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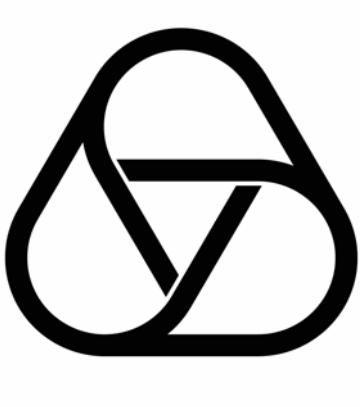

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NONCOMMUTATIVE TOPOLOGICAL ENTROPY OF ENDOMORPHISMS OF CUNTZ ALGEBRAS

ADAM SKALSKI AND JOACHIM ZACHARIAS

Abstract. Noncommutative topological entropy estimates are obtained for 'finite range' endomorphisms of Cuntz algebras, generalising known results for the canonical shift endomorphisms. Exact values are computed for a class of polynomial endomorphisms related to branching function systems introduced and studied by Bratteli, Jorgensen and Kawamura.

The notion of noncommutative topological entropy for an automorphism of a C^*-algebra was introduced by D. Voiculescu in [Vo] (we refer to the book [NS] for extensive discussion and many examples). In [BG] F. Boca and P. Goldstein showed that the noncommutative topological entropy of the canonical shift endomorphism of the Cuntz algebra $\mathcal{O}_N$ is equal to $\log N$. Their methods have been extended in [SZ] to determine the values of noncommutative entropy and pressure for the multidimensional shifts on C^*-algebras associated with higher-rank graphs. In both of these papers explicit descriptions of the shift endomorphisms were used.

Endomorphisms of Cuntz algebras are known to be in bijective correspondence with unitaries in the algebra ([Cu]). A particularly interesting class is formed by those which leave the UHF-subalgebra of $\mathcal{O}_N$ invariant. Besides the canonical shift this class contains recently introduced and intensively studied in [BJ] and [Ka] *permutative polynomial endomorphisms*, induced by 'permutation unitaries' in $\mathcal{O}_N$. We refer to the references above for description of the connections with branching function systems and permutative representations of $\mathcal{O}_N$.

In the first part of this paper we adapt the arguments from [BG] and [SZ] to give an upper bound of the topological entropy of polynomial endomorphisms leaving the UHF subalgebra invariant. We note also in passing that the same methods apply to 'finite-range' endomorphisms of Cuntz-Krieger algebras, graph and even higher-rank graph C^*-algebras.

In the second part we obtain exact values of the entropy for all polynomial endomorphisms of $\mathcal{O}_2$ coming from permutations of rank 2 (fully classified in [Ka]). They all leave the canonical maximal abelian subalgebra $\mathcal{C}_2$ of $\mathcal{O}_2$ invariant, but contrary to the case of the shift endomorphism it may happen that the entropy of the endomorphism is greater than the entropy of its restriction to $\mathcal{C}_2$. In general there are only few known examples of C^*-dynamical systems for which the noncommutative topological entropy is strictly greater than the supremum of entropies over all classical (commutative) subsystems. It can be deduced from the results of Chapter 12 in [NS] that a natural shift on a C^*-algebra of a bitstream has this property for a big class of bitstreams.

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Considerations above lead to the following question: is the entropy of an arbitrary endomorphism of $\mathcal{O}_N$ leaving $\mathcal{F}_N$ invariant always equal to the entropy of its restriction to $\mathcal{F}_N$? We conjecture that the answer is yes, but were unable to prove it.

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1. Basics on endomorphisms of Cuntz algebras

Fix $N \in \mathbb{N}$. For $k \in \mathbb{N}$ the set of multindices of length k with values in $\{1, \ldots, N\}$ will be denoted by $\mathcal{J}_k$, so that $\mathcal{J}_k = \{1, \ldots, N\}^k$; we write $\mathcal{J}_0 = \{\emptyset\}$ and $\mathcal{J} = \cup_{k \in \mathbb{N}_0} \mathcal{J}_k$. Sometimes we will also use letters μ, ν for multiindices. The canonical generators for $\mathcal{O}_N$ will be denoted by s_i $(i = 1, \ldots, N)$. For $J \in \mathcal{J}_k$, $J = (j_1, \ldots, j_k)$ we write $s_J = s_{j_1} \cdots s_{j_k}$ (and $s_\emptyset := 1$). The standard UHF subalgebra of $\mathcal{O}_N$ generated by $\{s_I s_J^* : k \in \mathbb{N}, I, J \in \mathcal{J}_k\}$ will be denoted by $\mathcal{F}_N$ and the standard masa (maximal abelian subalgebra) generated by $\{s_I s_I^* : J \in \mathcal{J}\}$ by $\mathcal{C}_N$.

Besides the standard masa $\mathcal{C}_N$ we will consider some other masas in $\mathcal{F}_N$. Recall that a masa $\mathcal{D}$ in $\mathcal{F}_N$ is called canonical if it is of the form $\bigotimes_{i=1}^\infty C_i$ where each C_i is a masa in M_n. Any two canonical masas are approximately unitarily equivalent but will be conjugate only in exceptional cases. Special canonical masas are of the type $\mathcal{C} = \bigotimes_{i=1}^\infty C_i$ where $C_i = C_j$ for all $i, j \in \mathbb{N}$. It is not hard to see that each such $\mathcal{C}$ is isomorphic to the algebra of continuous functions on the standard full subshift on N letters (further denoted by $\mathfrak{C}$), the canonical shift endomorphism θ leaves $\mathcal{C}$ invariant, the restriction is induced by the classical shift transformation (on $(\mathfrak{C})$ and thus the entropy of the restriction is equal $\log N$.

It is known ([Cu]) that there is a bijective correspondence between endomorphisms of the Cuntz algebra $\mathcal{O}_N$ and unitaries in $\mathcal{O}_N$. Given a unitary $u \in \mathcal{O}_N$ the corresponding endomorphism ρ_u is given by continuous *-preserving multiplicative linear extension of the formula:

$$\rho_u(s_i) = us_i, \quad i = 1, \ldots, N.$$

A description of the action of ρ_u on higher monomials can be given as follows: if $p \in \mathbb{N}$ and we write

$$u_p = u\theta(u) \ldots \theta^{p-1}(u) \tag{1.1}$$

(where θ is the canonical shift endomorphism), then $\rho_u(s_P) = u_p s_P$ for all $P \in \mathcal{J}_p$. If $x \in \mathcal{F}_N$ then $\rho_u(x) = \lim_{i \to \infty} u_i x u_i^*$; it is also known that ρ_u preserves $\mathcal{F}_N$ if and only if $u \in \mathcal{F}_N$ ([Cu]). It follows that $\tau \circ \mathbb{E} \circ \rho_u = \tau \circ \mathbb{E}$ where $\mathbb{E}$ the conditional on $\mathcal{F}_N$ and τ the unique trace on it, provided $u \in \mathcal{F}_N$.

We need to introduce some more notations. Put $(p, l \in \mathbb{N}_0)$

$$A_{p,l} = \{s_P s_L^* : P \in \mathcal{J}_p, L \in \mathcal{J}_l\}. \tag{1.2}$$

Recall that $\bigcup_{p,l \in \mathbb{N}_0} A_{p,l}$ is total in $\mathcal{O}_N$; write also

$$F_{p,l} = \mathrm{Lin} A_{p,l}. \tag{1.3}$$

Let also Ψ_k denote the canonical embedding of $\mathcal{O}_N$ into $M_{N^k} \otimes \mathcal{O}_N$:

$$\Psi_k(X) = \sum_{K,M \in \mathcal{J}_k} e_{K,M} \otimes s_K^* X s_M, \quad X \in \mathcal{O}_N,$$

where $e_{K,M}$ denote the matrix units in M_{N^k}.

The following lemma is very elementary, but also crucial for what follows. It shows that neither in Lemma 2 of [BG] nor in Lemma 2.2 of [SZ] was it essential that one dealt with the shift-type transformations.

Lemma 1.1. *Let $k, p, l \in \mathbb{N}$, $k \geq \max\{p, l\}$. Suppose that $X \in F_{p,l}$. If $p > l$ then*

$$\Psi_k(X) = \sum_{J \in \mathcal{J}_{p-l}} T_J \otimes s_J, \tag{1.4}$$

where $\|T_J\| \leq \|X\|$ for each $T_J \in M_{N^k}$; if $p < l$ then

$$\Psi_k(X) = \sum_{J \in \mathcal{J}_{l-p}} T_J \otimes s_J,$$

where $\|T_J\| \leq \|X\|$ for each $T_J \in M_{N^k}$. Finally if $p = l$ then

$$\Psi_k(X) = T \otimes 1,$$

where $\|T\| \leq \|X\|$ for $T \in M_{N^k}$,.

Proof. Suppose that $p > l$ and let $X = \sum_{P \in \mathcal{J}_p, L \in \mathcal{J}_l} \gamma_{P,L} s_P s_L^*$, where $\gamma_{P,L} \in \mathbb{C}$. Then

$$\begin{aligned}
\Psi_k(X) &= \sum_{K,M \in \mathcal{J}_k} \sum_{P \in \mathcal{J}_p, L \in \mathcal{J}_l} \gamma_{P,L} e_{K,M} \otimes s_K^* s_P s_L^* s_M \\
&= \sum_{P \in \mathcal{J}_p, L \in \mathcal{J}_l} \sum_{K' \in \mathcal{J}_{k-p}, M' \in \mathcal{J}_{k-l}} \gamma_{P,L} e_{PK',LM'} \otimes s_{K'}^* s_{M'} \\
&= \sum_{P \in \mathcal{J}_p, L \in \mathcal{J}_l} \sum_{K' \in \mathcal{J}_{k-p}, J \in \mathcal{J}_{p-l}} \gamma_{P,L} e_{PK',LK'L'} \otimes s_J.
\end{aligned}$$

This shows that (1.4) holds for some $T_J \in M_{N^k}$.

Observe now that for each $n, k \in \mathbb{N}$ and a family of complex $n \times n$ matrices $\{\alpha_J : J \in \mathcal{J}_m\}$ we have

$$\begin{aligned}
\| \sum_{J \in \mathcal{J}_m} \alpha_J \otimes s_J \|^2 &= \| (\sum_{J \in \mathcal{J}_m} \alpha_J \otimes s_J)^* (\sum_{K \in \mathcal{J}_m} \alpha_K \otimes s_K) \| \\
&= \| \sum_{J,K \in \mathcal{J}_m} \alpha_J^* \alpha_K \otimes s_J^* s_K \| = \| \sum_{J \in \mathcal{J}_m} \alpha_J^* \alpha_J \otimes 1 \| = \| \sum_{J \in \mathcal{J}_m} \alpha_J^* \alpha_J \|,
\end{aligned}$$

so in particular for any fixed $K \in \mathcal{J}_m$ we have

$$\|\alpha_K\| \leq \| \sum_{J \in \mathcal{J}_m} \alpha_J \otimes s_J \|.$$

Now connecting the above with the fact that Ψ_k is a *-homomorphism, we get for each $J \in \mathcal{J}_m$

$$\|T_J\| \leq \|\Psi_k(X)\| \leq \|X\|$$

and the proof is finished. The cases $p < l$ and $p = l$ follow in a similar way. □

Analogous result remains true in the context of the Cuntz-Krieger, graph and higher-rank graph C^*-algebras. It can be used to estimate the entropy of a completely positive contractive map on the C^*-algebra of one of the types listed above if only we have control on 'how far' the map sends the canonical matrix units. The last statement is made formal in the next lemma, where we show that an appropriate assumption is satisfied by endomorphisms of $\mathcal{O}_N$ associated to unitaries in the

finite part of the UHF subalgebra. Recall the notations (1.2) and (1.3) and write $F_k = F_{k,k}$.

Lemma 1.2. *Let $k \in \mathbb{N}$ and let $u \in F_k$ be a unitary. Then for all $m, p, l \in \mathbb{N}$*

$$\rho_u^m(F_{p,l}) \subset F_{p+m(k-1),l+m(k-1)}.$$

Proof. It is clearly enough to prove the inclusion above in the case $m = 1$. For $i \in \{1, \dots, N\}$ we have $\rho_u(s_i) = us_i \in F_k F_{1,0} \subset F_{k,k-1}$. Further $F_{k,k-1}F_{k,k-1} \subset F_{k+1,k-1}$ and inductively one can show that $\rho_u(s_P) \in F_{k+p-1,k-1}$ for $P \in \mathcal{J}_p$. In this case

$$\rho_u(s_P s_L^*) \in F_{k+p-1,k+l-1},$$

whenever $P \in \mathcal{J}_p, L \in \mathcal{J}_l$. □

Note that Lemma 1.2 also follows from the fact that $\rho_u(s_P s_L^*) = u_p s_P s_L^* u_l^*$, where $u_p \in F_{p+k-1}$ and $u_l \in F_{l+k-1}$ are defined as in (1.1).

A particular class of the endomorphisms of $\mathcal{O}_N$, the so-called permutative endomorphisms arises as follows. Suppose that σ is a permutation of the set $\mathcal{J}_k$. Put

$$u_\sigma = \sum_{J \in \mathcal{J}_k} s_{\sigma(J)} s_J^*.$$

It is easy to check that u_σ is a unitary in $\mathcal{O}_N$. The permutative endomorphism ρ_σ corresponding to σ is simply ρ_{u_σ}. Several examples will be discussed in detail in the last section of this note. Remark here only that identity, canonical shift endomorphism and quasi-free automorphisms exchanging the generators all fall into this class.

Lemma 1.3. *Let σ be a permutation of $\mathcal{J}_k$, $p, l, m \in \mathbb{N}$. Then*

$$\rho_\sigma^m(A_{p,l}) \subset F_{p+m(k-1),l+m(k-1)}. \tag{1.5}$$

Further ρ_σ leaves both $\mathcal{F}_N$ and $\mathcal{C}_N$ invariant.

Proof. Since $u_\sigma \in F_k$ all that remains to show is that $\rho_\sigma(\mathcal{C}_N) \subset \mathcal{C}_N$. This can be understood as follows: $u_{\sigma,i} = u_\sigma \theta(u_\sigma) \dots \theta^{i-1}(u_\sigma)$ describes a permutation of $\mathcal{J}_{k+i-1}$ which correspond to a permutation of points in the spectrum of $\mathcal{C}_N$. Thus $\rho_\sigma(s_I s_I *) = u_{\sigma,i} s_I s_I^* u_{\sigma,i}^* \in \mathcal{C}_N$, whenever $I \in \mathcal{J}_i$.

Here is a more explicit formula for this. We need to introduce some further notation: for $J = (j_1, \dots, j_k) \in \mathcal{J}_k$ write $J_k := j_k$, $J_1 := j_1$ and $J^- := (j_1, \dots, j_{k-1}) \in \mathcal{J}_{k-1}$. Moreover for $i \in \{1, \dots, N\}$ write $iJ = (i, j_1, \dots, j_k) \in \mathcal{J}_{k+1}$.

Fix $k \in \mathbb{N}$ and a permutation σ of $\mathcal{J}_k$. Let $l \in \{1, \dots, N\}$. Then

$$\rho_\sigma(s_l) = u_\sigma s_l = \sum_{J \in \mathcal{J}_k} s_{\sigma(J)} s_J^* s_l = \sum_{J \in \mathcal{J}_k; J_1 = l} s_{\sigma(J)} s_{J_-}^* = \sum_{L \in \mathcal{J}_{k-1}} s_{\sigma(lL)} s_L^*.$$

Further for $l, j \in \{1, \dots, N\}$

$$\rho_\sigma(s_l s_j) = \sum_{L \in \mathcal{J}_{k-1}} s_{\sigma(lL)} s_L^* \sum_{J \in \mathcal{J}_{k-1}} s_{\sigma(jJ)} s_J^* = \sum_{J \in \mathcal{J}_{k-1}} s_{\sigma(l\sigma(jJ)^-)} s_{\sigma(jJ)_k} s_J^*.$$

From this one deduces inductively that for any $p \in \mathbb{N}$, $i_1, \dots, i_p \in \{1, \dots, N\}$

$$\rho_\sigma(s_{i_1} \cdots s_{i_p}) = \sum_{J \in \mathcal{J}_{k-1}} s_{\sigma(i_1 J^1)} s_{\sigma(i_2 J^2)_k} \cdots s_{\sigma(i_{p-1} J^{p-1})_k} s_{\sigma(i_p J^p)_k} s_J^*,$$

where

$$J^p = J \quad \text{and} \quad J^r = \sigma(i_{r+1}J^{r+1})^-, \;\; r = 1, \ldots, p-1.$$

The last formula generalises immediately to the following ($p, l \in \mathbb{N}$, $i_1, \ldots, i_p, j_1, \ldots, j_l \in \{1, \ldots, N\}$)

$$\rho_\sigma(s_{i_1} \cdots s_{i_p}(s_{j_1} \cdots s_{j_l})^*) = \tag{1.6}$$

$$\sum_{J \in \mathcal{J}_{k-1}} s_{\sigma(i_1 J^1)} s_{\sigma(i_2 J^2)_k} \cdots s_{\sigma(i_{p-1} J^{p-1})_k} s_{\sigma(i_p J^p_k)} s^*_{\sigma(j_l T^l_k)} s^*_{\sigma(j_{l-1} T^{l-1})_k} \cdots s^*_{\sigma(j_2 T^2)_k} s^*_{\sigma(j_1 T^1)},$$

where J^r are as above and

$$T^l = J \quad \text{and} \quad T^r = \sigma(i_{r+1}T^{r+1})^-, \;\; r = 1, \ldots, m-1.$$

The above allow to see directly that $\rho_\sigma : \mathcal{C}_N \to \mathcal{C}_N$, $\rho_\sigma : \mathcal{F}_N \to \mathcal{F}_N$.

□

If a unitary $u \in F_k$ is not a permutation matrix, the endomorphism ρ_u need not leave $\mathcal{C}_N$ invariant.

2. Entropy estimate

Let A be a unital C^*-algebra. We say that (ϕ, ψ, C) is an approximating triple for A if C is a finite-dimensional C^*-algebra and both $\phi : C \to \mathsf{A}$, $\psi : \mathsf{A} \to C$ are unital and completely positive (ucp). This will be indicated by writing $(\phi, \psi, C) \in CPA(\mathsf{A})$. Whenever Ω is a finite subset of A ($\Omega \in FS(\mathsf{A})$) and $\varepsilon > 0$ the statement $(\phi, \psi, C) \in CPA(\mathsf{A}, \Omega, \varepsilon)$ means that $(\phi, \psi, C) \in CPA(\mathsf{A})$ and for all $a \in \Omega$

$$\|\phi \circ \psi(a) - a\| < \varepsilon.$$

Nuclearity of A is equivalent to the fact that for each $\Omega \in FS(\mathsf{A})$ and $\varepsilon > 0$ there exists a triple $(\phi, \psi, C) \in CPA(\mathsf{A}, \Omega, \varepsilon)$. For such algebras one can define

$$\mathrm{rcp}(\Omega, \varepsilon) = \min\{\mathrm{rank}\, C : \; (\phi, \psi, C) \in CPA(\mathsf{A}, \Omega, \varepsilon)\},$$

where $\mathrm{rank}\, C$ denotes the dimension of a maximal abelian subalgebra of C. Let us recall the definition of noncommutative topological entropy in nuclear unital C^*-algebras, due to D.Voiculescu ([Vo]). Assume that A is nuclear and $\gamma : \mathsf{A} \to \mathsf{A}$ is a ucp map. For any $\Omega \in FS(\mathsf{A})$ and $n \in \mathbb{N}$ let

$$\mathrm{orb}^n(\Omega) = \Omega^{(n)} = \bigcup_{j=0}^{n} \gamma^j(\Omega). \tag{2.1}$$

Then the (Voiculescu) noncommutative topological entropy is given by the formula:

$$\mathrm{ht}(\gamma) = \sup_{\varepsilon > 0, \, \Omega \in FS(\mathsf{A})} \limsup_{n \to \infty} \left(\frac{1}{n} \log \mathrm{rcp}(\Omega^{(n)}, \varepsilon) \right).$$

As shown in [Vo] Proposition 4.8 the approximation entropy coincides with classical topological entropy in the commutative case (see [Wa] for the definition of the latter). Another important property to be used in the sequel is the fact that the entropy decreases when passing to invariant subalgebras. More precisely, if $B \subset \mathsf{A}$ is a subalgebra with $\gamma(B) \subset B$ then $\mathrm{ht}(\gamma|_B) \leq \mathrm{ht}(\gamma)$.

For the details of the above, extensions to the case of exact C^*-algebras and various related topics we refer to [NS]. Note that $\mathcal{O}_N$ as a unital nuclear C^*-algebra falls into the class considered above.

The main general result of this note is the following theorem. The proof is a generalisation of that of Theorem 2.4 of [SZ] (see also [BG]), now using Lemmas 1.1 and 1.2 instead of Lemma 2.2 of that paper. We reproduce it for the convenience of the reader.

Theorem 2.1. *Let $k \in \mathbb{N}$ and $u \in F_k$ be a unitary. Then*

$$ht(\rho_u) \leq (k-1)\log N.$$

In particular if σ is a permutation of $\mathcal{J}_k$, then this estimate holds true for the corresponding permutation endomorphism ρ_σ.

Proof. Put for each $n \in \mathbb{N}$

$$\omega_l = \cup_{p,q=1}^{l} A_{p,q}.$$

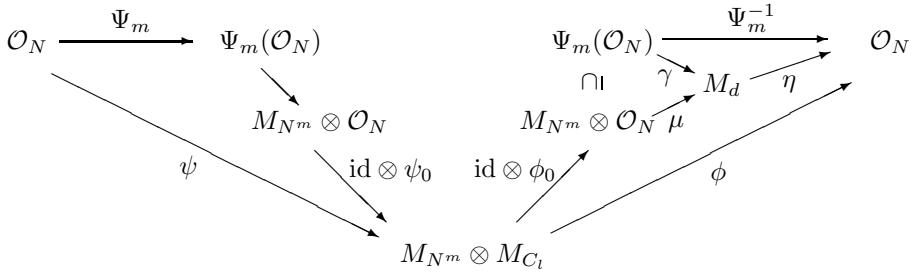
Fix $l \in \mathbb{N}$, $\delta > 0$. As $\mathcal{O}_N$ is nuclear, there exists a triple $(\phi_0, \psi_0, M_{C_l}) \in CPA(\mathcal{O}_N, \omega_l, \frac{1}{4N^l}\delta)$. Fix further $n \in \mathbb{N}$ and let

$$\omega_l^{(n)} = \bigcup_{p=0}^{n} \rho_u^p(\omega_l).$$

Put $m = n(k-1)+l$. Nuclearity of $\mathcal{O}_N$ implies that there exists $d \in \mathbb{N}$ and ucp maps $\gamma : \Psi_m(\mathcal{O}_N) \to M_d$ and $\eta : M_d \to \mathcal{O}_N$ such that for all $a \in \Psi_m(\rho_u^k(\omega_l))$, $k \leq n$,

$$\|\eta \circ \gamma(a) - \Psi_m^{-1}(a)\| < \frac{\delta}{2}.$$

Let $\mu : M_{N^m} \otimes \mathcal{O}_N \to M_d$ be a ucp extension of γ. Consider the following diagram:



Consider now any $X \in \omega_l$ and let $p \in \mathbb{N}$, $p \leq n$. Then

$$\begin{aligned}
&\|\phi \circ \psi(\rho_u^p(X)) - \rho_u^p(X)\| \\
&\qquad = \|\eta \circ \mu \circ (\mathrm{id} \otimes \phi_0 \circ \psi_0) \circ \Psi_m(\rho_u^p(X)) - (\Psi_m^{-1} \circ \Psi_m)(\rho_u^p(X))\| \\
&\qquad \leq \|(\mathrm{id} \otimes \phi_0 \circ \psi_0) \circ \Psi_m(\rho_u^p(X)) - \Psi_m(\rho_u^p(X))\| + \frac{\delta}{2}.
\end{aligned}$$

Lemma 1.2 implies that $\rho_u^p(X) \in F_{q,r}$, where $q, r \leq l + (k-1)p \leq m$. We can assume that for example $q > r$. Then Lemma 1.1 implies that

$$\begin{aligned}
&\|\phi \circ \psi(\rho_u^p(X)) - \rho_u^p(X)\| \\
&\qquad < \Big\| \sum_{J \in \mathcal{J}_{q-r}} T_J \otimes \big((\phi_0 \circ \psi_0)(s_J) - s_J\big) \Big\| + \frac{\delta}{2} \\
&\qquad < 2N^m \frac{\delta}{4N^m} + \frac{\delta}{2} = \delta,
\end{aligned}$$

and we proved that

$$(\phi, \psi, M_{N^m} \otimes M_{C_l}) \in CPA(\mathcal{O}_N, \omega_l^{(n)}, \delta). \tag{2.2}$$

This shows that $\mathrm{rcp}(\omega_l^{(n)}, \delta) \leq C_l N^m$,

$$\log \mathrm{rcp}(\omega_l^{(n)}, \delta) \leq C_l + m \log N = C_l + ((k-1)n + l) \log N$$

and finally

$$\limsup_{n\to\infty} \left(\frac{1}{n} \log \mathrm{rcp}(\omega_l^{(n)}, \delta) \right) \leq (k-1) \log N.$$

The Kolmogorov-Sinai property for noncommutative entropy (Theorem 6.2.4 of [NS]) ends the proof. □

The proof above remains valid for any unital completely positive map on $\mathcal{O}_N$ which satisfies the conclusions of Lemma 1.2, and again can be suitably adapted to the context of (higher-rank) graph algebras O_Λ. As we are not aware of any interesting and natural examples of such ucp maps for O_Λ (apart from the canonical shifts in various directions analysed in [SZ]), we decided to present the result in the context of specific endomorphisms of $\mathcal{O}_N$.

If the endomorphism ρ_u leaves the canonical masa $\mathcal{C}_N$ invariant, it is also possible to obtain, exactly as in [SZ], estimates for the noncommutative pressure ([KP]) of any selfadjoint element of $\mathcal{C}_N$. Note that in this case however the estimate will not necessary be optimal nor will it have to coincide with the pressure of the corresponding element computed in $\mathcal{C}_N$ viewed as the commutative subalgebra. This can be deduced from results in the next section.

It is easy to see that it is both possible to have $\mathrm{ht}(\rho_u) = 0$ (for the identity endomorphism) and $\mathrm{ht}(\rho_u) = (k-1) \log N$ (for the endomorphism given by $(k-1)$-th iterate of the canonical shift). Note that as ρ_u need not leave the canonical masa in $\mathcal{O}_N$ invariant, we cannot always use the classical topological entropy to obtain the estimates from below (as was done in [BG] and [SZ]).

Let us stress once more that the examples below will show that even if ρ_u leaves $\mathcal{C}_N$ invariant it is not necessarily true that $\mathrm{ht}(\rho_u) = \mathrm{ht}(\rho_u|_{\mathcal{C}_N})$.

3. Examples for $\mathcal{O}_2$

This section is devoted to computing entropy of the endomorphisms $\rho_\sigma : \mathcal{O}_2 \to \mathcal{O}_2$ for σ being a permutation of $\mathcal{J}_2$. These endomorphisms were listed and classified up to unitary equivalence in [Ka]. Theorem 2.1 implies that $\mathrm{ht}(\rho_\sigma) \leq \log 2$. Below we will compute the actual value of $\mathrm{ht}(\rho_\sigma)$ (and of $\mathrm{ht}(\rho_\sigma|_{\mathcal{F}_2})$, $\mathrm{ht}(\rho_\sigma|_{\mathcal{C}_2})$). The notation will coincide with that of [Ka]; we identify $\mathcal{J}_2 = \{(1,1), (1,2), (2,1), (2,2)\}$ with $\{1,2,3,4\}$. The subscripts in the symbols denoting permutations (e.g. $\sigma_{(12),(34)}$) will then correspond to the cycle decomposition of the permutation.

By Lemma 1.3 each ρ_σ leaves the canonical masa invariant and we always have a 'commutative model' for our endomorphism. To be more precise, the algebra $\mathcal{C}_2$ is $^*-$isomorphic to the algebra $C(\mathfrak{C})$ of continuous functions on the full shift $\mathfrak{C} = \{(w_k)_{k=1}^\infty : w_k \in \{1,2\}\}$ (equipped with the usual metric making it a 0-dimensional compact space). The standard isomorphism is given by the linear extension of the map $s_I s_I^* \mapsto \chi_{Z_I}$, where Z_I denotes the set of these sequences in $\mathfrak{C}$ which begin with the finite sequence I. Each of the endomorphisms ρ_σ restricted to $\mathcal{C}_2$ corresponds therefore to a continuous map $T_\sigma : \mathfrak{C} \to \mathfrak{C}$. Note that if we want to determine for a given σ what exactly T_σ looks like we need to analyse the values

$\rho_\sigma(s_I s_I^*)$ for $I \in \mathcal{J}$; if $\rho_\sigma(s_I s_I^*) = s_{I_1} s_{I_1}^* + \cdots + s_{I_m} s_{I_m}^*$ then $T_\sigma : \bigcup_{j=1}^m Z_{I_j} \to Z_I$. Thus for each $k \in \mathbb{N}$, $i \in \{1, 2\}$

$$(T_\sigma(w))_k = i \text{ if and only if } w \in \bigcup_{l=0}^{m} Z_{I_l},$$

where

$$\rho_\sigma(\sum_{J \in \mathcal{J}_k, j_k = i} s_J s_J^*) = \sum_{l=0}^{m} s_{I_l} s_{I_l}^*.$$

This will be used to identify what the entropy of $\rho_\sigma|_{\mathcal{C}_2}$ is. We will often also make use of the following remark: suppose that $T : \mathfrak{C} \to \mathfrak{C}$ is a continuous map such that if $k \in \mathbb{N}$, $v, w \in \mathfrak{C}$ and $w|_{k+1]} \neq v|_{k+1]}$ then either $(Tw)|_{k]} \neq (Tv)|_{k]}$ or $w|_{k]} \neq v|_{k]}$. Then the usual argument with the (n, ϵ) separated subets of $\mathfrak{C}$ (see [Wa]) shows that $h_{\text{top}}(T) \geq \log 2$.

The following table modelled on that of [Ka] summarises the results of the entropy computations. The rest of the section will be devoted to explaining how the values in the table were obtained. In all cases $\text{ht}(\rho_\sigma|_{\mathcal{F}_2}) = \text{ht}(\rho_\sigma)$.

Table 1. Entropy of the 'rank 2' permutative endomorphisms of $\mathcal{O}_2$.

ρ_σ	$\rho_\sigma(s_1)$	$\rho_\sigma(s_2)$	$\text{ht}(\rho_\sigma)$	$\text{ht}(\rho_\sigma\|_{\mathcal{C}_2})$
ρ_{id}	s_1	s_2	0	0
ρ_{12}	$s_{12,1} + s_{11,2}$	s_2	$\log 2$	0
ρ_{13}	$s_{21,1} + s_{12,2}$	$s_{11,1} + s_{22,2}$	$\log 2$	$\log 2$
ρ_{14}	$s_{22,1} + s_{12,2}$	$s_{21,1} + s_{11,2}$	$\log 2$	$\log 2$
ρ_{23}	$s_{11,1} + s_{21,2}$	$s_{12,1} + s_{22,2}$	$\log 2$	$\log 2$
ρ_{24}	$s_{11,1} + s_{22,2}$	$s_{21,1} + s_{12,2}$	$\log 2$	$\log 2$
ρ_{34}	s_1	$s_{22,1} + s_{21,2}$	$\log 2$	0
ρ_{123}	$s_{12,1} + s_{21,2}$	$s_{11,1} + s_{22,2}$	$\log 2$	$\log 2$
ρ_{132}	$s_{21,1} + s_{11,2}$	$s_{12,1} + s_{22,2}$	$\log 2$	$\log 2$
ρ_{124}	$s_{12,1} + s_{22,2}$	$s_{21,1} + s_{11,2}$	$\log 2$	$\log 2$
ρ_{142}	$s_{22,1} + s_{11,2}$	$s_{21,1} + s_{12,2}$	$\log 2$	$\log 2$
ρ_{134}	$s_{21,1} + s_{12,2}$	$s_{22,1} + s_{11,2}$	$\log 2$	$\log 2$
ρ_{143}	$s_{22,1} + s_{12,2}$	$s_{11,1} + s_{21,2}$	$\log 2$	$\log 2$
ρ_{234}	$s_{11,1} + s_{21,2}$	$s_{22,1} + s_{12,2}$	$\log 2$	$\log 2$
ρ_{243}	$s_{11,1} + s_{22,2}$	$s_{12,1} + s_{21,2}$	$\log 2$	$\log 2$
ρ_{1234}	$s_{12,1} + s_{21,2}$	$s_{22,1} + s_{11,2}$	$\log 2$	$\log 2$
ρ_{1243}	$s_{12,1} + s_{22,2}$	$s_{11,1} + s_{21,2}$	$\log 2$	$\log 2$
ρ_{1324}	s_2	$s_{12,1} + s_{11,2}$	$\log 2$	0
ρ_{1342}	$s_{21,1} + s_{11,2}$	$s_{22,1} + s_{12,2}$	$\log 2$	$\log 2$
ρ_{1423}	$s_{22,1} + s_{21,2}$	s_1	$\log 2$	0
ρ_{1432}	$s_{22,1} + s_{11,2}$	$s_{12,1} + s_{21,2}$	$\log 2$	$\log 2$
$\rho_{(12)(34)}$	$s_{12,1} + s_{11,2}$	$s_{22,1} + s_{21,2}$	0	0
$\rho_{(13)(24)}$	s_2	s_1	0	0
$\rho_{(14)(23)}$	$s_{22,1} + s_{21,2}$	$s_{12,1} + s_{11,2}$	0	0

Identity map. Entropy 0.

Shift endomorphism. Arises from σ_{23}. The entropy is equal to $\log 2$ (see [BG]). The shift θ leaves the canonical masa invariant and $\text{ht}(\theta) = \text{ht}(\theta|_{\mathcal{F}_2}) = \text{ht}(\theta|_{\mathcal{C}_2})$.

Flip transformation. Arises from $\sigma_{(13)(24)}$, acts as $\rho_\sigma(s_1) = s_2$, $\rho_\sigma(s_2) = s_1$. The entropy is 0; this follows immediately from general results of [DS], but can be also easily deduced directly.

The transformation induced by σ_{12}. Denote it simply by ψ. It is defined by

$$\psi(s_1) = s_1 s_2 s_1^* + s_1 s_1 s_2^*, \quad \psi(s_2) = s_2.$$

We have $(n, m \in \mathbb{N})$

$$\psi(s_1^n) = s_1 s_2^n s_1^* + s_1 s_2^{n-1} s_1 s_2^*, \;\; \psi(s_2^m) = s_2^m$$

(the first formula can be easily shown inductively, the second is obvious). This leads to

$$\psi(s_1^{i_1} s_2^{j_1} \cdots s_2^{j_{k-1}} s_1^{i_k}) = s_1 s_2^{i_1 - 1} s_1 s_2^{j_1 - 1} \cdots s_1 s_2^{j_{k-1} - 1} s_1 (s_2^{i_k - 1} s_1 s_2^* + s_2^{i_k} s_1^*),$$

$$\psi(s_1^{i_1} s_2^{j_1} \cdots s_2^{j_{k-1}} s_1^{i_k} s_2^{j_k}) = s_1 s_2^{i_1 - 1} s_1 s_2^{j_1 - 1} \cdots s_1 s_2^{j_{k-1} - 1} s_1 s_2^{i_k - 1} s_1 s_2^{j_k - 1}$$

$(k \in \mathbb{N},\, i_1, \ldots, i_k, j_1, \ldots, j_k \in \mathbb{N})$. Further if $s_\nu = s_1^{i_1} s_2^{j_1} \cdots s_2^{j_{k-1}} s_1^{i_k}$ then

$$\psi(s_\nu s_\nu^*) = s_{\nu_1} s_{\nu_1}^* + s_{\nu_2} s_{\nu_2}^*,$$

where

$$s_{\nu_1} = s_1 s_2^{i_1 - 1} s_1 s_2^{j_1 - 1} \cdots s_1 s_2^{j_{k-1} - 1} s_1 s_2^{i_k - 1} s_1,$$

$$s_{\nu_2} = s_1 s_2^{i_1 - 1} s_1 s_2^{j_1 - 1} \cdots s_1 s_2^{j_{k-1} - 1} s_1 s_2^{i_k}.$$

This shows immediately that

$$\psi(s_\nu s_\nu^*) = s_{\widetilde{\nu}} s_{\widetilde{\nu}}^*,$$

where

$$s_{\widetilde{\nu}} = s_1 s_2^{i_1 - 1} s_1 s_2^{j_1 - 1} \cdots s_1 s_2^{j_{k-1} - 1} s_1 s_2^{i_k - 1}.$$

If we have an index ending with 2, so that $s_\mu = s_1^{i_1} s_2^{j_1} \cdots s_2^{j_{k-1}} s_1^{i_k} s_2^{j_k}$, then

$$\psi(s_\mu s_\mu^*) = s_{\widetilde{\mu}} s_{\widetilde{\mu}}^*,$$

where

$$s_{\widetilde{\mu}} = s_1 s_2^{i_1 - 1} s_1 s_2^{j_1 - 1} \cdots s_1 s_2^{j_{k-1} - 1} s_1 s_2^{i_k - 1} s_1 s_2^{j_k - 1}.$$

Note that each occurrence of s_1 in $\psi(s_\mu s_\mu^*)$ is caused by a 'change' in the sequence represented by μ (if we assume that all sequences μ have s_2 as the 0-th element). This observation implies that any sequence μ ending in 2 gives an output sequence with an even number of 1's (even number of changes), so that $\psi|_{\mathcal{C}_2}$ is induced by the transformation of $\mathfrak{C}$ given by

$$(T_\psi(w))_k = \begin{cases} 1 & \text{if } \sharp\{j \le k : w_j = 1\} \text{ is odd,} \\ 2 & \text{if } \sharp\{j \le k : w_j = 1\} \text{ is even.} \end{cases}$$

It is easy to see that $h_{\mathrm{top}}(T_\psi) = 0$, so also $\mathrm{ht}(\psi|_{\mathcal{C}_2}) = 0$.

We will see that there is another masa in $\mathcal{O}_N$ which is left invariant by ψ and such that the corresponding restriction has entropy $\log 2$. Let $X = s_1 s_2^* + s_2 s_1^*$. Then

$$\psi(X) = s_1 s_2 s_1^* s_2^* + s_1 s_1 s_2^* s_2^* + s_2 s_1 s_2^* s_1^* + s_2 s_2 s_1^* s_1^* = s_1 X s_2^* + s_2 X s_1^*.$$

Moreover if $\theta : \mathcal{O}_2 \to \mathcal{O}_2$ denotes the canonical shift endomorphism then

$$\psi(\theta(X)) = \psi(s_1 X s_1^* + s_2 X s_2^*) = \psi(s_1)(s_1 X s_2^* + s_2 X s_1^*)\psi(s_1^*) + s_2 \psi(X) s_2^* = \theta(\psi(X)).$$

We will now show that for each $k \in \mathbb{N}$

$$\theta^k(\psi(X)) = \psi(\theta^k(X)) \tag{3.1}$$

Suppose we have shown for some $n \in \mathbb{N}$ that $\theta^n(\psi(X)) = \psi(\theta^n(X))$. Then

$$\begin{aligned}
\psi(\theta^{n+1}(X)) &= \sum_{\mu \in \mathcal{J}_{n+1}} \psi(s_\mu X s_\mu^*) \\
&= \sum_{\nu \in \mathcal{J}_n} (\psi(s_1)\psi(s_\nu X s_\nu^*)\psi(s_1^*) + s_2\psi(s_\nu X s_\nu^*)s_2^*) \\
&= \psi(s_1) \sum_{\nu \in \mathcal{J}_n} s_\nu \psi(X) s_\nu^* \psi(s_1)^* + s_2 \sum_{\nu \in \mathcal{J}_n} s_\nu \psi(X) s_\nu^* s_2^* \\
&= \sum_{\mu \in \mathcal{J}_{n+1}} s_\mu \psi(X) s_\mu^* = \theta^{n+1}(\psi(X)).
\end{aligned}$$

The second last equality follows if we notice that $\psi(s_1)s_\nu = s_1 s_{\widetilde{\nu}}$, where $\widetilde{\nu}$ equals ν but with the first letter 'switched'.

The formula (3.1) will become useful when we view the UHF algebra $\mathcal{F}_2$ as the tensor product $\bigotimes_{i=1}^{\infty} M_2^{(i)}$. Define first

$$E = \frac{1}{2}(I + X), \quad F = \frac{1}{2}(I + X).$$

It is clear that E and F are minimal projections in the first matrix factor of the UHF algebra. Thus the algebra generated by $\{\theta^n(E), \theta^n(F) : n \in \mathbb{N}_0\}$ is a masa, further denoted by $\mathcal{C}_{E,F}$. Because of (3.1) we immediately see that also

$$\theta^k(\psi(E)) = \psi(\theta^k(E)), \quad \theta^k(\psi(F)) = \psi(\theta^k(F))$$

for all $k \in \mathbb{N}$. As in the tensor picture $\psi(X) = X \otimes X$, it is easy to see that

$$\psi(E) = E \otimes E + F \otimes F, \quad \psi(F) = F \otimes F + E \otimes E.$$

In conjunction with the previous statement this implies that ψ leaves $\mathcal{C}_{E,F}$ invariant. The algebra $\mathcal{C}_{E,F}$ is isomorphic to $C(\mathfrak{C})$. The isomorphism may be given for example by identifying E with χ_{Z_1} and F with χ_{Z_2} so that for example $E \otimes F \otimes E \otimes E$ is mapped to $\chi_{Z_{1211}}$. It is easy to show that $T_{E,F}$, the induced continuous map on $\mathfrak{C}$, is given by the formula:

$$(T_{E,F}(w))_k = \begin{cases} 1 & \text{if } w_k = w_{k+1}, \\ 2 & \text{if } w_k \neq w_{k+1}. \end{cases}$$

A comparison of this formula with the remarks in the beginning of this section shows that $\mathrm{ht}(\psi_{\mathcal{C}_{E,F}}) = h_{\mathrm{top}}(T_{E,F}) = \log 2$. Together with Theorem 2.1 this implies that

$$\mathrm{ht}(\psi) = \mathrm{ht}(\psi|_{\mathcal{F}_2}) = \log 2.$$

The transformation induced by σ_{1324}. Let ψ denote again the endomorphism induced by σ_{12} and let ψ' denote the one induced by σ_{1324}. Then

$$\psi'(s_1) = \psi(s_2), \quad \psi'(s_2) = \psi'(s_1).$$

Note that this implies in particular that on the masa $\mathcal{C}_{E,F}$ introduced earlier the endomorphisms ψ' and ψ coincide. Indeed, $\psi(E) = \psi'(E)$ and also

$$\psi'(\theta^n(E)) = \psi'(\sum_{\mu\in\mathcal{J}_n} s_\mu E s_\mu^*) = \sum_{\mu\in\mathcal{J}_n} \psi'(s_\mu)\psi'(E)\psi'(s_\mu^*)$$
$$= \sum_{\mu\in\mathcal{J}_n} \psi(s_\mu)\psi(E)\psi(s_\mu^*) = \psi(\theta^n(E)).$$

Thus $\mathrm{ht}(\psi') \geq \mathrm{ht}(\psi'|_{\mathcal{C}_{E,F}}) = \mathrm{ht}(\psi|_{\mathcal{C}_{E,F}}) = \log 2$ and we obtain

$$\mathrm{ht}(\psi) = \mathrm{ht}(\psi|_{\mathcal{F}_2}) = \log 2.$$

Note that as $\psi'(\sum_{J\in\mathcal{J}_k, j_k=1} s_J s_J^*) = \psi(\sum_{J\in\mathcal{J}_k, j_k=2} s_J s_J^*)$, the restriction of ψ' to $\mathcal{C}_2$ is isomorphic to the map given by

$$(T_{\psi'}(w))_k = \begin{cases} 2 & \text{if } \sharp\{j \leq k : w_j = 1\} \text{ is odd} \\ 1 & \text{if } \sharp\{j \leq k : w_j = 1\} \text{ is even} \end{cases}$$

and thus $\mathrm{ht}(\psi'|_{\mathcal{C}_2}) = 0$.

The transformation induced by $\sigma_{(14),(23)}$. It is shown in [Ka] that this endomorphism is given by the conjugation with the unitary $s_1 s_2^* + s_2 s_1^*$. Thus it leaves each of the subalgebras $F_{p,l}$ $(p, l \in \mathbb{N})$ invariant and one can easily deduce using the Kolmogorov-Sinai property that its entropy is 0.

The transformation induced by $\sigma_{(12),(34)}$. This one is the composition of the flip automorphism and of ρ_σ for $\sigma_{(14),(23)}$. As they both leave $F_{p,l}$ invariant, the entropy is 0.

The transformations induced by $\sigma_{14}, \sigma_{132}, \sigma_{124}, \sigma_{143}, \sigma_{234}, \sigma_{1243}, \sigma_{1342}$. Let σ be one of the permutations from the above list. It is easy to show inductively that for any $k \in \mathbb{N}$ and $J \in \mathcal{J}_k$ there is

$$\rho_\sigma(s_J) = s_{J_1} s_1^* + s_{J_2} s_2^*, \tag{3.2}$$

where J_1, J_2 are certain multiindices in $\mathcal{J}_k$. This implies, as we will show below, that in some special cases the formula for the map $T_{\rho_\sigma} : \mathfrak{C} \to \mathfrak{C}$ induced by the restriction of ψ to $\mathcal{C}_2$ is determined already by the value of $\psi(s_1 s_1^*)$. Let us formulate it in a lemma:

Lemma 3.1. *Suppose that the endomorphism $\rho_\sigma : \mathcal{O}_2 \to \mathcal{O}_2$ satisfies the condition* (3.2). *Then if $\rho_\sigma(s_1 s_1^*) = s_1 s_2 s_2^* s_1^* + s_2 s_2 s_2^* s_2^*$, then*

$$(T_{\rho_\sigma}(w))_k = \begin{cases} 1 & \text{if } w_{k+1} = 2, \\ 2 & \text{if } w_{k+1} = 1. \end{cases}$$

If $\rho_\sigma(s_1 s_1^*) = s_1 s_1 s_1^* s_1^* + s_2 s_1 s_1^* s_2^*$, then

$$(T_{\rho_\sigma}(w))_k = \begin{cases} 1 & \text{if } w_{k+1} = 1, \\ 2 & \text{if } w_{k+1} = 2. \end{cases}$$

Proof. It is enough to consider the first case, the second follows in an analogous way. Suppose that $J \in \mathcal{J}$ and J_1, J_2 are as in the formula (3.2). Then

$$\rho_\sigma(s_J s_1 s_1^* s_J^*) = (s_{J_1} s_1^* + s_{J_2} s_2^*)(s_1 s_2 s_2^* s_1^* + s_2 s_2 s_2^* s_2^*)(s_1 s_{J_1}^* + s_2 s_{J_2}^*)$$
$$= s_{J_1} s_2 s_2^* s_{J_1}^* + s_{J_2} s_2 s_2^* s_{J_2}^*.$$

This implies that

$$\rho_\sigma(\sum_{J\in\mathcal{J}_k,j_k=1} s_J s_J^*) = \sum_{I\in\mathcal{J}_{k+1},I_{k+1}=2} s_I s_I^*$$

and the considerations from the beginning of this section end the proof. □

The analysis of the values at $s_1 s_1^*$ together with Theorem 2.1, Lemma 3.1 and remarks in the beginning of this section show that for any σ from the following list: $\sigma_{14}, \sigma_{132}, \sigma_{124}, \sigma_{143}, \sigma_{234}, \sigma_{1243}, \sigma_{1342}$ there is

$$\mathrm{ht}(\rho_\sigma) = \mathrm{ht}(\rho_\sigma|_{\mathcal{C}_2}) = \log 2.$$

The transformation induced by σ_{13}. Let $\sigma = \sigma_{13}$. It is easy to see that actually in this case the formula (3.2) can be made more precise so that we obtain for any $k \in \mathbb{N}$ and $J \in \mathcal{J}_k$

$$\rho_\sigma(s_J) = s_{J_1} s_1 s_1^* + s_{J_2} s_2 s_2^*,$$

where now $J_1, J_2 \in \mathcal{J}_{k-1}$. Moreover we have

$$\rho_\sigma(s_1 s_1^*) = s_1 s_2 s_2^* s_1^* + s_2 s_1 s_1^* s_2^*,$$

so that

$$\begin{aligned}\rho_\sigma(s_J s_1 s_1^* s_J^*) &= (s_{J_1} s_1 s_1^* + s_{J_2} s_2 s_2^*)(s_1 s_2 s_2^* s_1^* + s_2 s_1 s_1^* s_2^*)(s_{J_1} s_1 s_1^* + s_{J_2} s_2 s_2^*)^* \\ &= s_{J_1} s_1 s_2 s_2^* s_1^* s_{J_1}^* + s_{J_2} s_2 s_1 s_1^* s_2^* s_{J_2}^*.\end{aligned}$$

This implies easily that ρ_σ restricted to $\mathcal{C}_2$ is isomorphic to the map

$$(T_{\rho_\sigma}(w))_k = \begin{cases} 1 & \text{if } w_k \neq w_{k+1}, \\ 2 & \text{if } w_k = w_{k+1}. \end{cases}$$

As in the last subsection we obtain

$$\mathrm{ht}(\rho_\sigma) = \mathrm{ht}(\rho_\sigma|_{\mathcal{C}_2}) = \log 2.$$

The transformation induced by σ_{1432}. The endomorphism is the composition of the inner automorphism $\rho_{\sigma_{1243}}$ with $\rho_{\sigma_{13}}$. This implies that ρ_σ restricted to $\mathcal{C}_2$ is isomorphic to the map

$$(T_{\rho_\sigma}(w))_k = \begin{cases} 1 & \text{if } k \geq 2 \text{ and } w_k \neq w_{k+1} \text{ or } k = 1 \text{ and } w_1 = w_2, \\ 2 & \text{if } k \geq 2 \text{ and } w_k = w_{k+1} \text{ or } k = 1 \text{ and } w_1 \neq w_2, \end{cases}$$

and we obtain

$$\mathrm{ht}(\rho_\sigma) = \mathrm{ht}(\rho_\sigma|_{\mathcal{C}_2}) = \log 2.$$

The transformation induced by σ_{123}. The endomorphism is given by the formulas

$$\rho_\sigma(s_1) = s_1 s_2 s_1^* + s_2 s_1 s_2^*, \quad \rho_\sigma(s_2) = s_1 s_1 s_1^* + s_2 s_2 s_2^*,$$

so that

$$\begin{aligned}\rho_\sigma(s_1 s_1^*) &= s_1 s_2 s_2^* s_1^* + s_2 s_1 s_1^* s_2^*, \\ \rho_\sigma(s_2 s_2^*) &= s_1 s_1 s_1^* s_1^* + s_2 s_2 s_2^* s_2^*.\end{aligned}$$

It is also easy to check that it has the property described in (3.2). Moreover we have the following lemma holds:

Lemma 3.2. *Let $k \in \mathbb{N}_0$, $J \in \mathcal{J}_k$. Then*

$$\rho_\sigma(s_J s_1 s_1^* s_J^*) = s_{J_1} s_{J_1}^* + s_{J_2} s_{J_2}^*,$$

where J_1, J_2 are certain indices in $\mathcal{J}_{k+1}$ such that the number of constant sequences in J_1 is even and the number of constant sequences in J_2 is even.

Proof. The statement will be proved by the induction on k. The case $k = 0$ follows from the explicit formulae before the lemma. Let then $J \in \mathcal{J}_k$ and compute

$$\rho_\sigma(s_1 s_J s_J^* s_1^*) = (s_1 s_2 s_1^* + s_2 s_1 s_2^*)(s_{J_1} s_{J_1}^* + s_{J_2} s_{J_2}^*)(s_1 s_2 s_1^* + s_2 s_1 s_2^*)^*.$$

Suppose first that $J_1 = 1K$ for some $K \in \mathcal{J}_k$. Then

$$(s_1 s_2 s_1^* + s_2 s_1 s_2^*)(s_{J_1} s_{J_1}^*)(s_1 s_2 s_1^* + s_2 s_1 s_2^*)^* = s_1 s_2 s_K s_K^* s_2^* s_1^*.$$

We want to count the constant sequences in the multiindex $12K$. A moment of thought shows that it has either equally many constant sequences as $1K$ (if K began with 2) or two more (if K began with 1). Similarly if $J_1 = s_2 s_K$ for some $K \in \mathcal{J}_{k-1}$ we have

$$(s_1 s_2 s_1^* + s_2 s_1 s_2^*)(s_{J_1} s_{J_1}^*)(s_1 s_2 s_1^* + s_2 s_1 s_2^*)^* = s_2 s_1 s_K s_K^* s_1^* s_2^*,$$

and again the multiindex $21K$ has either equally many constant sequences as $2K$ (if K originally began with 1) or two more (if K originally began with 2).

It remains to consider what happens when on the left we add 2 instead of 1:

$$\rho_\sigma(s_2 s_J s_J^* s_2^*) = (s_1 s_1 s_1^* + s_2 s_2 s_2^*)(s_{J_1} s_{J_1}^* + s_{J_2} s_{J_2}^*)(s_1 s_1 s_1^* + s_1 s_1 s_1^*)^*.$$

Suppose first that $J_1 = s_1 s_K$ for some $K \in \mathcal{J}_k$. Then

$$(s_1 s_1 s_1^* + s_2 s_2 s_2^*)(s_{J_1} s_{J_1}^*)(s_1 s_1 s_1^* + s_2 s_2 s_2^*)^* = s_1 s_1 s_K s_K^* s_1^* s_1^*.$$

The multindex $11K$ has obviously equally many constant sequences as $1K$. An analogous argument suffices if $J_1 = s_2 s_K$ for some $K \in \mathcal{J}_{k-1}$ and the inductive proof is finished - the parity of the number of constant sequences in the multiindices appearing in the $(k+1)$-th stage is the same as in those which appeared in the k-th stage. □

The lemma above implies that the map on $\mathfrak{C}$ induced by $\rho_\sigma|_{\mathcal{C}_2}$ is given by the formula:

$$(T_{\rho_\sigma}(w))_k = \begin{cases} 1 & \text{if the number of constant sequences in } w|_{k+1]} \text{ is even,} \\ 2 & \text{if the number of constant sequences in } w|_{k+1]} \text{ is odd.} \end{cases}$$

This implies again that

$$\mathrm{ht}(\rho_\sigma) = \mathrm{ht}(\rho_\sigma|_{\mathcal{C}_2}) = \log 2.$$

The transformation induced by σ_{142}. We will apply the method analogous to that used for σ_{123}. Here the endomorphism is given by the formulas

$$\rho_\sigma(s_1) = s_2 s_2 s_1^* + s_1 s_1 s_2^*, \quad \rho_\sigma(s_2) = s_2 s_1 s_1^* + s_1 s_2 s_2^*,$$

so that

$$\rho_\sigma(s_1 s_1^*) = s_2 s_2 s_2^* s_2^* + s_1 s_1 s_1^* s_1^*,$$
$$\rho_\sigma(s_2 s_2^*) = s_2 s_1 s_1^* s_2^* + s_1 s_2 s_2^* s_1^*.$$

It is also easy to check that it has the property described in (3.2).

The next lemma is analogous to Lemma 3.2.

Lemma 3.3. *Let $k \in \mathbb{N}_0$, $J \in \mathcal{J}_k$. Then*

$$\rho_\sigma(s_J s_1 s_1^* s_J^*) = s_{J_1} s_{J_1}^* + s_{J_2} s_{J_2}^*,$$

where J_1, J_2 are certain indices in $\mathcal{J}_{k+1}$ such that $k+$ the number of constant sequences in J_1 is even and $k+$ the number of constant sequences in J_2 is even.

Proof. Again the case $k = 0$ follows from the formulas listed before the lemma. Let then $J \in \mathcal{J}_k$ and compute

$$\rho_\sigma(s_1 s_J s_J^* s_1^*) = (s_2 s_2 s_1^* + s_1 s_1 s_2^*)(s_{J_1} s_{J_1}^* + s_{J_2} s_{J_2}^*)(s_2 s_2 s_1^* + s_1 s_1 s_2^*)^*.$$

Suppose first that $J_1 = s_1 s_K$ for some $K \in \mathcal{J}_k$. Then

$$(s_2 s_2 s_1^* + s_1 s_1 s_2^*)(s_{J_1} s_{J_1}^*)(s_2 s_2 s_1^* + s_1 s_1 s_2^*)^* = s_2 s_2 s_K s_K^* s_2^* s_2^*.$$

The multiindex $22K$ has either one more constant sequence then $1K$ (if K began with 1) or one less (if K began with 2). Similarly if $J_1 = s_2 s_K$ for some $K \in \mathcal{J}_{k-1}$ we have

$$(s_2 s_2 s_1^* + s_1 s_1 s_2^*)(s_{J_1} s_{J_1}^*)(s_2 s_2 s_1^* + s_1 s_1 s_2^*)^* = s_1 s_1 s_K s_K^* s_1^* s_1^*,$$

and again the multiindex $11K$ has either one more constant sequence then $2K$ (if K began with 2) or one less (if K began with 1).

Further

$$\rho_\sigma(s_2 s_J s_J^* s_2^*) = (s_2 s_1 s_1^* + s_1 s_2 s_2^*)(s_{J_1} s_{J_1}^* + s_{J_2} s_{J_2}^*)(s_2 s_1 s_1^* + s_1 s_1 s_2^*)^*.$$

Suppose first that $J_1 = s_1 s_K$ for some $K \in \mathcal{J}_k$. Then

$$(s_2 s_1 s_1^* + s_1 s_2 s_2^*)(s_{J_1} s_{J_1}^*)(s_2 s_1 s_1^* + s_1 s_2 s_2^*)^* = s_2 s_1 s_K s_K^* s_1^* s_2^*.$$

The multindex $21K$ has obviously one more constant sequence than $1K$. An analogous argument applies if $J_1 = s_2 s_K$ for some $K \in \mathcal{J}_{k-1}$. The inductive proof is finished, as now we have shown that at every step parity of the number of constant sequences changes. □

$$(T_{\rho_\sigma}(w))_k = \begin{cases} 1 & \text{if } k + \text{the number of constant sequences in } w|_{k+1]} \text{ is odd,} \\ 2 & \text{if } k + \text{the number of constant sequences in } w|_{k+1]} \text{ is even.} \end{cases}$$

Thus we once more obtain

$$\mathrm{ht}(\rho_\sigma) = \mathrm{ht}(\rho_\sigma|_{\mathcal{C}_2}) = \log 2.$$

The transformations induced by $\sigma_{34}, \sigma_{1423}, \sigma_{24}, \sigma_{1234}, \sigma_{243}$ and σ_{134}. Arise from respectively $\sigma_{12}, \sigma_{1324}, \sigma_{13}, \sigma_{1432}, \sigma_{123}$ and σ_{142} just by 1 and 2 switching places. The entropy values can be thus read from the earlier computations.

Remark 3.4. As in all the cases above the maximal value of the topological entropy is achieved on a commutative subalgebra, the variational principle has to hold for all ρ_σ. Recall that this means that $\mathrm{ht}(\rho_\sigma) = \sup_\phi h_\phi(\rho_\sigma)$, where the supremum is taken over all states on $\mathcal{O}_2$ left invariant by ρ_σ and $h_\phi(\rho_\sigma)$ denotes the dynamical state entropy of Connes and Størmer (see [NS]). It can be easily seen that in each case the supremum is realised by the state $\tau \circ \mathbb{E}$.

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