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On the L_2 Markov Inequality with Laguerre Weight

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On the L_2 Markov Inequality with Laguerre Weight

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Abstract Let $w_\alpha(t) = t^\alpha e^{-t}$, $\alpha > -1$, be the Laguerre weight function, and $\|\cdot\|_{w_\alpha}$ denote the associated L_2-norm, i.e.,

$$\|f\|_{w_\alpha} := \Big(\int_0^\infty w_\alpha(t) |f(t)|^2 \, dt \Big)^{1/2}.$$

Denote by $\mathscr{P}_n$ the set of algebraic polynomials of degree not exceeding n. We study the best constant $c_n(\alpha)$ in the Markov inequality in this norm,

$$\|p'\|_{w_\alpha} \le c_n(\alpha) \, \|p\|_{w_\alpha}, \quad p \in \mathscr{P}_n,$$

namely the constant

$$c_n(\alpha) = \sup_{\substack{p \in \mathscr{P}_n \\ p \not\equiv 0}} \frac{\|p'\|_{w_\alpha}}{\|p\|_{w_\alpha}},$$

and we are also interested in its asymptotic value

$$c(\alpha) = \lim_{n \to \infty} \frac{c_n(\alpha)}{n}.$$

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In this paper we obtain lower and upper bounds for both $c_n(\alpha)$ and $c(\alpha)$.
Note that according to a result of P. Dörfler from 2002, $c(\alpha) = [j_{(\alpha-1)/2,1}]^{-1}$, with $j_{\nu,1}$ being the first positive zero of the Bessel function $J_\nu(z)$, hence our bounds for $c(\alpha)$ imply bounds for $j_{(\alpha-1)/2,1}$ as well.

1 Introduction and Statement of the Results

The Markov inequality (or, to be more precise, the inequality of the brothers Markov) has proven to be one of the most important polynomial inequalities, with numerous applications in approximation theory, numerical analysis, and many other branches of mathematics. In its classical variant it reads as follows:

The inequality of the brothers Markov. *If $p \in \mathscr{P}_n$, then for $k = 1, \dots, n$,*

$$\|p^{(k)}\| \le T_n^{(k)}(1)\, \|p\|\,.$$

The equality is attained if and only if $p = c\,T_n$, where T_n is the n-th Chebyshev polynomial of the first kind, $T_n(x) = \cos n \arccos x, \;\; x \in [-1,1]$.

Here, $\mathscr{P}_n$ is the set of algebraic polynomials of degree not exceeding n and $\|\cdot\|$ is the uniform norm in $[-1,1]$, $\|f\| := \sup\{|f(x)| : x \in [-1,1]\}$.

Proved for $k = 1$ in 1889 by Andrey Markov [14], and for $k \ge 1$, in 1892, by his kid brother, Vladimir Markov [15], throughout the years Markov inequality has got many alternative proofs and various generalizations. For the intriguing story of Markov's inequality in the uniform norm, and twelve of its proofs, we refer the reader to the survey paper [27]. Another survey on the subject is [2]. For some recent developments, see [3, 17, 18, 20, 21, 22, 23, 24].

Generally, Markov-type inequalities provide upper bounds for a certain norm of a derivative of an algebraic polynomial $p \in \mathscr{P}_n$ in terms of some (usually the same) norm of p. Our subject here is Markov-type inequalities in L_2-norms for the first derivative of an algebraic polynomial. For a weight function w on the finite or infinite interval (a,b) with all moments finite, let $\|\cdot\|_w$ be the associated L_2-norm,

$$\|f\|_w := \Big(\int_a^b w(t) |f(t)|^2\, dt \Big)^{1/2},$$

and let $c_n(w)$ be the best (i.e., the smallest) constant in the L_2 Markov inequality

$$\|p'\|_w \le c_n(w)\, \|p\|_w, \qquad p \in \mathscr{P}_n\,.$$

This constant possesses a simple characterization: it is the largest singular value of a certain matrix, see, e.g., [7] or [16], however the exact values of the best Markov constants are generally unknown even in the cases of the classical weight functions of Laguerre and Jacobi, and, in particular, of Gegenbauer.

The Hermite weight $w_H(t) = e^{-t^2}$, $t \in \mathbb{R}$. This is the only case where both the sharp Markov constant and the extremal polynomial are known. Namely, in this case the sharp Markov constant is $c_n(w_H) = \sqrt{2n}$, and the unique (up to a constant factor) extremal polynomial is the n-th Hermite polynomial $H_n(t) = (-1)^n e^{t^2} \left(\frac{d}{dt}\right)^n e^{-t^2}$. The extremality of H_n persists in the L_2 Markov inequalities for higher order derivatives,

$$\|p^{(k)}\|_{w_H} \le c_n^{(k)}(w_H)\, \|p\|_{w_H}, \quad k = 1, \dots, n,$$

with the sharp Markov constants given by $c_n^{(k)}(w_H) = \left(2^k \frac{n!}{(n-k)!}\right)^{1/2}$. The reason for this case to be trivial comes from the fact that the derivatives of Hermite's polynomials are Hermite's polynomials of lower degrees [28, Chapt. 5], and as a result, the sharp Markov constant is simply the largest entry in a diagonal matrix.

The Gegenbauer weight $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$, $\lambda > -1/2$, $t \in [-1,1]$. Neither the sharp Markov constant nor the extremal polynomial are known explicitly in that case. For $\lambda = 1/2$ (a constant weight function) E. Schmidt [25] found tight estimates for the Markov constant, which in a slightly weaker form look like

$$\frac{1}{\pi}(n+3/2)^2 < c_n(w_{1/2}) < \frac{1}{\pi}(n+2)^2, \qquad n > 5.$$

Recently, A. Kroó [13] turned back to this case, identifying $c_n(w_{1/2})$ as the largest positive root of a polynomial of degree n. This polynomial was found explicitly (to some extent) by Kroó.

Nikolov [19] studied two further special cases $\lambda = 0$ and $\lambda = 1$; in particular, he obtained the following two-sided estimates for the corresponding best Markov constants:

$$0.472135\, n^2 \le c_n(w_0) \le 0.478849\, (n+2)^2,$$
$$0.248549\, n^2 \le c_n(w_1) \le 0.256861\, (n+\tfrac{5}{2})^2.$$

In [1] we obtained an upper bound for $c_n(w_\lambda)$, which is valid for all $\lambda > -1/2$:

$$c_n(w_\lambda) < \frac{(n+1)(n+2\lambda+1)}{2\sqrt{2\lambda+1}},$$

however it seems that the correct order with respect to λ should be $O(1/\lambda)$. Also, it has been shown in [1] that the extremal polynomial in the L_2 Markov inequality associated with w_λ, is even or odd when n is even or odd, accordingly (for $\lambda \ge 0$ this result was established, by a different argument, in [19]).

The Laguerre weight $w_\alpha(t) = t^\alpha e^{-t}$, $t \in (0,\infty)$, $\alpha > -1$. In the present paper we study the best constant in the Markov inequality for the first derivative of an algebraic polynomial in the L_2-norm, induced by the Laguerre weight function. We denote this norm by $\|\cdot\|_{w_\alpha}$,

$$\|f\|_{w_\alpha} := \left(\int_0^\infty t^\alpha e^{-t} |f(t)|^2\, dt\right)^{1/2}. \tag{1}$$

Further, we denote by $c_n(\alpha)$ the best constant in the Markov inequality in this norm,

$$c_n(\alpha) = \sup_{\substack{p\in\mathscr{P}_n \\ p\neq 0}} \frac{\|p'\|_{w_\alpha}}{\|p\|_{w_\alpha}}. \tag{2}$$

Before formulating our results, let us give a brief account on the known results on the Markov inequality in the L_2 norm induced by the Laguerre weight function. P. Turán [29] found the sharp Markov constant in the case $\alpha = 0$, namely,

$$c_n(0) = \Big(2\sin\frac{\pi}{4n+2}\Big)^{-1}. \tag{3}$$

In 1991, Dörfler [8] proved the inequalities

$$\frac{n^2}{(\alpha+1)(\alpha+3)} \le \big[c_n(\alpha)\big]^2 \le \frac{n(n+1)}{2(\alpha+1)}, \tag{4}$$

(the first one in a somewhat stronger form), and in 2002 he found [9] the sharp asymptotic of $c_n(\alpha)$, namely,

$$c(\alpha) := \lim_{n\to\infty} \frac{c_n(\alpha)}{n} = \frac{1}{j_{(\alpha-1)/2,1}}, \tag{5}$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

In a series of recent papers [4, 5, 6] A. Böttcher and P. Dörfler studied the asymptotic values of the best constants in L_2 Markov-type inequalities of a rather general form, namely 1) they include estimates for higher order derivatives and 2) different L_2-norms of Laguerre or Jacobi type are applied to the polynomial and its derivatives (i.e. at the two sides of their Markov inequalities).

Precisely, they proved that those asymptotic values are equal to the norms of certain Volterra operators. It seems, however, that finding the norms of these related Volterra operators explicitly is equally difficult task. They provide also some upper and lower bounds for the asymptotic values, but they do not match (they are similar to those given in (4)).

Our main goal is upper and lower bounds for the Markov constant $c_n(\alpha)$ which are valid for all n and α.

In this paper we prove the following.

Theorem 1. *For all $\alpha > -1$ and $n \in \mathbb{N}$, $n \ge 3$, the best constant $c_n(\alpha)$ in the Markov inequality*

$$\|p'\|_{w_\alpha} \le c_n(\alpha)\,\|p\|_{w_\alpha}, \qquad p \in \mathscr{P}_n$$

admits the estimates

$$\frac{2\big(n+\frac{2\alpha}{3}\big)\big(n-\frac{\alpha+1}{6}\big)}{(\alpha+1)(\alpha+5)} < \big[c_n(\alpha)\big]^2 < \frac{\big(n+1\big)\big(n+\frac{2(\alpha+1)}{5}\big)}{(\alpha+1)\big((\alpha+3)(\alpha+5)\big)^{\frac{1}{3}}},$$

where for the left-hand inequality it is additionally assumed that $n > (\alpha+1)/6$.

For $n = 1, 2$, the exact values are readily computable:

$$\big[c_1(\alpha)\big]^2 = \frac{1}{1+\alpha}, \qquad \big[c_2(\alpha)\big]^2 = \frac{3(\alpha+2)+\sqrt{(\alpha+2)(\alpha+10)}}{2(\alpha+1)(\alpha+2)}.$$

Compared to Dörfler's result (4), we improve the lower bound for $c_n(\alpha)$ by the factor of $\sqrt{2}$, and obtain for the upper bound the order $O(n/\alpha^{5/6})$ instead of $O(n/\alpha^{1/2})$.

As an immediate consequence of Theorem 1 we obtain the following

Corollary 1. *The asymptotic Markov constant* $c(\alpha) = \lim_{n\to\infty}\{n^{-1}c_n(\alpha)\}$ *satisfies the inequalities*

$$\underline{c}(\alpha) := \frac{\sqrt{2}}{\sqrt{(\alpha+1)(\alpha+5)}} \le c(\alpha) \le \frac{1}{\sqrt{\alpha+1}\sqrt[6]{(\alpha+3)(\alpha+5)}} =: \overline{c}(\alpha). \qquad (6)$$

Let us comment now on the bounds for $c(\alpha)$ given by Corollary 1. First of all,

$$\lim_{\substack{\alpha\to-1\\ \alpha>-1}} \frac{\overline{c}(\alpha)}{\underline{c}(\alpha)} = 1,$$

which indicates that for small α our bounds are pretty tight. In particular, in the case $\alpha = 0$, when we have $c(0) = 2/\pi$ (see (3)), the relative errors satisfy

$$\frac{c(0)}{\underline{c}(0)} = \frac{\sqrt{10}}{\pi} < 1.006585, \qquad \frac{\overline{c}(0)}{c(0)} = \frac{\pi}{2\sqrt[6]{15}} < 1.000242.$$

Second, Corollary 1 gives rise to the question: what is the right order of α in $c(\alpha)$ as $\alpha\to\infty$? The answer follows below:

Theorem 2. *For the asymptotic Markov constant* $c(\alpha)$ *we have* $c(\alpha) = O(\alpha^{-1})$ *as* $\alpha\to\infty$. *More precisely,* $c(\alpha)$ *satisfies the inequalities*

$$\frac{\sqrt{2}}{\sqrt{(\alpha+1)(\alpha+5)}} < c(\alpha) < \frac{2}{\alpha+2\pi-2}, \quad \alpha > 1. \qquad (7)$$

Proof. The lower bound for $c(\alpha)$ is simply $\underline{c}(\alpha)$ (in fact, the left-hand inequality in (7) holds for all $\alpha > -1$). For the right-hand inequality in (7), we recall that, by Dörfler's result (5), $c(\alpha) = \big[j_{(\alpha-1)/2,1}\big]^{-1}$, with $j_{\nu,1}$ being the first positive zero of the Bessel function $J_\nu(z)$. On using some lower bounds for the zeros of Bessel functions, obtained by Ifantis and Siafarikas [11] (see [10, eqn. (1.6)]), we get

$$\frac{1}{j_{(\alpha-1)/2,1}} < \frac{2}{\alpha+2\pi-2}, \quad \alpha > 1.$$

The inequalities in (7) imply that $c(\alpha) = O(\alpha^{-1})$ as $\alpha\to\infty$. □

Notice that the lower bound $\underline{c}(\alpha)$ has the right order with respect to α as $\alpha \to \infty$. Moreover, from (7) it follows that, roughly, this lower bound can only be improved by a factor of at most $\sqrt{2}$.

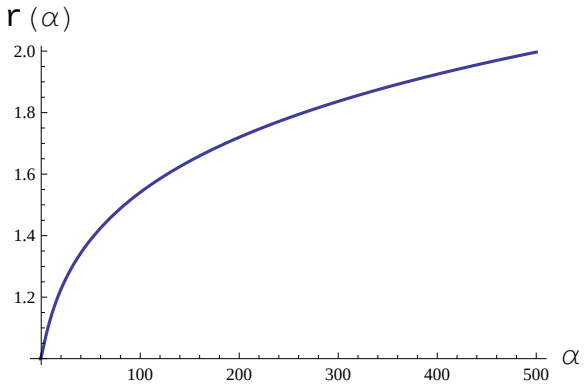

Fig. 1 The graph of the ratio $r(\alpha) = \dfrac{\overline{c}(\alpha)}{\underline{c}(\alpha)}$.

The upper bound $\overline{c}(\alpha)$ does not exhibit the right asymptotic of $c(\alpha)$ as $\alpha \to \infty$. Nevertheless, $\overline{c}(\alpha)$ is less than the upper bound in (7) for $\alpha \in [2.045, 47.762]$. Moreover, the ratio $r(\alpha) = \overline{c}(\alpha)/\underline{c}(\alpha)$ tends to infinity as $\alpha \to \infty$ rather slowly; for instance, $r(\alpha)$ is less than two for $-1 < \alpha < 500$ (see Fig. 1).

Finally, in view of (5), Corollary 1 provides bounds for $j_{\nu,1}$, the first positive zero of the Bessel function J_ν, which, for some particular values of ν, are better than some of the bounds known in the literature (e.g., the lower bound below is better than the one given in [10, eqn. (1.6)] for $\nu \in [0.53, 23.38]$).

Corollary 2. *The first positive zero $j_{\nu,1}$ of the Bessel function J_ν, $\nu > -1$, satisfies the inequalities*

$$2^{\frac{5}{6}} \sqrt{\nu+1}\, \sqrt[6]{(\nu+2)(\nu+3)} < j_{\nu,1} < \sqrt{2(\nu+1)(\nu+3)}.$$

The rest of the paper is organized as follows. In Sect. 2 we present some preliminary facts, which are needed for the proof of Theorem 1. In Sect. 2.1 we quote a known relation between the best Markov constant $c_n(\alpha)$ and the smallest (positive) zero of a polynomial $Q_n(x) = Q_n(x, \alpha)$ of degree n, defined by a three-term recurrent relation. By this definition, Q_n is identified as an orthogonal polynomial with respect to a measure supported on $\mathbb{R}_+$. In Sect. 2.2 we give lower and upper bounds for the largest zero of a polynomial, which has only positive zeros, in terms of a few of its highest degree coefficients. In Sect. 3 we prove formulae for the four lowest degree coefficients of the polynomial Q_n. The proof of our main result, Theorem 1, is given in Sect. 4. As the proof involves some lengthy tough straightforward calculations, for performing part of them we have used the assistance of a computer algebra system. Section 5 contains some final remarks.

2 Preliminaries

In this section we quote some known facts, and prove some results which will be needed for the proof of Theorem 1.

2.1 A Relation Between $c_n(\alpha)$ and an Orthogonal Polynomial

As was already said in the introduction, the best constant in a L_2 Markov inequality for polynomials of degree not exceeding n is equal to the largest singular value of a certain $n \times n$ matrix, say $\mathbf{A}_n$. The latter is equal to a square root of the largest eigenvalue of $\mathbf{A}_n^\top \mathbf{A}_n$ (or $\|\mathbf{A}_n\|_2$, the second matrix norm of $\mathbf{A}_n$). However, finding explicitly $\|\mathbf{A}_n\|_2$ (and for all $n \in \mathbb{N}$) is a fairly difficult task, and this explains the lack of many results on the sharp constants in the L_2 Markov inequalities. To avoid this difficulty, some authors simply try to estimate $\|\mathbf{A}_n\|_2$, or use other matrix norms, e.g., $\|\mathbf{A}_n\|_\infty$, the Frobenius norm, etc.

Our approach to the proof of Theorem 1 makes use of the following theorem:

Theorem 3 ([9, p. 85]). *The quantity $1/[c_n(\alpha)]^2$ is equal to the smallest zero of the polynomial $Q_n(x) = Q_n(x,\alpha)$, which is defined recursively by*

$$
\begin{aligned}
&Q_{n+1}(x) = (x-d_n)Q_n(x) - \lambda_n^2 Q_{n-1}(x), \quad n \geq 0;\\
&Q_{-1}(x) := 0, \;\; Q_0(x) := 1;\\
&d_0 := 1+\alpha, \;\; d_n := 2 + \frac{\alpha}{n+1}, \quad n \geq 1;\\
&\lambda_0 > 0 \;\text{ arbitrary, }\; \lambda_n^2 := 1 + \frac{\alpha}{n} \quad n \geq 1.
\end{aligned}
$$

By Favard's theorem, for any $\alpha > -1$, $\{Q_n(x,\alpha)\}_{n=0}^\infty$ form a system of monic orthogonal polynomials, and, in addition, we know that the support of their ortogonality measure is in $\mathbb{R}_+$. Theorem 3 transforms the problem of finding or estimating $c_n(\alpha)$ to a problem for finding or estimating the extreme zeros of orthogonal polynomials, or, equivalently, the extreme eigenvalues of certain tri-diagonal (Jacobi) matrices. For the latter problem one can apply numerous powerful methods such as the Gershgorin circles, the ovals of Cassini, etc. For more details on this kind of methods we refer the reader to the excellent paper of van Doorn [30].

However, we choose here a different approach for estimating the smallest positive zero of $Q_n(x,\alpha)$, which seems to be efficient, too.

2.2 Bounds for the Largest Zero of a Polynomial Having Only Positive Roots

In view of Theorem 3, we need to estimate the smallest (positive) zero of the polynomial $Q_n(x,\alpha)$. On using the three-term recurrence relation for $\{Q_m\}_{m=0}^{\infty}$, we can evaluate (at least theoretically) as many coefficients of $Q_n(x)$ as we wish (and thus coefficients of the reciprocal polynomial $x^n Q_n(x^{-1})$, too). Our proof of Theorem 1 makes use of the following statement.

Proposition 1. *Let $P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^{n-1} b_{n-1} x + (-1)^n b_n$ be a polynomial with positive roots $x_1 \le x_2 \le \cdots \le x_n$. Then the largest zero x_n of P satisfies the inequalities:*

(i) $\dfrac{b_1}{n} \le x_n < b_1$;

(ii) $b_1 - 2\dfrac{b_2}{b_1} \le x_n < (b_1^2 - 2b_2)^{\frac{1}{2}}$;

(iii) $\dfrac{b_1^3 - 3b_1 b_2 + 3b_3}{b_1^2 - 2b_2} \le x_n < (b_1^3 - 3b_1 b_2 + b_3)^{\frac{1}{3}}$.

Proof. Part (i) follows trivially from

$$\frac{b_1}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n} \le x_n < x_1 + x_2 + \cdots + x_n = b_1 .$$

For the proof of parts (ii) and (iii) we make use of Newton's identities to obtain

$$x_1^2 + x_2^2 + \cdots + x_n^2 = b_1^2 - 2b_2, \qquad x_1^3 + x_2^3 + \cdots + x_n^3 = b_1^3 - 3b_1 b_2 + 3b_3 .$$

Now (ii) follows from

$$\frac{b_1^2 - 2b_2}{b_1} = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \le x_n < (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} = (b_1^2 - 2b_2)^{\frac{1}{2}}$$

and (iii) follows from

$$\frac{b_1^3 - 3b_1 b_2 + 3b_3}{b_1^2 - 2b_2} = \frac{x_1^3 + \cdots + x_n^3}{x_1^2 + \cdots + x_n^2} \le x_n < (x_1^3 + \cdots + x_n^3)^{\frac{1}{3}} = (b_1^3 - 3b_1 b_2 + 3b_3)^{\frac{1}{3}} .$$

It is clear from the proof that the lower bounds for x_n are attained only when $x_1 = x_2 = \cdots = x_n$. □

3 The Lowest Degree Coefficients of the Polynomial $Q_{n,\alpha}$

Let us denote by $a_{k,n} = a_{k,n}(\alpha)$, $k = 0, \ldots, n$, the coefficients of the monic polynomial $Q_n(x) = Q_n(x,\alpha)$, introduced in Theorem 3, i.e.,

$$Q_n(x) = Q_n(x,\alpha) = x^n + a_{n-1,n}x^{n-2} + \cdots + a_{3,n}x^3 + a_{2,n}x^2 + a_{1,n}x + a_{0,n}\,.$$

For the sake of convenience, we set $a_{m,m} = 1$, $m \geq 0$, and

$$a_{k,m} = 0\,, \text{ if } k < 0 \text{ or } k > m\,.$$

From the recursive definition of Q_n we have $Q_0(x) = 1$, $Q_1(x) = x - \alpha - 1$, thus

$$a_{0,1} = -\alpha - 1\,,$$

and for $n \in \mathbb{N}$ we obtain a recurrence relations for the coefficients of Q_{n-1}, Q_n and Q_{n+1}:

$$a_{k,n+1} = a_{k-1,n} - \Big(2 + \frac{\alpha}{n+1}\Big)\, a_{k,n} - \Big(1 + \frac{\alpha}{n}\Big) a_{k,n-1}\,, \quad k = 0,\ldots,n\,. \tag{8}$$

Now recurrence relation (8) will be used to prove consecutively formulae for the coefficients $a_{k,n}$, $0 \leq k \leq 3$.

Proposition 2. *For all $n \in \mathbb{N}_0$, the coefficient $a_{0,n}$ of the polynomial Q_n is given by*

$$a_{0,n} = (-1)^n \prod_{k=1}^{n} \Big(1 + \frac{\alpha}{k}\Big)\,.$$

Proof. We apply induction with respect to n. Since $a_{0,0} = 1$ and $a_{0,1} = -(1+\alpha)$, Proposition 2 is true for $n = 0$ and $n = 1$. For $k = 0$ the recurrence relation (8) becomes

$$a_{0,n+1} = -\Big(2 + \frac{\alpha}{n+1}\Big)\, a_{0,n} - \Big(1 + \frac{\alpha}{n}\Big) a_{0,n-1}\,, \quad n \in \mathbb{N}\,.$$

Assuming Proposition 2 is true for $m \leq n$, for $m = n+1$ we obtain

$$\begin{aligned} a_{0,n+1} &= -\Big(2 + \frac{\alpha}{n+1}\Big)\,(-1)^n \prod_{k=1}^{n} \Big(1 + \frac{\alpha}{k}\Big) - \Big(1 + \frac{\alpha}{n}\Big)\,(-1)^{n-1} \prod_{k=1}^{n-1} \Big(1 + \frac{\alpha}{k}\Big) \\ &= (-1)^{n+1} \prod_{k=1}^{n+1} \Big(1 + \frac{\alpha}{k}\Big)\,, \end{aligned}$$

hence the induction step is done, and Proposition 2 is proved. □

Before proceeding with the proof of the formulae for $a_{k,n}$, $1 \leq k \leq 3$, let us point out to the relation

$$a_{0,m+1} = -\Big(1 + \frac{\alpha}{m+1}\Big)\, a_{0,m}\,, \quad m \in \mathbb{N}_0\,, \tag{9}$$

which follows from Proposition 2, and will be used in the proof of the next propositions.

Proposition 3. *For all $n\in\mathbb{N}_0$, the coefficient $a_{1,n}$ of the polynomial Q_n is given by*

$$a_{1,n}=-\frac{n(n+1)}{2(\alpha+1)}\,a_{0,n}\,.$$

Proof. Again, we apply induction on n. Proposition 3 is true for $n=0$ and $n=1$. Indeed, by our convention, $a_{1,0}=0$, and $a_{1,1}=1$ also obeys the desired representation, as $a_{0,1}=-(1+\alpha)$. Assume that Proposition 3 is true for $m\le n$, $m\in\mathbb{N}$. From the recurrence relation (8) (with $k=1$), the induction hypothesis and (9) we obtain

$$\begin{aligned}a_{1,n+1}&=a_{0,n}-\Big(2+\frac{\alpha}{n+1}\Big)\,a_{1,n}-\Big(1+\frac{\alpha}{n}\Big)\,a_{1,n-1}\\&=a_{0,n}+\Big(2+\frac{\alpha}{n+1}\Big)\,\frac{n(n+1)}{2(\alpha+1)}\,a_{0,n}+\Big(1+\frac{\alpha}{n}\Big)\,\frac{(n-1)n}{2(\alpha+1)}\,a_{0,n-1}\\&=a_{0,n}\Big[1+\Big(2+\frac{\alpha}{n+1}\Big)\,\frac{n(n+1)}{2(\alpha+1)}-\frac{(n-1)n}{2(\alpha+1)}\Big]\end{aligned}$$

$$\begin{aligned}&=\frac{a_{0,n}}{2(\alpha+1)}\big[n^2+(\alpha+3)n+2(\alpha+1)\big]=a_{0,n}\,\frac{(n+2)(n+\alpha+1)}{2(\alpha+1)}\\&=\frac{(n+1)(n+2)}{2(\alpha+1)}\Big(1+\frac{\alpha}{n+1}\Big)\,a_{0,n}=-\frac{(n+1)(n+2)}{2(\alpha+1)}\,a_{0,n+1}\,.\end{aligned}$$

Hence, the induction step is done, and the proof of Proposition 3 is complete. □

Proposition 4. *For all $n\in\mathbb{N}_0$, the coefficient $a_{2,n}$ of the polynomial Q_n is given by*

$$a_{2,n}=\frac{(n-1)n(n+1)}{24(\alpha+1)(\alpha+2)(\alpha+3)}\,\big[3(\alpha+2)n+2(\alpha+6)\big]\,a_{0,n}\,.$$

Proof. The claim is true for $n=0,1$ (according to our convention), and also for $n=2$, as in this case, taking into account that $a_{0,2}=1/\big((1+\alpha)(1+\alpha/2)\big)$, the above formula produces $a_{2,2}=1$. Assume now that the proposition is true for $m\le n$, where $n\in\mathbb{N}$, $n\ge 2$. We shall prove that it is true for $m=n+1$, too, thus proving Proposition 4 by induction. On using the recurrence relation (8) (with $k=2$), the inductional hypothesis, Proposition 3 and (9) we obtain

$$\begin{aligned}a_{2,n+1}&=a_{1,n}-\Big(2+\frac{\alpha}{n+1}\Big)\,a_{2,n}-\Big(1+\frac{\alpha}{n}\Big)\,a_{2,n-1}\\&=-\frac{n(n+1)}{2(\alpha+1)}\,a_{0,n}-\Big(2+\frac{\alpha}{n+1}\Big)\,\frac{(n-1)n(n+1)\big[3(\alpha+2)n+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\,a_{0,n}\\&\quad+\frac{(n-2)(n-1)n\big[3(\alpha+2)(n-1)+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\,a_{0,n}\end{aligned}$$

$$= \frac{n(n+1)}{n+\alpha+1}\left[\frac{n+1}{2(\alpha+1)} + \left(2+\frac{\alpha}{n+1}\right)\frac{(n^2-1)\left[3(\alpha+2)n+2(\alpha+6)\right]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\right.$$
$$\left. - \frac{(n-2)(n-1)\left[3(\alpha+2)(n-1)+2(\alpha+6)\right]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\right] a_{0,n+1}\,.$$

After some calculations the expression in the big brackets simplifies to

$$\frac{(n+2)(n+\alpha+1)\left[(3(\alpha+2)(n+1)+2(\alpha+6)\right]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\,.$$

and substitution of this expression yields the desired formula for $a_{2,n+1}$. The induction proof of Proposition 4 is complete. □

Proposition 5. *For all $n \in \mathbb{N}_0$, the coefficient $a_{3,n}$ of the polynomial Q_n is given by the expression*

$$-\frac{(n-2)(n-1)n(n+1)\left[5(\alpha+2)(\alpha+4)n(n+1)+8(7\alpha+20)n+12(\alpha+20)\right]}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\,a_{0,n}.$$

Proof. Again, induction is applied with respect to n. The formula for $a_{3,n}$ is easily verified to be true for $0 \le n \le 3$. Then, assuming that this formula is true for $m \le n$, where $n \in \mathbb{N}$, $n \ge 3$, we prove that it is true also for $m = n+1$, too. The induction step is performed along the same lines as the one in the proof of Proposition 4. First, we make use of the recurrence relation (8) with $k = 3$ to express $a_{3,n+1}$ as a linear combination of $a_{2,n}$, $a_{3,n}$ and $a_{3,n-1}$. Next, we apply the inductional hypothesis and (9) to represent $a_{3,n+1}$ in the form

$$a_{3,n+1} = \frac{-(n-1)n(n+1)}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\,\frac{r(n)}{n+\alpha+1}\,a_{0,n+1}\,,$$

where $r(n) = r(n,\alpha)$ is a polynomial of 4-th degree. With some lengthy though straightforward calculation (we used a computer algebra program for verification) we obtain that

$$r(n)=(n+2)(n+\alpha+1)\left[5(\alpha+2)(\alpha+4)(n+1)(n+2)+8(7\alpha+20)(n+1)+12(\alpha+20)\right]$$

and this expression substituted in the above formula implies the desired representation of $a_{3,n+1}$. To keep the paper condensed, we omit the details. □

4 Proof of Theorem 1

For the proof of Theorem 1 we prefer to work with the (constant multiplier of) reciprocal polynomial of Q_n

$$P_n(x) = P_n(x,\alpha) = (-1)^n\left(a_{0,n}\right)^{-1} x^n\, Q_n\left(x^{-1}\right).$$

Clearly, P_n is a monic polynomial of degree n,

$$P_n(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - b_3 x^{n-3} + \cdots$$

and, in view of Propositions 2–5, its coefficients b_1, b_2 and b_3 are

$$b_1 = \frac{n(n+1)}{2(\alpha+1)}, \quad b_2 = \frac{(n-1)n(n+1)}{24(\alpha+1)(\alpha+2)(\alpha+3)}\left[3(\alpha+2)n+2(\alpha+6)\right],$$
$$b_3 = \frac{(n-2)(n-1)n(n+1)\left[5(\alpha+2)(\alpha+4)n(n+1)+8(7\alpha+20)n+12(\alpha+20)\right]}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}.$$

As was indicated in Sect. 2.1, $Q_n(x,\alpha)$ is identified an orthogonal polynomial with positive and distinct zeros. Therefore, the same can be said for the zeros of P_n (as reciprocal of Q_n). If x_n is the largest zero of P_n, then, according to Theorem 3, we have $\left[c_n(\alpha)\right]^2 = x_n$.

Now Proposition 1 (iii) applied to $P = P_n$ yields immediately the following

Proposition 6. *For all $n \in \mathbb{N}$, $n \geq 3$, the best Markov constant $c_n(\alpha)$ satisfies*

$$\frac{b_1^3 - 3b_1 b_2 + 3b_3}{b_1^2 - 2b_2} < \left[c_n(\alpha)\right]^2 < (b_1^3 - 3b_1 b_2 + 3b_3)^{\frac{1}{3}}$$

with b_1, b_2 and b_3 as given above.

The estimates for $c_n(\alpha)\big]$ in Theorem 1 are a consequence of Proposition 6. For the proof of the lower bound, we obtain that

$$b_1^3 - 3b_1 b_2 + 3b_3 - \frac{2}{(\alpha+3)(\alpha+5)}\left(n+\frac{2\alpha}{3}\right)\left(n-\frac{\alpha+1}{6}\right)(b_1^2-2b_2)$$
$$= \frac{1}{(\alpha+1)^3(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\sum_{j=1}^{5}\kappa_j(\alpha)\,n^j,$$

with

$$\kappa_1(\alpha) = \frac{1}{270}(1+\alpha)^2(10\alpha^3+100\alpha^2+321\alpha+1620),$$
$$\kappa_2(\alpha) = \frac{1}{36}(1+\alpha)(4\alpha^4+35\alpha^3+166\alpha^2+417\alpha+660),$$
$$\kappa_3(\alpha) = \frac{1}{54}(4\alpha^5+36\alpha^4+192\alpha^3+625\alpha^2+1527\alpha+1332),$$
$$\kappa_4(\alpha) = \frac{1}{36}(\alpha^4-\alpha^3+157\alpha^2+579\alpha+780),$$
$$\kappa_5(\alpha) = \frac{1}{30}(\alpha^3+7\alpha^2+136\alpha+280).$$

Obviously, $\kappa_j(\alpha) > 0$ for $\alpha > -1$, $1 \leq j \leq 5$, and hence the lower bound holds:

$$\left[c_n(\alpha)\right]^2 > \frac{b_1^3 - 3b_1 b_2 + 3b_3}{b_1^2 - 2b_2} > \frac{2}{(\alpha+3)(\alpha+5)}\left(n+\frac{2\alpha}{3}\right)\left(n-\frac{\alpha+1}{6}\right).$$

For the proof of the upper bound for $c_n(\alpha)$ in Theorem 1, we find that

$$\frac{1}{(\alpha+1)^3(\alpha+3)(\alpha+5)}(n+1)^3\Big(n+\frac{2(\alpha+1)}{5}\Big)^3 - \left(b_1^3 - 3b_1 b_2 + 3b_3\right) = \frac{1}{(\alpha+1)^2(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\sum_{j=0}^{5}\nu_j(\alpha)\,n^j,$$

where

$$\begin{aligned}
\nu_0(\alpha) &= \frac{8}{125}(1+\alpha)^2(2+\alpha)(4+\alpha);\\
\nu_1(\alpha) &= \frac{3}{250}(1+\alpha)(16\alpha^3+152\alpha^2+439\alpha-52),\\
\nu_2(\alpha) &= \frac{1}{500}(96\alpha^4+1363\alpha^3+5656\alpha^2+9167\alpha+2828),\\
\nu_3(\alpha) &= \frac{1}{250}(16\alpha^4+363\alpha^3+2506\alpha^2+7167\alpha+4708),\\
\nu_4(\alpha) &= \frac{1}{100}(23\alpha^3+446\alpha^2+1657\alpha+2164),\\
\nu_5(\alpha) &= \frac{3}{5}(5\alpha+16).
\end{aligned}$$

We shall show now that

$$\sum_{j=0}^{5}\nu_j(\alpha)\,n^j \geq 0, \qquad n\geq 2,\ \ \alpha>-1. \tag{10}$$

Notice that, unlike the case with the coefficients $\{\kappa_j(\alpha)\}_{j=1}^5$, which are all positive for all admissible values of α, i.e., $\alpha>-1$, here the coefficients $\nu_j(\alpha)$, $1\leq j\leq 3$, assume negative values for some $\alpha\in(-1,0)$ ($\nu_1(\alpha)$ is negative also for some $\alpha>0$).

Since $\nu_4(\alpha)$ and $\nu_5(\alpha)$ are positive for $\alpha>-1$, for $n\geq 2$ we have

$$\sum_{j=3}^{5}\nu_j(\alpha)\,n^j \geq \big(4\nu_5(\alpha)+2\nu_4(\alpha)+\nu_3(\alpha)\big)n^3 =: \widetilde{\nu}_3(\alpha)\,n^3,$$

where

$$\widetilde{\nu}_3(\alpha) = \frac{1}{125}(8\alpha^4+239\alpha^3+2368\alpha^2+9226\alpha+12564).$$

Since $\widetilde{\nu}_3(\alpha)>0$ for $\alpha>-1$, we have

$$\sum_{j=2}^{5} \nu_j(\alpha)\, n^j \geq \big(2\widetilde{\nu}_3(\alpha) + \nu_2(\alpha)\big)\, n^2 =: \widetilde{\nu}_2(\alpha)\, n^2\,, \qquad n \geq 2\,,$$

where

$$\widetilde{\nu}_2(\alpha) = \frac{1}{100}\,(32\,\alpha^4 + 655\,\alpha^3 + 4920\,\alpha^2 + 16595\,\alpha + 20668)\,.$$

Now, from $\widetilde{\nu}_2(\alpha) > 0$ for $\alpha > -1$, we obtain

$$\sum_{j=1}^{5} \nu_j(\alpha)\, n^j \geq \big(2\widetilde{\nu}_2(\alpha) + \nu_1(\alpha)\big)\, n =: \widetilde{\nu}_1(\alpha)\, n\,, \qquad n \geq 2\,,$$

with

$$\widetilde{\nu}_1(\alpha) = \frac{1}{250}\,(160\,\alpha^4 + 3323\,\alpha^3 + 25056\,\alpha^2 + 84292\,\alpha + 103184) > 0\,, \quad \alpha > -1\,.$$

Hence, $\sum_{j=0}^{5} \nu_j(\alpha)\, n^j \geq \widetilde{\nu}_1(\alpha)\, n + \nu_0(\alpha) > 0$, and (10) is proved. From (10) we conclude that

$$\frac{1}{(\alpha+1)^3(\alpha+3)(\alpha+5)}\,(n+1)^3\Big(n + \frac{2(\alpha+1)}{5}\Big)^3 > b_1^3 - 3b_1 b_2 + 3b_3\,,$$

In view of Proposition 6, the latter inequality proves the upper bound for $c_n(\alpha)$ in Theorem 1.

5 Concluding Remarks

1. Our main concern here is the major terms in the bounds for the best Markov constant $c_n(\alpha)$, obtained through Proposition 1. We did not care much about the lower degree terms, where perhaps some improvement is possible.

2. Obviously, Dörfler's upper bound for $c_n(\alpha)$ in (4) is a consequence of Proposition 1 (i). Dörfler's lower bound for $c_n(\alpha)$ in [8], which is slightly better than the one given in (4), is obtained from Proposition 1 (ii). Both our lower and upper bounds for the asymptotic constant $c(\alpha)$, given in Corollary 1, are superior for all $\alpha > -1$ to Dörfler's bounds obtained from (4) .

3. The upper bounds for the largest zero x_n of a polynomial having only real and positive zeros in Proposition 1 (ii) and (ii) admit some improvement. For instance, in Proposition 1 (ii) one can apply the quadratic mean – arithmetic mean inequality to obtain

$$b_1^2 - 2b_2 = x_n^2 + \sum_{i=1}^{n-1} x_i^2 \geq x_n^2 + \frac{\Big(\sum_{i=1}^{n-1} |x_i|\Big)^2}{n-1} \geq x_n^2 + \frac{(b_1 - x_n)^2}{n-1}\,,$$

which yields a (slightly stronger) quadratic inequality for x_n (actually, for any of the zeros of the polynomial P),

$$n x_n^2 - 2b_1 x_n + 2(n-1)b_2 - (n-2)b_1^2 \leq 0\,.$$

The solution of the latter inequality,

$$\frac{1}{n}\left[b_1 - \sqrt{(n-1)^2 b_1^2 - 2(n-1)n\,b_2}\right] \leq x_n \leq \frac{1}{n}\left[b_1 + \sqrt{(n-1)^2 b_1^2 - 2(n-1)n\,b_2}\right],$$

provides lower and upper bounds for the zeros of an arbitrary real-root monic polynomial of degree n in terms of its two leading coefficients b_1 and b_2. This result, due to Laguerre, is known also as Laguerre-Samuelson inequality (for more details, see e.g. [12] and the references therein).

In a similar way one can obtain a slight improvement for the upper bound in Proposition 1 (iii). However, in our case this improvement is negligible (it affects only the lower degree terms in the upper bound for $c_n(\alpha)$).

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