}^{p} \big(\lambda_l u_0 + \frac{1}{2} V_{\lambda_l}\big)$. Let $A = \cup_{l=1}^{p} A_{\lambda_l}$, $\delta = \min\{\frac{1}{2}\delta_{\lambda_l} : l = 1, \ldots, p\}$ and denote by $V := \{y \in B_X : \sup_{x^* \in A} |x^*(y)| < \delta\}$. Notice $V \subset \frac{1}{2} V_{\lambda_l}$ for $l = 1, \ldots, p$.

Given a complex number λ of modulus one there exists $l \in \{1, \ldots, p\}$ with

$$\lambda u_0 \in \lambda_l u_0 + \frac{1}{2} V_{\lambda_l}.$$

Therefore $(x_0 + \lambda u_0) - (x_0 + \lambda_l u_0) = \lambda u_0 - \lambda_l u_0 \in \frac{1}{2} V_{\lambda_l} \subset V_{\lambda_l}$. Hence by equation (3.1) we have

$$|P(x_0 + \lambda u_0) - P(x_0 + \lambda_l u_0)| < \epsilon/2. \tag{3.2}$$

Also, for every $u \in B_X$ such that $u - u_0 \in V$, we have $\lambda u - \lambda u_0 = \lambda(u - u_0) \in \lambda V = V \subset \frac{1}{2}V_{\lambda_l} \subset V_{\lambda_l}$. Hence $(x_0 + \lambda u) - (x_0 + \lambda_l u_0) = \lambda u - \lambda_l u_0 \in V_{\lambda_l}$, and by equation (3.1) again

$$|P(x_0 + \lambda u) - P(x_0 + \lambda_l u_0)| < \epsilon/2. \tag{3.3}$$

Then, by the triangle inequality and equations (3.2) and (3.3)

$$|P(x_0 + \lambda u) - P(x_0 + \lambda u_0)| < \epsilon.$$

Therefore for every $u \in B_X(0, r)$

$$\begin{aligned} |P_n(u) - P_n(u_0)| &= \frac{1}{2\pi i}\Big|\int_{\lambda=1} \frac{P(x_0 + \lambda u) - P(x_0 + \lambda u_0)}{\lambda^{n+1}} d\lambda\Big| \\ &\leq \max_{|\lambda|=1} |P(x_0 + \lambda u) - P(x_0 + \lambda u_0)| \leq \epsilon. \end{aligned}$$

Thus the n-homogeneous polynomial $P_n|_{B_X(0,r)} = \frac{\hat{d}^n P}{n!}(x_0)$ is weakly continuous at u_0, for all u_0 in $B_X(0, r)$. Since P_n is a continuous n-homogeneous polynomial, we have that the restriction of P_n to bounded sets is weakly continuous. Therefore $P(x) = P(x_0) + P_1(x - x_0) + \cdots + P_m(x - x_0)$ restricted to bounded sets is weakly continuous. $\square$

To finish our study of weakly continuity of polynomials we would like to point out the following example, that shows that even if the homogeneous parts of a polynomial are non-weakly continuous, the polynomial itself need not be non-weakly continuous.

Example 3.9. Consider a Banach space X and a k-homogeneous polynomial Q such that the restriction of Q to B_X is not weakly continuous at some $x_0 \neq 0$. Then, there are non-homogeneous polynomials $P = P_1 + \cdots + P_m$ with $P_j \in \mathcal{P}(^{n_j}X)$, $n_1 < n_2 < \cdots < n_m$ such that $P|_{B_X}$ is weakly continuous at x_0 but none of the $P_j|_{B_X}$ are weakly continuous at x_0.

Proof. Consider x_0^* in X^* such that $x_0^*(x_0) = 1$ and an arbitrary but fixed natural number h. Define the polynomial

$$P(x) := \big(1 - x_0^*(x)\big)^h Q(x) = \sum_{j=0}^{h} \binom{h}{j} (-x_0^*(x))^j Q(x), \quad x \in X.$$

Then P is a non-homogeneous polynomial of degree $k + h$ with homogeneous parts $\binom{h}{j}(-x_0^*(x))^j Q(x) \in \mathcal{P}(^{j+k}X)$, for $j = 1, \ldots, h$, such that their restrictions to B_X are not weakly continuous at x_0.

However,

$$|P(x) - P(x_0)| = |\big(1 - x_0^*(x)\big)^h Q(x)| \leq \big|1 - x_0^*(x)\big|^h \|Q\|.$$

Hence if a net $(x_\alpha) \subset B_X$ weakly converges to x_0 we have that $\big|1 - x_0^*(x)\big|^h$ converges to zero. Thus $P|_{B_X}$ is weakly continuous at x_0. $\square$

We are now ready to prove the main theorem of this section.

Proof of Theorem 3.1. By Corollary 3.7, there exists an m-homogeneous continuous polynomial Q such that the restriction of its canonical extension $\tilde{Q}$ to $B_{X^{**}}$ is not $w(X^{**}, X^*)$-continuous at any point of $z \in B_{X^{**}}$.

Fix $z_0 \in B_{X^{**}}$ and $0 < r < 1 - \|z_0\|$. By Proposition 3.4, $\tilde{Q}|_{B_{X^{**}}(z_0,r)}$ is not w^*-continuous at z_0. Hence there is a net $(z_\lambda)_{\lambda\in\Lambda} \subset B_{X^{**}}(z_0, r)$ that is w^*-convergent to z_0 such that

$$\tilde{Q}(z_\lambda) \nrightarrow \tilde{Q}(z_0)\ .$$

Taking a subnet, and not changing the notation, we have for some $\varepsilon > 0$ that

$$|\tilde{Q}(z_\lambda) - \tilde{Q}(z_0)| \geq \varepsilon\ , \tag{3.4}$$

for all $\lambda \in \Lambda$. Let $\mathcal{U}$ be an ultrafilter on Λ such that $\mathcal{U} \supset \{\{\mu \in \Lambda : \mu \geq \lambda\} : \lambda \in \Lambda\}$ and define

$$\Phi : \overline{\mathbb{D}} \longrightarrow \mathcal{M}_{z_0}(\mathcal{A}_u(B_X))$$

by

$$\Phi(t)(f) := \lim_{\lambda\in\mathcal{U}} \tilde{f}(z_0 + t(z_\lambda - z_0))$$

for all $t \in \overline{\mathbb{D}}$ and all $f \in \mathcal{A}_u(B_X)$.

It is routine to verify that $\Phi(t) \in \mathcal{M}_{z_0}(\mathcal{A}_u(B_X))$ for all $t \in \overline{\mathbb{D}}$.

Since Q is polynomial of degree m

$$\tilde{Q}(z_0 + t(z_\lambda - z_0)) = \tilde{Q}(z_0) + \sum_{j=1}^{m} \tilde{Q}_j(t(z_\lambda - z_0)) = \tilde{Q}(z_0) + \sum_{j=1}^{m} t^j \tilde{Q}_j(z_\lambda - z_0).$$

Hence

$$\Phi(t)(Q) = \lim_{\lambda\in\mathcal{U}} \tilde{Q}(z_0 + t(z_\lambda - z_0)) = \tilde{Q}(z_0) + \sum_{j=1}^{m} t^j \lim_{\lambda\in\mathcal{U}} \tilde{Q}_j(z_\lambda - z_0) = \sum_{j=0}^{m} a_j t^j\ ,$$

where $a_0 := \tilde{Q}(z_0)$ and $a_j := \lim_{\lambda\in\mathcal{U}} \tilde{Q}_j(z_\lambda - z_0)$ for $j = 1, \ldots, m$.

Thus, the mapping $t \rightsquigarrow \Phi(t)(Q)$ $(t \in \overline{\mathbb{D}})$ is the restriction to $\overline{\mathbb{D}}$ of the polynomial $\sum_{j=0}^{m} a_j t^j$. But $\Phi(0)(Q) = \tilde{Q}(z_0)$ and by (3.4) $|\Phi(1)(Q) - \tilde{Q}(z_0)| \geq \varepsilon$. So the one variable polynomial $t \rightsquigarrow \Phi(t)(Q)$ is non-constant and has degree less than or equal to m. Therefore we can find a point $t_0 \in \mathbb{D}$ and $s > 0$ such that $\overline{\mathbb{D}}(t_0, s) := t_0 + s\overline{\mathbb{D}} \subset \mathbb{D}$ and $g(t) : \overline{\mathbb{D}}(t_0, s) \longrightarrow \mathbb{C}$, defined by $g(t) = \Phi(t)(Q)$ for all $t \in \overline{\mathbb{D}}(t_0, s)$ is injective.

We claim that $h : \overline{\mathbb{D}} \longrightarrow \mathcal{M}_{z_0}(\mathcal{A}_u(B_X))$ defined by $h(u) = \Phi(t_0 + su)$, for all $u \in \overline{\mathbb{D}}$, is injective and analytic on $\mathbb{D}$. By the above it is injective. Let us show that it is also analytic. Given $f \in \mathcal{A}_u(B_X)$, we have

$$\hat{f} \circ \Phi(t) = \Phi(t)(f) = \lim_{\lambda \in \mathcal{U}} \tilde{f}(z_0 + t(z_\lambda - z_0)) \ ,$$

where $\hat{f}$ stands for the Gelfand transform of f. For $\lambda \in \Lambda$, define $f_\lambda : \mathbb{D} \longrightarrow \mathbb{C}$ by $f_\lambda(t) := \tilde{f}(z_0 + t(z_\lambda - z_0))$ $(t \in \mathbb{D})$. Consider now the family $\mathcal{F} := \{f_\lambda : \lambda \in \Lambda\}$. This is a subset of $\|f\|_\infty \overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$ which is a relatively compact subset of the space $\mathcal{H}(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$, with respect to the compact-open topology. Hence there exists $g \in \|f\|_\infty \overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$ such that $g = \lim_{\lambda \in \mathcal{U}} f_\lambda$. Obviously $g(t) = \lim_{\lambda \in \mathcal{U}} f_\lambda(t) = \hat{f} \circ \Phi(t)$, for all $t \in \mathbb{D}$. Thus Φ is analytic on $\mathbb{D}$ and therefore h is analytic on $\mathbb{D}$ as well. □

In certain special cases, the conclusion of Theorem 3.1 can be considerably strengthened. One such situation follows.

Proposition 3.10. *For every $1 < p < \infty$ the closed unit ball of ℓ_p can be continuously injected into $\mathcal{M}_0(\mathcal{A}_u(B_{\ell_p}))$. Moreover, the restriction of that injection to the open unit ball of ℓ_p is analytic.*

Proof. We split $\mathbb{N} = \bigcup_{n=1}^\infty J_n$ with $\mathrm{Card}(J_n) = \mathrm{Card}(\mathbb{N})$ for all $n \in \mathbb{N}$, $J_n \bigcap J_m = \emptyset$ for $n \neq m$ and $J_n \subset \{n, n+1, \ldots\}$ for all $n \in \mathbb{N}$.

Now we write $J_n = (i_{(n,k)})_{k=1}^\infty$ such that $n \leq i_{(n,1)} < i_{(n,2)} < \ldots < i_{(n,k)} < i_{(n,k+1)} < \ldots$, and for a fixed $(\lambda_n)_{n\in\mathbb{N}} \in \overline{B}_{\ell_p}$ we define

$$\varphi_k := \delta_{\sum_{n=1}^\infty \lambda_n e_{i_{(n,k)}}} \quad (k \in \mathbb{N}) \ .$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and define

$$\varphi_{(\lambda_n)_{n\in\mathbb{N}}}(f) := \lim_{k \in \mathcal{U}} \varphi_k(f) \quad (f \in \mathcal{A}_u(B_{\ell_p})) \ .$$

The sequence $\left(\sum_{n=1}^\infty \lambda_n e_{i_{(n,k)}} \right)_{k=1}^\infty$ is weakly convergent to 0, as it is contained in the closed unit ball of ℓ_p and the support of each $\sum_{n=1}^\infty \lambda_n e_{i_{(n,k)}}$ is contained in $\{k, k+1, \ldots\}$. Hence, $\varphi_{(\lambda_n)_{n\in\mathbb{N}}} \in \mathcal{M}_0(\mathcal{A}_u(B_{\ell_p}))$ for every $(\lambda_n)_{n\in\mathbb{N}} \in \overline{B}_{\ell_p}$. Thus the mapping

$$\Phi : \overline{B}_{\ell_p} \longrightarrow \mathcal{M}_0(\mathcal{A}_u(B_{\ell_p}))$$

defined by $\Phi((\lambda_n)_{n\in\mathbb{N}}) = \varphi_{(\lambda_n)_{n\in\mathbb{N}}}$ $((\lambda_n)_{n\in\mathbb{N}} \in \overline{B}_{\ell_p}))$, is well-defined. Let us now see that it is injective.

Fix $n_0 \in \mathbb{N}$ and define the two polynomials

$$P(x) = \sum_{k=1}^{\infty} x_{i_{(n_0,k)}}^{\lfloor p\rfloor+1} \quad , \qquad Q(x) = \sum_{k=1}^{\infty} x_{i_{(n_0,k)}}^{\lfloor p\rfloor+2} \quad , \quad (x = (x_n)_{n\in\mathbb{N}} \in \ell_p)$$

where $\lfloor p\rfloor$ is the integer part of p. Since for $r \geq p$ and every $x = (x_n)_{n\in\mathbb{N}} \in \overline{B}_{\ell_p}$ we have that $\sum_{k=1}^{\infty} \left|x_{i_{(n_0,k)}}\right|^r \leq \sum_{k=1}^{\infty} \left|x_{i_{(n_0,k)}}\right|^p \leq 1$, the polynomials P and Q are in $\mathcal{A}_u(B_{\ell_p})$.

Suppose $\Phi((\lambda_n)_{n\in\mathbb{N}}) = \Phi((\mu_n)_{n\in\mathbb{N}})$ for some $(\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}} \in \overline{B}_{\ell_p}$. Then

$$\varphi_{(\lambda_n)_{n\in\mathbb{N}}}(P) = \varphi_{(\mu_n)_{n\in\mathbb{N}}}(P) \ ,$$

but

$$\begin{aligned}\varphi_{(\lambda_n)_{n\in\mathbb{N}}}(P) &= \lim_{k\in\mathcal{U}} \varphi_k(P) = \lim_{k\in\mathcal{U}} \delta_{\sum_{n=1}^{\infty} \lambda_n e_{i_{(n,k)}}}(P) \\ &= \lim_{k\in\mathcal{U}} P\Big(\sum_{n=1}^{\infty} \lambda_n e_{i_{(n,k)}}\Big) = \lim_{k\in\mathcal{U}} \lambda_{n_0}^{\lfloor p\rfloor+1} = \lambda_{n_0}^{\lfloor p\rfloor+1}.\end{aligned}$$

Analogously, $\varphi_{(\mu_n)_{n\in\mathbb{N}}}(P) = \mu_{n_0}^{\lfloor p\rfloor+1}$ and also $\varphi_{(\lambda_n)_{n\in\mathbb{N}}}(Q) = \lambda_{n_0}^{\lfloor p\rfloor+2}$, $\varphi_{(\mu_n)_{n\in\mathbb{N}}}(Q) = \mu_{n_0}^{\lfloor p\rfloor+2}$ and $\varphi_{(\lambda_n)_{n\in\mathbb{N}}}(Q) = \varphi_{(\mu_n)_{n\in\mathbb{N}}}(Q)$. Therefore

$$\begin{cases} 0 = \lambda_{n_0}^{\lfloor p\rfloor+1} - \mu_{n_0}^{\lfloor p\rfloor+1} = (\lambda_{n_0} - \mu_{n_0})(\lambda_{n_0}^{\lfloor p\rfloor} + \lambda_{n_0}^{\lfloor p\rfloor-1}\mu_{n_0} + \cdots + \mu_{n_0}^{\lfloor p\rfloor}), \\ 0 = \lambda_{n_0}^{\lfloor p\rfloor+2} - \mu_{n_0}^{\lfloor p\rfloor+2} = (\lambda_{n_0} - \mu_{n_0})\Big(\lambda_{n_0}(\lambda_{n_0}^{\lfloor p\rfloor} + \lambda_{n_0}^{\lfloor p\rfloor-1}\mu_{n_0} + \cdots + \mu_{n_0}^{\lfloor p\rfloor}) + \mu_{n_0}^{\lfloor p\rfloor+1}\Big). \end{cases}$$

These equalities imply that $\lambda_{n_0} = \mu_{n_0}$. Since n_0 is arbitrary the conclusion follows.

We still must prove that for every $f \in \mathcal{A}_u(B_{\ell_p})$ the mapping $\hat{f} \circ \Phi : \overline{B}_{\ell_p} \to \mathbb{C}$ is uniformly continuous on $\overline{B}_{\ell_p}$ and holomorphic on B_{ℓ_p}, where $\hat{f}$ stands for the Gelfand transform of f. The uniformly continuity of $\hat{f} \circ \Phi$ is obvious, since given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in \overline{B}_{\ell_p}$ satisfy $\|x - y\| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Since $\|\sum_{n=1}^{\infty} x_n e_{i_{(n,k)}} - \sum_{n=1}^{\infty} y_n e_{i_{(n,k)}}\| = \|x - y\|$, for every k, we obtain that

$$|\Phi((x_n))(f) - \Phi((y_n))(f)| \leq \varepsilon,$$

for every $x, y \in \overline{B}_{\ell_p}$. As $\hat{f} \circ \Phi$ is continuous on B_{ℓ_p}, in order to prove that it is holomorphic it is enough to check that it is Gâteaux holomorphic on B_{ℓ_p} (see, e.g. [14, Proposition 3.7, pag. 153]). Let $x_0 = (x_n) \in B_{\ell_p}$, $y_0 = (y_n) \in \ell_p \setminus \{0\}$, and let $r > 0$ be such that $\|x_0\| + r\|y_0\| < 1$. Given k, one sees that

$$\|\sum_{n=1}^{\infty} x_n e_{i_{(n,k)}}\| + r\|\sum_{n=1}^{\infty} y_n e_{i_{(n,k)}}\| = \|x_0\| + r\|y_0\| < 1.$$

If we denote by $(P_m(z))$ the sequence of m-homogeneous polynomials that are the Taylor coefficients of f at z in the open unit ball, we will have

$$f(\sum_{n=1}^{\infty} x_n e_{i_{(n,k)}} + t\sum_{n=1}^{\infty} y_n e_{i_{(n,k)}}) = \sum_{m=0}^{\infty} t^m P_m(\sum_{n=1}^{\infty} x_n e_{i_{(n,k)}})(\sum_{n=1}^{\infty} y_n e_{i_{(n,k)}}),$$

for every $t \in \mathbb{C}$ with $|t| < r$. Moreover, by the Cauchy inequalities,

$$r^m |P_m(\sum_{n=1}^{\infty} x_n e_{i_{(n,k)}})(\sum_{n=1}^{\infty} y_n e_{i_{(n,k)}})| \leq \sup\{|f(x)| : \|x\| \leq \|x_0\| + r\|y_0\|\} < \infty,$$

for every m and k. Hence,

$$\Phi(x_0 + ty_0)(f) = \sum_{m=0}^{\infty} t^m \lim_{k\in\mathcal{U}} P_m(\sum_{n=1}^{\infty} x_n e_{i_{(n,k)}})(\sum_{n=1}^{\infty} y_n e_{i_{(n,k)}}),$$

for every $t \in \mathbb{C}$ with $|t| < r$ and the conclusion follows. $\square$

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