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RESOLUTION OF SYMPLECTIC CYCLIC ORBIFOLD SINGULARITIES

KLAUS NIEDERKRÜGER AND FEDERICA PASQUOTTO

Abstract. In this paper we present a method to obtain resolutions of symplectic orbifolds arising as quotients of pre-symplectic semi-free $\mathbb{S}^1$–actions. This includes, in particular, all orbifolds arising from symplectic reduction of Hamiltonian $\mathbb{S}^1$–manifolds at regular values, as well as all isolated cyclic orbifold singularities.

As an application, we show that pre-quantisations of symplectic orbifolds are symplectically fillable by a smooth manifold.

1. Introduction

Symplectic quotients are an important source of new symplectic manifolds: they appear as *symplectic reductions* in the context of Hamiltonian actions and the associated moment maps. More generally, reduced spaces corresponding to regular values of the moment map turn out to be *symplectic orbifolds*, but one can still look for a closed smooth symplectic manifold which is isomorphic to the orbifold outside a neighbourhood of the singular set: we call this object a *symplectic orbifold resolution*.

Even in the case of a reduced space corresponding to a singular value of the moment map, where singularities of a more complicated type can occur, Kirwan's "partial desingularisation" method [Kir85] can be applied to obtain a resolution which has only orbifold singularities.

A technique involving quotients and resolutions via blow-ups is used in [FM06] to construct an example of an 8–dimensional non-formal simply connected symplectic manifold. Unfortunately, the desingularisation method they develop works only for very special cyclic orbifold singularities. With the construction described in this paper, we are able to find resolutions for all isolated cyclic orbifold singularities, and also for reduced spaces obtained from symplectic reduction at the regular level sets of a Hamiltonian function generating an $\mathbb{S}^1$–action. More precisely, we construct resolutions for symplectic orbifolds which are the quotients of pre-symplectic semi-free $\mathbb{S}^1$–actions.

As a direct application, we are able to show that a Seifert manifold with a contact structure that is $\mathbb{S}^1$–invariant and transverse to the fibres is symplectically fillable by a smooth manifold.

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1.1. Isolated singularities in dimension four.

We start our paper by discussing singularities of the simplest form: we hope that this will provide the reader with some motivation and will serve as the right introduction to the difficulties arising when considering more general examples.

In the four-dimensional case, we can give a very explicit description of the resolution of an isolated orbifold singularity. In order to do so, we use weighted blow-ups of isolated symplectic orbifold singularities, as defined in [God01].

Let $x \in \left(M^{(4)}, \omega\right)$ be an isolated orbifold singularity with structure group $\mathbb{Z}_p = \{1, \xi, \xi^2, \dots, \xi^{p-1}\}$. The orbifold chart around x is $\mathbb{Z}_p$–equivariantly symplectomorphic to a neighbourhood of 0 in $\mathbb{C}^2$ with $\mathbb{Z}_p$ acting by

$$(z_1, z_2) \mapsto (\xi^m z_1, \xi z_2), \quad 0 < m < p, \quad \gcd(m,p) = 1 \ .$$

If m and p were not coprime, the orbifold singularity would not be isolated. We can define an $\mathbb{S}^1$–action on $\mathbb{C}^2$ by setting $\lambda\cdot(z_1, z_2) = (\lambda^m z_1, \lambda z_2)$ for $\lambda \in \mathbb{S}^1$. The actions of $\mathbb{Z}_p$ and $\mathbb{S}^1$ commute, but the induced circle action on $\mathbb{C}^2/\mathbb{Z}_p$ is not effective. Instead we have to go to the $\mathbb{S}^1/\mathbb{Z}_p$–action obtained from the following exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{S}^1 \longrightarrow \hat{\mathbb{S}}^1 \longrightarrow 0 ,$$

with the homomorphism of the circle given by $\lambda \mapsto \lambda^p$. This defines now a symplectic $\hat{\mathbb{S}}^1$–action on $\mathbb{C}^2/\mathbb{Z}_p$ by

$$\mu \cdot [z_1, z_2] = [\lambda \cdot (z_1, z_2)] = [\lambda^m z_1, \lambda z_2]$$

for $\mu \in \hat{\mathbb{S}}^1$ and a $\lambda \in \mathbb{S}^1$ such that $\lambda^p = \mu$.

The weighted blow-up of $\mathbb{C}^2/\mathbb{Z}_p$ at the origin can be represented as a symplectic cut with respect to this $\hat{\mathbb{S}}^1$–action: Take the product manifold $\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}$ with the symplectic form given by $(\omega, -i\, dw \wedge d\bar{w})$ and the effective $\hat{\mathbb{S}}^1$–action

$$\mu \cdot \big([z_1, z_2], w\big) = \big(\mu \cdot [z_1, z_2], \mu^{-1} w\big) = \big([\lambda^m z_1, \lambda z_2], \lambda^{-p} w\big), \quad \lambda^p = \mu .$$

The blow-up is the symplectic reduction of this space. The Hamiltonian function that corresponds to this action is

$$H\big([z_1, z_2], w\big) = m|z_1|^2 + |z_2|^2 - p|w|^2 ,$$

and the ε–level set $H^{-1}(\varepsilon)$ is diffeomorphic to the manifold $\mathbb{S}^3/\mathbb{Z}_p \times \mathbb{C}$. The last step consists in taking the quotient $H^{-1}(\varepsilon)/\hat{\mathbb{S}}^1$ to obtain a symplectic orbifold that can be glued to $\mathbb{C}^2/\mathbb{Z}_p$ after removing a neighbourhood of the origin.

There is only one singular point in $H^{-1}(\varepsilon)/\hat{\mathbb{S}}^1$, namely $\big([1, 0], 0\big) \in \mathbb{S}^3/\mathbb{Z}_p \times \mathbb{C}$, with stabiliser $\mathbb{Z}_m$. By the slice theorem, a neighbourhood of this point admits an orbifold chart equivalent to $\mathbb{C}^2$ with structure group $\mathbb{Z}_m$ acting by $\eta \cdot (w_1, w_2) = (\eta^{-p} w_1, \eta w_2)$.

Now choose $a_1 \in \mathbb{Z}$ such that $0 < a_1 m - p < m$ and set $m_1 = a_1 m - p$ and $p_1 = m$: then the new singularity can also be modelled by $\mathbb{Z}_{p_1}$ acting by $\eta \cdot (z_1, z_2) = (\eta^{m_1} z_1, \eta z_2)$ with $\gcd(m_1, p_1) = 1$, because if b divides both p_1 and m_1 then it also divides m and p. We are thus in the initial type of situation, but we have managed to reduce the order of the singularity.

If we blow up once more, we replace this by a new singularity, this time with structure group $\mathbb{Z}_{m_1}$ acting by $\zeta \cdot (z_1, z_2) = \big(\zeta^{a_2 m_1 - p_1} z_1, \zeta z_2\big)$. If we iterate this blow-up process, at each step we replace the previous singularity by a new one with structure group $\mathbb{Z}_{p_i}$ acting by $(\xi^{m_i} z_1, z_2)$, where the pair (p_i, m_i) is recursively given by

$$\begin{pmatrix} p_i \\ m_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix} \begin{pmatrix} p_{i-1} \\ m_{i-1} \end{pmatrix} ,$$

with each a_i corresponding to the "roundup" of $\frac{p_{i-1}}{m_{i-1}}$, that is, the least integer $\geq \frac{p_{i-1}}{m_{i-1}}$. The sequence $[a_1, a_2, \dots]$ corresponds to the continued fraction of $\frac{p}{m}$, so in particular it is a finite sequence (a description of resolutions in terms of continued fractions is contained, for example, in Miles Reid's lecture notes [Rei]). After sufficiently many blow-ups, in other words, we get a pair of the form $(p_N, 1)$ and thus an orbifold chart which is in fact smooth. This is then the resolution of our initial singularity.

Notice that we can think of each weighted blow-up as taking the connected sum with a suitable orbifold. In dimension 4, if we define the *weighted projective space* to be

$$\mathbb{CP}(a_0, a_1, a_2) = \Big\{[z_0 : z_1 : z_2] \sim [\lambda^{a_0} z_0 : \lambda^{a_1} z_1 : \lambda^{a_2} z_2] \;\Big|\; (z_0, z_1, z_2) \in \mathbb{C}^3 - \{0\} \text{ and } \lambda \in \mathbb{C}^*\Big\},$$

the weighted blow-up of a singular point with structure group $\mathbb{Z}_p$ acting by $(z_1, z_2) \mapsto (\xi^a z_1, \xi^b z_2)$ can be described as taking the connected sum, around this point, with the orbifold $\mathbb{CP}(a, b, p)$ with reversed orientation. We can use this description to represent the resolution of a four-dimensional cyclic singularity as in Picture 1.1.

In higher dimension (≥ 6), the method just described is not suitable: even if we start with an isolated singularity, after the first blow-up the singular set is not necessarily discrete any more.

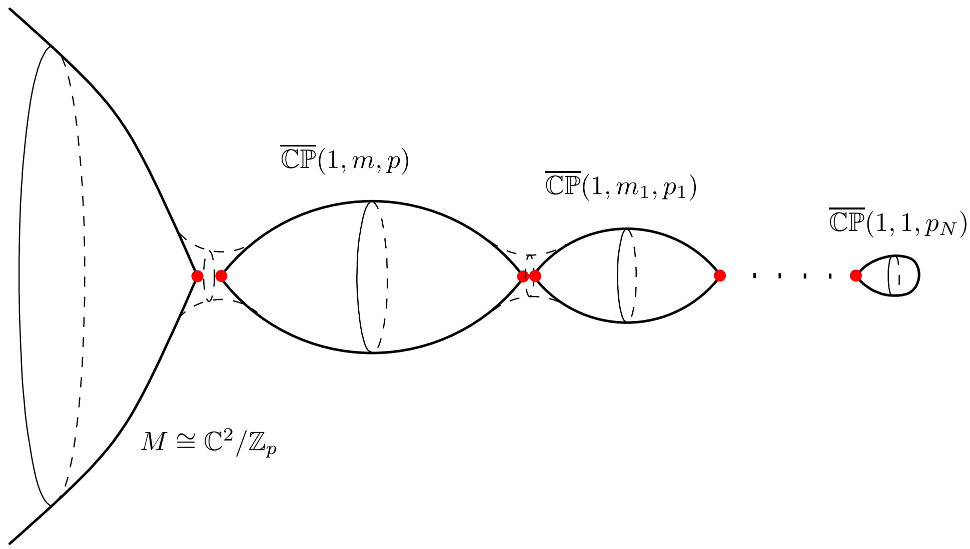

FIGURE 1. The resolution obtained via a sequence of blow-up can also be thought of as the connected sum $M\#\overline{\mathbb{CP}}(1,m,p)\#\overline{\mathbb{CP}}(1,m_1,p_1)\#\cdots\#\overline{\mathbb{CP}}(1,1,p_N)$.

Example 1. Consider an isolated orbifold singularity modelled on a neighbourhood of 0 in $\mathbb{C}^3$ with $\mathbb{Z}_p$ acting by

$$(z_1,z_2,z_3)\mapsto(\xi^{m_1}z_1,\xi^{m_2}z_2,\xi z_3),\quad 0<m_1<m_2<p,\quad \gcd(m_i,p)=1, i=1,2\ .$$

We can blow up this singularity with the method used in the four-dimensional case, that is, performing a symplectic cut with respect to a suitable circle action. If $\gcd(m_1,m_2)=d\neq 1$, though, after blowing up the singular set will be a two-dimensional suborbifold with stabiliser $\mathbb{Z}_d$ at generic points.

One could still hope to obtain a resolution of general cyclic orbifold singularities by using weighted blow-ups along suborbifolds (cf. [MS99]), but for this we would need to find a suitable circle action on the fibres of the normal orbibundle to singular strata. While this is not difficult in charts, we were not able to define a global action.

In the rest of this paper we thus adopt a different desingularisation method, which has the advantage of applying to a larger class of orbifold singularities than just cyclic isolated ones.

2. Symplectic orbifolds as quotients of semi-free actions

In this section, G will denote a compact connected Lie group, and $\mathfrak{g}$ its Lie algebra. If G acts on a manifold M, then the infinitesimal action is defined by the vector fields

$$X_M(p) := \left.\frac{d}{dt}\right|_{t=0}\exp(tX)*p$$

for every $X\in\mathfrak{g}$ and $p\in M$.

Definition. Let P be a $(2n+k)$–dimensional smooth manifold and assume that P admits a G–action

$$G\times P\to P,\ (g,x)\mapsto g*x\ ,$$

such that all orbits are k–dimensional, and a closed G–invariant 2–form ω_P, such that $\omega_P^n\neq 0$ and the infinitesimal action of G spans the kernel of ω_P. We call P a **pre-symplectic** G–manifold.

Let M be the quotient P/G and denote by π the orbit map. If S_x is a slice at the point $x\in P$, then by the slice theorem a neighbourhood of $\pi(x)\in M$ is homeomorphic to $S_x/\operatorname{Stab}(x)$. Since S_x is transverse to $\ker\omega_P(x)$, it follows that $\omega_P(x)$ has maximal rank on S_x, which is thus a $\operatorname{Stab}(x)$–invariant symplectic vector space. The fixed point set of $\operatorname{Stab}(x)$ on S_x is a symplectic subspace, which then necessarily has codimension at least 2. If $\mathrm{Stab}(x)$ is a finite group, this gives M locally the structure of a symplectic orbifold. If $\mathrm{Stab}(x)$ is not a discrete group, then its action is far from being effective. The stabiliser $\mathrm{Stab}(y)$ for $y \in S_x$ is a Lie subgroup of $\mathrm{Stab}(x)$. Under our assumption that all orbits have the same dimension, though, we get that $\dim \mathrm{Stab}(y) = \dim \mathrm{Stab}(x)$ and this means that the identity components of $\mathrm{Stab}(y)$ and $\mathrm{Stab}(x)$ are the same. Hence it follows that

$$H_0 := \bigcap_{y \in S_x} \mathrm{Stab}(y)$$

contains the identity component of $\mathrm{Stab}(x)$, and we can still find a homeomorphism from a neighbourhood of a point $\pi(x)$ in the orbit space to the quotient of S_x by a finite group, namely $\mathrm{Stab}(x)/H_0$. Finally, all these local structures are compatible and fit together to give a global symplectic orbifold structure: one can argue as in [Wei77], where the case $G = \mathbb{S}^1$ is considered.

Theorem 1. *The quotient of a pre-symplectic G–manifold by the given G–action is a symplectic orbifold.*

Example 2. The **weighted projective space** $\mathbb{CP}(a_0, a_1, \dots, a_n)$ is the $2n$–dimensional orbifold obtained as the quotient of $\mathbb{S}^{2n+1}$ by the $\mathbb{S}^1$–action

$$\lambda * (z_0, z_1, \dots, z_n) = (\lambda^{a_0} z_0, \dots, \lambda^{a_n} z_n) \ .$$

It is in fact a symplectic orbifold with the structure induced by the canonical symplectic form on $\mathbb{C}^{n+1}$, namely $\omega_0 = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$.

2.1. Symplectic resolutions.

Definition. Let (M, ω) be a symplectic orbifold with singular set X. A **symplectic ε–resolution** of M consists of a smooth symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ and a continuous map $p : (\widetilde{M}, \widetilde{\omega}) \to (M, \omega)$ for which there is an orbifold tubular neighbourhood U of X of size ε, such that

$$p|_{\widetilde{M} - p^{-1}(U)} : \widetilde{M} - p^{-1}(U) \to M - U$$

is a symplectic diffeomorphism.

In this paper we prove the following result.

Theorem 2. *Let M be a symplectic orbifold arising as the quotient of a pre-symplectic $\mathbb{S}^1$–manifold P by the given circle action. Then there exists a symplectic ε–resolution of M for any arbitrarily small $\varepsilon > 0$.*

2.2. **Isolated cyclic orbifold singularities.** Any isolated cyclic orbifold singularity can be represented as $\mathbb{C}^n/\mathbb{Z}_k$ with symplectic form $\omega = i \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, where the generator $\xi = e^{2\pi i/k}$ of $\mathbb{Z}_k$ acts by

$$\xi \cdot \mathbf{z} = (e^{2\pi i/k} z_1, e^{2\pi i a_2/k} z_2, \dots, e^{2\pi i a_n/k} z_n) \ ,$$

and $a_2, \dots, a_n \in \mathbb{N}$ are all coprime with k.

To find a resolution of the singularity using Theorem 2, just note that $\mathbb{C}^n/\mathbb{Z}_k$ can be regarded as the quotient of the pre-symplectic manifold $\mathbb{S}^1 \times \mathbb{C}^n$ by the $\mathbb{S}^1$–action given by

$$e^{i\varphi} \cdot (e^{i\vartheta}, z_1, \dots, z_n) := (e^{i(k\varphi + \vartheta)}, e^{i\varphi} z_1, e^{i a_2 \varphi} z_2, \dots, e^{i a_n \varphi} z_n) \ ,$$

with pre-symplectic 2–form $\omega + \frac{1}{k} d\vartheta \wedge \sum_{j=1}^n a_j \, d|z_j|^2$. Theorem 2 then immediately implies the following generalisation of the desingularisation result obtained in dimension 4:

Corollary 3. *Every symplectic orbifold with only isolated cyclic singularities admits a symplectic resolution.*

2.3. **The stratification of the singular set.** Let P be a pre-symplectic $\mathbb{S}^1$–manifold and denote by $X_P \subset P$ the singular set of the circle action, that is,

$$X_P = \left\{ x \in P \;\middle|\; \operatorname{Stab}(x) \neq \{1\} \right\} .$$

For a given isotropy group $\mathbb{Z}_k$, let P_k denote the union of orbits whose isotropy group is $\mathbb{Z}_k$, namely $P_k = \{x \in P \,|\, \operatorname{Stab}(x) \cong \mathbb{Z}_k\}$. Then X_P is stratified by singular strata, i.e., connected components of P_k for all $k \neq 1$. If $\pi : P \to M = X/\mathbb{S}^1$ denotes the orbit map, $X = \pi(X_P)$ is the set of orbifold singularities of M and the stratification of X_P descends to a stratification of X.

Our desingularisation method works by induction on the order of the stabilisers of the strata P_k. Namely, we start with the stratum with largest stabiliser (*minimal stratum*), remove a small neighbourhood of it and glue in a smooth manifold, in such a way that both the pre-symplectic 2–form and the $\mathbb{S}^1$–action extend to the manifold resulting from this surgery. The new singular set will also carry a stratification, but the order of the stabiliser of the minimal stratum will be strictly less than k. If we successively repeat this procedure sufficiently often, we will eventually reduce this maximal stabiliser to the trivial group. In particular, the quotient will then be a smooth symplectic manifold which, by construction, gives a resolution of the orbifold M.

After working out this desingularisation method for symplectic orbifolds, we were told by one of the authors of [GGK99] that they, and others before them, had already used the same construction in the smooth category.

3. Construction of the resolution

3.1. **A local model for the action.** Suppose P is a pre-symplectic $\mathbb{S}^1$–manifold. In a first step, we will construct a model for the neighbourhood of the minimal stratum, which then allows us to describe explicitely all the steps of the resolution. This section may seem overly detailed, but we preferred to give the complete arguments, because we are dealing with pre-symplectic instead of proper symplectic manifolds.

Choose first an auxiliary $\mathbb{S}^1$–invariant metric on P to split the tangent bundle into two invariant subbundles

$$TP = \ker \omega_P \oplus \Omega^P \;,$$

where Ω^P is the orthogonal complement of $\ker \omega_P$. Since Ω^P is a symplectic subbundle, it admits an $\mathbb{S}^1$–invariant almost complex structure J that is compatible with $\omega_P|_{\Omega^P}$ (see for example [MS98, Section 5.5]). Now we replace the auxiliary metric used above by a new one, which coincides on $\ker \omega_P$ with the old one, is given on Ω^P by $g = \omega_P(J\cdot, \cdot)$, and is such that $\ker \omega_P \perp \Omega^P$.

Consider the stratification $\{P_k\}$ of the singular set X_P given by the isotropy groups. If k is maximal, that is, there are no points in P of order larger than k, then P_k is a closed, $\mathbb{S}^1$–invariant submanifold of P, of codimension at least 2. By restricting to one component, we may further assume P_k to be connected. The restriction of the quotient map $\pi : P \to M$ to P_k gives it the structure of an $\mathbb{S}^1$–bundle (strictly speaking an $\mathbb{S}^1/\mathbb{Z}_k$–bundle) over S_k, the set of orbifold singularities of M of order k. A model of a neighbourhood of P_k is given by a neighbourhood of the zero section in the total space of its normal bundle ν_k. The linearisation of the circle action on P defines an $\mathbb{S}^1$–action on ν_k that is equivalent to the given one on P (in the sense that the exponential map defines an equivariant diffeomorphism from a neighbourhood of the zero section in ν_k to a neighbourhood of P_k in P). Later it will be necessary to introduce a second circle action. To avoid confusions, from now on we will call the action discussed above the β–action and we will write it as $\lambda *_\beta v$ for $\lambda \in \mathbb{S}^1$ and $v \in \nu_k$.

Let $x \in P_k$ be a singular point in the minimal stratum. The stabiliser $\operatorname{Stab}(x) \cong \mathbb{Z}_k$ acts by isometric J–linear transformations on Ω^P_x, hence the $\mathbb{Z}_k$–action is equivalent to

$$\lambda *_\beta \mathbf{z} = (\lambda^{\widetilde{a}_1} z_1, \ldots, \lambda^{\widetilde{a}_n} z_n), \quad \lambda \in \mathbb{Z}_k,\ \mathbf{z} \in \Omega^P_x,\ \widetilde{a}_1, \ldots, \widetilde{a}_n \in \mathbb{Z} \;.$$

Without loss of generality we may assume that $0 = \widetilde{a}_1 = \cdots = \widetilde{a}_m < \widetilde{a}_{m+1} \leq \cdots \leq \widetilde{a}_n < k$ for some m. The first m directions span the space $\Omega^P_x \cap T_x P_k$, and since the others are orthogonal, they coincide with the fibre $\nu_k(x)$. In particular, it follows that ν_k is a J–complex bundle with a fibrewise J–linear $\mathbb{Z}_k$–action.

Denote by $a_1 < \cdots < a_l$ the *distinct* exponents occurring in the normal form for the action: ν_k splits thus into a direct sum of subbundles

$$\nu_k = E_1 \oplus \cdots \oplus E_l \ ,$$

where $E_i(x)$ denotes the eigenspace corresponding to the eigenvalue λ^{a_i} in the fibre at the point x. This splitting is well defined for each component of P_k. This allows us to extend the $\mathbb{Z}_k$–action to a second circle action by setting for any $\lambda \in \mathbb{S}^1$

$$\lambda *_\varphi v := \lambda^{a_1} v_1 + \cdots + \lambda^{a_l} v_l \ ,$$

where $v = v_1 + \cdots + v_l$ is a splitting with respect to the eigenspaces defined above. This circle action is fibrewise and J–linear, and commutes with the original β–action. Unfortunately it does not need to respect the pre-symplectic form ω_P. By averaging ω_P over the φ–action, we obtain a closed 2–form ω on ν_k that is invariant with respect to both the β– and the φ–action, and such that the β–orbits still lie in the kernel of ω. There is a small neighbourhood of the zero section of ν_k where we also have $\omega^n \neq 0$ and hence ω is a pre-symplectic form with respect to the β–action. To prove this it suffices to show that ω_P is φ–invariant on the zero section. In fact, at the zero section P_k of ν_k, there is a well defined splitting of the tangent bundle $T\nu_k|_{P_k} = TP_k \oplus \nu_k$ and we can write

$$T\nu_k|_{P_k} = TP_k \oplus E_1 \oplus \cdots \oplus E_l \ .$$

The E_j's are J–linear subspaces and orthogonal to each other, so that they are also symplectically orthogonal. The linearised φ–action on a vector $w + v_1 + \cdots + v_l \in T\nu_k$ with $w \in T_xP_k$ and $v_j \in E_j(x)$ is given by

$$\lambda *_\varphi (w + v_1 + \cdots + v_l) = w + \lambda^{a_1} v_1 + \cdots + \lambda^{a_l} v_l$$

and using the orthogonality relations, it is easy to check that

$$\omega_P(\lambda *_\varphi v, \lambda *_\varphi v') = \omega_P(v, v') \ .$$

Hence ω_P is φ–invariant on the zero section of ν_k and we do not change it there by averaging. It follows that there is a small neighbourhood of the zero section, where ω will be pre-symplectic.

The proposition below shows that a neighbourhood of P_k can be represented by a neighbourhood of the zero section in ν_k with β–action and pre-symplectic form ω, and an auxiliary action φ.

Proposition 4. *There exist neighbourhoods U_1, U_2 of P_k in ν_k and a β–equivariant diffeomorphism $\Psi : U_1 \to U_2$ such that*

$$\Psi^*\omega = \omega_P \ .$$

Proof. The restrictions of ω and ω_P coincide along the zero section of ν_k. In particular, if we denote by $i_0 : P_k \hookrightarrow \nu_k$ the inclusion of P_k as the zero section of ν_k, we have

$$i_0^*(\omega - \omega_P) = 0 \ .$$

This implies that there exists a 1–form α on ν_k such that $\omega - \omega_P = d\alpha$ and moreover α vanishes on $T_x\nu_k$ for every $x \in P_k$ (see for example [CdS01, Theorem 6.8] or [MS98, Lemma 3.14]). Furthermore, we may assume α to be β–invariant, because if it were not, we could replace it by

$$\alpha' = \int_{\mathbb{S}^1} (\lambda^*\alpha) \ d\lambda \ ,$$

which still satisfies

$$d\alpha' = \int_{\mathbb{S}^1} \big(\lambda^* d\alpha\big) \, d\lambda = \int_{\mathbb{S}^1} \lambda^*(\omega - \omega_P) \, d\lambda = \omega - \omega_P$$

and $\alpha' = 0$ on points of P_k. Notice that from the β–invariance, we also obtain that $i_{X_\beta}\alpha = \text{const}$, but since $i_{X_\beta}\alpha = 0$ on the zero section, if follows that X_β lies everywhere in the kernel of α.

Now define the 1–parameter family of pre-symplectic 2–forms

$$\omega_s := s\,\omega + (1 - s)\,\omega_p \ .$$

Assume there exists a time-dependent vector field X_s such that $\omega = \left(\Phi_s^{X_s}\right)^* \omega_s$, where $\Phi_s^{X_s}$ denotes the flow of X_s. Then we have

$$0 = \frac{d}{ds}\left(\Phi_s^{X_s}\right)^* \omega_s = \left(\Phi_s^{X_s}\right)^* \left(\mathcal{L}_{X_s}\omega_s + \frac{d}{ds}\omega_s\right)$$

which is equivalent to $\mathcal{L}_{X_s}\omega_s + d\alpha = 0$, or

$$d\big(i_{X_s}\omega_s + \alpha\big) = 0\ .$$

In order to show that ω and ω_P are β–equivariantly isomorphic, we need to find X_s satisfying the last equation, integrating to a time–1 flow in a neighbourhood of P_k and commuting with X_β.

Let X_s be the unique vector field determined by

$$\begin{cases} g(X_\beta, X_s) = 0 \Leftrightarrow X_s \in \Omega^P = (\ker\omega_P)^\perp \\ \big(i_{X_s}\omega_s + \alpha\big)\big|_{\Omega^P} = 0 \end{cases}$$

It is easy to see that $i_{X_\beta}\big(i_{X_s}\omega_s + \alpha\big) = 0$, so it follows that $i_{X_s}\omega_s + \alpha = 0$. To see that X_s commutes with X_β, compute

$$0 = \mathcal{L}_{X_\beta}\big(i_{X_s}\omega_s + \alpha\big) = i_{\mathcal{L}_{X_\beta}X_s}\omega_s$$

and

$$0 = \mathcal{L}_{X_\beta}\big(g(X_s, X_\beta)\big) = g(X_\beta, \mathcal{L}_{X_\beta}X_s)\ .$$

Combining this with the fact that ω_s is nondegenerate on Ω^P we get that $\mathcal{L}_{X_\beta}X_s = 0$.

Finally notice that, since α vanishes at all points of the zero section of ν_k, X_s also does and hence its flow is defined in a neighbourhood of this section up to time one. □

Proposition 5. *There exists a neighbourhood U of P_k in ν_k and a non negative Morse-Bott function $\mu_\varphi : U \to \mathbb{R}$ such that*

- $i_{X_\varphi}\omega = d\mu_\varphi$
- *μ_φ vanishes only on the zero section of ν_k, and it is strictly increasing in radial fibre direction.*

Proof. Since ω is φ–invariant, one has that $di_{X_\varphi}\omega = \mathcal{L}_{X_\varphi}\omega = 0$. For the time being, let U be any tubular neighbourhood of P_k, where ω is defined. The closed 1–form $i_{X_\varphi}\omega$ represents a class in $H^1(U)$ which vanishes if we pull it back to the zero section P_k: Given that $H^1(U) \cong H^1(P_k)$, it follows that $i_{X_\varphi}\omega$ is exact on U, i.e., there exists a function μ_φ such that $i_{X_\varphi}\omega = d\mu_\varphi$. The function μ_φ is uniquely defined up to an additive constant (which we may choose such that $\mu_\varphi \equiv 0$ on P_k) and is β–invariant.

Recall that a function $f : M \to \mathbb{R}$ is called Morse-Bott if $\operatorname{Crit}(f)$ is a submanifold of M and $T_x \operatorname{Crit}(f) = \ker \operatorname{Hess}_x(f)$ for all $x \in \operatorname{Crit}(f)$, where $\operatorname{Hess}_x(f) : T_xM \to T_xM$ denotes the Hessian of f at the point x.

In the case we are considering $\operatorname{Crit}(\mu_\varphi) = P_k$: in fact, if $v = (x, 0) \in P_k$, $X_\varphi(v) = 0$, and hence $d\mu_\varphi = i_{X_\varphi}\omega = 0$. Conversely, assume $d\mu_\varphi = 0$ i.e. $X_\varphi \in \ker\omega = \langle X_\beta\rangle$. But X_β is transverse to the fibre of ν_k, whereas X_φ always lies in the fibre, so $X_\varphi \in \langle X_\beta\rangle$ can only occur on P_k, where one has $X_\varphi = 0$. It is easy to show that if $v \in P_k$, the inclusion $T_v \operatorname{Crit}(\mu_\varphi) \leq \ker \operatorname{Hess}_v(\mu_\varphi)$ holds.

To see that equality holds one needs to show that $\dim\ker\operatorname{Hess}_v(\mu_\varphi) \leq \dim P_k$ or, equivalently, that $\operatorname{rank}\operatorname{Hess}_v \mu_\varphi$ is at least equal to the rank of ν_k. Restricting ω, φ and μ_φ to one fibre of ν_k we are in a proper symplectic situation and we may conclude that $\mu_\varphi|_{\nu_k(x)}$ is Morse (see [MS98, Section 5.5]), hence in particular it has full rank. Introducing bundle coordinates on U and computing the matrix of second derivatives of μ_φ, which represent the Hessian in these coordinates, we see that it always contains a non singular block, corresponding to the above restriction of μ_φ to one fibre, having rank equal to the rank of ν_k. This proves that μ_φ is Morse-Bott, and it only remains to show that it is positive outside P_k.

Let ∂_r be the radial vector field on ν_k given by

$$\partial_r(v) = \frac{d}{dt}\bigg|_{t=1} t \cdot v$$

for $v \in \nu_k$. We will show that μ_φ strictly increases in radial direction, more precisely that $\mathcal{L}_{\partial_r}\mu_\varphi \geq 0$ in U (possibly after shrinking U) with equality only at the zero section. By definition of μ_φ, one has $i_{\partial_r} d\mu_\varphi = \omega(X_\varphi, \partial_r)$, so it will suffice to show that there exists a neighbourhood of P_k where $\omega(X_\varphi, \partial_r) \geq 0$.

With π denoting the bundle projection $\nu_k \to P_k$, the vertical bundle $V(\nu_k)$ of ν_k can be identified with the pull-back

$$\pi^*\nu_k = \left\{(v,w) \in \nu_k \times \nu_k \;\middle|\; \pi(v) = \pi(w)\right\} .$$

The identification of $\pi^*(\nu_k)$ and $V(\nu_k)$ goes as follows

$$\pi^*(\nu_k) \to V(\nu_k), \quad (v,w) \mapsto \left.\frac{d}{dt}\right|_{t=0} (v + tw) .$$

Let $v \in \nu_k$, and write it as $v = v_1 + \cdots + v_l$ with respect to the splitting $\nu_k = E_1 \oplus \cdots \oplus E_l$. Then the vectors X_φ and ∂_r are given by

$$X_\varphi(v) = (v, a_1 J v_1 + \cdots + a_l J v_l) \quad \text{and} \quad \partial_r(v) = (v,v)$$

as elements of $V(\nu_k) \cong \pi^*(\nu_k)$. We introduce on the vertical bundle a complex structure

$$\begin{aligned} J' : V(\nu_k) &\to V(\nu_k) \\ (v,w) &\mapsto (v, Jw) \end{aligned}$$

where J is the complex structure on ν_k. With the 2–form defined by

$$\omega'\big((v,w_1),(v,w_2)\big) := \omega_P(w_1,w_2) \text{ for all } w_1, w_2 \in T_{\pi(v)}P ,$$

the bundle $(\pi^*\nu_k, \omega', J')$ is Hermitian. Now assume $v = (x,0)$ lies in the zero section. Then

$$\begin{aligned} \omega\big((v, \textstyle\sum a_i J w_i), (v, w_1 + \cdots + w_l)\big) &= \omega'\big((v, \textstyle\sum a_i J w_i), (v, w_1 + \cdots + w_l)\big) \\ &= \sum_{j=1}^{l} a_j\, \omega'\big(J'(v,w_j),(v,w_j)\big) > 0 \end{aligned}$$

if $w \neq 0$, since the eigenspaces E_j's are ω_P–orthogonal. By continuity this also holds for all v in a neighbourhood of the zero section and all $w \neq 0$. Hence in particular $\omega(X_\varphi, \partial_r) > 0$ on $U - P_k$. □

Our aim is to describe a procedure to replace a neighbourhood of (one component of) P_k by a smooth manifold in a suitable way, namely so that we can extend both the $\mathbb{S}^1$–action and the closed 2–form ω to the manifold resulting from this surgery.

3.2. **Surgery along the minimal stratum.** As in the previous section, consider the minimal singular stratum P_k, denote by ν_k its normal bundle in P and take now the product $\nu_k \times \mathbb{C}$. It admits a first circle action, which is just the extension of the original $\mathbb{S}^1$–action β by the trivial action on the $\mathbb{C}$–factor, namely, for $v \in \nu_k(x)$, $w \in \mathbb{C}$

$$\lambda *_\beta (v,w) := (\lambda *_\beta v, w) ,$$

and we can define a second circle action on $\nu_k \times \mathbb{C}$ by setting

$$\lambda *_\varphi (v,w) = (\lambda *_\varphi v, \lambda^{-k} w) = \big((\lambda^{a_1} v_1 + \cdots + \lambda^{a_l} v_l), \lambda^{-k} w\big) ,$$

where $v = v_1 + \cdots + v_l$ is the splitting with respect to the eigenspaces defined above. These two actions commute and therefore we can combine them and define a new $\mathbb{S}^1$–action

$$\lambda *_\tau (v,w) := \lambda *_\varphi \left(\lambda^{-1} *_\beta (v,w)\right) .$$

This τ–action is not effective, because the φ– and the β–action coincide for elements in $\mathbb{Z}_k$. Hence consider the short exact sequence

$$0 \to \mathbb{Z}_k \to \mathbb{S}^1 \to \hat{\mathbb{S}}^1 \to 0 ,$$

with the homomorphism of the circle given by $\lambda \mapsto \lambda^k$, and let $\hat{\mathbb{S}}^1$ act on $\nu_k \times \mathbb{C}$ by $\sigma *_{\hat\tau} (v,w) = \lambda *_\tau (v,w)$ for some $\lambda \in \mathbb{S}^1$ such that $\lambda^k = \sigma$. This new action, which we denote by $\hat\tau$, is not only effective but even free and the quotient $(\nu_k \times \mathbb{C})/\hat\tau$ is a smooth manifold. It still carries an $\mathbb{S}^1$–action induced by the φ–action on $\nu_k \times \mathbb{C}$, and this is well defined because φ commutes with $\hat\tau$.

We define a 2–form $\Omega = (\omega, -i\,dw\wedge d\bar{w})$ on $\nu_k\times\mathbb{C}$, which is invariant with respect to the $\hat{\tau}$–action. By construction, the infinitesimal generator of this action can be written as $X_{\hat{\tau}} = -X_\beta + X_\varphi$. The "Hamiltonian" for the φ–action, given by

$$H_\varphi = \mu_\varphi(v) - k|w|^2 \ ,$$

satisfies

$$i_{X_{\hat{\tau}}}\Omega = -i_{X_\beta}\Omega + i_{X_\varphi}\Omega = dH_\varphi \ .$$

It follows that if we restrict to a regular level set of H_φ, we have $i_{X_{\hat{\tau}}}\Omega = 0$. In other words, on such a level set the generator of the $\hat{\tau}$–action is contained in the kernel of the 2–form. Hence the quotient $P_\varepsilon := H_\varphi^{-1}(\varepsilon)/\hat{\tau}$ (ε a regular value), with the structure induced by Ω and φ, is a smooth pre-symplectic $\mathbb{S}^1$–manifold.

Notice that $H_\varphi^{-1}(\varepsilon)$ can be written as the disjoint union of two $\hat{\tau}$–invariant manifolds (cf. [Ler95]):

$$H_\varphi^{-1}(\varepsilon) = \left\{ (v,w) \;\middle|\; \mu_\varphi(v) > \varepsilon,\ |w|^2 = \frac{\mu_\varphi(v)-\varepsilon}{k} \right\} \sqcup \left\{ (v,0) \;\middle|\; \mu_\varphi(v) = \varepsilon \right\} .$$

Choose $\delta > 0$ such that $\mu_\varphi^{-1}(\delta)$ is contained in U, the neighbourhood of P_k constructed in Proposition 5. Notice that $\mu_\varphi^{-1}(\delta)$ has the structure of a sphere bundle over P_k. For $0 < \varepsilon < \delta$, denote by $\nu_k(\varepsilon)$ the subset of ν_k given by $\{\mu_\varphi(v) < \varepsilon\}$, by $\nu_k(\varepsilon,\delta)$ the interior of the difference $\nu_k(\delta) - \nu_k(\varepsilon)$, and consider the map

$$\Phi :\ \nu_k(\varepsilon,\delta) \to P_\varepsilon \ , \qquad v \mapsto \left[v, \sqrt{\frac{\mu_\varphi(v)-\varepsilon}{k}} \right] \ .$$

This is a diffeomorphism (onto its image), equivariant with respect to the β–action on ν_k and the φ–action on P_ε. Its inverse can be constructed as follows: given $[v,w]$ with $w \neq 0$, we first represent the same class by an element (v',w') such that w' is a real positive number, and then define

$$\Phi^{-1}([v,w]) := v' \ .$$

Moreover, since Φ factors through a map $\nu_k(\varepsilon,\delta) \to H^{-1}(\varepsilon)$ which is the identity in the first component and a real function in the second one, we have

$$\Phi^*(\omega, -i\,dw\wedge d\bar{w}) = \omega \ ,$$

hence Φ gives in fact an equivariant pre-symplectic identification of $\nu_k(\varepsilon,\delta)$ with its image under Φ. More precisely we have

$$\Phi(\nu_k(\varepsilon,\delta)) = \left\{ [v,w] \in P_\varepsilon \;\middle|\; \varepsilon < \mu_\varphi(v) < \delta \right\} .$$

We can now remove a tubular ε–neighbourhood of P_k in ν_k and glue in the smooth manifold

$$V(\delta) := \left\{ (v,w) \;\middle|\; \varepsilon \le \mu_\varphi(v) < \delta,\ |w|^2 = \frac{\mu_\varphi(v)-\varepsilon}{k} \right\} \Big/ \hat{\tau}$$

along the open "collar" $\nu_k(\varepsilon,\delta)$, using the map Φ. In this way we define the new manifold

$$\widetilde{P} = \left(P - \overline{\nu_k(\varepsilon)}\right) \cup_\Phi V(\delta) \ .$$

Since Φ is equivariant, the β–action on $P - \overline{\nu_k(\varepsilon)}$ and the φ–action on $V(\delta)$ fit together to give a circle action $\tilde{\beta}$ on $\widetilde{P}$, which by construction coincides with β outside a δ–neighbourhood of P_k. Moreover, Φ identifies the given closed 2–forms on the two sides of the gluing, so $\widetilde{P}$ also admits a closed 2–form $\widetilde{\omega}$ with the property that $\widetilde{\omega} = \omega_P$ on $P - \nu_k(\delta)$. With the action $\tilde{\beta}$ and the 2–form $\widetilde{\omega}$ just defined, $\widetilde{P}$ is a pre-symplectic manifold.

We denote by $\widetilde{M}$ the orbit space $\widetilde{P}/\mathbb{S}^1$ of the $\tilde{\beta}$–action. We need to analyse the singular points of the $\tilde{\beta}$–action on the "patch" $V(\delta)$. They satisfy the relation $\lambda *_\varphi [v,w] = [v,w]$ for some $\lambda \in \mathbb{S}^1$ and this in turn means that there exist $\kappa \in \hat{\mathbb{S}}^1$ and $\sigma \in \mathbb{S}^1$, $\sigma^k = \kappa$ such that

$$\lambda *_\varphi (v,w) = \kappa *_{\hat{\tau}} (v,w) = \sigma *_\varphi \sigma^{-1} *_\beta (v,w) \ .$$

In particular, $\lambda *_\varphi v = \sigma *_\varphi \sigma^{-1} *_\beta v$. Since φ acts fibrewise, whereas β only leaves fibres invariant for elements in $\mathbb{Z}_k$, it follows that the above identity can only hold if $\sigma \in \mathbb{Z}_k$. Hence $\kappa = 1$ and singular points are characterised by $\lambda *_\varphi (v,w) = (v,w)$. Since v is never zero on $V(\delta)$, the isotropy groups are cyclic of order $a_1, \dots, a_l$ or certain divisors of these values, hence strictly smaller than k. In other words, the order of the worst singularities on $\widetilde{M}$ is lower than on M.

Moreover, there exists a map $f : \widetilde{M} \to M$, which is a symplectic orbifold isomorphism outside an arbitrarily small neighbourhood of P_k (and in fact coincides with the identity map outside a slightly larger neighbourhood). We shall describe how to define f. On $(P - \overline{\nu_k(\delta)})/\beta$ it is simply the identity. In order to define it on $V(\delta)/\varphi$ a little more work is required. First of all, denote by S_ε the quotient $\big\{(v,0) \in H_\varphi^{-1}(\varepsilon) \;\big|\; \mu_\varphi(v) = \varepsilon\big\}/\hat\tau$. Then the inverse of the gluing map Φ gives us a diffeomorphism $\Phi^{-1} : V(\delta) - S_\varepsilon \to \nu_k(\varepsilon,\delta)$. Since Φ is equivariant with respect to the φ– and β–actions, this descends to a symplectic orbifold isomorphism $\big(V(\delta) - S_\varepsilon\big)/\varphi \to \nu(\varepsilon,\delta)/\beta$. Let $h : [0,\infty) \to [0,\infty)$ be a smooth monotone real function, which satisfies the following three conditions:

(i) $h(t) = 0$ for all $t \le \varepsilon$;
(ii) h is strictly increasing on (ε,δ);
(iii) $h(t) = 1$ for all $t \ge \delta$.

If $v \in \nu_k(\varepsilon,\delta)$, define

$$\widetilde{S} : \nu_k(\varepsilon,\delta) \to \nu_k(\delta) - P_k \;, \qquad v \mapsto h \circ \mu_\varphi(v) \cdot v \;.$$

The map $\widetilde{S}$ is β–equivariant so it descends to a well-defined map

$$S : \nu_k(\varepsilon,\delta)/\beta \to (\nu_k(\delta) - P_k)/\beta \;.$$

The composition of Φ^{-1} with the "stretching" map S yields a map

$$f : \; (V(\delta) - S_\varepsilon)/\varphi \to (\nu_k(\delta) - P_k)/\beta \;.$$

Because of the boundary conditions on h, the map f glues on the outer side with the identity map on $(P - \nu_k(\delta))/\beta$, on the other side with the map $S_\varepsilon/\varphi \to P_k/\beta$, which sends the φ–orbit of $[v,0]$ to the β–orbit of $\pi(v)$ (π being the projection of ν_k to its zero section). To see that f is continuous in a neighbourhood of S_ε/φ, one has to show that for any sequence $[v_k,w_k]/\varphi \subset V(\delta)/\varphi$ that converges to some element $[v,0]/\varphi$, it follows that $f\big([v_k,w_k]/\varphi\big)$ converges to $f\big([v,0]/\varphi\big)$. We can find representatives $(v_k',w_k') \in H^{-1}(\varepsilon)$ for the sequence that converge to $(v',0)$, and hence $f\big([v_k',w_k']/\varphi\big) = \big(h \circ \mu_\varphi(v_k') \cdot v_k'\big)/\beta$ converges to $\pi(v')/\beta = \pi(v)/\beta$.

4. Generalised Boothby-Wang fibrations are fillable

Definition. A **Boothby-Wang fibration** is a closed contact manifold (P,α) with a free $\mathbb{S}^1$–action which is given by the flow of the Reeb field X_{Reeb}. A **generalised Boothby-Wang fibration** is a closed contact manifold (P,α), where the Reeb field induces a semi-free $\mathbb{S}^1$–action.

Remark 1. A Boothby-Wang fibration (P,α) defines an $\mathbb{S}^1$–principal bundle over the manifold $B = P/\mathbb{S}^1$ with connection 1–form α. The curvature form ω is the unique 2–form on B that satisfies $\pi^*\omega = d\alpha$. The base manifold (B,ω) is a symplectic manifold and ω represents an integral cohomology class.

Conversely, for any symplectic manifold (B,ω) with integral symplectic form, one can construct a Boothby-Wang fibration (P,α) over it, the so-called **pre-quantisation**. This is the inverse of the previous construction.

Remark 2. A generalised Boothby-Wang fibration can be considered as the pre-quantisation of the symplectic *orbifold* $(P/\mathbb{S}^1,\omega)$, and all the statements made in Remark 1 can be translated to this setting.

Proposition 6. *A generalised Boothby-Wang fibration (P,α) has a natural convex filling by a symplectic orbifold.*

Proof. These computations were obtained with the help of H. Geiges. Consider the (complex) "line bundle" L associated to P, i.e. the bundle obtained from $P \times \mathbb{C}$ by identifying (p, z) with $(e^{-i\varphi} * p, e^{i\varphi}\, z)$ for every $e^{i\varphi} \in \mathbb{S}^1$. The manifold P embeds naturally via

$$P \hookrightarrow L, \quad p \mapsto [p, 1]\ .$$

The two forms

$$\frac{1}{2}\, \Big(|z|^2\, \alpha + x\, dy - y\, dx\Big) \qquad \text{and} \qquad \frac{1}{2}\, d\alpha$$

on $P \times \mathbb{C}$ induce well-defined forms on L. By adding the differential of the first form to the second one, we obtain a pre-symplectic form

$$\omega := \frac{1}{2}\, d(|z|^2) \wedge \alpha + dx \wedge dy + \frac{1 + |z|^2}{2}\, d\alpha\ ,$$

because $2^n\, \omega^n = n\, (1+|z|^2)^{n-1}\, (d\alpha)^{n-1} \wedge \Big(d|z|^2 \wedge \alpha + 2dx \wedge dy\Big)$ has only a one-dimensional kernel on $P \times \mathbb{C}$ that is generated by $-Z_P + x\, \partial_y - y\, \partial_x$. It follows that $(P \times \mathbb{C}, \omega)$ is a pre-symplectic $\mathbb{S}^1$–manifold, and hence L is a symplectic orbifold where all orbifold singularities sit along the zero section.

Finally, the following field

$$X := \frac{1 + r^2}{2r}\, \partial_r = \frac{1 + x^2 + y^2}{2\,(x^2 + y^2)}\, (x\, \partial_x + y\, \partial_y)$$

is a Liouville vector field for the manifold (P, α), and (L, ω) is hence a convex filling of P. □

All the orbifold singularities lie in the interior of L. By passing to a symplectic resolution of L whose existence is guaranteed by Theorem 2, we can obtain a smooth symplectic filling.

Corollary 7. *Generalised Boothby-Wang fibrations are symplectically fillable by a* smooth *manifold.*

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