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Vector Bundles on Degenerations of Elliptic Curves and Yang-Baxter Equations

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VECTOR BUNDLES ON DEGENERATIONS OF ELLIPTIC CURVES AND YANG-BAXTER EQUATIONS

IGOR BURBAN AND BERND KREUSSLER

Abstract. In this paper we introduce the notion of a geometric associative r-matrix attached to a genus one fibration with a section and irreducible fibres. It allows us to study degenerations of solutions of the classical Yang-Baxter equation using the approach of Polishchuk. We also calculate certain solutions of the classical, quantum and associative Yang-Baxter equations obtained from moduli spaces of (semi-)stable vector bundles on Weierstraß cubic curves.

1. Introduction

There are many indications (for example from homological mirror symmetry) that the formalism of derived categories provides a compact way to formulate and solve complicated non-linear analytical problems. However, one would like to have more concrete examples, in which one can follow the full path starting from a categorical set-up and ending with an analytical output. In this article we study the interplay between the theory of the classical Yang-Baxter equation and properties of vector bundles on projective curves of arithmetic genus one, following the approach of Polishchuk [40].

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and $A = U(\mathfrak{g})$ its universal enveloping algebra. The classical Yang-Baxter equation (CYBE) is

$$[r^{12}(x), r^{13}(x+y)] + [r^{13}(x+y), r^{23}(y)] + [r^{12}(x), r^{23}(y)] = 0,$$

where $r(z)$ is the germ of a meromorphic function of one variable in a neighbourhood of 0 taking values in $\mathfrak{g} \otimes \mathfrak{g}$. The upper indices in this equation indicate various embeddings of $\mathfrak{g} \otimes \mathfrak{g}$ into $A \otimes A \otimes A$. For example, the function r^{13} is defined as

$$r^{13} : \mathbb{C} \xrightarrow{r} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\tau_{13}} A \otimes A \otimes A,$$

where $\tau_{13}(x \otimes y) = x \otimes 1 \otimes y$. Two other maps r^{12} and r^{23} have a similar meaning.

In the physical literature, solutions of (CYBE) are frequently called *r-matrices*. They play an important role in mathematical physics, representation theory, integrable systems and statistical mechanics.

By a famous result of Belavin and Drinfeld [7], there exist exactly three types of non-degenerate solutions of the classical Yang-Baxter equation: elliptic (two-periodic), trigonometric (one-periodic) and rational. This trichotomy corresponds to three models in statistical mechanics: XYZ (elliptic), XXZ (trigonometric) and XXX (rational), see [6].

Belavin and Drinfeld have also obtained a complete classification of elliptic and trigonometric solutions, see [7, Proposition 5.1 and Theorem 6.1]. A certain classification of rational solutions was obtained by Stolin [46, Theorem 1.1].

This article is devoted to a study of degenerations of elliptic r-matrices into trigonometric and then into rational ones. We hope that this sort of questions will be interesting from the point of view of applications in mathematical physics. In order to attack this problem we use a construction of Polishchuk [40]. After certain modifications of his original presentation, the core of this method can be described as follows.

Let E be a Weierstraß cubic curve, $M = M_E(n,d)$ the moduli space of stable bundles of rank n and degree d, assumed to be coprime, and let $\mathcal{P} = \mathcal{P}(n,d) \in \mathsf{VB}(E \times M)$ be a universal family of the moduli functor $\underline{\mathsf{M}}_E^{(n,d)}$. For a point $v \in M$ we denote by $\mathcal{V} = \mathcal{P}|_{E \times v}$ the corresponding vector bundle on E. Consider the following data

- two different points $v_1, v_2 \in M$ in the moduli space;
- two distinct smooth points $y_1, y_2 \in E_{\mathrm{reg}}$ such that $\mathcal{V}_1(y_2) \not\cong \mathcal{V}_2(y_1)$.

Using Serre Duality, the triple Massey product

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2}),$$

induces a linear map

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2}).$$

This map can be rewritten as the germ of a tensor-valued meromorphic function in four variables, defined in a neighbourhood of a smooth point o of the moduli space $M \times M \times E \times E$ (the choice of o will be explained in Corollary 6.9)

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : (\mathbb{C}^2 \times \mathbb{C}^2, 0) \cong \big((M \times M) \times (E \times E), o\big) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$$

and satisfying the so-called *associative Yang-Baxter equation* (AYBE)

$$\big(r^{\mathcal{V}_3,\mathcal{V}_2}_{y_1,y_2}\big)^{12}\big(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_1,y_3}\big)^{13} - \big(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_2,y_3}\big)^{23}\big(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}\big)^{12} + \big(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_3}\big)^{13}\big(r^{\mathcal{V}_2,\mathcal{V}_3}_{y_2,y_3}\big)^{23} = 0$$

viewed as a map

$$\begin{aligned} &\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_3}) \longrightarrow \\ &\longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_3}).\end{aligned}$$

Since the complex manifold $\mathsf{M}_E(n,d) \times E_{\mathrm{reg}}$ is a homogeneous space over the algebraic group $\mathsf{Pic}^0(E) \times \mathsf{Aut}(E)$, it turns out that

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} =: r(v_1, v_2, y_1, y_2) \sim r(v_1 - v_2, y_1 - y_2) = r(v, y),$$

with respect to a certain equivalence relation on the set of solutions.

Let $G = \mathsf{Pic}^0(E)$ and e be the neutral element of G. It was shown by Polishchuk [40, Lemma 1.2] that the function of one variable

$$\bar{r}(y) = \lim_{v \to e} (\mathrm{pr} \otimes \mathrm{pr}) r(v, y) \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$$

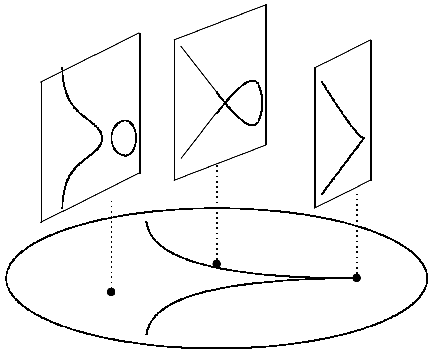
is a non-degenerate unitary solution of the classical Yang-Baxter equation. Moreover, under certain restrictions (which are always fulfilled at least for elliptic curves and Kodaira cycles of projective lines), for any fixed value $g \neq e$ from a small neighbourhood $U_e \subseteq G$ of the neutral element e, the tensor-valued function

$$r : (\{g\} \times G, e) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$$

satisfies the *quantum* Yang-Baxter equation, see [41, Theorem 1.4].

The aim of our article is to study a *relative* version of Polishchuk's construction. Although most of the results can be generalised on the case of arbitrary reduced curves of arithmetic genus one having trivial dualizing sheaf, in this article we shall concentrate mainly on the case of irreducible curves.

Let E be Weierstraß cubic curve, i.e. a plane projective curve given by the equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$. It is singular if any only if $\Delta := g_2^3 - 27g_3^2 = 0$.



Unless $g_2 = g_3 = 0$, the singularity is a node, whereas for $g_2 = g_3 = 0$ it is a cusp.

A connection between the theory of vector bundles on cubic curves and exactly solvable models of mathematical physics was observed a long time ago, see for example [33, Chapter 13] and [35] for a link with KdV equation, [18] for applications to integrable systems and [9] for an interplay with Calogero-Moser systems. In particular, the correspondence

elliptic	elliptic
trigonometric	nodal
rational	cuspidal

was discovered at the very beginning of the algebraic theory of completely integrable systems.

In this article we follow another strategy. Instead of looking at each curve of arithmetic genus one individually, we consider the relative case, so that all solutions will be considered as specialisations of one *universal* solution. Our main result can be stated as follows.

Let $E \to T$ be a genus one fibration with a section having reduced and irreducible fibres, $M = M_{E/T}(n,d)$ the moduli space of relatively stable vector bundles of rank n and degree d. We construct a meromorphic function

$$r : (M \times_T \times M \times_T E \times_T E, o) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$$

in a neighbourhood of a smooth point o of $M \times_T \times M \times_T E \times_T E$, which satisfies the associative Yang-Baxter equation for each fixed value $t \in T$ and $(v_1, v_2, y_1, y_2) \in \big((M_{E_t} \times M_{E_t}) \times (E_t \times E_t), o\big)$. Moreover, $r_t(v_1, v_2, y_1, y_2)$ depends analytically on t, is compatible with base change of the given family $E \to T$ and the corresponding solution of the classical Yang-Baxter equation $\bar{r}_t(y)$ is

- *elliptic if E_t is smooth;*
- *trigonometric if E_t is nodal;*
- *rational if E_t is cuspidal.*

We also carry out explicit calculations for vector bundles of rank two and degree one on irreducible Weierstraß cubic curves. In the case of an elliptic curve $E = E_\tau$ the corresponding solution is

$$r_{\mathrm{ell}}(v;y) = \frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\left[\frac{\theta_1(y+v|\tau)}{\theta_1(v|\tau)}\mathbb{1}\otimes\mathbb{1} + \frac{\theta_2(y+v|\tau)}{\theta_2(v|\tau)}h\otimes h + \right.$$
$$\left. + \frac{\theta_3(y+v|\tau)}{\theta_3(v|\tau)}\sigma\otimes\sigma + + \frac{\theta_4(y+v|\tau)}{\theta_4(v|\tau)}\tau\otimes\tau\right],$$

where $\mathbb{1} = e_{11} + e_{22}$, $h = e_{11} - e_{22}, \sigma = i(e_{21} - e_{12})$ and $\tau = e_{21} + e_{12}$.

In the case of a nodal cubic curve we get

$$r_{\mathrm{trg}}(v;y) = \frac{\sin(y+v)}{\sin(y)\sin(v)}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) - \frac{1}{\cos(v)}(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}) +$$
$$+ \frac{1}{\sin(y)}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \sin(y+v)e_{21}\otimes e_{12}$$

and in the case of a cuspidal cubic curve, the associative r–matrix is

$$r_{\mathrm{rat}}(v;y_1,y_2) = \frac{1}{v}\mathbb{1}\otimes\mathbb{1} + \frac{2}{y_2 - y_1}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) +$$
$$+ (v - y_1)e_{21}\otimes h + (v + y_2)h\otimes e_{21} + v(v - y_1)(v + y_2)e_{21}\otimes e_{21}.$$

Our results imply that up to a gauge transformation the trigonometric and rational solutions $r_{\mathrm{trg}}(v;y)$ and $r_{\mathrm{rat}}(v;y_1,y_2)$ are degenerations of $r_{\mathrm{ell}}(v;y)$, which seems to be difficult to show by a direct computation.

Moreover, for a generic v the tensors $r_{\mathrm{ell}}(v;y)$, $r_{\mathrm{trg}}(v;y)$ and $r_{\mathrm{rat}}(v;y_1,y_2)$ satisfy the quantum Yang-Baxter equation and are quantisations of the following classical r–matrices:

• Elliptic solution of Belavin:

$$\bar{r}_{\mathrm{ell}}(y) = \frac{\mathrm{cn}(y)}{\mathrm{sn}(y)} h\otimes h + \frac{1+\mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12}\otimes e_{21}+e_{21}\otimes e_{12}) + \frac{1-\mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12}\otimes e_{12}+e_{21}\otimes e_{21}).$$

• Trigonometric solution of Cherednik:

$$\bar{r}_{\mathrm{trg}}(y) = \frac{1}{2}\cot(y) h\otimes h + \frac{1}{\sin(y)}(e_{12}\otimes e_{21}+e_{21}\otimes e_{12}) + \sin(y) e_{21}\otimes e_{21}.$$

• Rational solution of Stolin:

$$\bar{r}_{\mathrm{rat}}(y_1,y_2) = \frac{1}{y_2-y_1}\left(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21}+e_{21}\otimes e_{12}\right) + y_2 h\otimes e_{21} - y_1 e_{21}\otimes h.$$

This paper is organised as follows. In Section 2 we briefly collect some known material about various types of Yang-Baxter equations. Section 3 gives a short introduction into a construction of Polishchuk which provides a method to obtain solutions of Yang-Baxter equations from triple Massey products in a derived category. In order to be able to calculate solutions explicitly, this construction has to be translated into another language, involving residue maps. In Section 4 we explain the corresponding result of Polishchuk whereby we provide some details which are only implicit in [40]. The understanding of these details is crucial for the study of the relative case, which is carried out in Sections 5 and 6. Theorem 6.6 is the main result of this article. We also explain in this section how to trivialise the universal bundle in order to obtain what we call the *geometric associative r-matrix*.

In Section 7 we recall some classical results about holomorphic vector bundles on a smooth elliptic curve. Using the methods described before, we explicitly compute the solution of the associative Yang-Baxter equation and the classical r–matrix corresponding to a universal family of stable vector bundles of rank two and degree one. These solutions were computed by Polishchuk in [40, Section 2] using homological mirror symmetry and formulas for higher products in the Fukaya category of an elliptic curve. Our direct computation, however, is independent of homological mirror symmetry. We are lead directly to express the resulting associative r–matrix in terms of Jacobi's theta-functions and the corresponding classical r–matrix in terms of the elliptic functions $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$. Sections 8 and 9 are devoted to similar calculations for nodal and cuspidal Weierstraß curves. Our computations are based on the description of vector bundles on such curves in terms of so-called matrix problems, which was given by Drozd and Greuel [20] and Burban [13]. We conclude this article with a brief summary in Section 10 and with an appendix (Section 11) in which we provide a proof of the representability of a relative moduli functor in the analytic case for which we were unable to find a reference.

Notation. Throughout this paper we work in the category of analytic spaces over the field of complex numbers $\mathbb{C}$, see [39]. However, most of the results remain valid in the category of algebraic varieties over an algebraically closed field $\boldsymbol{k}$ of characteristic zero. If V, W are two complex vector spaces, $\mathsf{Lin}(V, W)$ denotes the vector space of complex linear maps from V to W.

If X is a complex projective variety, we denote by $\mathsf{Coh}(X)$ the category of coherent $\mathcal{O}_X$-modules and by $\mathsf{VB}(X)$ its full subcategory of locally free sheaves (holomorphic vector bundles). The torsion sheaf of length one, supported at a closed point $y \in X$, is always denoted by $\mathbb{C}_y$. By $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ we denote the full subcategory of derived category of the abelian category of all $\mathcal{O}_X$-modules whose objects are those complexes which have bounded and coherent cohomology. The notation $\mathsf{Perf}(X)$ is used for the full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ whose objects are isomorphic to bounded complexes of locally free sheaves. If $\mathcal{F}_1, \mathcal{F}_2$ are coherent sheaves on X, we denote by $\mathsf{add}\big(\mathcal{F}_1 \oplus \mathcal{F}_2\big)$ the full subcategory of $\mathsf{Coh}(X)$ whose objects are isomorphic to $\mathcal{F}_1^{\oplus n_1} \oplus \mathcal{F}_2^{\oplus n_2}$ for some non-negative integers n_1, n_2.

A Weierstraß curve is a plane cubic curve given in homogeneous coordinates by an equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$. Such a curve is always irreducible. It is a smooth elliptic curve if and only if $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$.

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2. Yang-Baxter equations

In this section we are going to recall some standard results on Yang-Baxter equations. Let $\mathfrak{g}$ be a simple complex Lie algebra (throughout this paper $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$), $\langle\ \ ,\ \ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form. The classical Yang-Baxter equation is

$$(1)\quad [r^{12}(y_1,y_2), r^{23}(y_2,y_3)] + [r^{12}(y_1,y_2), r^{13}(y_1,y_3)] + [r^{13}(y_1,y_3), r^{23}(y_2,y_3)] = 0,$$

where $r(x,y)$ is the germ of a meromorphic function of two complex variables in a neighbourhood of 0, taking values in $\mathfrak{g} \otimes \mathfrak{g}$. A solution of (1) is called *unitary* if

$$r^{12}(y_1,y_2) = -r^{21}(y_2,y_1)$$

and *non-degenerate* if $r(y_1,y_2) \in \mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g}^* \otimes \mathfrak{g} \cong \mathsf{End}(\mathfrak{g})$ is invertible for generic (y_1,y_2). On the set of solutions of (1) there is a natural action of the algebra of holomorphic function germs $\phi : (\mathbb{C},0) \longrightarrow \mathrm{Aut}(\mathfrak{g})$ given by the rule

$$(2)\qquad r(y_1,y_2) \mapsto \big(\phi(y_1) \otimes \phi(y_2)\big) r(y_1,y_2).$$

Proposition 2.1 (see [8]). *Modulo the equivalence relation (2) any non-degenerate solution of the equation (1) is equivalent to a solution $r(u,v) = r(u-v)$ depending only on the difference (or the quotient) of spectral parameters.*

This means that equation (1) is essentially equivalent to the equation

$$(3)\qquad [r^{12}(x), r^{13}(x+y)] + [r^{13}(x+y), r^{23}(y)] + [r^{12}(x), r^{23}(y)] = 0.$$

Although the classical Yang-Baxter equation with one spectral parameter is better adapted for applications in mathematical physics, it seems that from a geometric point of view equation (1) is more natural.

Theorem 2.2 (see Proposition 2.1 and Proposition 4.1 in [7]). *Let $r(z)$ be a non-degenerate solution of (3). Then*

- *If $r(z)$ is non-constant, then it has a simple pole at* 0*. Moreover,*

$$\mathrm{res}_{z=0}(r(z)) = \alpha\Omega \in \mathfrak{g} \otimes \mathfrak{g}$$

  where $\alpha \in \mathbb{C}^*$ and Ω is the so-called Casimir element.
- *r is automatically unitary, i.e. $r^{12}(z) = -r^{21}(-z)$.*

As it was already mentioned in the introduction, there is the following classification of non-degenerate solutions of (CYBE) due to Belavin and Drinfeld.

Theorem 2.3 (see Proposition 4.5 and Proposition 4.7 in [7]). *There are three types of non-degenerate solutions of the classical Yang-Baxter equation (3): elliptic, trigonometric and rational.*

Let us now consider some examples. Fix the following basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Note that $\Omega = \frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}$ is the Casimir element of $\mathfrak{sl}_2(\mathbb{C})$.

- Historically, the first solution ever found was the rational solution of Yang

$$r_{\mathrm{rat}}(z) = \frac{1}{z}\left(\frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right).$$

- Next, in 1978 Baxter discovered the trigonometric solution

$$r_{\mathrm{trg}}(z) = \frac{1}{2}\cot(z) h \otimes h + \frac{1}{\sin(z)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}).$$

- A few years later, Belavin found a solution of elliptic type:

$$r_{\mathrm{ell}}(z) = \frac{\mathrm{cn}(z)}{\mathrm{sn}(z)} h \otimes h + \frac{1 + \mathrm{dn}(z)}{\mathrm{sn}(z)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \frac{1 - \mathrm{dn}(z)}{\mathrm{sn}(z)}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}),$$

where $\mathrm{cn}(z), \mathrm{sn}(z)$ and $\mathrm{dn}(z)$ are doubly periodic meromorphic functions on $\mathbb{C}$ with periods 2 and 2τ. These functions also satisfy identities of the form $f(z+1) = \varepsilon f(z)$ and $f(z+\tau) = \varepsilon f(z)$ with $\varepsilon = \pm 1$.

At first glance, all these solutions seem to be completely different. However, it is easy to see that

$$\lim_{t \to \infty} \frac{1}{t} r_{\mathrm{trg}}\left(\frac{z}{t}\right) = r_{\mathrm{rat}}(z),$$

hence the solution of Yang is a degeneration of Baxter's solution. Moreover, there exist degenerations $\mathrm{dn}(z) \to 1$, $\mathrm{cn}(z) \to \cos(z)$ and $\mathrm{sn}(z) \to \sin(z)$, when the imaginary period τ tends to infinity, see for example [32, Section 2.6].

Hence, both solutions of Baxter and Yang are degenerations of Belavin's solution. However, as we shall see later, the theory of degenerations of r–matrices is more complicated as it might look like at first sight.

In this article we deal with a new type of Yang-Baxter equation, called *associative Yang-Baxter equation* (AYBE) introduced by Aguiar [1] and Polishchuk [40].

Definition 2.4. An associative r-matrix is the germ of a meromorphic function in four variables

$$r : (\mathbb{C}^4, 0) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$$

satisfying the equation

$$r(v_1, v_3; y_1, y_3)^{13} r(v_3, v_2; y_1, y_2)^{12} - r(v_1, v_2; y_1, y_2)^{12} r(v_1, v_3; y_2, y_3)^{23} + \\ + r(v_2, v_3; y_2, y_3)^{23} r(v_1, v_2; y_1, y_3)^{13} = 0. \tag{4}$$

Such a matrix is called unitary if

$$r(v_1, v_2; y_1, y_2)^{12} = -r(v_2, v_1; y_2, y_1)^{21}.$$

On the set of solutions of the equation (4) there exists a natural equivalence relation.

Definition 2.5 (see section 1.2 in [40])**.** Let $\phi : (\mathbb{C}^2, 0) \to \mathsf{GL}_n(\mathbb{C})$ be the germ of a holomorphic function and $r(v_1, v_2; y_1, y_2)$ be a solution of (AYBE) then

$$r'(v_1, v_2; y_1, y_2) = \bigl(\phi(v_1, y_1) \otimes \phi(v_2, y_2)\bigr) r(v_1, v_2; y_1, y_2) \bigl(\phi(v_2, y_1)^{-1} \otimes \phi(v_1, y_2)^{-1}\bigr)$$

is a new solution of (4). Two such tensors r and r' are called equivalent.

Assume we have a unitary solution r of (AYBE) such that

$$r(v_1, v_2; y_1, y_2) = r(v_2 - v_1; y_1, y_2) = r(v; y_1, y_2).$$

Then the equation (4) can be rewritten as

$$r(u+v; y_1, y_3)^{13} r(-v; y_1, y_2)^{12} - r(u; y_1, y_2)^{12} r(u+v; y_2, y_3)^{23} + \\ + r(v; y_2, y_3)^{23} r(u; y_1, y_3)^{13} = 0. \tag{5}$$

If, in addition, it holds $r(v_1, v_2; y_1, y_2) = r(v_2 - v_1; y_2 - y_1)$, then the associative Yang-Baxter equation reduces to the form

$$r(u+v; x+y)^{13} r(-v; x)^{12} - r(u; x)^{12} r(u+v; y)^{23} + \\ + r(v; y)^{23} r(u; x+y)^{13} = 0. \tag{6}$$

For the sake of convenience we reprove the following lemma.

Lemma 2.6 (see Lemma 1.2 in [40])**.** *Let $r(v; y_1, y_2)$ be a unitary solution of the associative Yang-Baxter equation (5), and let* $\mathrm{pr} : \mathsf{Mat}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ *be the projection along the scalar matrices, i.e.* $\mathrm{pr}(A) = A - \frac{1}{n}\mathrm{tr}(A) \cdot \mathbb{1}$. *Assume that* $(\mathrm{pr} \otimes \mathrm{pr})(r(v; y_1, y_2))$ *has a limit as $v \to 0$. Then*

$$\bar{r}(y_1, y_2) := \lim_{v \to 0} (\mathrm{pr} \otimes \mathrm{pr}) r(v, y_1, y_2)$$

is a unitary solution of (1).

Proof. Let τ be the automorphism of $\mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$ given by the formula $\tau(a \otimes b) = b \otimes a$. Applying $\tau \otimes 1$ to the equation (5) we get

$$r(u+v; y_1, y_3)^{23} r(-v; y_1, y_2)^{21} - r(u; y_1, y_2)^{21} r(u+v; y_2, y_3)^{13} + \\ + r(v; y_2, y_3)^{13} r(u; y_1, y_3)^{23} = 0.$$

Moreover, by unitarity of r it holds

$$-r(u+v; y_1, y_3)^{23} r(v; y_2, y_1)^{12} + r(-u; y_2, y_1)^{12} r(u+v; y_2, y_3)^{13} + \\ + r(v; y_2, y_3)^{13} r(u; y_1, y_3)^{23} = 0.$$

Since this equation is fulfilled for all values of u, v and y_1, y_2, y_3 taken in a neighbourhood of zero, we can use a change of variables $u \leftrightarrow v$, $y_1 \leftrightarrow y_2$, $y_3 \leftrightarrow y_3$. This gives us the relation

$$-r(u+v; y_2, y_3)^{23} r(u; y_1, y_2)^{12} + r(-v; y_1, y_2)^{12} r(u+v; y_1, y_3)^{13} + \\ + r(u; y_1, y_3)^{13} r(v; y_2, y_3)^{23} = 0.$$

Subtracting this equation from (5) we get

$$[r(-v;y_1,y_2)^{12}, r(u+v;y_1,y_3)^{13}] + [r(u;y_1,y_2)^{12}, r(u+v;y_2,y_3)^{23}]+$$
$$+[r(u;y_1,y_3)^{13}, r(v;y_2,y_3)^{23}] = 0.$$

The claim follows by applying $\mathrm{pr}\otimes\mathrm{pr}$ to this equation and taking the limit $u, v \to 0$. □

Let $U \subset \mathbb{C}^3_{(v;y_1,y_2)}$ be a small open neighbourhood of 0 and denote by M the algebra of meromorphic functions on U holomorphic on $\overset{\bullet}{U} = U \setminus \{v\cdot(y_2-y_1)=0\}$. With $A = \mathsf{Mat}_n(\mathsf{M})$, a *quantum* R-matrix is a tensor $R \in A\otimes A$ such that

$$R^{12}(v,y_1,y_2)R^{13}(v,y_1,y_3)R^{23}(v,y_2,y_3) = R^{23}(v,y_2,y_3)R^{13}(v,y_1,y_3)R^{12}(v,y_1,y_2). \tag{7}$$

Definition 2.7. A solution $r(y_1,y_2)$ of the equation (3) has an infinitesimal symmetry, if there exists an element $a\in\mathfrak{g}$ such that

$$[r(z), a\otimes \mathbb{1} + \mathbb{1}\otimes a] = 0.$$

For example, let $r(z) = r_{\mathrm{rat}}(z) = \frac{1}{z}\Omega$ be Yang's solution for $\mathfrak{sl}_2(\mathbb{C})$, then

$$[r(z), a\otimes \mathbb{1} + \mathbb{1}\otimes a] = 0$$

for any $a\in\mathfrak{g}$.

The *raison d'être* for the equation (6) is explained by the following theorem of Polishchuk.

Theorem 2.8 (see Theorem 1.4 of [41] and Theorem 6 of [40]). *Let $r(v;y)$ be a non-degenerate unitary solution of (6) and assume there exists a Laurent expansion of the form*

$$r(v;y) = \frac{1}{v}\mathbb{1}\otimes\mathbb{1} + r_0(y) + v r_1(y) + v^2 r_2(y) + \dots. \tag{8}$$

Then the following holds.

- *The function*
$$\bar{r}_0(y) := (\mathrm{pr}\otimes\mathrm{pr})(r_0(y))$$
is a non-degenerate unitary solution of the classical Yang-Baxter equation.
- *If $\bar{r}_0(y)$ is either periodic (elliptic or trigonometric), or without infinitesimal symmetries, then $r(v;y)$ satisfies the quantum Yang-Baxter equation (7).*
- *If $\bar{r}_0(y)$ does not have infinitesimal symmetries and if $r'(v;y)$ is another such solution of (6) of the form (8) and such that $\bar{r}_0(y) = (\mathrm{pr}\otimes\mathrm{pr})(r'_0(y))$, then there exist $\alpha_1\in\mathbb{C}^*$ and $\alpha_2\in\mathbb{C}$ such that $r'(v;y) = \alpha_1\exp(\alpha_2 vy)r(v;y)$. In other words, under these conditions $r(v;y)$ is determined by $\bar{r}_0(y)$ up to rescaling.*

Remark 2.9. It was proved by Polishchuk in [40] that any elliptic solution of the classical Yang-Baxter equation (3) can be lifted to a solution of (6) having a Laurent expansion of the form (8). However, Schedler showed in [43] that there exist trigonometric solutions of (3), which can not be lifted to a solution of (6).

The following proposition is straightforward.

Proposition 2.10. *Let $r : (\mathbb{C}^3, 0) \to \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$ be a solution of the associative Yang-Baxter equation having a Laurent expansion of the form*

$$r(v; y_1, y_2) = \frac{1}{v}\mathbb{1} \otimes \mathbb{1} + r_0(y_1, y_2) + v r_1(y_1, y_2) + v^2 r_2(y_1, y_2) + \dots,$$

and $\bar{r}_0(y_1, y_2)$ the corresponding solution of the classical Yang-Baxter equation. If $\phi : (\mathbb{C}^2, 0) \to \mathsf{GL}_n(\mathbb{C})$ is a holomorphic function such that

$$r'(v_1, v_2; y_1, y_2) := (\phi(v_1, y_1) \otimes \phi(v_2, y_2)) r(v; y_1, y_2)(\phi(v_2, y_1)^{-1} \otimes \phi(v_1, y_2)^{-1})$$

is again a function of $v = v_2 - v_1$, then

$$r'(v; y_1, y_2) = \frac{1}{v}\mathbb{1} \otimes \mathbb{1} + r'_0(y_1, y_2) + v r'_1(y_1, y_2) + v^2 r'_2(y_1, y_2) + \dots$$

and moreover, $\bar{r}_0(y_1, y_2)$ and $\bar{r}'_0(y_1, y_2)$ are equivalent under (2).

This means, that the requirement for a solution $r(v; y_1, y_2)$ of the associative Yang-Baxter equation (6) to have a Laurent expansion of the form (8) is very natural from the point of view of applications to the theory of the classical Yang-Baxter equation.

3. Polishchuk's construction

Let X be a reduced Gorenstein projective curve, $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ the bounded derived category of coherent sheaves and $\mathsf{Perf}(X)$ its full subcategory of perfect complexes. We denote by ω_X the dualising sheaf on X. This means (see for example [26, Section III.7]) that we have an isomorphism $t : H^1(\omega_X) \to \mathbb{C}$, also called a *trace map*, such that for any coherent sheaf $\mathcal{F} \in \mathsf{Coh}(X)$ the pairing

$$H^1(\mathcal{F}) \times \mathsf{Hom}_X(\mathcal{F}, \omega_X) \longrightarrow H^1(\omega_X) \xrightarrow{t} \mathbb{C}$$

is non-degenerate.

Remark 3.1. Such a map t is defined only up to a non-zero constant. However, it will be explained later that in the case of reduced Gorenstein projective curves there exists a "canonical" choice for t.

By [2, Chapter VIII] or by [19, Appendix B] the dualising sheaf ω_X is isomorphic to the sheaf of *regular* or *Rosenlicht's* differential 1-forms $\Omega_X = \Omega^{1,R}_X$. If X is smooth, then Ω_X coincides with the sheaf of holomorphic 1-forms. For E singular the definition is as follows.

Definition 3.2. Let X be a reduced Gorenstein projective curve, $n : \widetilde{X} \to X$ its normalisation. Denote by Ω_X^M and $\Omega_{\widetilde{X}}^M$ the sheaves of meromorphic differential 1-forms on X and $\widetilde{X}$ respectively. Observe that $\Omega_X^M = n_*(\Omega_{\widetilde{X}}^M)$. Then Ω_X is defined to be the subsheaf of Ω_X^M such that for any open subset $U \subseteq X$ one has

$$\Omega_X(U) = \left\{ \omega \in \Omega_{\widetilde{X}}^M(n^{-1}(U)) \,\middle|\, \forall p \in U, \forall f \in \mathcal{O}_X(U) : \sum_{i=1}^t \mathrm{res}_{p_i}((f \circ n)\omega) = 0 \right\},$$

where $\{p_1, p_2, \dots, p_t\} = n^{-1}(p)$.

A reduced projective curve E whose canonical sheaf Ω_E is isomorphic to the structure sheaf has arithmetic genus one. For example, reduced plane cubics, Kodaira cycles and generic configurations of $n+1$ lines in $\mathbb{P}^n$ passing through a given point are of this type. In what follows, for such a curve E we fix a global section $\omega \in H^0(\Omega_E)$ giving an isomorphism $\omega : \mathcal{O} \to \Omega_E$ and a trace map $H^1(\mathcal{O}) \to H^1(\Omega_E) \xrightarrow{t} \mathbb{C}$, also denoted by t.

A characteristic property of reduced projective curves with trivial dualising sheaf is a very special form of the Serre duality.

Proposition 3.3. *Let E be a reduced projective curve with trivial dualising sheaf and $\mathcal{E}, \mathcal{F} \in \mathsf{Perf}(E)$. Then the map*

$$\langle\ ,\ \rangle_{\mathcal{E},\mathcal{F}} : \mathsf{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathsf{Hom}(\mathcal{F}, \mathcal{E}[1]) \xrightarrow{\circ} \mathsf{Hom}(\mathcal{E}, \mathcal{E}[1]) \xrightarrow{\mathrm{Tr}_{\mathcal{E}}} H^1(\mathcal{O}) \xrightarrow{t} \mathbb{C}$$

where

$$\mathrm{Tr}_{\mathcal{E}} : \mathsf{Hom}(\mathcal{E}, \mathcal{E}[1]) \xrightarrow{\cong} \mathsf{Hom}(\mathcal{O}, \mathcal{E}^\vee \overset{\mathbb{L}}{\otimes} \mathcal{E}[1]) \longrightarrow \mathsf{Hom}(\mathcal{O}, \mathcal{O}[1]) = H^1(\mathcal{O}),$$

is a non-degenerate pairing.

Remark 3.4. The pairing $\langle\ ,\ \rangle_{\mathcal{E},\mathcal{F}}$ coincides with the composition

$$\langle\ ,\ \rangle_{\mathcal{F},\mathcal{E}} : \mathsf{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathsf{Hom}(\mathcal{F}, \mathcal{E}[1]) \xrightarrow{\cong} \mathsf{Hom}(\mathcal{F}, \mathcal{E}[1]) \otimes \mathsf{Hom}(\mathcal{E}[1], \mathcal{F}[1]) \xrightarrow{\circ}$$
$$\longrightarrow \mathsf{Hom}(\mathcal{F}, \mathcal{F}[1]) \xrightarrow{\mathrm{Tr}_{\mathcal{F}}} H^1(\mathcal{O}) \xrightarrow{t} \mathbb{C}.$$

Remark 3.5. The choice of non-degenerate pairings $\langle\ ,\ \rangle_{\mathcal{F},\mathcal{E}}$ is actually not unique, see the proof of Proposition I.2.3 in [42]. In particular, $\langle\ ,\ \rangle_{\mathcal{F},\mathcal{E}}$ depends on the choice of a global section of the dualising sheaf Ω_E.

The bounded derived category of coherent sheaves $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E)$ can be identified with the bounded from the left homotopy category $\mathsf{Hot}^{+,\mathsf{b}}_{\mathsf{coh}}(\mathsf{Inj}(E))$ of injective $\mathcal{O}_E$-modules having bounded coherent cohomology. Because the curve E is Gorenstein, the triangulated category of perfect complexes $\mathsf{Perf}(E)$ is equivalent to the bounded homotopy category $\mathsf{Hot}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{Inj}(E))$. Note that

$$\mathsf{Hot}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{Inj}(E)) = H^*\big(\mathsf{Com}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{Inj}(E))\big)$$

where $\mathsf{Com}^{\mathsf{b}}_{\mathsf{coh}}\big(\mathsf{Inj}(E)\big)$ is the dg-category of the bounded complexes of injective modules with coherent cohomology. By the homological perturbation theory (see [24, 28]) the category $\mathsf{Perf}(E) \cong \mathsf{Hot}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{Inj}(E))$ is equipped with an A_∞–structure such that $m_1 = 0$.

Proposition 3.6. *The constructed A_∞ –structure on $\mathsf{Perf}(E)$ is cyclic, see* [30]. *In particular, this means*

$$\langle m_3(f_1, g_1, f_2), g_2\rangle = -\langle f_1, m_3(g_1, f_2, g_2)\rangle = -\langle m_3(f_2, g_2, f_1), g_1\rangle$$

for any objects $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{F}_1, \mathcal{F}_2 \in \mathsf{Perf}(E)$, and any morphisms

$$f_i \in \mathsf{Hom}(\mathcal{E}_i, \mathcal{F}_i) \text{ and } g_i \in \mathsf{Hom}(\mathcal{F}_i, \mathcal{E}_{3-i}[1]), i = 1, 2.$$

Now we recall the main construction of [40]. Take a reduced projective curve E with trivial dualising sheaf and fix the following data:

- Two vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ of the same rank n such that $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0 = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$.
- Two distinct smooth points $y_1, y_2 \in E_{\mathrm{reg}}$ lying on the same irreducible component of E and such that $\mathsf{Hom}_E\big(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\big) = 0 = \mathsf{Ext}^1_E\big(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\big)$.

Remark 3.7. This "orthogonality" assumption on vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ might seem to be quite artificial. The natural example of such data is the following. Let $(n, d) \in \mathbb{N} \times \mathbb{Z}$ be a pair of coprime integers, $M_E(n, d)$ the moduli space of stable vector bundles of rank n and degree d on a Weierstraß curve E. Let $\mathcal{P}(n, d)$ be a universal family on $E \times M_E(n, d)$. For points $v_i \in M_E(n, d)$ denote by $\mathcal{V}_i$ the corresponding stable vector bundle $\mathcal{P}(n, d)|_{E\times v_i}$ on the curve E. Then, for any two distinct points $v_1, v_2 \in M_E(n, d)$, we have $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0 = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$.

Actually, one can also consider a more general situation. Namely, for any pair $(n, d) \in \mathbb{N} \times \mathbb{Z}$, not necessarily coprime, one can take indecomposable semi-stable vector bundles of rank n and degree d having *locally free* Jordan-Hölder factors. The orthogonality condition between non-isomorphic bundles of this type follows from the following lemma.

Lemma 3.8 (see [16]). *Let (n, d) be as above, $m = \gcd(n, d)$ and $n = mn', d = md'$. Let $\mathcal{V}$ be an indecomposable semi-stable vector bundle of rank n and degree d on a Weierstraß curve E with locally free Jordan-Hölder factors. Then all these factors are isomorphic to a single stable vector bundle $\mathcal{V}' \in M_E(n', d')$. Moreover, it holds $\mathcal{V} \cong \mathcal{V}' \otimes \mathcal{A}_m$, where $\mathcal{A}_m$ is the indecomposable vector bundle of rank m and degree 0 defined recursively by the non-split extension sequences*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A}_{m+1} \longrightarrow \mathcal{A}_m \longrightarrow 0 \quad m \geq 1,$$

where $\mathcal{A}_1 = \mathcal{O}$.

Let us return to Polishchuk's construction. Since $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0 = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$, we have a linear map

$$m_3 : \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})$$

called the *triple Massey product* and defined as follows. Let $a \in \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)$, $g \in \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1})$, $h \in \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2})$ and let

$$0 \longrightarrow \mathcal{V}_2 \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathbb{C}_{y_1} \longrightarrow 0$$

be an exact sequence representing the element a. The vanishing of $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2)$ and $\mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$ implies that we can uniquely lift the morphisms g and h to morphisms $\tilde{g} : \mathcal{V}_1 \longrightarrow \mathcal{A}$ and $\tilde{h} : \mathcal{A} \longrightarrow \mathbb{C}_{y_2}$. So, we obtained a diagram

$$\begin{tikzcd}
 & & & & \mathcal{V}_1 \arrow[dl, "\tilde{g}"'] \arrow[d, "g"] & \\
a : 0 \arrow[r] & \mathcal{V}_2 \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{A} \arrow[dl, "\tilde{h}"] \arrow[r, "\beta"] & & \mathbb{C}_{y_1} \arrow[r] & 0 \\
 & \mathbb{C}_{y_2} & & & &
\end{tikzcd}$$

and the triple Massey product is defined as $m_3(g \otimes a \otimes h) = \tilde{h}\tilde{g}$. Note that a determines an extension only up to an automorphism of the middle term, but the action of $\mathsf{Aut}(\mathcal{A})$ leads to the same answer for $m_3(g \otimes a \otimes h) = m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(g \otimes a \otimes h)$.

Now one can use a sequence of canonical isomorphisms in order to rewrite $m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ in another form:

$$\mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big) \cong$$

$$\mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)^* \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big) \cong$$

$$\mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big),$$

where we use the Serre duality formula $\mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)^* \cong \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1})$ given by the bilinear form $\langle\ ,\ \rangle_{\mathcal{V}_2,\mathbb{C}_{y_1}}$ from Proposition 3.3. Let $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ be the image of $m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ under this chain of isomorphisms.

Theorem 3.9 (see Theorem 1 in [40])**.** *The tensor $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ satisfies the following "triangle equation" (associative Yang-Baxter equation)*

$$(9) \qquad (\widetilde{m}^{\mathcal{V}_3,\mathcal{V}_2}_{y_1,y_2})^{12}(\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_3}_{y_1,y_3})^{13} - (\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_3}_{y_2,y_3})^{23}(\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2})^{12} + (\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_3})^{13}(\widetilde{m}^{\mathcal{V}_2,\mathcal{V}_3}_{y_2,y_3})^{23} = 0.$$

The left-hand side of this equation is a linear map

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_3}) \longrightarrow$$

$$\longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_3}).$$

Moreover, the tensor $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ *is non-degenerate and skew-symmetric:*

$$\tau(\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}) = -\widetilde{m}_{y_2,y_1}^{\mathcal{V}_2,\mathcal{V}_1},$$

where τ *is the isomorphism*

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1})$$

given by $\tau(f \otimes g) = g \otimes f$.

Idea of the proof. This equality is a consequence of the A_∞ - constraint

$$m_3 \circ (m_3 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_3) = 0,$$

and skew-symmetry of $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ follows from the cyclicity of the A_∞ - structure. □

Note that for a vector bundle $\mathcal{V}$ and a smooth point $y \in E$ we have canonical isomorphisms

$$\mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_y) \cong \mathsf{Hom}_E(\mathcal{V} \otimes \mathbb{C}_y, \mathbb{C}_y) = \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}|_y, \mathbb{C}) = \mathcal{V}|_y^*.$$

In these terms $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ is a linear map

$$\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2} : \mathcal{V}_1|_{y_1}^* \otimes \mathcal{V}_2|_{y_2}^* \longrightarrow \mathcal{V}_2|_{y_1}^* \otimes \mathcal{V}_1|_{y_2}^*.$$

Now we use the canonical isomorphism

$$\alpha : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathcal{V}_1|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}^* \otimes \mathcal{V}_2|_{y_2}^*, \mathcal{V}_2|_{y_1}^* \otimes \mathcal{V}_1|_{y_2}^*)$$

mapping a simple tensor $f_1 \otimes f_2$ to $f_1^t \otimes f_2^t$. Then the tensor

$$r_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2} := \alpha^{-1}(\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}) \in \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathcal{V}_1|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})$$

satisfies the equation

$$(r_{y_1,y_3}^{\mathcal{V}_1,\mathcal{V}_3})^{13}(r_{y_1,y_2}^{\mathcal{V}_3,\mathcal{V}_2})^{12} - (r_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2})^{12}(r_{y_2,y_3}^{\mathcal{V}_1,\mathcal{V}_3})^{23} + (r_{y_2,y_3}^{\mathcal{V}_2,\mathcal{V}_3})^{23}(r_{y_1,y_3}^{\mathcal{V}_1,\mathcal{V}_2})^{13} = 0 \tag{10}$$

and the unitarity condition

$$\tau(r_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}) = -r_{y_2,y_1}^{\mathcal{V}_2,\mathcal{V}_1}.$$

On the set of solutions of this equation we have an equivalence relation as in Definition 2.5.

Remark 3.10. Since the functorial isomorphism of vector spaces $\mathsf{Hom}_{\mathbb{C}}(U, V) \to \mathsf{Hom}_{\mathbb{C}}(V^*, U^*)$ is contravariant, the tensors $r_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ and $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ appear in inverse order in Equations (9) and (10).

Note that the bilinear map

$$\mathrm{tr} : \mathsf{Hom}_{\mathbb{C}}(U, V) \times \mathsf{Hom}_{\mathbb{C}}(V, U) \longrightarrow \mathbb{C}, \quad (f, g) \mapsto \mathrm{tr}(f \circ g)$$

is non-degenerate and induces an isomorphism

$$\mathsf{Hom}_{\mathbb{C}}(U, V)^* \cong \mathsf{Hom}_{\mathbb{C}}(V, U).$$

Using this we get a chain of canonical isomorphisms

$$\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathcal{V}_1|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \cong \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1})^* \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \cong$$
$$\cong \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}), \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})\big).$$

We let $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} \in \mathsf{Lin}\big(\mathsf{Hom}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}), \mathsf{Hom}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})\big)$ be the image of $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$.

Remark 3.11. Note that the triple Massey product m_3 is canonical, however the tensor $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ depends on the choice of a global section $\omega \in H^0(\Omega_E)$.

Our next aim is to answer the following questions:

- **Q1** How does the tensor $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ depend on trivialisations of $\mathcal{V}_1$ and $\mathcal{V}_2$?
- **Q2** What is a geometrical interpretation of the equivalence relation given in Definition 2.5?
- **Q3** How can we view $\mathcal{V}_1$ and $\mathcal{V}_2$ as variables?
- **Q4** Is there a practical way to compute $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$?

4. Geometric description of Massey products

Let E be a reduced projective curve with trivial dualising sheaf. As in the previous section, we fix the following data:

- two vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ of rank n such that $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0$ and $\mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2) = 0$.
- two distinct smooth points $y_1, y_2 \in E_{\mathrm{reg}}$ lying on the same irreducible component of E such that $\mathsf{Hom}_E\big(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\big) = \mathsf{Ext}^1_E\big(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\big) = 0$.

The main goal of this section is to get an alternative description of the linear map

$$\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})$$

introduced in Section 3. Let $\Omega_E = \Omega^{1,R}_E$ be the sheaf of regular differential 1-forms on E. For any smooth point $x \in E$ we have an exact sequence

$$0 \longrightarrow \Omega_E \longrightarrow \Omega_E(x) \xrightarrow{\mathrm{res}_x} \mathbb{C}_x \longrightarrow 0,$$

where res_x is the residue map for differential 1-forms. It induces the coboundary map

$$\delta_x : H^0(\mathbb{C}_x) \to H^1(\Omega_E)$$

which is an isomorphism. Let $w_x = \delta_x(1_x) \in H^1(\Omega_E)$. By a result of Kunz it holds

Theorem 4.1 (see Satz 4.1 in [31])**.** *The element w_x does not depend on x.*

Let $w = w_x$ and $t : H^1(\Omega_E) \to \mathbb{C}$ be the isomorphism mapping w to 1. We fix a global regular differential form $\omega : \mathcal{O}_E \to \Omega_E$, which induces for any two perfect complexes $\mathcal{E}, \mathcal{F} \in \mathsf{Perf}(E)$ a non-degenerate pairing (see Proposition 3.3)

$$\langle\ ,\ \rangle_{\mathcal{E},\mathcal{F}} : \mathsf{Hom}_E(\mathcal{E}, \mathcal{F}) \otimes \mathsf{Hom}_E(\mathcal{F}, \mathcal{E}[1]) \longrightarrow \mathbb{C}.$$

Recall that, when passing from the triple Massey product $m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ to the tensor $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$, we have already used these bilinear forms.

The alternative description of $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ involves two isomorphisms. In order to describe the first of them, we consider the composition map

$$\mathcal{O}(y_1) \xrightarrow{\omega} \Omega_E(y_1) \xrightarrow{\mathrm{res}_{y_1}} \mathbb{C}_{y_1},$$

which we also denote by res_{y_1}. The exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(y_1) \xrightarrow{\mathrm{res}_{y_1}} \mathbb{C}_{y_1} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_2(y_1) \xrightarrow{\mathrm{res}^{\mathcal{V}_2}_{y_1}} \mathcal{V}_2 \otimes \mathbb{C}_{y_1} \longrightarrow 0.$$

Since $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2) = 0$, all maps in the commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2(y_1)) \arrow[r] \arrow[dr, "\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1}(\omega)"'] & \mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2 \otimes \mathbb{C}_{y_1}) \arrow[d] \\
 & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1})
\end{tikzcd}$$

are isomorphisms. In what follows, we shall frequently abuse notation and write res_{y_1} instead of $\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1}(\omega)$.

In order to describe the second isomorphism needed below, we start with the standard exact sequence

$$0 \longrightarrow \mathcal{V}_2(y_1 - y_2) \longrightarrow \mathcal{V}_2(y_1) \longrightarrow \mathcal{V}_2(y_1) \otimes \mathbb{C}_{y_2} \longrightarrow 0.$$

Because $\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1 - y_2)\big) = \mathsf{Ext}^1_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1 - y_2)\big) = 0$, the horizontal arrow in the following commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \arrow[r] \arrow[ddr, "\mathrm{ev}^{\mathcal{V}_1,\mathcal{V}_2}_{y_2}"'] & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1) \otimes \mathbb{C}_{y_2}\big) \arrow[d, "(r_{y_2})_*"] \\
 & \mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2 \otimes \mathbb{C}_{y_2}) \arrow[d] \\
 & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}),
\end{tikzcd}$$

is an isomorphism. The vertical arrows in this diagram are isomorphisms as well, where $(r_{y_2})_*$ is induced by the embedding $\mathcal{O} \to \mathcal{O}(y_1)$.

The following theorem [40, Theorem 4] is the key statement to explicitly compute the tensor $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ describing triple Massey products.

Theorem 4.2. *The diagram*

$$\begin{tikzcd}
 & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \arrow[dl, "\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1}(\omega)"'] \arrow[dr, "\mathrm{ev}^{\mathcal{V}_1,\mathcal{V}_2}_{y_2}"] & \\
\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \arrow[rr, "\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(\omega)"] & & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})
\end{tikzcd}$$

is commutative.

Since this result plays a crucial role in our approach to degeneration problems, we decided to give a detailed proof of this statement, stressing those points which are implicit in [40]. Before we start, several technical lemmas have to be proved.

Lemma 4.3. *Let E be a reduced projective curve with trivial dualising sheaf, $x \in E$ a smooth point. Then we have an isomorphism of functors* $\mathsf{VB}(E) \longrightarrow \mathsf{Vect}_{\mathbb{C}}$*:*

$$T_x : \mathsf{Ext}^1_E(\mathbb{C}_x, -) \longrightarrow \mathsf{Hom}_E(\mathbb{C}_x, - \otimes \mathbb{C}_x)$$

Proof. Let $\mathcal{V}$ be a vector bundle on E of rank n. ¿From the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(x) \xrightarrow{\mathrm{res}_x} \mathbb{C}_x \longrightarrow 0$$

we get

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}(x) \longrightarrow \mathcal{V} \otimes \mathbb{C}_x \longrightarrow 0$$

and then

$$0 \to \mathsf{Hom}_E(\mathbb{C}_x, \mathcal{V}\otimes\mathbb{C}_x) \xrightarrow{\delta_x} \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}) \to \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}(x)) \to \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}\otimes\mathbb{C}_x) \to 0.$$

Because $\mathsf{Ext}^1_E\big(\mathbb{C}_x, \mathcal{V}(x)\big) \cong H^0\big(\mathcal{E}xt^1(\mathbb{C}_x, \mathcal{V}(x))\big)$ and $\mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V} \otimes \mathbb{C}_x)$ are both of dimension $n = \mathrm{rank}(\mathcal{V})$, we conclude that δ_x is an isomorphism. Moreover, this map is functorial and we can put $T_x = \delta_x^{-1}$. □

Remark 4.4. By the construction of the functor T_x we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & \mathbb{C}_x \arrow[r] \arrow[d, "T_x(a)"] & 0 \\
0 \arrow[r] & \mathcal{V} \arrow[r] & \mathcal{V}(x) \arrow[r, "\mathrm{res}_x^{\mathcal{V}}"] & \mathcal{V} \otimes \mathbb{C}_x \arrow[r] & 0,
\end{tikzcd}$$

where the upper exact sequence corresponds to the element $a \in \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V})$.

In order to justify our calculations in Sections 7 and 9 we need to establish an explicit link between the "categorical trace map" of Proposition 3.3 and the usual trace from linear algebra.

Let X be a reduced Gorenstein projective curve, $x \in X$ a smooth point, $\mathcal{V}$ a vector bundle on X. From the exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(x) \xrightarrow{\mathrm{res}_x} \mathbb{C}_x \longrightarrow 0$$

we get a commutative diagram

$$\begin{tikzcd}
\mathsf{Ext}^1_X(\mathcal{V}, \mathcal{V} \otimes \Omega_X) \arrow[r, "\mathrm{Tr}_{\mathcal{V}}"] & H^1(\Omega_X) \arrow[r, "t"] & \mathbb{C} \\
\mathsf{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathbb{C}_x) \arrow[u, "\delta_x"] \arrow[r, "\mathrm{Tr}_{\mathcal{V}}"] & H^0(\mathbb{C}_x) \arrow[u, "\delta_x"'] \arrow[ur] & \\
\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}|_x, \mathcal{V}|_x) \arrow[u] \arrow[r, "\mathrm{tr}"] \arrow[drr, "\mathrm{tr}"'] & \mathsf{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \arrow[u] \arrow[dr] & \\
& & \mathbb{C} \arrow[uuu, "="']
\end{tikzcd}$$

where t is the trace map from Theorem 4.1 and tr is the ordinary trace of an endomorphism of the vector space $\mathcal{V}|_x$. The commutativity of this diagram gives us the following result.

Lemma 4.5. *For an element $f \in \mathsf{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathbb{C}_x)$ we have:*

$$t\big(\mathrm{Tr}_{\mathcal{V}}(\delta_x(f))\big) = \mathrm{tr}(f_x),$$

which is the required link between the categorical trace and the usual trace for vector spaces.

Lemma 4.6. *Let E be a reduced projective curve with trivial dualising sheaf, $x \in E$ a smooth point, $\mathcal{V} \in \mathsf{VB}(E)$ a vector bundle and*

$$S : \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_x)^*$$

the isomorphism induced by the bilinear form $\langle\ ,\ \rangle_{\mathcal{V},\mathbb{C}_x}$. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}) \arrow[r, "S"] \arrow[d, "T_x"'] & \mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_x)^* \arrow[d] \\
\mathsf{Hom}_E(\mathbb{C}_x, \mathcal{V}) \arrow[r, "\mathrm{tr}"] & \mathsf{Hom}_E(\mathcal{V} \otimes \mathbb{C}_x, \mathbb{C}_x)^*,
\end{tikzcd}$$

where tr *is induced by the canonical isomorphism of vector spaces*

$$\mathsf{Hom}_{\mathbb{C}}(U, V)^* \cong \mathsf{Hom}_{\mathbb{C}}(V, U).$$

Proof. Let $\xi \in \mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_x)$ and $a \in \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V})$. Then $T_x(a) : \mathbb{C}_x \to \mathcal{V} \otimes \mathbb{C}_x$ and $\xi_x : \mathcal{V} \otimes \mathbb{C}_x \to \mathbb{C}_x$ satisfy:

$$\mathrm{tr}(\xi_x \circ T_x(a)) = \mathrm{tr}(T_x(a) \circ \xi_x) = \mathrm{Tr}_{\mathcal{V}}(\delta_x(T_x(a) \circ \xi)),$$

where the last equality holds by Lemma 4.5. □

Now, after proving these preliminary statements we are ready to prove Theorem 4.2. Let $(\mathcal{V}_1, \mathcal{V}_2, y_1, y_2)$ be the data fixed at the beginning of the Section. Recall that we have to compare the triple Massey product

$$m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})$$

with the map

$$\tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} := \mathrm{ev}_{y_2} \circ \mathrm{res}^{-1}_{y_1} : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}).$$

Proposition 4.7. *If $g \in \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1})$, $h \in \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2})$ and $a \in \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)$, then*

$$h_{y_2} \circ \tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(T_{y_1}(a) g_{y_1}) = \big(m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(g \otimes a \otimes h)\big)_{y_2}.$$

Proof. Let us first explain our notation. We have a composition map

$$\mathcal{V}_1|_{y_1} \xrightarrow{g_{y_1}} \mathbb{C}_{y_1} \xrightarrow{T_{y_1}(a)} \mathcal{V}_2|_{y_1},$$

hence we may consider

$$\mathcal{V}_1|_{y_2} \xrightarrow{\tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}\left(T_{y_1}(a)g_{y_1}\right)} \mathcal{V}_2|_{y_2}.$$

Let $0 \to \mathcal{V}_2 \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathbb{C}_{y_1} \to 0$ be an exact sequence representing $a \in \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)$. Then we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathcal{V}_2 \arrow[r, "="] \arrow[d, "\alpha"'] & \mathcal{V}_2 \arrow[d, "r"] \\
\mathcal{V}_1 \arrow[r, "\tilde{g}"] \arrow[d] \arrow[dr, "g"] & \mathcal{A} \arrow[r, "\varepsilon"] \arrow[d, "\beta"'] & \mathcal{V}_2(y_1) \arrow[d, "\mathrm{res}^{\mathcal{V}_2}_{y_1}"] \\
\mathcal{V}_1 \otimes \mathbb{C}_{y_1} \arrow[r, "g_{y_1}"] & \mathbb{C}_{y_1} \arrow[r, "T_{y_1}(a)"] \arrow[d] & \mathcal{V}_2 \otimes \mathbb{C}_{y_1} \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

where $\tilde{g}$ is the unique lift of g and the two columns on the right form a transposed version of the diagram from Remark 4.4. Since

$$\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1} : \mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2(y_1)) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1 \otimes \mathbb{C}_{y_1}, \mathcal{V}_2 \otimes \mathbb{C}_{y_1})$$

is an isomorphism, by definition we have

$$\mathrm{res}^{-1}_{y_1}(T_{y_1}(a)g_{y_1}) = \varepsilon\tilde{g}.$$

Moreover, tensoring the whole diagram with $\mathbb{C}_{y_2}$ we obtain a new commutative diagram

$$\begin{tikzcd}
 & \mathcal{V}_2 \otimes \mathbb{C}_{y_2} \arrow[r, "="] \arrow[d, "\alpha_{y_2}"'] & \mathcal{V}_2 \otimes \mathbb{C}_{y_2} \arrow[d, "r_{y_2}"] \\
\mathcal{V}_1 \otimes \mathbb{C}_{y_2} \arrow[r, "\tilde{g}_{y_2}"] & \mathcal{A} \otimes \mathbb{C}_{y_2} \arrow[r, "\varepsilon_{y_2}"] & \mathcal{V}_2(y_1) \otimes \mathbb{C}_{y_2},
\end{tikzcd}$$

from which the identity $\mathrm{ev}_{y_2}(\varepsilon\tilde{g}) = \alpha_{y_2}^{-1}\tilde{g}_{y_2}$ follows. By the definition of Massey products we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_2 \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{A} \arrow[r] \arrow[dl, "\tilde{h}"'] & \mathbb{C}_{y_1} \arrow[r] & 0 \\
 & \mathbb{C}_{y_2} & \mathcal{V}_1 \arrow[l, "m_3(g\otimes a\otimes h)"] \arrow[u, "\tilde{g}"'] & &
\end{tikzcd}$$

which finally implies

$$\big(m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}(g\otimes a\otimes h)\big)_{y_2} = h_{y_2}\circ\alpha_{y_2}^{-1}\circ\tilde{g}_{y_2} = h_{y_2}\circ\tilde{\bar{r}}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}(T_{y_1}(a)g_{y_1}).$$

□

Now we are ready to finish the proof of Theorem 4.2. Our goal is to keep track of the linear map $m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ under a long chain of canonical isomorphisms. Let us do it step by step.

Each linear map

$$m \in \mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1,\mathbb{C}_{y_1})\otimes\mathsf{Ext}_E^1(\mathbb{C}_{y_1},\mathcal{V}_2)\otimes\mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_2}),\mathsf{Hom}_E(\mathcal{V}_1,\boldsymbol{k}_{y_2})\big)$$

corresponds to an element

$$n \in \mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1,\mathbb{C}_{y_1})\otimes\mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_1})^*,\mathsf{Lin}(\mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_2}),\mathsf{Hom}_E(\mathcal{V}_1,\mathbb{C}_{y_2}))\big)$$

which is related to m by the formula

$$n(g\otimes S(a))(h) = m(g\otimes a\otimes h),$$

where $S : \mathsf{Ext}_E^1(\mathbb{C}_{y_1},\mathcal{V}_2) \to \mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_1})^*$ is given by the bilinear form $\langle\ ,\ \rangle_{\mathcal{V}_2,y_1}$ from Proposition 3.3. By Lemma 4.6, the element $S(a)\in\mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_1})^*$ is mapped to $T(a)\in\mathsf{Hom}_{\mathbb{C}}(\mathbb{C},\mathcal{V}_2|_{y_1})$ under the chain of isomorphisms

$$\mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_1})^* \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2\otimes\mathbb{C}_{y_1},\mathbb{C}_{y_1})^* \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1},\mathbb{C})^* \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathbb{C},\mathcal{V}_2|_{y_1}).$$

This implies that the linear map n corresponds to

$$l \in \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1},\mathbb{C})\otimes\mathsf{Hom}(\mathbb{C},\mathcal{V}_2|_{y_1}),\mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_2},\mathbb{C}),\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2},\mathbb{C})\big)\big)$$

given by $l(g_{y_1}\otimes T(a))(h_{y_2}) = m(g\otimes a\otimes h)_{y_2}$. But since

$$\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1},\mathbb{C})\otimes\mathsf{Hom}(\mathbb{C},\mathcal{V}_2|_{y_1}) \xrightarrow{\circ} \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1},\mathcal{V}_2|_{y_1})$$

is an isomorphism and

$$\mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_2},\mathbb{C}),\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2},\mathbb{C})\big) \cong \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2},\mathcal{V}_2|_{y_2}),$$

we obtain a linear map

$$k \in \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1},\mathcal{V}_2|_{y_1}),\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2},\mathcal{V}_2|_{y_2})\big)$$

such that for any elements g, a and h the following diagram commutes

$$\begin{tikzcd}
\mathcal{V}_1|_{y_2} \arrow[r, "k(T(a)g_{y_1})"] \arrow[dr, "m(g\otimes a\otimes h)_{y_2}"'] & \mathcal{V}_2|_{y_2} \arrow[d, "h_{y_2}"] \\
& \mathbb{C}_{y_2}.
\end{tikzcd}$$

With $m = r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ we obtain now, using Proposition 4.7,

$$k = \tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} = \tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} = \mathrm{ev}_{y_2} \circ \mathrm{res}^{-1}_{y_1},$$

where the map $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ was defined in Remark 3.10. This completes the proof. □

5. On a relative construction of geometric triple Massey products

Our next goal is to extend the definition of the map $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ to case of genus one fibrations. We achieve this by generalising the construction of Theorem 4.2 to the relative case. Throughout this section we work either in the category of locally Noetherian algebraic schemes over an algebraically closed field $\boldsymbol{k}$ or in the category of complex analytic spaces.

5.1. **The relative residue map.** Let $p : X \longrightarrow S$ be a *smooth* map of complex analytic spaces or of algebraic schemes. Assume p has a section $i : S \longrightarrow X$, let D be the image of i equipped with the ringed space structure induced from S. Recall that the sheaf of relative differentials $\Omega^1_{X/S}$ is defined via the exact sequence

$$p^*\Omega^1_S \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0,$$

see [2, Chapter 7], [26, Section II.8 and Section III.10] and [39] for definitions and basic properties of smooth morphisms and Kähler differential forms. In particular, for any closed point $s \in S$ it holds: $\Omega^1_{X/S}|_{X_s} \cong \Omega^1_{X_s}$ and $\Omega^1_{X/S}$ is locally free.

Assume additionally that p has relative dimension one and X itself is smooth. Our aim is to define a canonical epimorphism of $\mathcal{O}_X$–modules

$$\mathrm{res}_D : \Omega^1_{X/S}(D) \longrightarrow \mathcal{O}_D,$$

later called *the residue map.* We shall explain our construction in the case of algebraic schemes, whereas its generalisation on the case of complex analytic spaces is straightforward.

Let $x \in D \subset X$ be a closed point, then we can find affine neighbourhoods $U = \mathsf{Spec}(B)$ of $x \in X$ and $V = \mathsf{Spec}(A)$ of $f(x) \in S$ such that the map $p|_U : U \to V$ is induced by the ring homomorphism $p^* : A \to B$:

$$\begin{tikzcd} U = \mathsf{Spec}(B) \arrow[r, hook] \arrow[d, "p|_U"'] & X \arrow[d, "p"] \\ V = \mathsf{Spec}(A) \arrow[r, hook] & S. \end{tikzcd}$$

Then the sheaf $\Omega^1_{X/S}|_U$ is isomorphic to the sheafification of the B–module of Kähler differentials $\Omega_{B/A}$.

Let $i^* : B \to A$ be the ring homomorphism corresponding to the section i and $I = \ker(i^*)$. Then the map $C := B/I \xrightarrow{i^*} A$ is an isomorphism and I is the ideal, locally defining the subscheme D. By Krull's Hauptidealsatz, since U is smooth and $V(I) \subset U$ has codimension one, shrinking the open sets U and V if necessary, we can achieve that I is generated by a single element $a \in A$. ¿From the exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

where $\delta([b]) = d(b) \otimes 1$ and the fact that $\Omega_{C/A} = 0$ it follows that the C–module $\Omega_{B/A} \otimes_B C$ is generated by a single element $d(a) \otimes 1$.

Definition 5.1. Let $p : X \longrightarrow S$ be a smooth map of relative dimension one, $i : S \longrightarrow X$ a section of p and $D = i(S)$. We define the sheaf homomorphism

$$\mathrm{res}_D : \Omega^1_{X/S}(D) \longrightarrow \mathcal{O}_D$$

to be the composition of the canonical map $\Omega^1_{X/S}(D) \longrightarrow \Omega^1_{X/S}|_D \otimes \mathcal{O}(D)|_D$ and the morphism $\Omega^1_{X/S}|_D \otimes \mathcal{O}(D)|_D \to \mathcal{O}_D$ locally defined as follows.

In the notation as above let $M = \{\frac{u}{a} \mid u \in B\} = \Gamma\big(U, \mathcal{O}_X(D)\big) \subset Q(B)$, where $Q(B)$ is the field of fractions of the domain B. The map

$$\mathrm{res}_D : (\Omega_{B/A} \otimes_B C) \otimes (M \otimes_B C) \longrightarrow C$$

is given by the formula $\big(d(a) \otimes 1\big) \otimes \big(\frac{u}{a} \otimes 1\big) \mapsto u \otimes 1 = \overline{u} := u \mod I$.

It is easy to see that the morphism res_D is C-linear, surjective and does not depend on the choice of a generator of the ideal I.

Proposition 5.2. *Let $p : X \longrightarrow S$ be a smooth map as above, $i : S \longrightarrow X$ a section of p and $f : S' \longrightarrow S$ any morphism. Let $X' = X \times_S S'$ and $i' : S' \longrightarrow X'$ be the section defined by the universal property of pull-backs:*

$$\begin{tikzcd} S' \arrow[dr, dotted, "i'"] \arrow[drr, bend left, "if"] \arrow[ddr, bend right, "="'] & & \\ & X' \arrow[r, "g"] \arrow[d, "p'"] & X \arrow[d, "p"'] \\ & S' \arrow[r, "f"] & S \arrow[u, bend right, "i"'] \end{tikzcd}$$

and $D' = i'(S')$. Then the following diagram is commutative:

$$\begin{tikzcd} g^*\big(\Omega^1_{X/S}(D)\big) \arrow[r, "g^*(\mathrm{res}_D)"] \arrow[d, "\cong"'] & g^*(\mathcal{O}_D) \arrow[d, "\cong"] \\ \Omega^1_{X'/S'}(D') \arrow[r, "\mathrm{res}_{D'}"] & \mathcal{O}_{D'} \end{tikzcd}$$

where the vertical arrows are canonical isomorphisms.

Proof. The problem is local, so we can assume, without loss of generality, $X = \mathsf{Spec}(B), X' = \mathsf{Spec}(B'), S = \mathsf{Spec}(A)$ and $S' = \mathsf{Spec}(A')$. Then, we have a Cartesian diagram of rings and ring homomorphisms

$$\begin{tikzcd} B' \arrow[d, bend right, "i'^*"'] & B \arrow[l, "g^*"'] \arrow[d, bend left, "i^*"] \\ A' \arrow[u, "p'^*"] & A \arrow[l, "f^*"] \arrow[u, "p^*"'] \end{tikzcd}$$

where $B' = B \otimes_A A'$, $p'^*(a') = 1 \otimes a'$ and $g^*(b) = b \otimes 1$. Denote $C := B/\ker(i^*)$ and $C' := B'/\ker(i'^*)$ then we have an isomorphism of C'–modules $C \otimes_B B' \longrightarrow C'$.

Let $d : B \to \Omega_{B/A}$ and $d' : B' \to \Omega_{B'/A'}$ be the universal derivations from the definition of Kähler differentials. By the universal property we obtain a uniquely determined B–module homomorphism $\Omega_{B/A} \longrightarrow \Omega_{B'/A'}$ and an induced B'–module isomorphism $\tilde{g}^* : \Omega_{B/A} \otimes_B B' \longrightarrow \Omega_{B'/A'}$ making the following diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g^*"'] & \Omega_{B/A} \arrow[r] \arrow[d, dotted] & \Omega_{B/A} \otimes_B B' \arrow[dl, dotted, "\tilde{g}^*"] \\ B' \arrow[r] & \Omega_{B'/A'} & \end{tikzcd}$$

commutative, in particular $\tilde{g}^*\big(d(b) \otimes 1\big) = d\big(g^*(b)\big)$. Moreover, we have a canonical homomorphism $g^*_M : M \otimes_B B' \longrightarrow M'$, given by $g^*_M\Big(\frac{u}{a} \otimes 1\Big) = \frac{g^*(u)}{g^*(a)}$, where

$$M = \left\{ \frac{u}{a} \,\middle|\, u \in B \right\} \subset Q(B) \quad \text{and} \quad M' = \left\{ \frac{v}{g^*(a)} \,\middle|\, v \in B' \right\} \subset Q(B').$$

We know that the C–module $\Omega_{B/A} \otimes C$ is generated by the single element $a \otimes 1$. Hence the commutativity of the diagram

$$\begin{tikzcd} \big(\Omega_{B/A} \otimes_B B'\big) \otimes \big(M \otimes_B B'\big) \arrow[r, "g^*(\mathrm{res}_D)"] \arrow[d, "\tilde{g}^* \otimes g^*_M"'] & C \otimes_B B' \arrow[d] \\ \Omega_{B'/A'} \otimes M' \arrow[r, "\mathrm{res}_{D'}"] & C' \end{tikzcd}$$

can be checked on the generator:

$$\begin{tikzcd}
\big(d(a)\otimes 1\big)\otimes\big(\frac{u}{a}\otimes 1\big) \arrow[r, mapsto, "g^*(\mathrm{res}_D)"] \arrow[d, mapsto, "\tilde{g}^*\otimes g^*_M"'] & \bar{u}\otimes 1 \arrow[d, mapsto] \\
d\big(g^*(a)\big)\otimes\dfrac{g^*(u)}{g^*(a)} \arrow[r, mapsto, "\mathrm{res}_{D'}"] & \overline{g^*(u)}.
\end{tikzcd}$$

and the proposition is proved. □

5.2. **On the sheaf of relative differential forms of a Gorenstein fibration.** Let $p : X \longrightarrow S$ be a *proper* and *flat* morphism of relative dimension *one*, either in the category of complex analytic spaces or of algebraic schemes over an algebraically closed field $\boldsymbol{k}$. Assume additionally that for all closed points $s \in S$ the fibres X_s are reduced and we have an embedding

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

where $q : Y \longrightarrow S$ is a *proper* and *smooth* morphism of the relative dimension *two*.

Remark 5.3. Since for any $s \in S$ the surface Y_s is smooth and $X_s \subset Y_s$ has codimension one, the curve X_s has hypersurface singularities and is in particular Gorenstein.

Recall that for a smooth morphism q we have an exact sequence

$$q^*\Omega^1_S \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0$$

where $\Omega^1_{Y/S}$ is a locally free $\mathcal{O}_Y$–module of rank two.

Definition 5.4. The relative dualising sheaf is defined by the formula

$$\omega_{X/S} := \big(\wedge^2\Omega^1_{Y/S}\otimes\mathcal{O}_Y(X)\big)|_X.$$

Proposition 5.5 (see Chapter II in [5])**.** *For any $s \in S$ the sheaf $\omega_{X/S}|_{X_s}$ is the dualising sheaf of the projective curve X_s.*

Remark 5.6. It can be shown that up to the pull-back of a line bundle on S this definition of $\omega_{X/S}$ does not depend on the embedding $X \hookrightarrow Y$.

Let $\breve{X}$ be the regular locus of p, then $j : \breve{X} \longrightarrow X$ is an open embedding and the morphism $\breve{X} \xrightarrow{p} S$ is flat but in general not proper. Our aim is to define an injective map of $\mathcal{O}_X$–modules $\mathrm{cl}_S : \omega_{X/S} \longrightarrow j_*(\Omega^1_{\breve{X}/S})$.

For a closed point $x \in \breve{X}$ let $U \subset Y$ be an open neighbourhood of x and S_0 an open neighbourhood of $f(x)$ in S. Choose local coordinates (u, v, s) on U such that we have a commutative diagram

$$\begin{tikzcd}
(\mathbb{C}^2 \times S_0) = U \arrow[r] \arrow[d, "\mathrm{pr}"'] & Y \arrow[d, "q"] \\
S_0 \arrow[r] & S
\end{tikzcd}$$

where $\mathrm{pr}(u, v, s) = s$ and $du|_{\breve{X}} \not\equiv 0$, $dv|_{\breve{X}} \not\equiv 0$. Assume that the closed subset $X \cap U$ is given in U by an equation $f(u, v, s) = 0$. Then

$$\left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv\right) |_{\breve{X}} = 0$$

where the left-hand side of this equality is viewed as a local section of $\Omega^1_{\breve{X}/S}$.

Consider the composition map $i : \breve{X} \xrightarrow{j} X \longrightarrow Y$.

Definition 5.7 (see Section II.1 in [5]). The Poincaré residue map is the morphism of $\mathcal{O}_Y$–modules

$$\mathrm{res}^P : \wedge^2 \Omega^1_{Y/S}(X) \longrightarrow i_* \Omega^1_{\breve{X}/S}$$

locally defined as follows. Let $U \subseteq Y$ be an open neighbourhood of $x \in \breve{X}$ as above and $V := U \cap \breve{X}$, then the map

$$\mathrm{res}^P : \Gamma\big(U, \wedge^2 \Omega^1_{Y/S}(X)\big) \longrightarrow \Gamma\big(V, \Omega^1_{\breve{X}/S}\big) = \Gamma\big(U, i_* \Omega^1_{\breve{X}/S}\big)$$

is given by the formula

$$\frac{h\, du \wedge dv}{f} \mapsto \begin{cases} \dfrac{h du}{\partial_v f}|_V & \text{if } \dfrac{\partial f}{\partial v}(u, v, s) \neq 0, \\ -\dfrac{h dv}{\partial_u f}|_V & \text{if } \dfrac{\partial f}{\partial u}(u, v, s) \neq 0. \end{cases}$$

Remark 5.8. Since for any point $s \in S$ the fibre $\breve{X}_s$ is a smooth curve, the set $V(f, \partial_u f, \partial_v f) \subseteq \breve{X}_s$ is empty and the map res^P is well-defined. Moreover, res^P is independent of the choice of a local equation $f \in \mathcal{O}_Y(U)$ for $X \subset Y$ and also of the choice of local coordinates (u, v, s) on Y, see for example [5, Section II.1].

¿From what was said above it follows:

Corollary 5.9. *The commutative diagram of $\mathcal{O}_Y$–modules*

$$\begin{tikzcd}
0 \arrow[r] & \wedge^2 \Omega^1_{Y/S} \arrow[r] \arrow[dr, "0"'] & \wedge^2 \Omega^1_{Y/S}(X) \arrow[r] \arrow[d, "\mathrm{res}^P"] & \wedge^2 \Omega^1_{Y/S}(X)|_X \arrow[r] \arrow[dl, dotted] & 0 \\
& & i_* \Omega^1_{\breve{X}/S} & &
\end{tikzcd}$$

induces an injective morphism of $\mathcal{O}_X$–modules

$$\mathrm{cl}_S : \omega_{X/S} = \wedge^2\Omega^1_{Y/S}(X)|_X \longrightarrow j_*\Omega^1_{\breve{X}/S}.$$

Remark 5.10. In what follows the morphism cl_S will be called the *class map*. For a Gorenstein projective variety X of dimension n let $\mathcal{M}_X$ denote the sheaf of meromorphic functions on X. Angéniol and Lejeune-Jalabert construct a morphism $\Omega^n_X \longrightarrow \omega_X$ which induces an isomorphism

$$\Omega^n_X \otimes \mathcal{M}_X \xrightarrow{\cong} \omega_X \otimes \mathcal{M}_X$$

also called “class map”, see [3]. The relationship between this class map and the class map constructed above will be discussed elsewhere.

The following proposition can be shown on the lines of [5, Section II.1].

Proposition 5.11. *Let $p : X \longrightarrow S$ be a Gorenstein fibration of relative dimension one satisfying the conditions from the beginning of this subsection. If $g : S' \longrightarrow S$ is any base change, we obtain the Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ S' \arrow[r, "g"] & S. \end{tikzcd}$$

Then, the following diagram is commutative

$$\begin{tikzcd} f^*\big(\wedge^2\Omega^1_{Y/S}(X)|_X\big) \arrow[r, "\cong"] \arrow[d, "f^*(\mathrm{cl}_S)"'] & \wedge^2\Omega^1_{Y'/S'}(X')|_{X'} \arrow[d, "\mathrm{cl}_{S'}"] \\ f^*\big(j_*\Omega^1_{\breve{X}/S}\big) \arrow[r, "\cong"] & j'_*\Omega^1_{\breve{X}'/S'} \end{tikzcd}$$

where the upper horizontal isomorphism is canonical and the lower one is induced by the base-change property.

The reason to introduce the map cl_S is explained by the following proposition.

Proposition 5.12 (see Proposition 6.2 in [5])**.** *Let $p : X \longrightarrow S$ be as in Proposition 5.11, $t \in S$ a closed point and $\mathrm{cl}_t : \omega_{X_t} \longrightarrow j_{t*}\Omega^1_{\breve{X}_t}$ the class map constructed in Corollary 5.9. Then it holds*

1. *If the fibre X_t is smooth, then the image of cl_t is the sheaf $\Omega^1_{X_t}$ of holomorphic differential one-forms on X_t.*
2. *In the case X_t is singular, the image of cl_t is the sheaf of Rosenlicht's differential forms, see Definition 3.2. In particular, $\mathrm{im}(\mathrm{cl}_t)$ is a subsheaf of the sheaf of* meromorphic *differential one-forms on X_t regular at smooth points of X_t.*

The following definition is central for our construction of associative geometric r–matrices. Let $p : X \longrightarrow S$ be flat and proper morphism such that
• All fibres X_t, $t \in S$ are reduced projective Gorenstein curves.
• There exists an embedding

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where $q : Y \longrightarrow S$ is a proper and smooth morphism of relative dimension two.

Definition 5.13. Let $j : \breve{X} \longrightarrow X$ be the inclusion of the smooth locus of p, $i : S \longrightarrow \breve{X}$ a section of p and $D = i(S)$. Then the residue map

$$\mathrm{res}_D : \omega_{X/S}(D) \xrightarrow{\mathrm{cl}_S} j_*\big(\Omega^1_{\breve{X}/S}(D)\big) \longrightarrow \mathcal{O}_D$$

is defined as the composition of the class map cl_S from Corollary 5.9 and the residue map for smooth morphisms of relative dimension one from Definition 5.1.

Propositions 5.2 and 5.11 imply the following corollary.

Proposition 5.14. *Let $p : X \longrightarrow S$ and $i : S \longrightarrow X$ be as in Definition 5.13 and $g : S' \longrightarrow S$ be any base change. Denote $X' = X \times_S S'$, $f : X' \longrightarrow X$, $i' : S' \longrightarrow X'$ the pull-back of i and $D' = i'(S')$. Then the following diagram is commutative*

$$\begin{tikzcd} f^*\big(\omega_{X/S}(D)\big) \arrow[r, "f^*(\mathrm{res}_D)"] \arrow[d, "\cong"'] & f^*(\mathcal{O}_D) \arrow[d, "\cong"] \\ \omega_{X'/S'}(D') \arrow[r, "\mathrm{res}_{D'}"] & \mathcal{O}_{D'} \end{tikzcd}$$

where the vertical maps are canonical isomorphisms.

5.3. **Geometric triple Massey products.** Let $E \xrightarrow{p} S$ be a genus one fibration embedded into a smooth fibration of surfaces, i.e we have a commutative diagram

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where p is a proper and flat map, for all $t \in S$ the fibre E_t is a reduced projective curve with trivial dualising sheaf and q is a smooth and proper map of relative dimension two.

Let $\breve{E}$ be the regular locus of p. Assume S is chosen sufficiently small, so that $\omega_{E/S} \cong \mathcal{O}_E$. Fix the following data:

- A non-zero global section $\omega \in H^0(\omega_{E/S})$.

- Two holomorphic vector bundles $\mathcal{V}$ and $\mathcal{W}$ on the total space E having the same rank and such that for all $t \in S$ it holds:
$$\mathsf{Hom}_{E_t}\big(\mathcal{V}_{E_t}, \mathcal{W}_{E_t}\big) = 0 = \mathsf{Ext}^1_{E_t}\big(\mathcal{V}_{E_t}, \mathcal{W}_{E_t}\big).$$
- Two sections $h_1, h_2 : S \longrightarrow \breve{E}$ of p such that for all $t \in S$ it holds: $h_1(t) \neq h_2(t)$ and $h_1(t), h_2(t)$ belong to the same irreducible component of E_t. We additionally assume that
$$\mathsf{Hom}_{E_t}\big(\mathcal{V}_{E_t}(h_2(t)), \mathcal{W}_{E_t}(h_1(t))\big) = 0 = \mathsf{Ext}^1_{E_t}\big(\mathcal{V}_{E_t}(h_2(t)), \mathcal{W}_{E_t}(h_1(t))\big).$$

The main result of this section is the following theorem.

Theorem 5.15. *There exists an isomorphism of vector bundles on S*
$$\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}(\omega) = \tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2} : \quad h_1^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \longrightarrow h_2^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W})$$
such that for any base change diagram
$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
the following diagram is commutative:
$$\begin{tikzcd} g^*h_1^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \arrow[r, "g^*\left(\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}(\omega)\right)"] \arrow[d, "\cong"'] & g^*h_2^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \arrow[d, "\cong"] \\ h_1'^*\mathcal{H}om_E(f^*\mathcal{V}, f^*\mathcal{W}) \arrow[r, "\tilde{r}^{f^*\mathcal{V},f^*\mathcal{W}}_{h_1',h_2'}(\omega')"] & h_2'^*\mathcal{H}om_E(f^*\mathcal{V}, f^*\mathcal{W}) \end{tikzcd}$$
where $h_1', h_2' : S' \longrightarrow E'$ are sections of p' obtained as pull-backs of h_1 and h_2. The vertical arrows are canonical isomorphisms and the section $\omega' \in H^0(\omega_{E'/S'})$ is defined via the commutative diagram
$$\begin{tikzcd} f^*\mathcal{O}_E \arrow[r, "\cong"] \arrow[d, "f^*(\omega)"'] & \mathcal{O}_{E'} \arrow[d, "\omega'"] \\ f^*\omega_{E/S} \arrow[r, "\cong"] & \omega_{E'/S'}. \end{tikzcd}$$
Moreover, for any $s \in S$ the morphism $\tilde{r}^{\mathcal{V}_s,\mathcal{W}_s}_{h_1(s),h_2(s)}$ coincides with the morphism describing triple Massey products constructed in Theorem 4.2. Here, we denote $\mathcal{F}_s = \mathcal{F}|_{E_s}$ for any vector bundle $\mathcal{F}$ on E.

Proof. The construction of the morphism $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}$ is the following. Let $D_i = h_i(S)$, then the exact sequence
$$0 \longrightarrow \omega_{E/S} \longrightarrow \omega_{E/S}(D_1) \xrightarrow{\mathrm{res}_{D_1}} \mathcal{O}_{D_1} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathcal{W} \otimes \omega_{E/S} \longrightarrow \mathcal{W} \otimes \omega_{E/S}(D_1) \longrightarrow \mathcal{W} \otimes \mathcal{O}_{D_1} \longrightarrow 0.$$

Since $\mathcal{E}xt^1_E(\mathcal{V}, \mathcal{W}) = 0$ and $\omega_{E/S} \cong \mathcal{O}_E$, applying the functor $\mathcal{H}om_E(\mathcal{V}, -)$ we obtain the exact sequence

$$0 \to \mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \to \mathcal{H}om_E(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)) \to \mathcal{H}om_E(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}) \to 0. \tag{11}$$

Lemma 5.16. *In the notations of the theorem it holds:*

$$p_*\bigl(\mathcal{H}om_E(\mathcal{V}, \mathcal{W})\bigr) \cong \mathbb{R}^1 p_*\bigl(\mathcal{H}om_E(\mathcal{V}, \mathcal{W})\bigr) \cong 0.$$

Proof of the lemma. It suffices to show that $\mathbb{R}p_*\bigl(\mathcal{H}om_E(\mathcal{V}, \mathcal{W})\bigr) = 0$ viewed as an object of the derived category of coherent sheaves $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(S)$. Note that a complex $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(S)$ is zero if and only if for all points $t \in S$ it holds: $\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathbb{C}_t \cong 0$. Since the morphism p is flat, by a base-change isomorphism it holds

$$\mathbb{R}p_*\bigl(\mathcal{H}om_E(\mathcal{V}, \mathcal{W})\bigr) \overset{\mathbb{L}}{\otimes} \mathbb{C}_t \cong \mathsf{RHom}_{E_t}(\mathcal{V}|_{E_t}, \mathcal{W}|_{E_t}) \cong 0,$$

where the last equality follows from the assumption $\mathsf{Ext}^i_{E_t}(\mathcal{V}|_{E_t}, \mathcal{W}|_{E_t}) = 0$ for all $i \in \mathbb{Z}$ and $t \in S$. $\square$

Hence, applying the left-exact functor p_* to the exact Sequence (11) we obtain an isomorphism $p_*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\bigr) \xrightarrow{\cong} p_*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\bigr)$. Combining it with the canonical isomorphisms

$$\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\bigr) \xrightarrow{\cong} h_{1*}\mathcal{H}om_S\bigl(h_1^*\mathcal{V}, h_1^*\mathcal{W}\bigr) \xrightarrow{\cong} h_{1*}h_1^*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W}\bigr)$$

we obtain an isomorphism

$$\mathrm{res}^{\mathcal{V},\mathcal{W}}_{h_1} : p_*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\bigr) \xrightarrow{\cong} h_1^*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W}\bigr).$$

Moreover, the choice of a global section $\mathcal{O}_E \xrightarrow{\omega} \omega_{E/S}$ induces an isomorphism

$$\mathrm{res}^{\mathcal{V},\mathcal{W}}_{h_1}(\omega) : \quad p_*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W}(D_1)\bigr) \xrightarrow{\cong} h_1^*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W}\bigr),$$

which we shall frequently denoted by $\mathrm{res}^{\mathcal{V},\mathcal{W}}_{h_1}$.

The construction of another isomorphism

$$\mathrm{ev}^{\mathcal{V},\mathcal{W}}_{h_2} : p_*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W}(D_1)\bigr) \xrightarrow{\cong} h_2^*\mathcal{H}om_E\bigl(\mathcal{V}, \mathcal{W}\bigr)$$

is similar. We start with the exact sequence

$$0 \longrightarrow \mathcal{O}_E(D_1 - D_2) \longrightarrow \mathcal{O}_E(D_1) \longrightarrow \mathcal{O}_E(D_1) \otimes \mathcal{O}_{D_2} \longrightarrow 0. \tag{12}$$

For any Weil divisor $D \subset E$ view the line bundle $\mathcal{O}_E(D)$ as a subsheaf of the sheaf of meromorphic functions $\mathcal{M}_E$. Then there exists a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_E(D_1) \xrightarrow{\mathrm{ev}_{D_1}} \mathcal{O}_{D_1}(D_1) \longrightarrow 0$$

inducing an isomorphism $\mathcal{O}_{D_2} \longrightarrow \mathcal{O}_E(D_1)\otimes\mathcal{O}_{D_2}$. Tensoring the exact sequence (12) with the vector bundle $\mathcal{W}$ and applying $\mathcal{H}om_E(\mathcal{V}, -)$ we obtain an exact sequence

$$0 \to \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1 - D_2)\big) \to \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1)\big) \to \mathcal{H}om_E(\mathcal{V}, \mathcal{W}\otimes\mathcal{O}_{D_2}) \to 0.$$

By the same argument as in Lemma 5.16 one can show that

$$p_*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1 - D_2)\big) \cong \mathbb{R}^1 p_*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1 - D_2)\big) \cong 0$$

which implies that we obtain an isomorphism of vector bundles on S

$$\mathrm{ev}_{h_2}^{\mathcal{V},\mathcal{W}}: \quad p_*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1)\big) \xrightarrow{\cong} h_2^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}).$$

The isomorphism of vector bundles

$$\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}} = \tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega): \quad h_1^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \longrightarrow h_2^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W})$$

is defined by the commutative diagram of vector bundles on S

$$\begin{tikzcd} & p_*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1)\big) \arrow[dl, "\mathrm{res}_{h_1}^{\mathcal{V},\mathcal{W}}(\omega)"'] \arrow[dr, "\mathrm{ev}_{h_2}^{\mathcal{V},\mathcal{W}}"] & \\ h_1^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \arrow[rr, "\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega)"] & & h_2^*\mathcal{H}om_E(\mathcal{V}, \mathcal{W}). \end{tikzcd}$$

Now let us prove the compatibility of $\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}$ with respect to base-change. We start with the commutative diagram of coherent sheaves on E'

$$\begin{tikzcd} f^*\big(\omega_{E/S}(D_1)\big) \arrow[r, "f^*(\mathrm{res}_{D_1})"] \arrow[d, "\cong"'] & f^*(\mathcal{O}_{D_1}) \arrow[d, "\cong"] \\ \omega_{E'/S'}(D') \arrow[r, "\mathrm{res}_{D_1'}"] & \mathcal{O}_{D_1'} \end{tikzcd}$$

obtained in Proposition 5.14 and then apply the functor

$$p'_*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W}\otimes -) : \mathsf{Coh}(E') \longrightarrow \mathsf{Coh}(S').$$

This yields a commutative diagram

$$\begin{tikzcd} p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\otimes f^*(\omega_{E/S}(D_1))\big) \arrow[r] \arrow[d, "\cong"'] & p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\otimes f^*\mathcal{O}_{D_1}\big) \arrow[d, "\cong"] \\ p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\otimes \omega_{E'/S'}(D_1')\big) \arrow[r] & p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\otimes \mathcal{O}_{D_1'}\big) \end{tikzcd}$$

in $\mathsf{Coh}(S')$. Next, we have an isomorphism of functors

$$\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\otimes f^*(-)\big) \longrightarrow f^*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}\otimes -\big)$$

between the categories of coherent sheaves $\mathsf{Coh}(E)$ and $\mathsf{Coh}(E')$. Composing these functors with p'_* and then applying them to the residue map $\omega_{E/S}(D_1) \longrightarrow \mathcal{O}_{D_1}$ we obtain a commutative diagram

$$\begin{tikzcd}
p'_* f^* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big) \arrow[r] \arrow[d, "\cong"'] & p'_* f^* \mathcal{H}om_{E'}\big(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\big) \arrow[d, "\cong"] \\
p'_* \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*(\omega_{E/S}(D_1))\big) \arrow[r] & p'_* \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*\mathcal{O}_{D_1}\big).
\end{tikzcd}$$

Finally, there exists a natural transformation of functors (base-change) $g^* p_* \longrightarrow p'_* f^*$, which is an isomorphism of functors on the category of S–flat coherent sheaves on E. Since both sheaves $\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big)$ and $\mathcal{H}om_E(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}) \cong h_{1*}\mathcal{H}om_S(h_1^*\mathcal{V}, h_1^*\mathcal{W})$ are flat over S, we obtain a commutative diagram

$$\begin{tikzcd}
g^* p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big) \arrow[r] \arrow[d, "\cong"'] & g^* p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\big) \arrow[d, "\cong"] \\
p'_* \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \omega_{E'/S'}(D'_1)\big) \arrow[r] & p'_* \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \mathcal{O}_{D'_1}\big).
\end{tikzcd}$$

Using similar arguments one can show that the following diagram is commutative:

$$\begin{tikzcd}
g^* p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\big) \arrow[r, "\cong"] \arrow[d, "\cong"'] & g^* h_1^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \arrow[d, "\cong"] \\
p'_* \mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W} \otimes \mathcal{O}_{D'_1}) \arrow[r, "\cong"] & {h'_1}^* \mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W}),
\end{tikzcd}$$

in which all arrows are canonical isomorphisms. Summing everything up we obtain the compatibility of $\mathrm{res}_{h_1}^{\mathcal{V},\mathcal{W}}$ with base change, i.e. the diagram

$$\begin{tikzcd}
g^* p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big) \arrow[r, "g^*(\mathrm{res}_{h_1}^{\mathcal{V},\mathcal{W}})"] \arrow[d, "\cong"'] & g^* h_1^* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}\big) \arrow[d, "\cong"] \\
p'_* \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \omega_{E'/S'}(D'_1)\big) \arrow[r, "\mathrm{res}_{h'_1}^{f^*\mathcal{V}, f^*\mathcal{W}}"] & {h'_1}^* \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\big)
\end{tikzcd}$$

in which the vertical arrows are compositions of the natural isomorphisms constructed above, is commutative. The proof of naturality of $\mathrm{res}_{h_1}^{\mathcal{V},\mathcal{W}}(\omega)$ and $\mathrm{ev}_{h_2}^{\mathcal{V},\mathcal{W}}$ is completely analogous and therefore is left to the reader. This finishes the proof of the theorem. $\square$

Let $r^{\mathcal{V},\mathcal{W}}_{h_1,h_2} = r^{\mathcal{V},\mathcal{W}}_{h_1,h_2}(\omega)$ denote the image of $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}$ under the canonical isomorphism

$$\begin{array}{c} \mathsf{Hom}_S\bigl(h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}), h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W})\bigr) \\ \big\downarrow \cong \\ \Gamma\bigl(S, h_1^*\mathcal{H}om_E(\mathcal{W},\mathcal{V}) \otimes h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W})\bigr). \end{array}$$

¿From Theorem 5.15 we immediately obtain the following corollary.

Corollary 5.17. *In the notation of Theorem 5.15 let* $\eta_{\mathcal{V},\mathcal{W}} : g^*\bigl(h_1^*\mathcal{H}om_E(\mathcal{W},\mathcal{V}) \otimes h_2^*\mathcal{H}om(\mathcal{V},\mathcal{W})\bigr) \longrightarrow {h'_1}^*\mathcal{H}om_E(f^*\mathcal{W}, f^*\mathcal{V}) \otimes {h'_2}^*\mathcal{H}om_E(f^*\mathcal{V}, f^*\mathcal{W})$ *be the canonical isomorphism of bifunctors. Then it holds:*

$$\eta_{\mathcal{V},\mathcal{W}}\bigl(g^*(r^{\mathcal{V},\mathcal{W}}_{h_1,h_2})\bigr) = r^{f^*\mathcal{V},f^*\mathcal{W}}_{h'_1,h'_2}.$$

The following properties of the morphism $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}$ can be proved in a similar way.

Proposition 5.18. *Let* $E \xrightarrow{p} S$, $\mathcal{V},\mathcal{W}$ *and* h_1, h_2 *be as in Theorem 5.15. Then the isomorphism* $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2} : h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \longrightarrow h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W})$ *is functorial with respect to isomorphisms* $\mathcal{V} \longrightarrow \mathcal{V}'$ *and* $\mathcal{W} \longrightarrow \mathcal{W}'$. *Moreover, for any line bundle* $\mathcal{L}$ *on* E *the following diagram is commutative*

$$\begin{tikzcd} h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[r, "\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}"] \arrow[d, "\cong"'] & h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[d, "\cong"] \\ h_1^*\mathcal{H}om_E(\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}) \arrow[r, "\tilde{r}^{\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}}_{h_1,h_2}"] & h_2^*\mathcal{H}om_E(\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}), \end{tikzcd}$$

where the vertical arrows are induced by the canonical isomorphism

$$\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \xrightarrow{\cong} \mathcal{H}om_E(\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}).$$

6. Geometric associative r–matrix

The main goal of this section is to define the so-called geometric associative r-matrix attached to a genus one fibration. We start with the following geometric data.

- Let $E \xrightarrow{p} T$ be a flat *projective* morphism of complex spaces of relative dimension one and denote by $\breve{E}$ the smooth locus of p.
- Assume there exists section $i : T \longrightarrow \breve{E}$ of p.
- Moreover, we assume that for all points $t \in T$ the fibre E_t is a reduced and *irreducible* projective curve of arithmetic genus one.
- The fibration $E \xrightarrow{p} T$ is embeddable into a smooth fibration of projective surfaces over T and $\omega_{E/T} \cong \mathcal{O}_E$.

For a pair of coprime integers $(n,d) \in \mathbb{N} \times \mathbb{Z}$ let $M = M_{E/T}(n,d)$ be the moduli space of relatively stable vector bundles of rank n and degree d. It is well-known that $M \cong \breve{E}$, see Appendix 11. Let $\mathcal{P} = \mathcal{P}(n,d) \in \mathsf{VB}(M \times_T E)$ be a universal family and denote $\mathcal{P}_t = \mathcal{P}|_{M_{E_t}(n,d)\times E_t}$ for any $t \in T$.

In these notations we have the following Cartesian diagram of complex spaces

$$\begin{tikzcd} M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \arrow[r, "f"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ M \times_T M \times_T \breve{E} \times_T \breve{E} \arrow[r, "g"] & T. \end{tikzcd}$$

Observe that $q : M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E}$ is also a genus one fibration satisfying all the conditions above. Let us fix some notation.

Definition 6.1. The diagonal map $\Delta : \breve{E} \longrightarrow \breve{E} \times_T E$ induces two *canonical* sections

$$h_1, h_2 : \quad M \times_T M \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E$$

of the morphism q, given by the rule $h_i(v_1, v_2, y_1, y_2) = (v_1, v_2, y_1, y_2, y_i)$ for $i = 1, 2$. Let divisor D_i denote the image of h_i. Next, consider two projection maps

$$pr_i : M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \longrightarrow M \times_T E$$

given by $pr_i(v_1, v_2, y_1, y_2, y) = (v_i, y)$ for $i = 1, 2$ and denote $\mathcal{V}_i = pr_i^*\mathcal{P}$. Then both vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ are relatively stable and for any base point $x = (v_1, v_2, y_1, y_2) \in M \times_T M \times_T \breve{E} \times_T \breve{E}$ it holds: $\mathcal{V}_i|_{q^{-1}(x)} \cong \mathcal{P}_t|_{\{v_i\}\times E_t} =: \mathcal{P}^{v_i}$, where $t = g(x)$.

Lemma 6.2. *The set of points*

$$\Delta_1 = \{x \in M \times_T M \times_T \breve{E} \times_T \breve{E} \mid \mathcal{V}_1(D_2)|_{q^{-1}(x)} \cong \mathcal{V}_2(D_1)|_{q^{-1}(x)}\}$$

is a closed analytic subset of $M \times_T M \times_T \breve{E} \times_T \breve{E}$.

Proof. Since the morphism q is projective, the sheaf $q_*\mathcal{H}om\big(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\big)$ is coherent. Moreover, if $\mathcal{V}$ and $\mathcal{W}$ are two stable vector bundles on an irreducible projective curve E_t of arithmetic genus one having the same rank and degree, then $\mathsf{Hom}_{E_t}(\mathcal{V}, \mathcal{W}) \neq 0$ if and only if $\mathcal{V} \cong \mathcal{W}$. Since the sheaf $\mathcal{H}om\big(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\big)$ is locally free, it is flat over $M \times_T M \times_T \breve{E} \times_T \breve{E}$ and the base-change formula implies that for a point $x = (v_1, v_2; y_1, y_2) \in M \times_T M \times_T \breve{E} \times_T \breve{E}$ and $t = g(t)$ it holds:

$$q_*\mathcal{H}om\big(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\big) \otimes \mathbb{C}_x \cong \mathsf{Hom}_{E_t}\big(\mathcal{V}_1|_{E_t}(y_2), \mathcal{V}_2|_{E_t}(y_1)\big).$$

Therefore, the set Δ_1 coincides with the reduced support of $q_*\mathcal{H}om\big(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\big)$, hence it is a closed analytic subset. $\square$

For further purposes we shall need some more notation. Let

$$\begin{aligned}\Delta_2 &= \{x \in M \times_T M \times_T \breve{E} \times_T \breve{E} \mid \mathcal{V}_1|_{q^{-1}(x)} \cong \mathcal{V}_2|_{q^{-1}(x)}\},\\ \Delta_0 &= \{x = (v_1, v_2, y_1, y_2) \in M \times_T M \times_T \breve{E} \times_T \breve{E} \mid y_1 = y_2\}\end{aligned}$$

and $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$. Consider the open set $B = M \times_T M \times_T \breve{E} \times_T \breve{E} \setminus \Delta$ and the induced genus one fibration:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "q|_X"'] & M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \arrow[r, "f"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B \arrow[r, hook] & M \times_T M \times_T \breve{E} \times_T \breve{E} \arrow[r, "g"] & T. \end{tikzcd}$$

By abuse of notation we shall write $\mathcal{V}_i = \mathcal{V}_i|_X$ for $i = 1, 2$ and denote the two canonical sections $h_1, h_2 : B \longrightarrow X$ and the corresponding divisors D_1, D_2 by the same letters as for $M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E}$.

Definition 6.3. Let $\omega \in H^0(\omega_{E/T})$ be a nowhere vanishing section of the dualising sheaf $\omega_{E/T}$ and $f^*(\omega) \in H^0(\omega_{X/B})$ its pull-back on X. Then we have a canonical isomorphism of vector bundles on B:

$$\tilde{r} = \tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) := \tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(f^*(\omega)) : \quad h_1^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2) \longrightarrow h_2^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)$$

and a canonical section

$$r = r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) \in H^0\big(h_1^*\mathcal{H}om_X(\mathcal{V}_2, \mathcal{V}_1) \otimes h_2^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)\big)$$

constructed in Theorem 5.15.

Remark 6.4. Let φ_{ij} denote the composition map

$$B \hookrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E} \xrightarrow{h_i} M \times_T M \times_T \breve{E} \times_T \breve{E} \times E \xrightarrow{pr_j} M \times_T E$$

then r is an element in $\Gamma(B, \mathcal{H}om_B\big(\varphi_{12}^*\mathcal{P}, \varphi_{11}^*\mathcal{P}) \otimes \mathcal{H}om_B(\varphi_{21}^*\mathcal{P}, \varphi_{22}^*\mathcal{P})\big)$.

The constructed section r has the following base-change property. Let $\tilde{g} : T' \longrightarrow T$ be any morphism and

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ T' \arrow[r, "\tilde{g}"] & T \end{tikzcd}$$

the corresponding base-change diagram. Since the functor $\underline{\mathsf{M}}^{(n,d)}_{E/T}$ is representable (see Appendix 11), there exists a unique morphism

$$M' = M_{E'/T'}(n, d) \xrightarrow{m} M = M_{E/T}(n, d)$$

making the following diagram

$$\begin{tikzcd} M' \times_{T'} E' \arrow[r, "m\times\tilde{f}"] \arrow[d] & M \times_T E \arrow[d] \\ T' \arrow[r, "\tilde{g}"] & T \end{tikzcd}$$

commutative and such that $\mathcal{P}' := (m\times\tilde{f})^*\mathcal{P}$ is a universal vector bundle on $M'\times_{T'}E'$. In fact, $M' \cong M \times_T T'$ and m is the projection and the above diagram is Cartesian. Denote $\hat{g} = m \times m \times \tilde{f} \times \tilde{f} \times \tilde{f} : X' \longrightarrow X$ and $\tilde{g}_B = m \times m \times \tilde{f} \times \tilde{f} : B' \longrightarrow B$, then we obtain a new Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\hat{g}"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ B' \arrow[r, "\tilde{g}_B"] & B. \end{tikzcd}$$

Note that there exists canonical isomorphisms $\phi_i : \hat{g}^*\mathcal{V}_i \longrightarrow \mathcal{V}'_i$, where $\mathcal{V}'_i = pr_i'^*\mathcal{P}'$ in the notation of Definition 6.1.

Proposition 6.5. *Let $\omega \in H^0(\omega_{E/T})$ and $\omega' = f^*(\omega) \in H^0(\omega_{E'/T'})$, then the image of the section*

$$r = r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) \in H^0\big(h_1^*\mathcal{H}om_X(\mathcal{V}_2,\mathcal{V}_1) \otimes h_2^*\mathcal{H}om_X(\mathcal{V}_1,\mathcal{V}_2)\big)$$

under the chain of morphisms

$$\begin{tikzcd} H^0\big(h_1^*\mathcal{H}om_X(\mathcal{V}_2,\mathcal{V}_1) \otimes h_2^*\mathcal{H}om_X(\mathcal{V}_1,\mathcal{V}_2)\big) \arrow[d] \\ H^0\big(\tilde{g}_B^*(h_1^*\mathcal{H}om_X(\mathcal{V}_2,\mathcal{V}_1) \otimes h_2^*\mathcal{H}om_X(\mathcal{V}_1,\mathcal{V}_2))\big) \arrow[d] \\ H^0\big(h_1'^*\mathcal{H}om_{X'}(\hat{g}^*\mathcal{V}_2,\hat{g}^*\mathcal{V}_1) \otimes h_2'^*\mathcal{H}om_{X'}(\hat{g}^*\mathcal{V}_1,\hat{g}^*\mathcal{V}_2)\big) \arrow[d] \\ H^0\big(h_1'^*\mathcal{H}om_{X'}(\mathcal{V}'_2,\mathcal{V}'_1) \otimes h_2'^*\mathcal{H}om_{X'}(\mathcal{V}'_1,\mathcal{V}'_2)\big) \end{tikzcd}$$

is $r' = r^{\mathcal{V}'_1,\mathcal{V}'_2}_{h'_1,h'_2}(\omega')$, where the first arrow is induced by the functor $\tilde{g}_B^*$, the second by the canonical isomorphisms of functors $\tilde{g}_B^* h_i^* \cong h_i'^*\hat{g}^*$ and the third by the isomorphisms of vector bundles $\phi_i : \hat{g}^*\mathcal{V}_i \longrightarrow \mathcal{V}'_i$, $i = 1, 2$.

Proof. This proposition is an immediate consequence of Corollary 5.17. $\square$

Now we introduce further notation. Let

$$p^{ij}_{kl} : M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E}$$

be the projection $p^{ij}_{kl}(v_1, v_2, v_3, y_1, y_2, y_3) = (v_i, v_j, y_k, y_l)$ and

$$\psi_{ij} : M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T E$$

be the composition

$$M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \xrightarrow{h_i} M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \times_T E \xrightarrow{pr_j} M \times_T E,$$

so $\psi_{ij}(v_1, v_2, v_3, y_1, y_2, y_3) = (v_j, y_i)$. We also take

$$C := \bigcap_{i,j,k,l=1}^{3} (p^{ij}_{kl})^{-1}(B) \subseteq M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E}.$$

Note that

$$\varphi_{12} \circ p^{ij}_{kl} = \psi_{kj}, \quad \varphi_{11} \circ p^{ij}_{kl} = \psi_{ki}$$

and

$$\varphi_{21} \circ p^{ij}_{kl} = \psi_{li}, \quad \varphi_{22} \circ p^{ij}_{kl} = \psi_{lj}.$$

Hence, the section $r = r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) \in \Gamma\big(B, \mathcal{H}om_B\big(\varphi_{12}^*\mathcal{P}, \varphi_{11}^*\mathcal{P}\big) \otimes \mathcal{H}om_B(\varphi_{21}^*\mathcal{P}, \varphi_{22}^*\mathcal{P})\big)$ defines elements $r^{ij}_{kl} := (p^{ij}_{kl})^*(r)$ lying in

$$\Gamma\big(C, \mathcal{H}om(\psi_{kj}^*\mathcal{P}, \psi_{ki}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{li}^*\mathcal{P}, \psi_{lj}^*\mathcal{P})\big).$$

Let $x = (v_1, v_2, v_3, y_1, y_2, y_3)$ be a point in C, then the fibre $r^{ij}_{kl}(x)$ is an element of the tensor product $\mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_j}|_{y_k}, \mathcal{P}^{v_i}|_{y_k}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_i}|_{y_l}, \mathcal{P}^{v_j}|_{y_l})$.

The following theorem is the main result of this article.

Theorem 6.6. *The section*

$$r = r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) \in \Gamma\big(B, \mathcal{H}om_B\big(\varphi_{12}^*\mathcal{P}, \varphi_{11}^*\mathcal{P}\big) \otimes \mathcal{H}om_B(\varphi_{21}^*\mathcal{P}, \varphi_{22}^*\mathcal{P})\big)$$

satisfies the geometric associative Yang-Baxter equation

$$(r^{13}_{13})^{13}(r^{32}_{12})^{12} - (r^{12}_{12})^{12}(r^{13}_{23})^{23} + (r^{23}_{23})^{23}(r^{12}_{13})^{13} = 0, \tag{13}$$

where the left-hand side of this equation takes values in

$$\Gamma\Big(C, \mathcal{H}om(\psi_{12}^*\mathcal{P}, \psi_{11}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{23}^*\mathcal{P}, \psi_{22}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{31}^*\mathcal{P}, \psi_{33}^*\mathcal{P})\Big).$$

Moreover, this section is unitary, which means that

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) = -\sigma^*\big(r^{\mathcal{V}_2,\mathcal{V}_1}_{h_2,h_1}(\omega)\big),$$

where $\sigma : M \times_T M \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E}$ *is defined by the rule* $\sigma(v_1, v_2, y_1, y_2) = (v_2, v_1, y_2, y_1)$.

Proof. In our notation, r_{13}^{13} is a section of $\mathcal{H}om(\psi_{13}^*\mathcal{P}, \psi_{11}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{31}^*\mathcal{P}, \psi_{33}^*\mathcal{P})$ and r_{12}^{32} is a section of $\mathcal{H}om(\psi_{12}^*\mathcal{P}, \psi_{13}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{23}^*\mathcal{P}, \psi_{22}^*\mathcal{P})$, so their composition $(r_{13}^{13})^{13}(r_{12}^{32})^{12}$ is indeed a section of

$$\mathcal{H}om(\psi_{12}^*\mathcal{P}, \psi_{11}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{23}^*\mathcal{P}, \psi_{22}^*\mathcal{P}) \otimes \mathcal{H}om(\psi_{31}^*\mathcal{P}, \psi_{33}^*\mathcal{P}).$$

The arguments for the remaining two summands of the left-hand side of equation (13) are analogous.

To prove the Yang-Baxter relation, take any point $x = (v_1, v_2, y_1, y_2) \in B$ and denote $t = g(x) \in T$. Then we obtain a commutative diagram

$$\begin{tikzcd}
E_t \arrow[r, "j_x"] \arrow[d] & X \arrow[r] \arrow[d, "q"] & E \arrow[d, "p"] \\
\mathsf{Specan}(\mathbb{C}) \arrow[r, "x"] & B \arrow[r, "g"] & T,
\end{tikzcd}$$

where $E_t = E|_{p^{-1}(t)}$. By the base-change property of r (see Proposition 6.5) we obtain:

$$j_x^*(r) = r_{y_1,y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}} \in \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_2}|_{y_1}, \mathcal{P}^{v_1}|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_1}|_{y_2}, \mathcal{P}^{v_2}|_{y_2}).$$

Moreover, we know that $j_x^*(r)$ satisfy the associative Yang-Baxter equation, hence the equality

$$j_x^*\Big((r_{13}^{13})^{13}(r_{12}^{32})^{12} - (r_{12}^{12})^{12}(r_{23}^{13})^{23} + (r_{23}^{23})^{23}(r_{13}^{12})^{13}\Big) = 0 \tag{14}$$

holds for any point $x = (v_1, v_2, y_1, y_2) \in B$.

Since equation (13) holds for all points $x \in B$ and B contains an open ball in $\mathbb{C}^n$, $n = \dim(B)$, Lemma 6.7 below implies that the geometric associative Yang-Baxter equation (13) also holds globally. The unitarity of r can be shown in a similar way. $\square$

The proof of the following lemma is straightforward.

Lemma 6.7. *Let $U \subseteq \mathbb{C}^n$ be an open ball, $\mathcal{E}$ a holomorphic vector bundle on U and $s \in H^0(U, \mathcal{E})$. If for all $x \in U$ the element $s(x) \in \mathcal{E}|_x$ is zero, then $s = 0$.* $\square$

Our next goal is to explain the dependence of the tensor r on the trivialisation of the universal family $\mathcal{P} = \mathcal{P}(n, d)$.

Proposition 6.8. *Different trivialisations of the universal bundle $\mathcal{P}$ lead to equivalent associative r–matrices in the sense of Definition 2.5.*

Proof. Recall that for each $i, j \in \{1, 2\}$ we have defined the map φ_{ij} to be the composition

$$M \times_T M \times \breve{E} \times \breve{E} \xrightarrow{h_i} M \times_T M \times \breve{E} \times \breve{E} \times_T E \xrightarrow{pr_j} M \times_T E$$

and r is a distinguished section in $\Gamma\Big(B, \mathcal{H}om_B(\varphi_{12}^*\mathcal{P}, \varphi_{11}^*\mathcal{P}) \otimes \mathcal{H}om_B(\varphi_{21}^*\mathcal{P}, \varphi_{22}^*\mathcal{P})\Big)$.

Consider a small open set $V \subset M \times_T \breve{E}$ such that there exists an isomorphism of vector bundles $\eta : \mathcal{P}|_V \xrightarrow{\cong} V \times \mathbb{C}^n$ and an isomorphism $V \cong \mathbb{C}^2 \times T_0$ such that the following diagram, where $T_0 \subseteq T$ is an open subset, is commutative

$$\begin{tikzcd}
M \times_T E \arrow[drr] & & V \arrow[ll, hook'] \arrow[r, "\cong"] \arrow[d] & \mathbb{C}^2 \times T_0 \arrow[d, "pr"] \\
& & T & T_0. \arrow[l]
\end{tikzcd}$$

Let $B_0 = \cap_{i,j=1}^2 \varphi_{ij}^{-1}(V)$, then the trivialisation η induces trivialisations $\varphi_{ij}^*(\eta)$: $\varphi_{ij}^*\mathcal{P} \longrightarrow B_0 \times \mathbb{C}^n$ and the section r can be written as a tensor

$$r = r_t(v_1, v_2; y_1, y_2) = \sum_{\nu} a_t^{\nu}(v_1, v_2; y_1, y_2) \otimes b_t^{\nu}(v_1, v_2; y_1, y_2) \in \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C}),$$

where the functions $a_t = a_t^{\nu}(v_1, v_2; y_1, y_2)$ and $b_t = b_t^{\nu}(v_1, v_2; y_1, y_2)$ are holomorphic on B_0 and are defined via the following commutative diagrams

$$\begin{tikzcd}
\varphi_{12}^*\mathcal{P}|_{B_0} \arrow[r] \arrow[d, "\varphi_{12}^*(\eta)"'] & \varphi_{11}^*\mathcal{P}|_{B_0} \arrow[d, "\varphi_{11}^*(\eta)"] \\
B_0 \times \mathbb{C}^n \arrow[r, "a_t"] & B_0 \times \mathbb{C}^n
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\varphi_{21}^*\mathcal{P}|_{B_0} \arrow[r] \arrow[d, "\varphi_{21}^*(\eta)"'] & \varphi_{22}^*\mathcal{P}|_{B_0} \arrow[d, "\varphi_{22}^*(\eta)"] \\
B_0 \times \mathbb{C}^n \arrow[r, "b_t"] & B_0 \times \mathbb{C}^n.
\end{tikzcd}$$

Let $\mathcal{P}|_V \xrightarrow{\eta'} V \times \mathbb{C}^n$ be another trivialisation of $\mathcal{P}$, then we obtain a commutative diagram

$$\begin{tikzcd}
& \mathcal{P}|_V \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\
V \times \mathbb{C}^n \arrow[rr, "id \times \phi_t(v;y)"] & & V \times \mathbb{C}^n,
\end{tikzcd}$$

where $\phi_t(v; y) : V \longrightarrow \mathsf{GL}_n(\mathbb{C})$ is a holomorphic function. Identifying trivialisations $\eta : \mathcal{P}|_V \xrightarrow{\cong} V \times \mathbb{C}^n$ and $\varphi_{ij}^*(\eta) : \varphi_{ij}^*\mathcal{P} \xrightarrow{\cong} B_0 \times \mathbb{C}^n$ we obtain a commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "a_t"] \arrow[dd, bend right=60, "\phi_t(v_2, y_1)"'] & \mathbb{C}^n \arrow[dd, bend left=60, "\phi_t(v_1, y_1)"] \\
\mathcal{P}^{v_2}|_{y_1} \arrow[r] \arrow[u, "\eta(v_2, y_1)"] \arrow[d, "\eta'(v_2, y_1)"] & \mathcal{P}^{v_1}|_{y_1} \arrow[u, "\eta(v_1, y_1)"'] \arrow[d, "\eta'(v_1, y_1)"'] \\
\mathbb{C}^n \arrow[r, "a'_t"] & \mathbb{C}^n
\end{tikzcd}$$

and similarly a diagram for b_t:

$$\begin{tikzcd}
\mathbb{C}^n \arrow[rr, "b_t"] & & \mathbb{C}^n \\
\mathcal{P}^{v_1}|_{y_2} \arrow[u, "\eta(v_1,y_2)"'] \arrow[d, "\eta'(v_1,y_2)"] \arrow[rr] & & \mathcal{P}^{v_2}|_{y_2} \arrow[u, "\eta(v_2,y_2)"] \arrow[d, "\eta'(v_2,y_2)"'] \\
\mathbb{C}^n \arrow[rr, "b'_t"] & & \mathbb{C}^n,
\arrow[from=1-1, to=3-1, bend right=60, "\phi_t(v_1,y_2)"']
\arrow[from=1-3, to=3-3, bend left=60, "\phi_t(v_2,y_2)"]
\end{tikzcd}$$

which imply the transformation rules

$$\begin{aligned} a'_t &= \phi_t(v_1, y_1) a_t \phi_t^{-1}(v_2, y_1), \\ b'_t &= \phi_t(v_2, y_2) b_t \phi_t^{-1}(v_1, y_2). \end{aligned}$$

This means that the choice of a different trivialisation of $\mathcal{P}$ leads to the transformation rule

$$r_t(v_1, v_2; y_1, y_2) \mapsto \big(\phi_t(v_1, y_1) \otimes \phi_t(v_2, y_2)\big) r_t(v_1, v_2; y_1, y_2) \big(\phi_t(v_2, y_1)^{-1} \otimes \phi_t(v_1, y_2)^{-1}\big).$$

Proposition is proved. $\square$

Corollary 6.9. *Let $E \xrightarrow{p} T$ be as at the beginning of the chapter, $\underline{\mathsf{M}}_{E/T}^{(n,d)}$ the moduli functor of relatively stable vector bundles of rank n and degree d, $\underline{\mathsf{Pic}}_{E/T}^{0}$ and $\underline{\mathsf{Pic}}_{E/T}^{d}$ the Picard functors. Then the section $i : T \longrightarrow \breve{E}$ induces an isomorphism $\underline{\mathsf{Pic}}_{E/T}^{0} \longrightarrow \underline{\mathsf{Pic}}_{E/T}^{d}$. Combining it with the isomorphism induced by the natural transformation of functors* det : $\underline{\mathsf{M}}_{E/T}^{(n,d)} \longrightarrow \underline{\mathsf{Pic}}_{E/T}^{d}$, *we obtain a section $\hat{\imath} : T \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E}$. Let o be any point belonging to the image of $\hat{\imath}$. Taking an arbitrary trivialisation of the universal family $\mathcal{P}(n, d)$ we obtain a germ of a holomorphic function*

$$r : \big(M \times_T M \times_T \breve{E} \times_T \breve{E} \setminus \Delta, o\big) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \times \mathsf{Mat}_n(\mathbb{C}).$$

satisfying the equation

$$r_t(v_3, v_2; y_1, y_2)^{12} r_t(v_1, v_3; y_1, y_3)^{13} - r_t(v_1, v_3; y_1, y_3)^{23} r_t(v_1, v_2; y_1, y_2)^{12} +$$
$$+ r_t(v_1, v_2; y_1, y_3)^{13} r_t(v_2, v_3; y_2, y_3)^{23} = 0.$$

The function $r_t(v_1, v_2; y_1, y_2)$ depends analytically on the parameter t and different choices of trivialisations of $\mathcal{P}(n, d)$ lead to equivalent solutions.

Remark 6.10. The construction of the geometric associative r–matrix can be carried out in the category of algebraic schemes over $\mathbb{C}$. This means that if the complex-analytic fibration $E \longrightarrow T$ is a complexification of an *algebraic fibration*, then the canonical section $r = r_{h_1,h_2}^{pr_1^*\mathcal{P},pr_2^*\mathcal{P}}$ leads not only to a holomorphic tensor-valued function on $B = \big(M \times_T M \times_T \breve{E} \times_T \breve{E}\big) \setminus \Delta$ but also to a *meromorphic* function

$$r : \big(M \times_T M \times_T \breve{E} \times_T \breve{E}, o\big) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \times \mathsf{Mat}_n(\mathbb{C}).$$

Our next goal is to show in the absolute case that there always exists a trivialisation of the universal family $\mathcal{P}$ such that the geometric associative r–matrix will depend only on the difference $v = v_2 - v_1$. Namely, we are going to prove that for any geometric r–matrix $r(v_1, v_2; y_1, y_2) \in \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$ attached to a Weierstraß cubic curve E and a pair of coprime integers $(n, d) \in \mathbb{N} \times \mathbb{Z}$ there exists a gauge transformation $\phi : (\mathbb{C}^2, 0) \to \mathsf{GL}_n(\mathbb{C})$ such that the function

$$\big(\phi(v_1, y_1) \otimes \phi(v_2, y_2)\big) r(v_1, v_2; y_1, y_2) \big(\phi(v_2, y_1)^{-1} \otimes \phi(v_1, y_2)^{-1}\big)$$

is invariant under transformations $(v_1, v_2) \mapsto (v_1 + v, v_2 + v)$. The key idea of the proof is to look at the behaviour of the geometric r–matrix under the action of the Jacobian $G = \mathsf{Pic}^0(E)$.

Let E be a Weierstraß cubic curve and Ans denote the category of complex analytic spaces. For a pair of coprime integers $(n, d) \in \mathbb{N} \times \mathbb{Z}$ let be $\mathsf{St}^{(n,d)}(E)$ be the set of stable vector bundles of rank n and degree d on E.

Recall that the moduli functors $\underline{\mathsf{M}}_E^{(n,d)}, \underline{\mathsf{Pic}}_E^0 : \mathsf{Ans} \longrightarrow \mathsf{Sets}$ are given by

$$\underline{\mathsf{M}}_E^{(n,d)}(S) = \left\{ \text{ iso-classes of } \quad \mathcal{F} \in \mathsf{Coh}(E \times S) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is } S\text{-flat} \\ \mathcal{F}|_{E \times s} \in \mathsf{St}^{(n,d)}(E) \end{array} \right\} / \sim$$

where $\mathcal{F} \sim \mathcal{F} \otimes pr_S^*(\mathcal{L})$ for any $\mathcal{L} \in \mathsf{Pic}(S)$. In our notation it holds $\underline{\mathsf{Pic}}_E^d = \underline{\mathsf{M}}_E^{(1,d)}$.

For any pair of coprime integers (n, d) the functors $\underline{\mathsf{M}}_E^{(n,d)}$ and $\underline{\mathsf{Pic}}_E^0$ are representable by the complex curve $M \cong E_{\mathrm{reg}} \cong G$, see Appendix 11. This means that we have two isomorphisms of functors

$$\alpha : \underline{\mathsf{M}}_E^{(n,d)} \longrightarrow \mathsf{Mor}(\,-\,, M) \quad \text{and} \quad \beta : \underline{\mathsf{Pic}}_E^0 \longrightarrow \mathsf{Mor}(\,-\,, G).$$

Let $\mathcal{P} = \mathcal{P}(r, d)$ be a vector bundle on $M \times E$ representing the equivalence class of the element $\alpha(M)^{-1}(1_M)$. In a similar way, let $\mathcal{L} = \beta(M)^{-1}(1_M) \in \mathsf{VB}(G \times E)$ be a universal family of line bundles on E having degree zero.

Recall that for two functors $F, G : \mathsf{Ans} \longrightarrow \mathsf{Sets}$ we can take their product $F \times G : \mathsf{Ans} \longrightarrow \mathsf{Sets}$ defined by $(F \times G)(S) = F(S) \times G(S)$. Taking the tensor product provides us with a natural transformation of functors

$$\tau : \underline{\mathsf{Pic}}_E^0 \times \underline{\mathsf{M}}_E^{(n,d)} \longrightarrow \underline{\mathsf{M}}_E^{(n,d)}$$

given by $\tau_S(\mathcal{N}, \mathcal{F}) = \mathcal{N} \otimes \mathcal{F}$ for a complex space S and $\mathcal{N} \in \underline{\mathsf{Pic}}_E^0(S), \mathcal{F} \in \underline{\mathsf{M}}_E^{(n,d)}(S)$.

Since for any pair of complex spaces $S, T \in \mathsf{Ans}$ it holds

$$\mathsf{Mor}(\,-\,, S) \times \mathsf{Mor}(\,-\,, T) \cong \mathsf{Mor}(\,-\,, S \times T),$$

the natural transformation τ induces a map of complex spaces $\tau : G \times M \longrightarrow M$ making the following diagram commutative:

$$\begin{tikzcd}
\underline{\mathsf{Pic}}^0_E \times \underline{\mathsf{M}}^{(r,d)}_E \arrow[r, "\beta\times\alpha"] \arrow[d, "\tau"] & \mathsf{Mor}(-,G) \times \mathsf{Mor}(-,M) \arrow[r] & \mathsf{Mor}(-,G\times M) \arrow[d, "\tau_*"] \\
\underline{\mathsf{M}}^{(r,d)}_E \arrow[rr, "\alpha"] & & \mathsf{Mor}(-,M),
\end{tikzcd}$$

where we have used Yoneda's Lemma

$$\mathsf{Hom}(\mathsf{Mor}(-,G\times M),\, \mathsf{Mor}(-,M)) \cong \mathsf{Mor}(G\times M, M).$$

Moreover, we have an isomorphism of functors $\det : \underline{\mathsf{M}}^{(r,d)}_E \longrightarrow \underline{\mathsf{Pic}}^d_E$, see Appendix 11. If $G^d \cong E_{\mathrm{reg}}$ is the complex space representing $\underline{\mathsf{Pic}}^d_E$ then the following lemma holds.

Lemma 6.11. *Using the notation introduced above, the following diagram is commutative*

$$\begin{tikzcd}
G \times M \arrow[r, "\tau"] \arrow[d, "1\times\det"'] & M \arrow[d, "\det"] \\
G \times G^d \arrow[r, "\sigma"] & G^d,
\end{tikzcd}$$

where $\sigma : G \times G^d \longrightarrow G^d$ is induced by the natural transformation of functors $\sigma : \underline{\mathsf{Pic}}^0_E \times \underline{\mathsf{Pic}}^d_E \longrightarrow \underline{\mathsf{Pic}}^d_E$ which sends $(\mathcal{L}', \mathcal{L}'')$ to $\mathcal{L}'^{\otimes n} \otimes \mathcal{L}''$.

Proof. This result follows from Theorem 7.1, Remark 8.16 and Remark 8.31. $\square$

Recall that the Jacobian $G = \mathsf{Pic}^0(E)$ has the following description:

$$G \cong \begin{cases} \mathbb{C}/\Gamma & \text{if E is elliptic,} \\ \mathbb{C}^* & \text{if E is nodal,} \\ \mathbb{C} & \text{if E is cuspidal.} \end{cases}$$

Let $o \in M$ be any point and e the neutral element of G. If we identify appropriate open neighbourhoods of o and e with open subsets of G as described above, the map σ from Lemma 6.11 takes the form $\sigma(g, v) = g^n \circ v$, where $\circ$ is the group operation of G.

Lemma 6.12. *Consider the following diagram of complex spaces:*

$$\begin{tikzcd}
 & M \times E & \\
 & G \times M \times E \arrow[u, "\tau\times 1"'] \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\
M \times E & G \times E & G \times M
\end{tikzcd}$$

where p_i are natural projections, $i = 1, 2, 3$. Then $p_1^*\mathcal{P} \otimes p_2^*\mathcal{L} \sim (\tau \times 1)^*\mathcal{P}$, i.e. there exists a line bundle $\mathcal{N} \in \mathsf{Pic}(G \times M)$ such that

$$p_1^*\mathcal{P} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{N} \cong (\tau \times 1)^*\mathcal{P}.$$

Replacing G and M by small open neighbourhoods of $e \in G$ we even obtain

$$p_1^*\mathcal{P} \otimes p_2^*\mathcal{L} \cong (\tau \times 1)^*\mathcal{P}.$$

Proof. Note that we have the commutative diagram

$$\begin{tikzcd}
 & \mathsf{Mor}(G \times M, G) \times \mathsf{Mor}(G \times M, M) \arrow[dr] & \\
\underline{\mathsf{Pic}}^0_E(G \times M) \times \underline{\mathsf{M}}^{(n,d)}_E(G \times M) \arrow[ur, "\beta \times \alpha"'] \arrow[d, "\tau_{G\times M}"'] & & \mathsf{Mor}(G \times M, G \times M) \arrow[d, "\tau_*"] \\
\underline{\mathsf{M}}^{(n,d)}_E(G \times M) \arrow[rr, "\alpha(G\times M)"] & & \mathsf{Mor}(G \times M, M) \\
\underline{\mathsf{M}}^{(n,d)}_E(M) \arrow[u, "\tau^*"] \arrow[rr, "\alpha(M)"] & & \mathsf{Mor}(M, M). \arrow[u, "\tau^*"']
\end{tikzcd}$$

Since $\tau_*(1_{G\times M}) = \tau^*(1_M) = \tau$, we obtain $p_1^*\mathcal{P} \otimes p_2^*\mathcal{L} \sim (\tau \times 1)^*\mathcal{P}$. □

The following proposition is crucial.

Proposition 6.13. *Let E be a Weierstraß cubic curve, $M = M_E(n, d)$ the moduli space of stable vector bundles of rank n and degree d, $G = \mathsf{Pic}^0(E)$ the Jacobian of E. Let $U \subseteq G$ be an open neighbourhood of the neutral element $e \in G$ and V a small open set in the moduli space M such that we have an isomorphism*

$$\varphi : \big(p_1^*\mathcal{P} \otimes p_2^*\mathcal{L}\big)|_{V\times U\times E} \cong p_1^*(\mathcal{P}|_{V\times E}) \otimes p_2^*(\mathcal{L}|_{U\times E}) \xrightarrow{\cong} (\tau \times 1)^*\mathcal{P}|_{V\times U\times E}$$

constructed in Lemma 6.12. Identify V with an open subset of G and such that $\tau : U \times V \longrightarrow V$ is given by the rule $\tau(l, v) = l^n \circ v$, where $\circ$ is the group operation of G. Then there exists a small neighbourhood $W \subseteq E$ and trivialisations $\mathcal{P}|_{V\times W} \xrightarrow{\cong} (V \times W) \times \mathbb{C}^n$ and $\mathcal{L}|_{U\times W} \xrightarrow{\cong} (U \times W) \times \mathbb{C}$ with respect to which the morphism φ is given by the identity *map.*

Proof. This proposition is follows from a case-by-case analysis made below for each of three types of Weierstraß cubic curves, see Corollary 7.4, Remark 8.20 and Remark 8.30. □

Theorem 6.14. *Let E be a Weierstraß cubic curve, $\breve{E}$ its smooth part, $(n, d) \in \mathbb{N}\times\mathbb{Z}$ a pair of coprime integers, $M = M_E(n, d)$ the moduli space of stable vector bundles of rank n and degree d, $G = \mathsf{Pic}^0(E)$ the Jacobian of E and*

$$r : (M \times M \times \breve{E} \times \breve{E}, o) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \times \mathsf{Mat}_n(\mathbb{C})$$

the geometric associative r–matrix. Then there exists a trivialisation of $\mathcal{P}$ such that

$$r(v_1, v_2; y_1, y_2) \sim r(v; y_1, y_2),$$

where $v = v_2 - v_1$ in G and there exists the limit

$$\bar{r}(y_1, y_2) = \lim_{v \to e} \bigl(\mathrm{pr} \otimes \mathrm{pr}\bigr) r(v; y_1, y_2) \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$$

satisfying the classical Yang-Baxter equation.

Proof. Let $\mathcal{P} = \mathcal{P}(n, d)$ be a universal family on $M \times E$ and $\mathcal{L} \in \mathsf{VB}(G \times E)$ a universal line bundle,

$$\mathrm{pr}_1, \mathrm{pr}_2 : M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow M \times E$$

the two projection maps,

$$h_1, h_2 : M \times M \times \breve{E} \times \breve{E} \longrightarrow M \times M \times \breve{E} \times \breve{E} \times E$$

the two canonical sections and $\mathcal{V}_i = \mathrm{pr}_i^* \mathcal{P}$. In a similar way, denote

$$\widehat{\mathrm{pr}}_1, \widehat{\mathrm{pr}}_2 : G \times M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow M \times E$$

and

$$\hat{h}_1, \hat{h}_2 : G \times M \times M \times \breve{E} \times \breve{E} \longrightarrow G \times M \times M \times \breve{E} \times \breve{E} \times E$$

and put $\widehat{\mathcal{V}}_i = \widehat{\mathrm{pr}}_i^* \mathcal{P}$. Let

$$p : G \times M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow G \times E$$

and

$$\hat{\pi} : G \times M \times M \times \breve{E} \times \breve{E} \longrightarrow M \times M \times \breve{E} \times \breve{E}$$

be the canonical projections. Then we have a commutative diagram

$$\begin{tikzcd}
G \times M \times M \times \breve{E} \times \breve{E} \times E \arrow[r] \arrow[d] & M \times M \times \breve{E} \times \breve{E} \times E \arrow[r] \arrow[d] & E \arrow[d] \\
G \times M \times M \times \breve{E} \times \breve{E} \arrow[r, "\hat{\pi}"] & M \times M \times \breve{E} \times \breve{E} \arrow[r] & \mathsf{Specan}(\mathbb{C}),
\end{tikzcd}$$

where all arrows are canonical projections. Moreover, it induces a commutative diagram of vector bundles on $G \times M \times M \times \breve{E} \times \breve{E}$:

$$\begin{tikzcd}
\hat{\pi}^*\bigl(h_1^* \mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)\bigr) \arrow[r, "\hat{\pi}^*(\tilde{r}^{\mathcal{V}_1, \mathcal{V}_2}_{h_1, h_2})"] \arrow[d] & \hat{\pi}^*\bigl(h_2^* \mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)\bigr) \arrow[d] \\
\hat{h}_1^* \mathcal{H}om(\widehat{\mathcal{V}}_1, \widehat{\mathcal{V}}_2) \arrow[r, "\tilde{r}^{\widehat{\mathcal{V}}_1, \widehat{\mathcal{V}}_2}_{\hat{h}_1, \hat{h}_2}"] \arrow[d] & \hat{h}_2^* \mathcal{H}om(\widehat{\mathcal{V}}_1, \widehat{\mathcal{V}}_2) \arrow[d] \\
\hat{h}_1^* \mathcal{H}om(\widehat{\mathcal{V}}_1 \otimes p^*\mathcal{L}, \widehat{\mathcal{V}}_2 \otimes p^*\mathcal{L}) \arrow[r, "\tilde{r}^{\widehat{\mathcal{V}}_1 \otimes p^*\mathcal{L}, \widehat{\mathcal{V}}_2 \otimes p^*\mathcal{L}}_{\hat{h}_1, \hat{h}_2}"] & \hat{h}_2^* \mathcal{H}om(\widehat{\mathcal{V}}_1 \otimes p^*\mathcal{L}, \widehat{\mathcal{V}}_2 \otimes p^*\mathcal{L}),
\end{tikzcd}$$

where all vertical morphisms are canonical isomorphisms from Theorem 5.15 and Proposition 5.18. Let

$$\hat{\tau}: G \times M \times M \times \breve{E} \times \breve{E} \longrightarrow M \times M \times \breve{E} \times \breve{E}$$

be given by $\hat{\tau}(l, v_1, v_2, y_1, y_2) = \big(\tau(l, v_1), \tau(l, v_2), y_1, y_2\big)$. Taking open sets U and V in G and M as in Proposition 6.13 we obtain a new commutative diagram

$$\begin{tikzcd}
\hat{h}_1^* \mathcal{H}om(\widehat{\mathcal{V}}_1 \otimes p^*\mathcal{L}, \widehat{\mathcal{V}}_2 \otimes p^*\mathcal{L}) \arrow[r, "\tilde{r}^{\widehat{\mathcal{V}}_1 \otimes p^*\mathcal{L}, \widehat{\mathcal{V}}_2 \otimes p^*\mathcal{L}}_{\hat{h}_1, \hat{h}_2}"] \arrow[d] & \hat{h}_2^* \mathcal{H}om(\widehat{\mathcal{V}}_1 \otimes p^*\mathcal{L}, \widehat{\mathcal{V}}_2 \otimes p^*\mathcal{L}) \arrow[d] \\
\hat{h}_1^* \mathcal{H}om(\hat{\tau}^*\widehat{\mathcal{V}}_1, \hat{\tau}^*\widehat{\mathcal{V}}_2) \arrow[r, "\tilde{r}^{\hat{\tau}^*\widehat{\mathcal{V}}_1, \hat{\tau}^*\widehat{\mathcal{V}}_2}_{\hat{h}_1, \hat{h}_2}"] & \hat{h}_2^* \mathcal{H}om((\hat{\tau}^*\widehat{\mathcal{V}}_1, \hat{\tau}^*\widehat{\mathcal{V}}_2)
\end{tikzcd}$$

where the vertical maps are induced by the isomorphism

$$p_1^*(\mathcal{P}|_{V \times E}) \otimes p_2^*(\mathcal{L}|_{U \times E}) \xrightarrow{\cong} (\tau \times 1)^*\mathcal{P}|_{V \times U \times E}.$$

Since we have assumed $\tau : U \times V \longrightarrow V$ is given by $\tau(l, v) = l^n \circ v$, the geometric associative r–matrix $\tilde{r}^{\mathcal{V}_1, \mathcal{V}_2}_{h_1, h_2}$ is given by a holomorphic function $\tilde{r}(v_1, v_2; y_1, y_2)$: $\mathsf{Mat}_n(\mathbb{C}) \longrightarrow \mathsf{Mat}_n(\mathbb{C})$ such that the following diagram

$$\begin{tikzcd}
\mathsf{Mat}_n(\mathbb{C}) \arrow[r, "\tilde{r}(v_1, v_2; y_1, y_2)"] \arrow[d, "\mathrm{id}"'] & \mathsf{Mat}_n(\mathbb{C}) \arrow[d, "\mathrm{id}"] \\
\mathsf{Mat}_n(\mathbb{C}) \arrow[r, "\tilde{r}(l^n \circ v_1, l^n \circ v_2; y_1, y_2)"] & \mathsf{Mat}_n(\mathbb{C})
\end{tikzcd}$$

is commutative for all l, v_1, v_2, y_1, y_2. But this implies that $r(v_1, v_2; y_1, y_2) = r(v_2 - v_1; y_1, y_2)$. By [40, Theorem 2] there exists a limit

$$\bar{r}(y_1, y_2) = \lim_{v \to e} \big(\mathrm{pr} \otimes \mathrm{pr}\big) r(v; y_1, y_2)$$

satisfying the classical Yang-Baxter equation. Theorem is proved. □

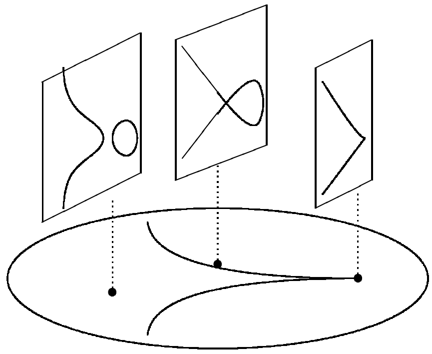
Remark 6.15. Unfortunately, we have not found a "conceptual way" to prove Proposition 6.13, without going to a case-by-case analysis. In particular, we do not know whether this result generalises to the relative case.

It is also natural to conjecture, that a similar statement holds for the other pair of spectral variables (y_1, y_2) and a geometric associative r–matrix can always be transformed into the form $r(v_1, v_2; y_1, y_2) = r(v_2 - v_1; y_2 - y_1)$; compare with the corresponding result for the classical r–matrices [8].

Definition 6.16. Let $r(v; y_1, y_2) \in \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C})$ be a non-degenerate unitary solution of the associative Yang-Baxter equation such that there exists a limit $\bar{r}(y_1, y_2) = \lim_{v \to e} \big(\mathrm{pr} \otimes \mathrm{pr}\big) r(v; y_1, y_2)$. We say that r is of elliptic type if $\bar{r}$ is an elliptic classical r-matrix, trigonometric if $\bar{r}$ is trigonometric and rational if $\bar{r}$ is rational.

It was shown by Polishchuk [40, 41] that in the case of elliptic curves one always gets an associative r–matrix of elliptic type and in the case of Kodaira cycles a solution of trigonometric type.

Corollary 6.17. *Let $E \subset \mathbb{P}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the elliptic fibration given by the equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, $\Delta(g_2, g_3) = g_2^3 - 27g_3^2$ the discriminant of this family.*



This fibration has a section $(g_2, g_3) \mapsto \big((0 : 1 : 0), (g_2, g_3)\big)$. Fix a pair of coprime integers (n, d), then there exists a meromorphic function

$$r = r^{(n,d)} : (\mathbb{C}^4 \times \mathbb{C}^2, 0) \longrightarrow \mathsf{Mat}_n(\mathbb{C}) \times \mathsf{Mat}_n(\mathbb{C})$$

such that

- *The tensor $r := r_t(v_1, v_2; y_1, y_2) := r(v_1, v_2; y_1, y_2; t)$ satisfies the associative Yang-Baxter equation with respect to the variables $v_1, v_2; y_1, y_2$ for any fixed value of $t = (g_2, g_3)$ in a neighbourhood of $(0, 0)$.*
- *The poles of r lie on the divisors $v_1 = v_2$ and $y_1 = y_2$. Moreover, r is holomorphic with respect to the parameters g_1 and g_2.*
- *The tensor r is of elliptic type if $\Delta(g_2, g_3) \neq 0$ and of trigonometric type if the fibre is nodal.*

Remark 6.18. It is natural to conjecture that for any pair of coprime integers (n, d) the geometric r–matrix corresponding to a cuspidal cubic curve is always of rational type.

The goal of the following three sections is to get an explicit form of the geometric r–matrix attached to the Weierstraß fibration $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$ and the pair $(n, d) = (2, 1)$ at any fixed point $(g_2, g_3) \in \mathbb{C}^2$.

7. Elliptic solutions of the associative Yang-Baxter equation

In this section we are going to compute the solution of the associative Yang-Baxter equation and the corresponding classical r–matrix, obtained from the universal family of stable vector bundles of rank two and degree one on a smooth elliptic curve. In [40, Section 2] Polishchuk computed the corresponding triple Massey products using homological mirror symmetry and formulas for higher products in the Fukaya category of an elliptic curve.

It is very instructive, however, to carry out a direct computation of the geometric triple Massey products, independent of homological mirror symmetry for an elliptic curve. This approach allows us to express the resulting associative r–matrix in terms of Jacobi's theta-functions and the corresponding classical r–matrix in terms of the elliptic functions $\mathrm{cn}(z)$, $\mathrm{sn}(z)$ and $\mathrm{dn}(z)$.

In order to carry out the necessary calculations we recall some standard results about holomorphic vector bundles on one-dimensional complex tori, a description of morphisms between them in terms of theta-functions etc.

7.1. **Vector bundles on a one-dimensional complex torus.** We start with some classical results about vector bundles on smooth elliptic curves.

Theorem 7.1 (Atiyah, Theorem 7 in [4]). *Let E be a smooth elliptic curve over $\mathbb{C}$ and $\mathcal{V}$ a vector bundle on E.*

- *If $\mathsf{End}_E(\mathcal{V}) = \mathbb{C}$ then $\gcd\big(\mathrm{rk}(V), \deg(V)\big) = 1$, $\mathcal{V}$ is* stable *and is determined by $\big(\mathrm{rk}(V), \deg(V), \det(\mathcal{V})\big) \in \mathbb{N} \times \mathbb{Z} \times E$, where we use the isomorphism $\mathsf{Pic}^d(E) \cong E$.*
- *If $\mathcal{V}$ is indecomposable and $m = \gcd\big(\mathrm{rk}(V), \deg(V)\big)$ then there exists a unique stable vector bundle $\mathcal{V}'$ such that $\mathcal{V} = \mathcal{V}' \otimes \mathcal{A}_m$, where $\mathcal{A}_m$ is the indecomposable vector bundle of rank m and degree 0 recursively defined by non-split the exact sequences*

$$0 \longrightarrow \mathcal{A}_{m-1} \longrightarrow \mathcal{A}_m \longrightarrow \mathcal{O} \longrightarrow 0, \quad \mathcal{A}_1 = \mathcal{O}.$$

In the complex-analytic case one can give an explicit description of the stable holomorphic vector bundles on a one-dimensional complex torus.

Theorem 7.2 (Oda, Theorem 1.2 in [38]). *Let $E = E_\tau = \mathbb{C}/\langle 1, \tau\rangle$ be an elliptic curve and $\pi_n : E_{n\tau} \to E_\tau$ the étale covering of degree n induced by the inclusion of lattices.*

- *If $\mathcal{V}$ is a stable vector bundle on E_τ of rank n and degree d, then there exists a line bundle $\mathcal{L} \in \mathsf{Pic}^d(E_{n\tau})$ such that $\mathcal{V} \cong \pi_{n*}(\mathcal{L})$. Conversely, if $\gcd(n, d) = 1$, then for any $\mathcal{L} \in \mathsf{Pic}^d(E_{n\tau})$ the vector bundle $\mathcal{V} \cong \pi_{n*}(\mathcal{L})$ is stable.*
- *If $\mathcal{L}, \mathcal{N} \in \mathsf{Pic}^d(E_{n\tau})$ satisfy $\pi_{n*}(\mathcal{L}) \cong \pi_{n*}(\mathcal{N})$, then*
$$(\mathcal{L} \otimes \mathcal{N}^{-1})^{\otimes n} \cong \mathcal{O}_{E_{n\tau}}.$$

A very convenient way to carry out calculations with vector bundles on complex tori is provided by the theory of automorphy factors, see [34] or [37, Section I.2]. Let $\tau \in \mathbb{C}$ be a complex number such that $\mathrm{Im}(\tau) \neq 0$, $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}^2$ the corresponding lattice, $E = E_\tau = \mathbb{C}/\Lambda_\tau$ and $\pi : \mathbb{C} \to E_\tau$ the universal covering of E_τ.

The holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}^*$ which sends z to $\exp(2\pi i z)$ identifies $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}$. It sends τ to q^2, where $q = \exp(\pi i \tau)$. Hence, it induces an isomorphism $E_\tau \cong \mathbb{C}^*/q^2$ where the quotient is formed modulo the multiplicative subgroup generated by q^2.

Holomorphic vector bundles of rank n on E_τ are described by pairs (V, Φ), where $V \cong \mathbb{C}^n$ is a vector space and $\Phi : \mathbb{C} \to \mathsf{GL}(V)$ a holomorphic function (also called *automorphy factor*) such that $\Phi(z+1) = \Phi(z)$. The corresponding holomorphic vector bundle $\mathcal{E}(V,\Phi)$ is defined as the quotient $\mathbb{C} \times V/\sim$, where the equivalence relation is $(z, v) \sim (z+1, v) \sim (z+\tau, \Phi(z)v)$. Using this description, we have the following commutative diagram of complex manifolds

$$\begin{tikzcd}
\mathbb{C} \times V \arrow[r] \arrow[d, "\mathrm{pr}_V"'] & \mathcal{E}(V,\Phi) = \mathbb{C} \times V/\sim \arrow[d] \\
\mathbb{C} \arrow[r, "\pi"] & E_\tau.
\end{tikzcd}$$

In these terms

$$\mathsf{Hom}_{E_\tau}\big(\mathcal{E}(V,\Phi), \mathcal{E}(W,\Psi)\big) = \left\{ A : \mathbb{C} \to \mathsf{Lin}(V,W) \;\middle|\; \begin{array}{l} A \quad \textit{is holomorphic} \\ A(z+1) = A(z) \\ A(z+\tau)\Phi(z) = \Psi(z)A(z) \end{array} \right\}.$$

In particular, if $\Phi(z)$ is an automorphy factor and $A : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ is a holomorphic function such that $A(z+1) = A(z)$, then $\Psi(z) = A(z+\tau)^{-1}\Phi(z)A(z)$ defines an isomorphic vector bundle $\mathcal{E}(\mathbb{C}^n, \Phi) \cong \mathcal{E}(\mathbb{C}^n, \Psi)$.

In the case of line bundles, automorphy factors are holomorphic functions $\varphi : \mathbb{C} \longrightarrow \mathbb{C}^*$ which satisfy $\varphi(z+1) = \varphi(z)$. We simplify notation and write $\mathcal{L}(\varphi) := \mathcal{E}(\mathbb{C}, \varphi)$.

Line bundles of degree zero can be given by constant automorphy factors. Because the function $a(z) = \exp(2\pi i z)$ satisfies $a(z+1) = a(z)$ and $a(z+\tau) = q^2 a(z)$ with $q = \exp(\pi i \tau)$ as above, the constants $\varphi \in \mathbb{C}^*$ and $q^2\varphi$ define isomorphic line bundles on E_τ. In fact, the map

$$E_\tau := \mathbb{C}^*/q^2 \longrightarrow \mathsf{Pic}^0(E_\tau)$$

which sends $\varphi \in \mathbb{C}^*$ to $\mathcal{L}(\varphi) \in \mathsf{Pic}^0(E_\tau)$, is an isomorphism.

To describe line bundles of non-zero degree, we denote $p_0 = \dfrac{1+\tau}{2} \in \mathbb{C}$ and let $[p_0]$ be the corresponding point in E_τ. The automorphy factor

$$\varphi_0(z) = \exp(-\pi i \tau - 2\pi i z)$$

satisfies $\mathcal{L}(\varphi_0) = \mathcal{O}_E([p_0])$. To see this, recall that, by the definition,

$$H^0\big(\mathcal{L}(\varphi_0)\big) = \mathsf{Hom}\big(\mathcal{L}(1), \mathcal{L}(\varphi_0)\big) = \left\{ f : \mathbb{C} \to \mathbb{C} \;\middle|\; \begin{array}{l} f \quad \textit{is holomorphic} \\ f(z+1) = f(z) \\ f(z+\tau) = \varphi_0(z)f(z) \end{array} \right\}$$

and that this one-dimensional vector space is generated by the basic theta function

$$\theta(z|\tau) = \theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z),$$

see for example [36]. It is well-known that $\theta(z|\tau)$ vanishes at $p_0 = \frac{1+\tau}{2}$ and that this is the only zero in the fundamental parallelogram of Λ_τ.

Because $\theta\left(z + \frac{1+\tau}{2} - x \middle| \tau\right)$ has its unique zero at $[x] \in E_\tau$, we obtain

$$\mathcal{O}_E(x) \cong \mathcal{L}\left(\varphi_0\left(z + \frac{1+\tau}{2} - x\right)\right).$$

This gives a complete description of $\mathsf{Pic}^1(E_\tau)$.

Finally, any line bundle of degree d can be written as $\mathcal{O}_{E_\tau}\big([(d-1)p_0] + [p_0 - x]\big)$ for some point $x \in E_\tau$. To complete the description of $\mathsf{Pic}(E_\tau)$, it remains to observe that

$$\mathcal{O}_E\big([p_0 - x] + (d-1)[p_0]\big) \cong \mathcal{L}\big(t_x^*\varphi_0 \cdot \varphi_0^{d-1}\big),$$

where $t_x^*\varphi_0(z) := \varphi_0(z+x)$.

The following properties of vector bundles on a one-dimensional complex torus E_τ are very useful in our calculations below. They follow by a direct calculation.

(1) $\mathcal{E}(V,\Phi) \otimes \mathcal{E}(W,\Psi) \cong \mathcal{E}(V\otimes W, \Phi\otimes\Psi)$.

(2) If $\pi_n : E_{n\tau} \to E_n$ is the étale covering given by the inclusion of lattices $\Lambda_{n\tau} \subset \Lambda_\tau$, then $\pi_n^*\big(\mathcal{E}(V,\Phi)\big) \cong \mathcal{E}(V,\widetilde{\Phi})$, where

$$\widetilde{\Phi}(z) := \Phi(z + (n-1)\tau)\cdot \ldots \cdot \Phi(z+\tau)\Phi(z).$$

(3) The direct image $\pi_{n*}\big(\mathcal{E}(V,\Phi)\big) \cong \mathcal{E}(V^{\oplus n}, \widetilde{\Phi})$ of a vector bundle is given by[1]

$$\widetilde{\Phi} = \begin{pmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ \Phi & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Our next aim is to get a formula for an automorphy factor describing the set of stable vector bundles of rank n and degree d on E, where $\gcd(n,d) = 1$. Interpreting Oda's description from Theorem 7.2 in terms of automorphy factors we immediately obtain the following proposition.

[1] we thank to Oleksandr Iena for helping us at this point

Proposition 7.3. *Let* $(n,d) \in \mathbb{N}\times\mathbb{Z}$ *and* $\tilde{\varphi}_{n,d,x}(z) := \exp(-\pi i n d\tau - 2\pi i d z - 2\pi i x)$, *where* $x \in \mathbb{C}/\langle 1, n\tau\rangle$ *and* $\gcd(n,d)=1$. *Then, the automorphy factor*

$$\widetilde{\Phi}_{n,d,x}(z) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \tilde{\varphi}_{n,d,x} & 0 & 0 & \dots & 0 \end{pmatrix}$$

describes the set of stable vector bundles of rank n *and degree* d *on the complex torus* E_τ. *Note that* $\mathcal{E}\big(\mathbb{C}^n, \widetilde{\Phi}_{n,d,x}\big) \cong \mathcal{E}\big(\mathbb{C}^n, \widetilde{\Phi}_{n,d,x'}\big)$ *if and only if* $x - x' \in \Lambda_\tau$.

However, this automorphy factor is not compatible with the action of the Jacobian $\mathsf{Pic}^0(E_\tau)$. In order to overcome this problem we denote $q_{\frac{x}{n}} = \exp(-\frac{2\pi i x}{n})$ and let

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q_{\frac{x}{n}} & 0 & \dots & 0 \\ 0 & 0 & q^2_{\frac{x}{n}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q^{n-1}_{\frac{x}{n}} \end{pmatrix}.$$

Then

$$\Phi_{n,d,x}(z) := A^{-1}\widetilde{\Phi}_{n,d,x}(z)A = \begin{pmatrix} 0 & q_{\frac{x}{n}} & 0 & \dots & 0 \\ 0 & 0 & q_{\frac{x}{n}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q_{\frac{x}{n}} \\ q_{\frac{x}{n}}\varphi_{n,d} & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\varphi_{n,d}(z) = \exp(-\pi i n d\tau - 2\pi i d z)$.

Corollary 7.4. *We have* $\exp(-2\pi i y)\Phi_{n,d,x}(z) = \Phi_{n,d,x+ny}(z)$, *hence the trivialisation of the universal family* $\mathcal{E}(V, \Phi_{n,d,x})_{x\in E_\tau}$ *induced by the isomorphism*

$$\pi^*\big(\mathcal{E}(\mathbb{C}^n, \Phi_{n,d,x}(z))\big) \longrightarrow \mathbb{C}\times\mathbb{C}^n$$

is compatible with the action of $\mathsf{Pic}^0(E_\tau)$.

This leads to the following result about the associative r-matrices of elliptic type.

Corollary 7.5. *Let* r *be an associative* r*–matrix obtained from a universal family of stable vector bundles on an elliptic curve, then it always holds* $r(v_1, v_2; y_1, y_2) \sim r'(v_2 - v_1; y_1, y_2)$, *where* r' *is an equivalent associative* r*–matrix.*

7.2. **Rules to calculate the evaluation and the residue maps.** Let $E = E_\tau$ be a complex torus, Ω_E the sheaf of holomorphic differential 1-forms, $\omega = dz \in H^0(\Omega_E)$ a global section. Recall that for two non-isomorphic stable holomorphic vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ of rank n and degree d and two distinct smooth points y_1 and y_2 we have defined a holomorphic function $\tilde{r}(v_1, v_2; y_1, y_2) : \mathsf{Mat}_n(\mathbb{C}) \longrightarrow \mathsf{Mat}_n(\mathbb{C})$ using the following commutative diagram

$$\begin{tikzcd}
 & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \arrow[dl, "\mathrm{res}_{y_1}^{\mathcal{V}_1,\mathcal{V}_2}"'] \arrow[dr, "\mathrm{ev}_{y_2}^{\mathcal{V}_1,\mathcal{V}_2}"] & \\
\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \arrow[rr, "\tilde{r}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}"] \arrow[d, "\cong"'] & & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \arrow[d, "\cong"] \\
\mathsf{Mat}_n(\mathbb{C}) \arrow[rr, "\tilde{r}(v_1,v_2;y_1,y_2)"] & & \mathsf{Mat}_n(\mathbb{C}),
\end{tikzcd}$$

where $\mathrm{res}_{y_1}^{\mathcal{V}_1,\mathcal{V}_2}$ and $\mathrm{ev}_{y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ are canonical maps and the vertical isomorphisms are induced by a trivialisation of the universal bundle $\mathcal{P}(n, d) \in \mathsf{VB}\big(E \times M_E(n, d)\big)$.

Recall that the exact sequence $0 \longrightarrow \Omega_E \longrightarrow \Omega_E(y) \xrightarrow{\mathrm{res}_y} \mathbb{C}_y \longrightarrow 0$ and the differential form ω induce a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2 \otimes \Omega_E(y)\big) \arrow[r] & \mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2 \otimes \mathbb{C}_y) \arrow[r] & 0 \\
 & \mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2(y)) \arrow[u, "\cdot\omega"] \arrow[r, "\mathrm{res}_y^{\mathcal{V}_1,\mathcal{V}_2}"] & \mathsf{Hom}_E(\mathcal{V}_1 \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y). \arrow[u, "\cong"'] &
\end{tikzcd}$$

In order to compute the horizontal map $\mathrm{res}_y^{\mathcal{V}_1,\mathcal{V}_2}$, observe that we have a commutative diagram of coherent sheaves

$$\begin{tikzcd}
\mathcal{H}om_E\big(\mathcal{V}_1, \mathcal{V}_2 \otimes \Omega_E(y)\big) \arrow[r, "\cong"] & \mathcal{H}om_E(\mathcal{V}_1, \mathcal{V}_2) \otimes \Omega_E \otimes \mathcal{O}_E(y) \arrow[d, "\mathrm{res}'_y"] \\
\mathcal{H}om_E(\mathcal{V}_1, \mathcal{V}_2(y)) \arrow[u, "\cdot\omega"] \arrow[r, "\widetilde{res}_y^{\mathcal{V}_1,\mathcal{V}_2}"] & (i_y)_*\mathcal{H}om_E(\mathcal{V}_1 \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y),
\end{tikzcd}$$

where, on the level of presheaves, the map

$$\mathrm{res}'_y : \mathcal{H}om_E(\mathcal{V}_1, \mathcal{V}_2) \otimes \Omega_E \otimes \mathcal{O}_E(y) \longrightarrow (i_y)_*\mathcal{H}om_E(\mathcal{V}_1 \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y)$$

sends $F \otimes \omega \otimes f$ to $\bigl(v \mapsto \mathrm{res}_y(f\omega)F(v)\bigr)$. Moreover $H^0(\widetilde{res}_y^{\mathcal{V}_1,\mathcal{V}_2}) = \mathrm{res}_y^{\mathcal{V}_1,\mathcal{V}_2}$ and we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Hom}_E\bigl(\mathcal{V}_1, \mathcal{V}_2(y)\bigr) \arrow[r, "\mathrm{res}_y^{\mathcal{V}_1,\mathcal{V}_2}"] \arrow[d] & \mathsf{Hom}_E(\mathcal{V}_1 \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y) \\
\mathsf{Hom}_{\mathcal{O}_y}\bigl((\mathcal{V}_1)_y, (\mathcal{V}_2)_y\bigr) \otimes_{\mathcal{O}_y} (\Omega_E)_y \otimes_{\mathcal{O}_y} \bigl(\mathcal{O}(y)\bigr)_y \arrow[ur, "\mathrm{res}'_y"'] &
\end{tikzcd}$$

where $\mathcal{F}_y$ denotes the stalk of a coherent sheaf $\mathcal{F}$ at the point y.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ have the automorphy factors $\Phi_1(z)$ and $\Phi_2(z)$ respectively, and let

$$\psi_y(z) = -\exp(-2\pi i z + 2\pi i y - 2\pi i \tau)$$

be the automorphy factor of $\mathcal{O}_E(y)$. The theta function $\theta_y(z) = \theta\left(z + \frac{1+\tau}{2} - y \,\middle|\, \tau\right)$ is a global section of $\mathcal{O}_E(y)$. Using this notation, we can describe the space of morphisms from $\mathcal{V}_1$ to $\mathcal{V}_1(y)$ as follows

$$\mathsf{Hom}_E\bigl(\mathcal{V}_1, \mathcal{V}_2(y)\bigr) = \left\{ F : \mathbb{C}^n \to \mathbb{C}^n \;\middle|\; \begin{array}{l} F \quad \textit{is holomorphic} \\ F(z+1) = F(z) \\ F(z+\tau)\Phi_1(z) = \psi_y(z)\Phi_2(z)F(z) \end{array} \right\}$$

Using the trivialisations of $\mathcal{V}_1$ and $\mathcal{V}_2$ induced by the maps $\pi^*\mathcal{V}_i \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}^n$ we obtain that

$$\mathrm{res}_y^{\mathcal{V}_1,\mathcal{V}_2} : \mathsf{Hom}_E\bigl(\mathcal{V}_1, \mathcal{V}_2(y)\bigr) \longrightarrow \mathsf{Mat}_n(\mathbb{C})$$

is given by

$$F(z) \mapsto \mathrm{res}_y\left(\frac{F(z)}{\theta_y(z)}dz\right) = \frac{F(y)}{\theta'_y(y)} = \frac{F(y)}{\theta'\left(\frac{1+\tau}{2}|\tau\right)}$$

In a similar way one can show that the evaluation map

$$\mathrm{ev}_{y_2}^{\mathcal{V}_1,\mathcal{V}_2} : \mathsf{Hom}_E\bigl(\mathcal{V}_1, \mathcal{V}_2(y_1)\bigr) \longrightarrow \mathsf{Mat}_n(\mathbb{C})$$

is given by the formula

$$F(z) \mapsto \frac{F(y_2)}{\theta_{y_1}(y_2)} = \frac{F(y_2)}{\theta(y + \frac{1+\tau}{2}|\tau)}.$$

Remark 7.6. Note that the resulting map $\tilde{r}(v_1, v_2; y_1, y_2) : \mathsf{Mat}_n(\mathbb{C}) \to \mathsf{Mat}_n(\mathbb{C})$ is invariant under the rescaling $\theta_y(z) \mapsto c\theta_y(z), c \in \mathbb{C}^*$ of the global section of the line bundle $\mathcal{O}_E(y)$.

7.3. **Calculation of the elliptic r–matrix corresponding to $M_E(2,1)$.** Let $\mathcal{V}_1 = \mathcal{V}_{x_1}$ and $\mathcal{V}_2 = \mathcal{V}_{x_2}$ be two non-isomorphic simple vector bundles of rank two and degree one on the elliptic curve $E = E_\tau$, y_1 and y_2 two distinct points. In what follows we denote $q = \exp(\pi i\tau)$, $q_x = \exp(-\pi i x)$, $e(z) = \exp(-2\pi i z)$, $\varphi(z) = e(z+\tau)$, $x = x_2 - x_1$ and $y = y_2 - y_1$.

As we have seen in the previous subsection, one can write $\mathcal{V}_i \cong \mathcal{E}\big(\mathbb{C}^2, \Phi_{2,1,x_i}(z)\big)$, where

$$\Phi_{2,1,x_i}(z) = q_{x_i}\begin{pmatrix} 0 & 1 \\ \varphi(z) & 0 \end{pmatrix} =: q_{x_i}\Phi(z),$$

and the line bundle $\mathcal{O}_E(y_1)$ corresponds to the automorphy factor

$$\psi_{y_1}(z) = -e(z+\tau-y_1).$$

Recall that $\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) =$

$$\left\{ A(z) = \begin{pmatrix} u(z) & v(z) \\ w(z) & t(z) \end{pmatrix} \middle| \; A(z+1) = A(z),\; A(z+\tau)\Phi(z) = q_x\psi_{y_1}(z)\Phi(z)A(z) \right\}$$

This leads to two systems of functional equations

$$\begin{cases} u(z+\tau) &= q_x\psi_{y_1}(z)t(z) \\ t(z+\tau) &= q_x\psi_{y_1}(z)u(z) \end{cases} \quad \text{and} \quad \begin{cases} \varphi(z)v(z+\tau) &= q_x\psi_{y_1}(z)w(z) \\ w(z+\tau) &= q_x\varphi(z)\psi_{y_1}(z)v(z) \end{cases}$$

which are equivalent to

$$\begin{cases} u(z+2\tau) &= a(z)u(z) \\ u(z+1) &= u(z) \\ t(z) &= \dfrac{u(z+\tau)}{q_x\psi_{y_1}(z)} \end{cases} \quad \text{and} \quad \begin{cases} v(z+2\tau) &= b(z)v(z) \\ v(z+1) &= v(z) \\ w(z) &= \dfrac{\varphi(z)}{q_x\psi_{y_1}(z)}v(z+\tau) \end{cases}$$

where

$$a(z) = \exp\left(-2\pi i\tau - 2\pi i\left(z + \frac{x+\tau}{2} - y_1\right)\right)^2 \quad \text{and}$$
$$b(z) = \exp\left(-2\pi i\tau - 2\pi i\left(z + \frac{x}{2} - y_1\right)\right)^2.$$

Lemma 7.7. *Let $E = E_\tau$ be an elliptic curve, $\varphi_0(z) = \exp(-\pi i\tau - 2\pi i z)$, $l \in \mathbb{N}$. Then $H^0\big(\mathcal{L}(\varphi_0^l)\big)$ has a basis $\left\{\theta\left[\frac{a}{l}, 0\right](lz|l\tau) \mid 0 \le a < l, a \in \mathbb{Z}\right\}$, where we use Mumford's notation*

$$\theta[a,b](z|\tau) = \sum_{n\in\mathbb{Z}} \exp\big(\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)\big).$$

In the particular case of bundles of rank two and degree one it is convenient to use instead the four classical theta-functions of Jacobi:

$$\begin{aligned}
\theta_1(z|\tau) &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin\bigl((2n+1)\pi z\bigr),\\
\theta_2(z|\tau) &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n(n+1)} \cos\bigl((2n+1)\pi z\bigr),\\
\theta_3(z|\tau) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2\pi n z),\\
\theta_4(z|\tau) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2\pi n z).
\end{aligned}$$

Remark 7.8. In Mumford's notation it holds:

$$\begin{aligned}
\theta_1(z|\tau) &= -\theta\left[\tfrac{1}{2},\tfrac{1}{2}\right](z|\tau) & \qquad \theta_2(z|\tau) &= \theta\left[\tfrac{1}{2},0\right](z|\tau)\\
\theta_3(z|\tau) &= \theta\left[0,0\right](z|\tau) & \qquad \theta_4(z|\tau) &= \theta\left[0,\tfrac{1}{2}\right](z|\tau).
\end{aligned}$$

In what follows we shall express all our computations in terms of Jacobi's theta-functions. From Lemma 7.7 and Remark 7.8 we immediately obtain:

Corollary 7.9. *If we let*

$$\begin{aligned}
u_1(z) &= \theta_3\left(2\left(z - y_1 + \frac{x+\tau}{2}\right)\Big| 4\tau\right) & \qquad v_1(z) &= \theta_3\left(2\left(z - y_1 + \frac{x}{2}\right)\Big| 4\tau\right)\\
u_2(z) &= \theta_2\left(2\left(z - y_1 + \frac{x+\tau}{2}\right)\Big| 4\tau\right) & \qquad v_2(z) &= \theta_2\left(2\left(z - y_1 + \frac{x}{2}\right)\Big| 4\tau\right)
\end{aligned}$$

and

$$F_k(z) = \begin{pmatrix} u_k(z) & 0 \\ 0 & \dfrac{u_k(z+\tau)}{q_x \psi_{y_1}(z)} \end{pmatrix}, G_k(z) = \begin{pmatrix} 0 & v_k(z) \\ \dfrac{\varphi(z)}{q_x \psi_{y_1}(z)} v_k(z) & 0 \end{pmatrix}, \quad k = 1, 2,$$

then $F_1(z), F_2(z), G_1(z), G_2(z)$ *is a basis of* $\mathsf{Hom}_E\bigl(\mathcal{V}_1, \mathcal{V}_2(y_1)\bigr)$.

The following proposition sums up the main properties of Jacobi's theta-functions which we need in our calculation of the associative r–matrix corresponding to the universal family of stable vector bundles of rank two and degree one.

Proposition 7.10 (see [21] and Section I.4 in [32])**.** *The transformation rules for shifts of theta-functions are given by the table*

$\theta(z)$	$\theta(-z)$	$\theta(z+1)$	$\theta(z+\tau)$	$\theta(z+1+\tau)$	$\theta(z+\frac{1}{2})$	$\theta(z+\frac{\tau}{2})$
$\theta_1(z)$	$-\theta_1(z)$	$-\theta_1(z)$	$-p(z)\theta_1(z)$	$p(z)\theta_1(z)$	$\theta_2(z)$	$iq(z)\theta_4(z)$
$\theta_2(z)$	$\theta_2(z)$	$-\theta_2(z)$	$p(z)\theta_2(z)$	$-p(z)\theta_2(z)$	$-\theta_1(z)$	$q(z)\theta_3(z)$
$\theta_3(z)$	$\theta_3(z)$	$\theta_3(z)$	$p(z)\theta_3(z)$	$p(z)\theta_3(z)$	$\theta_4(z)$	$q(z)\theta_2(z)$
$\theta_4(z)$	$\theta_4(z)$	$\theta_4(z)$	$-p(z)\theta_4(z)$	$-p(z)\theta_4(z)$	$\theta_3(z)$	$iq(z)\theta_1(z)$

where $p(z) = \exp\big(-\pi i(2z+\tau)\big)$ *and* $q(z) = \exp\big(-\pi i(z+\frac{\tau}{4})\big)$. *Moreover, Jacobi's theta-functions satisfy the so-called Watson's determinantal identities:*

$$\begin{aligned}
\theta_3(2x|2\tau)\theta_2(2y|2\tau) - \theta_3(2y|2\tau)\theta_2(2x|2\tau) &= \theta_1(x+y|\tau)\theta_1(x-y|\tau),\\
\theta_1(2x|2\tau)\theta_4(2y|2\tau) - \theta_1(2y|2\tau)\theta_4(2x|2\tau) &= \theta_2(x+y|\tau)\theta_1(x-y|\tau),\\
\theta_1(2x|2\tau)\theta_4(2y|2\tau) + \theta_1(2y|2\tau)\theta_4(2x|2\tau) &= \theta_1(x+y|\tau)\theta_2(x-y|\tau),\\
\theta_4(2x|2\tau)\theta_4(2y|2\tau) - \theta_1(2y|2\tau)\theta_1(2x|2\tau) &= \theta_3(x+y|\tau)\theta_4(x-y|\tau),\\
\theta_4(2x|2\tau)\theta_4(2y|2\tau) + \theta_1(2y|2\tau)\theta_1(2x|2\tau) &= \theta_4(x+y|\tau)\theta_3(x-y|\tau).
\end{aligned}$$

By Corollary 7.9 any element of $\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big)$ can be written as a sum

$$A(z) = \alpha F_1(z) + \beta F_2(z) + \gamma G_1(z) + \delta G_2(z)$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. In order to calculate the geometric associative r–matrix $r(x_1, x_2; y_1, y_2)$ we have to solve the system of linear equations

$$\mathrm{res}_{y_1}\big(A(z)\big) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the linear map $\tilde{r}(x_1, x_2; y_1, y_2) : \mathsf{Mat}_2(\mathbb{C}) \to \mathsf{Mat}_2(\mathbb{C})$ is given by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\mathrm{res}_{y_1}^{-1}} A(z) \xmapsto{\mathrm{ev}_{y_2}} \frac{1}{\theta_3(y+\frac{1+\tau}{2}|\tau)} A(y_2).$$

It is easy to see that the system of linear equations

$$\mathrm{res}_{y_1}\big(\alpha F_1(z) + \beta F_2(z) + \gamma G_1(z) + \delta G_2(z)\big) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

splits into two independent systems

$$\mathrm{res}_{y_1}\big(F(z)\big) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad \mathrm{res}_{y_1}\big(G(z)\big) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

where $F(z) = \alpha F_1(z) + \beta F_2(z)$ and $G(z) = \gamma G_1(z) + \delta G_2(z)$.

Computation of the "diagonal terms". The system of linear equations

$$\mathrm{res}_{y_1}\big(F(z)\big) := \frac{1}{\theta_3'(\frac{1+\tau}{2})} F(y_1) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

reads as

$$\begin{cases} \theta_3(x+\tau|4\tau)\alpha + \theta_2(x+\tau|4\tau)\beta &= \theta_3'(\frac{1+\tau}{2}|\tau)a \\ \theta_3(x+3\tau|4\tau)\alpha + \theta_2(x+3\tau|4\tau)\beta &= -e(\tau+\frac{x}{2})\theta_3'(\frac{1+\tau}{2}|\tau)d. \end{cases}$$

By a Watson's identity the determinant of this system is

$$\Delta_1 = \begin{vmatrix} \theta_3(x+\tau|4\tau) & \theta_2(x+\tau|4\tau) \\ \theta_3(x+3\tau|4\tau) & \theta_2(x+3\tau|4\tau) \end{vmatrix} = \theta_1(x+2\tau|2\tau)\theta_1(-\tau|2\tau) =$$

$$= e(x+\tau)\theta_1(x|2\tau)\theta_1(\tau|2\tau)$$

and we obtain:

$$\begin{cases} \alpha & = & \frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\Delta_1}\bigl(\theta_2(x+3\tau|4\tau)a + e(\tau+\frac{x}{2})\theta_2(x+\tau|4\tau)d\bigr) \\ \beta & = & -\frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\Delta_1}\bigl(\theta_3(x+3\tau|4\tau)a + e(\tau+\frac{x}{2})\theta_3(x+\tau|4\tau)d\bigr). \end{cases}$$

This implies:

$$\tilde{r}(x_1,x_2;y_1,y_2)\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] =$$

$$\frac{\theta_3'(\frac{1+\tau}{2})|\tau}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_1} \times \left[p_1(z)\begin{pmatrix} \theta_3(2y+x+\tau|4\tau) & 0 \\ 0 & -\frac{\theta_3(2y+x+3\tau|4\tau)}{e(x/2+y+\tau)} \end{pmatrix}\right.$$

$$\left. -p_2(z)\begin{pmatrix} \theta_2(2y+x+\tau|4\tau) & 0 \\ 0 & -\frac{\theta_2(2y+x+3\tau|4\tau)}{e(x/2+y+\tau)} \end{pmatrix}\right].$$

where

$$p_1(z) = \theta_2(x+3\tau|4\tau)a + e(x/2+\tau)\theta_2(x+\tau|4\tau)d,$$
$$p_2(z) = \theta_3(x+3\tau|4\tau)a + e(x/2+\tau)\theta_3(x+\tau|4\tau)d.$$

In order to calculate the "diagonal part" of the corresponding tensor $r(x_1,x_2;y_1,y_2)$ we use the inverse of the canonical isomorphism

$$\mathsf{Mat}_2(\mathbb{C}) \otimes \mathsf{Mat}_2(\mathbb{C}) \longrightarrow \mathsf{Lin}\bigl(\mathsf{Mat}_2(\mathbb{C}), \mathsf{Mat}_2(\mathbb{C})\bigr)$$

given by the formula $X \otimes Y \mapsto \mathrm{tr}(XY)Y$. It is easy to see that under the map

$$\mathsf{Lin}\bigl(\mathsf{Mat}_2(\mathbb{C}), \mathsf{Mat}_2(\mathbb{C})\bigr) \longrightarrow \mathsf{Mat}_2(\mathbb{C}) \otimes \mathsf{Mat}_2(\mathbb{C})$$

a linear function $e_{ij} \mapsto \alpha_{ij}^{kl}e_{kl}, \alpha_{ij}^{kl} \in \mathbb{C}^*$ corresponds to the tensor $\alpha_{ij}^{kl}e_{ji} \otimes e_{kl}$.

Again, Watson's identities imply:

- The coefficient at $e_{11} \otimes e_{11}$ is

$$\theta_3(2y+x+\tau|4\tau)\theta_2(x+3\tau|4\tau) - \theta_2(2y+x+\tau|4\tau)\theta_3(x+3\tau|4\tau) =$$
$$\theta_1(x+y+2\tau|2\tau)\theta_1(y-\tau|2\tau).$$

- The coefficient at $e_{22} \otimes e_{22}$ is

$$e(-y)\bigl(\theta_3(x+\tau|4\tau)\theta_2(2y+x+3\tau|4\tau) - \theta_2(x+\tau|4\tau)\theta_3(2y+x+3\tau|4\tau) =\bigr)$$
$$= \theta_1(x+y+2\tau|2\tau)\theta_1(y-\tau|2\tau).$$

- The coefficient at $e_{22} \otimes e_{11}$ is

$$e(x/2+\tau)\bigl(\theta_2(x+\tau|4\tau)\theta_3(2y+x+\tau) - \theta_3(x+\tau|4\tau)\theta_2(2y+x+\tau)\bigr) =$$
$$= e(x/2+\tau)\theta_1(y+x+\tau|2\tau)\theta_1(y|2\tau).$$

- The coefficient at $e_{11} \otimes e_{22}$ is

$$e(-y-x/2-\tau)\bigl(\theta_2(2y+x+3\tau|4\tau)\theta_3(x+3\tau|4\tau) - \theta_3(2y+x+3\tau|4\tau)\theta_3(x+3\tau|4\tau)\bigr) =$$

$$= e(x/2+\tau)\theta_1(y+x+\tau|2\tau)\theta_1(y|2\tau).$$

Now observe that

$$\theta_1(x+y+2\tau|2\tau)\theta_1(y-\tau|2\tau) = ie(x+y/2+5\tau/4)\theta_1(x+y|2\tau)\theta_4(y|2\tau)$$

and

$$e(x/2+\tau)\theta_1(y+x+\tau|2\tau)\theta_1(y|2\tau) = ie(x+y/2+5\tau/4)\theta_4(x+y|2\tau)\theta_1(y|2\tau).$$

Hence, the "diagonal part" of $r(x_1,x_2;y_1,y_2)$ is

$C\big[\theta_1(x+y|2\tau)\theta_4(y|2\tau)(e_{11}\otimes e_{11}+e_{22}\otimes e_{22})+\theta_4(x+y|2\tau)\theta_1(y|2\tau)(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})\big]$,

where

$$C = \frac{\theta_3'(\frac{1+\tau}{2}|\tau)e(x+\frac{y}{2}+\frac{5\tau}{4})}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_1}.$$

¿From the identities $\theta_3(y+\frac{1+\tau}{2}|\tau) = i\exp(-\pi i(y+\tau/4))\theta_1(y|\tau)$ and $\theta_1(0|\tau)=0$ it follows: $\theta_3'(\frac{1+\tau}{2}|\tau) = ie(\frac{\tau}{8})\theta'(0|\tau)$. Using the transformation rules from Proposition 7.10 we get:

$$C = \frac{\theta_1'(0|\tau)}{\theta_4(0|2\tau)\theta_1(x|2\tau)\theta_1(y|\tau)} = \frac{\theta_1'(0|\tau)}{\theta_1\left(\frac{x}{2}\middle|\tau\right)\theta_2\left(\frac{x}{2}\middle|\tau\right)\theta_1\left(y|\tau\right)},$$

where we have used the Landen's transform

$$\theta_4(0|2\tau)\theta_1(2x|2\tau) = \theta_1(x|\tau)\theta_2(x|\tau).$$

It remains to observe that $A(e_{11}\otimes e_{11}+e_{22}\otimes e_{22})+B(e_{11}\otimes e_{22}+e_{22}\otimes e_{11}) =$

$$\frac{1}{2}(A+B)(e_{11}+e_{22})\otimes(e_{11}+e_{22}) - \frac{1}{2}(A-B)(e_{11}-e_{22})\otimes(e_{11}-e_{22}),$$

and that by Watson's identities it holds

$$\theta_1(x+y|2\tau)\theta_4(y|2\tau)+\theta_4(x+y|2\tau)\theta_1(y|2\tau) = \theta_1\left(y+\frac{x}{2}\middle|\tau\right)\theta_2\left(\frac{x}{2}\middle|\tau\right)$$

and

$$\theta_1(x+y|2\tau)\theta_4(y|2\tau)-\theta_4(x+y|2\tau)\theta_1(y|2\tau) = \theta_2\left(y+\frac{x}{2}\middle|\tau\right)\theta_1\left(\frac{x}{2}\middle|\tau\right),$$

so the contribution of the "diagonal terms" is

$$\frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\frac{\theta_1(y+\frac{x}{2}|\tau)}{\theta_1(\frac{x}{2}|\tau)}\mathbb{1}\otimes\mathbb{1} + \frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\frac{\theta_2(y+\frac{x}{2}|\tau)}{\theta_2(\frac{x}{2}|\tau)}h\otimes h.$$

Contribution of the "skew terms". We have to solve the system of linear equations

$$\mathrm{res}_{y_1}\big(\gamma G_1(z)+\delta G_2(z)\big) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

In an explicit form this system reads as

$$\left\{\begin{array}{rcl} \theta_3(x|4\tau)\gamma + \theta_2(x|4\tau)\delta & = & \theta_3'(\frac{1+\tau}{2}|\tau)b \\ \theta_3(x+2\tau|4\tau)\gamma + \theta_2(x+2\tau|4\tau)\delta & = & -e(x/2-y_1)\theta_3'(\frac{1+\tau}{2}|\tau)c. \end{array}\right.$$

By Watson's formulas the determinant of this system is

$$\Delta_2 = \begin{vmatrix} \theta_3(x|4\tau) & \theta_2(x|4\tau) \\ \theta_3(x+2\tau|4\tau) & \theta_2(x+2\tau|4\tau) \end{vmatrix} = -\theta_1(x+\tau|2\tau)\theta_1(\tau|2\tau).$$

Hence, the solution of this system of equations is

$$\left\{\begin{array}{rcl} \gamma & = & \theta_3'(\frac{1+\tau}{2}|\tau)\big(\theta_2(x+2\tau|4\tau)b + e(x/2-y_1)\theta_2(x|4\tau)c\big) \\ \delta & = & \theta_3'(\frac{1+\tau}{2}|\tau)\big(\theta_3(x+2\tau|4\tau)b + e(x/2-y_1)\theta_3(x|4\tau)c.\big) \end{array}\right.$$

As a result, we obtain:

$$\tilde{r}(x_1,x_2;y_1,y_2)\left[\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right] =$$

$$\frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_2} \times \left[q_1(z)\begin{pmatrix} 0 & \theta_3(2y+x|4\tau) \\ -e(y_1-x/2)\theta_3(2y+x+2\tau|4\tau) & 0 \end{pmatrix}\right.$$

$$\left. -q_2(z)\begin{pmatrix} 0 & \theta_2(2y+x|4\tau) \\ -e(y_1-x/2)\theta_2(2y+x+2\tau|4\tau) & 0 \end{pmatrix}\right],$$

where

$$\begin{aligned} q_1(z) &= \theta_2(x+2\tau|4\tau)b + e(x/2-y_1)\theta_2(x|4\tau)c, \\ q_2(z) &= \theta_3(x+2\tau|4\tau)b + e(x/2-y_1)\theta_3(x|4\tau)c \end{aligned}$$

Again, Watson's identities imply:

- The coefficient at $e_{21} \otimes e_{12}$ is

$$\theta_2(x+2\tau|4\tau)\theta_3(2y+x|4\tau) - \theta_3(x+2\tau|4\tau)\theta_2(2y+x|4\tau) =$$
$$= e\left(\frac{1}{2}(x+\tau)\right)\theta_4(x+y|2\tau)\theta_4(y|2\tau).$$

- The coefficient at $e_{12} \otimes e_{21}$ is

$$\theta_3(x|4\tau)\theta_2(2y+x+2\tau|4\tau) - \theta_2(x|4\tau)\theta_3(2y+x+2\tau|4\tau) =$$
$$= e\left(\frac{1}{2}(x+\tau)+y\right)\theta_4(x+y|2\tau)\theta_4(y|2\tau).$$

- The coefficient at $e_{12} \otimes e_{12}$ is

$$e(x/2-y_1)\big(\theta_2(x|4\tau)\theta_3(2y+x+2\tau|4\tau) - \theta_3(x|4\tau)\theta_2(2y+x+2\tau|4\tau)\big) =$$
$$= e(x/2-y_1)\theta_1(x+y|2\tau)\theta_1(y|2\tau).$$

- The coefficient at $e_{21} \otimes e_{21}$ is

$$e(y_1-x/2)\big(\theta_3(x+2\tau|4\tau)\theta_2(2y+x+2\tau|4\tau) - \theta_2(x+2\tau|4\tau)\theta_3(2y+x+2\tau|4\tau)\big) =$$

$$= e(y_2 + x/2 + \tau)\theta_1(y + x|2\tau)\theta_1(y|2\tau).$$

Note that the coefficients of the tensors $e_{12} \otimes e_{12}$ and $e_{21} \otimes e_{21}$ are not functions of $y = y_2 - y_1$. In order to overcome this problem we take $\phi(y) = \begin{pmatrix} e(-y/2) & 0 \\ 0 & e(\tau/4) \end{pmatrix}$ and consider the gauge transformation

$$r(x; y_1, y_2) \mapsto \big(\phi(y_1) \otimes \phi(y_2)\big) r(x; y_1, y_2) \big(\phi^{-1}(y_1) \otimes \phi^{-1}(y_2)\big).$$

It is easy to see that the "diagonal tensors" $e_{kk} \otimes e_{ll} (k, l = 1, 2)$ remain unchanged (and, in particular, this gauge transformation does not influence the final answer for the "diagonal terms" obtained before) and the transformation rule for the "skew tensors" is the following:

$$\begin{array}{lcl} e_{12} \otimes e_{12} & \mapsto & e(\frac{\tau}{2})e(\frac{y_1+y_2}{2})e_{12} \otimes e_{12}, \\ e_{21} \otimes e_{21} & \mapsto & e(-\frac{\tau}{2})e(-\frac{y_1+y_2}{2})e_{21} \otimes e_{21}, \\ e_{12} \otimes e_{21} & \mapsto & e(-\frac{y}{2})e_{12} \otimes e_{21}, \\ e_{21} \otimes e_{12} & \mapsto & e(\frac{y}{2})e_{21} \otimes e_{12}. \end{array}$$

Hence, the new tensor of "skew terms" is

$$C\big[\theta_4(x+y|2\tau)\theta_4(y|2\tau)(e_{21}\otimes e_{12} + e_{12}\otimes e_{21}) + \theta_1(x+y|2\tau)\theta_1(y|2\tau)(e_{12}\otimes e_{12} + e_{21}\otimes e_{21})\big],$$

where

$$C = \frac{\theta_3'(\frac{1+\tau}{2}|\tau)e(\frac{1}{2}(x+y+\tau))}{\theta_3(y + \frac{1+\tau}{2}|\tau)\Delta_2} = \frac{\theta_1'(0|\tau)}{\theta_4(0|2\tau)\theta_4(x|2\tau)\theta_1(y|\tau)}$$
$$= \frac{\theta_1'(0|\tau)}{\theta_3(x/2|\tau)\theta_4(x/2|\tau)\theta_1(y|\tau)}.$$

Using the equality

$$A\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =$$

$$= (A+B)(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + (A-B)(e_{21} \otimes e_{21} + e_{12} \otimes e_{12})$$

and Watson's identities

$$\theta_4(y + x|2\tau)\theta_4(y|2\tau) + \theta_1(y + x|2\tau)\theta_1(y|2\tau) = \theta_4\left(y + \frac{x}{2}\Big|\,\tau\right)\theta_3\left(\frac{x}{2}\Big|\,\tau\right)$$

$$\theta_4(y + x|2\tau)\theta_4(y|2\tau) - \theta_1(y + x|2\tau)\theta_1(y|2\tau) = \theta_3\left(y + \frac{x}{2}\Big|\,\tau\right)\theta_4\left(\frac{x}{2}\Big|\,\tau\right)$$

it follows that the contribution of the "skew terms" is

$$\frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\left(\frac{\theta_3(y+\frac{x}{2}|\tau)}{\theta_3(\frac{x}{2}|\tau)}\sigma \otimes \sigma + \frac{\theta_2(y+\frac{x}{2}|\tau)}{\theta_2(\frac{x}{2}|\tau)}\tau \otimes \tau\right),$$

where
$$\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Summing everything up we obtain the following theorem.

Theorem 7.11. *The universal family of stable vector bundles of rank two and degree one on an elliptic curve E_τ gives the following solution of the associative Yang-Baxter equation:*
$$r(x;y) = \frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\left(\frac{\theta_1(y+\frac{x}{2}|\tau)}{\theta_1(\frac{x}{2}|\tau)}\mathbb{1}\otimes\mathbb{1} + \frac{\theta_2(y+\frac{x}{2}|\tau)}{\theta_2(\frac{x}{2}|\tau)}h\otimes h + \right.$$
$$\left. + \frac{\theta_3(y+\frac{x}{2}|\tau)}{\theta_3(\frac{x}{2}|\tau)}\sigma\otimes\sigma + \frac{\theta_4(y+\frac{x}{2}|\tau)}{\theta_4(\frac{x}{2}|\tau)}\tau\otimes\tau\right).$$

Recall that
$$\mathrm{cn}(z) = \frac{\theta_4(0|\tau)\theta_2(z|\tau)}{\theta_2(0|\tau)\theta_4(z|\tau)}, \quad \mathrm{sn}(z) = \frac{\theta_3(0|\tau)\theta_1(z|\tau)}{\theta_2(0|\tau)\theta_4(z|\tau)}, \quad \mathrm{dn}(z) = \frac{\theta_4(0|\tau)\theta_3(z|\tau)}{\theta_3(0|\tau)\theta_4(z|\tau)}$$
and
$$\theta_1'(0|\tau) = \theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau),$$
see [32, Sections I.5 and Section II.1]. Let $\bar{r}(y) = \lim_{x\to 0}(\mathrm{pr}\otimes\mathrm{pr})r(x;y)$ then we have:

Theorem 7.12. *The solution of the classical Yang-Baxter equation obtained from the universal family of stable vector bundles of rank two and degree one on a complex torus E_τ is*
$$\bar{r}(y) = \frac{1}{2}\left(\frac{\mathrm{cn}(y)}{\mathrm{sn}(y)}h\otimes h + \frac{1}{\mathrm{sn}(y)}\tau\otimes\tau + \frac{\mathrm{dn}(y)}{\mathrm{sn}(y)}\sigma\otimes\sigma\right).$$

Remark 7.13. Note that $\mathrm{res}_x\big(r(x;y)\big) = \frac{1}{4}\mathbb{1}\otimes\mathbb{1}$, hence the tensor $r_x(y) := r(x;y)$ also satisfies the quantum Yang-Baxter equation for $x \neq 0$.

Remark 7.14 (see for example Section VII.3 in [17])**.** Let
$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus\{0,0\}}\left(\frac{1}{(z-n\tau-m)^2} - \frac{1}{(n\tau+m)^2}\right)$$
be the Weierstraß $\wp$-function. Then $\wp'(\frac{1}{2}) = \wp'(\frac{\tau}{2}) = \wp'(\frac{1+\tau}{2}) = 0$ and $\frac{1}{2}, \frac{\tau}{2}$ and $\frac{1+\tau}{2}$ are the only branch points of $\wp(z)$ in the fundamental parallelogram of Λ_τ. Denote $e_1 = \wp(\frac{1}{2})$, $e_2 = \wp(\frac{\tau}{2})$ and $e_3 = \wp(\frac{1+\tau}{2})$. Then it holds:
$$\wp(z) - e_1 = \left(\frac{\mathrm{cn}(z)}{\mathrm{sn}(z)}\right)^2, \quad \wp(z) - e_2 = \left(\frac{1}{\mathrm{sn}(z)}\right)^2, \quad \wp(z) - e_3 = \left(\frac{\mathrm{dn}(z)}{\mathrm{sn}(z)}\right)^2.$$

8. Vector bundles on singular cubic curves

In this section we shall recall the approach of Drozd and Greuel to study torsion free sheaves on rational projective curves [20].

8.1. **The category of triples.** In order to fix the notation we start with some standard facts about vector bundles on a projective line.

Theorem 8.1 (Birkhoff-Grothendieck). *On the projective line $\mathbb{P}^1$, taking the degree gives an isomorphism $\mathsf{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ and any vector bundle of finite rank $\mathcal{E}$ on $\mathbb{P}^1$ splits into a direct sum of line bundles:*

$$\mathcal{E} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{m_n}.$$

Let (z_0, z_1) be coordinates on $V = \mathbb{C}^2$. They induce homogeneous coordinates $(z_0 : z_1)$ on the projective line $\mathbb{P}^1(V) = (V \setminus \{0\})/\sim$, where $v \sim \lambda v$ for all $\lambda \in \mathbb{C}^*$.

We set $U_0 = \{(z_0 : z_1)|z_0 \neq 0\}$ and $U_\infty = \{(z_0 : z_1)|z_1 \neq 0\}$ and put $0 := (1 : 0)$, $\infty := (0 : 1)$, $z = z_1/z_0$ and $w = z_0/z_1$. So, z is a coordinate in a neighbourhood of 0. If $U = U_0 \cap U_\infty$ and $w = 1/z$ is used as a coordinate on U_∞, then the transition function of the line bundle $\mathcal{O}(n)$ is

$$U_0 \times \mathbb{C} \supset U \times \mathbb{C} \xrightarrow{(z,v)\mapsto(\frac{1}{z},\frac{v}{z^n})} U \times \mathbb{C} \subset U_\infty \times \mathbb{C}.$$

The vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ is isomorphic to the sheaf of sections of the so-called tautological line bundle

$$\{(l, v) \mid v \in l\} \subset \mathbb{P}^1(V) \times V = \mathcal{O}^2_{\mathbb{P}^1}.$$

The choice of coordinates on $\mathbb{P}^1$ fixes two distinguished elements, z_0 and z_1, in the space $\mathsf{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1})$:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{C}^2 \arrow[dr] & \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[l, hook'] \arrow[r, "z_i"] \arrow[d] & \mathbb{P}^1 \times \mathbb{C} \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

where z_i maps $(l, (v_0, v_1))$ to (l, v_i) for $i = 0, 1$. It is clear that the section z_0 vanishes at ∞ and z_1 vanishes at 0. After making this choice we may write

$$\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\big) = \mathbb{C}[z_0, z_1]_{m-n} := \langle z_0^{m-n}, z_0^{m-n-1}z_1, \dots, z_1^{m-n}\rangle_{\mathbb{C}}.$$

Remark 8.2. In these terms we have the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\binom{z_0}{z_1}} \mathcal{O}^2_{\mathbb{P}^1} \xrightarrow{(-z_1\ \ z_0)} \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

To compute the associative r-matrices coming from the nodal and cuspidal Weierstraß cubic curves, we use a description of vector bundles on singular projective curves via the formalism of matrix problems [20], see also [13, 11].

Let X be a reduced singular projective curve, $\pi : \widetilde{X} \to X$ its normalisation, $\mathcal{I} := \mathcal{H}om_{\mathcal{O}}\bigl(\pi_*(\mathcal{O}_{\widetilde{X}}), \mathcal{O}\bigr) = \mathcal{A}nn_{\mathcal{O}}\bigl(\pi_*(\mathcal{O}_{\widetilde{X}})/\mathcal{O}\bigr)$ the conductor ideal sheaf. Denote by $\eta : Z = V(\mathcal{I}) \longrightarrow X$ the closed artinian subspace defined by $\mathcal{I}$ (its topological support is precisely the singular locus of X) and by $\tilde{\eta} : \widetilde{Z} \longrightarrow \widetilde{X}$ its preimage in $\widetilde{X}$, defined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\pi}"'] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\eta"] & X. \end{tikzcd}$$

In order to relate vector bundles on X and $\widetilde{X}$ we need the following definition.

Definition 8.3. The category of triples $\mathsf{Tri}(X)$ is defined as follows.

- Its objects are triples $(\widetilde{\mathcal{V}}, \mathcal{N}, \mu)$, where $\widetilde{\mathcal{V}} \in \mathsf{VB}(\widetilde{X})$, $\mathcal{N} \in \mathsf{VB}(Z)$ and
$$\mu : \tilde{\pi}^*\mathcal{N} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{V}}$$
is an isomorphism of $\mathcal{O}_{\widetilde{Z}}$–modules, also called *gluing map*.
- The set of morphisms $\mathsf{Hom}_{\mathsf{Tri}(X)}\bigl((\widetilde{\mathcal{V}}, \mathcal{N}, \mu), (\widetilde{\mathcal{V}}', \mathcal{N}', \mu')\bigr)$ consists of all pairs (F, f), where $F : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}'$ and $f : \mathcal{N} \to \mathcal{N}'$ are morphisms of vector bundles such that the following diagram is commutative
$$\begin{tikzcd} \tilde{\pi}^*\mathcal{N} \arrow[r, "\mu"] \arrow[d, "\tilde{\pi}^*(f)"'] & \tilde{\eta}^*\widetilde{\mathcal{V}} \arrow[d, "\tilde{\eta}^*(F)"] \\ \tilde{\pi}^*\mathcal{N}' \arrow[r, "\mu'"] & \tilde{\eta}^*\widetilde{\mathcal{V}}'. \end{tikzcd}$$

The category $\mathsf{Tri}(X)$ is endowed with a natural tensor product

$$(\widetilde{\mathcal{V}}, \mathcal{N}, \mu) \otimes (\widetilde{\mathcal{V}}', \mathcal{N}', \mu') = (\widetilde{\mathcal{V}} \otimes \widetilde{\mathcal{V}}', \mathcal{N} \otimes \mathcal{N}', \mu \otimes \mu')$$

and we have the following theorem.

Theorem 8.4 (Drozd-Greuel, Lemma 2.4 in [20], see also Theorem 1.3 in [13])**.** *The functor $\mathbb{F} : \mathsf{VB}(X) \longrightarrow \mathsf{Tri}(X)$ which assigns to a vector bundle $\mathcal{V}$ the triple $(\pi^*\mathcal{V}, \eta^*\mathcal{V}, \mu_{\mathcal{V}})$, where $\mu_{\mathcal{V}} : \tilde{\pi}^*\eta^*\mathcal{V} \longrightarrow \tilde{\eta}^*\pi^*\mathcal{V}$ is a canonical isomorphism, is an equivalence of categories. Moreover, for any $\mathcal{V}, \mathcal{V}' \in \mathsf{VB}(X)$ it holds $\mathbb{F}(\mathcal{V} \otimes \mathcal{V}') \cong \mathbb{F}(\mathcal{V}) \otimes \mathbb{F}(\mathcal{V}')$.*

Remark 8.5. Since the functor $\mathbb{F}$ commutes with tensor products, we also obtain that, if $\mathbb{F}(\mathcal{V}) = (\widetilde{\mathcal{V}}, \mathcal{N}, \mu)$ and n is the rank of $\mathcal{V}$, then

$$\mathbb{F}(\det(\mathcal{V})) = \bigl(\det(\widetilde{\mathcal{V}}), \det(\mathcal{N}), \wedge^n(\mu)\bigr).$$

Assume that our curve X is irreducible with normalisation $\widetilde{X} = \mathbb{P}^1$. Let $(\widetilde{\mathcal{V}}, \mathcal{N}, \mu)$ be an object of $\mathsf{Tri}(X)$ such that $\mathrm{rk}(\widetilde{\mathcal{V}}) = n$. Then it holds

$$\widetilde{\mathcal{V}} = \bigoplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{m_l} \quad \text{and} \quad \mathcal{N} \cong \mathcal{O}_Z^n, \quad \text{where } \sum_{l\in\mathbb{Z}} m_l = n.$$

In order to keep compatibility with tensor products in our description of vector bundles on singular Weierstraß curves, for each $l \in \mathbb{Z}$ we fix an isomorphism $\tau_l : \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(l) \to \mathcal{O}_{\widetilde{Z}}$ such that for all $k, l \in \mathbb{Z}$ the following diagram is commutative

$$\begin{tikzcd}
\tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(l) \arrow[r] \arrow[d, "\tau_k\otimes\tau_l"'] & \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(k+l) \arrow[d, "\tau_{k+l}"] \\
\mathcal{O}_{\widetilde{Z}} \otimes \mathcal{O}_{\widetilde{Z}} \arrow[r, "\text{mult}"] & \mathcal{O}_{\widetilde{Z}}.
\end{tikzcd}$$

Such a choice of $\{\tau_l\}_{l\in\mathbb{Z}}$ induces an isomorphism $\tau : \tilde{\eta}^*\widetilde{\mathcal{V}} \to \mathcal{O}_{\widetilde{Z}}^n$. Because $\mathcal{N} \cong \mathcal{O}_Z^n$, we get an isomorphism $\tilde{\pi}^*\mathcal{N} \to \mathcal{O}_{\widetilde{Z}}^n$. Since $\widetilde{Z}$ is an artinian complex space, the map $\mu : \tilde{\pi}^*\mathcal{N} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{V}}$ can be described as a matrix in $\mathsf{GL}_n(\mathcal{O}_{\widetilde{Z}})$. We have a natural action of the group $\mathsf{Aut}_{\mathbb{P}^1}(\widetilde{\mathcal{V}}) \times \mathsf{Aut}_Z(\mathcal{N})$ on the vector space $\mathsf{Hom}_{\widetilde{Z}}(\tilde{\pi}^*\mathcal{N}, \tilde{\eta}^*\widetilde{\mathcal{V}})$. The orbits of this action correspond precisely to the points in the fibre over $\widetilde{\mathcal{V}}$ of the functor $\pi^* : \mathsf{VB}(X) \to \mathsf{VB}(\widetilde{X})$.

Our aim is now to get a description of semi-stable vector bundles on singular irreducible Weierstraß cubic curves with locally free Jordan-Hölder factors in terms of triples. Let us start by collecting some standard results.

Lemma 8.6. *Let E be a singular Weierstraß curve, $\pi : \mathbb{P}^1 \to E$ its normalisation and $\mathcal{V}$ a vector bundle on E. Then* $\deg_E(\mathcal{V}) = \deg_{\mathbb{P}^1}(\pi^*\mathcal{V})$.

Proof. If $n = \mathrm{rk}(\mathcal{V})$ then $\pi^*\mathcal{V}$ is a vector bundle of rank n on $\mathbb{P}^1$. The canonical map $g : \mathcal{V} \to \pi_*\pi^*\mathcal{V}$ is generically injective and $\mathcal{V}$ is torsion free, hence $\ker(g) = 0$ and we have an exact sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{g} \pi_*\pi^*\mathcal{V} \longrightarrow \mathcal{T} \longrightarrow 0,$$

where $\mathcal{T}$ is a torsion sheaf supported at the singular point s of the curve E. Since g commutes with restrictions on an open set, we have

$$\mathcal{T} \cong \bigl(\mathrm{coker}(\mathcal{O}_E \to \pi_*(\mathcal{O}_{\mathbb{P}^1}))\bigr)^n.$$

Because s is either a node or a cusp, we obtain $h^0(\mathcal{T}) = n$. Using the Riemann-Roch formula, this implies

$$\deg_E(\mathcal{V}) = \chi(\mathcal{V}) = \chi(\pi_*\pi^*\mathcal{V}) - \chi(\mathcal{T}) = \chi(\pi^*\mathcal{V}) - n = \deg_{\mathbb{P}^1}(\pi^*\mathcal{V}).$$

□

Lemma 8.7. *Let E be a singular Weierstraß curve, $\pi : \mathbb{P}^1 \to E$ its normalisation and $\mathcal{V}$ a simple vector bundle of rank n on E. Then*

- *$\mathcal{V}$ is stable.*
- *$\pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(c)^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(c+1)^{n_2}$ for some integer $c \in \mathbb{Z}$ and some non-negative integers n_1, n_2 which satisfy $n = n_1 + n_2$.*

Proof. For the first statement see for example [16, Corollary 4.5]. To prove the second part assume

$$\widetilde{\mathcal{V}} = \pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(c) \oplus \mathcal{O}_{\mathbb{P}^1}(d) \oplus \widetilde{\mathcal{V}}'',$$

where $d - c > 1$. Let $(\widetilde{\mathcal{V}}, \mathcal{O}_Z^n, \mu)$ be a triple corresponding to $\mathcal{V}$. Because the length of $\widetilde{Z}$ is two and $d - c \geq 2$, we can find a non-zero homogeneous form $p = p(z_0, z_1) \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(c), \mathcal{O}_{\mathbb{P}^1}(d)\big)$ such that $\tilde{\eta}^*(p) = 0$. This gives us a non-scalar endomorphism of $\mathcal{V}$ by taking the endomorphism (F, f) of the corresponding triple given by

$$F = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $f = \mathrm{id}$. This contradicts our assumption that $\mathcal{V}$ was simple. □

Remark 8.8. Note that the map

$$\tilde{\eta}^* : \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow \mathsf{Hom}_{\widetilde{Z}}(\mathcal{O}_{\widetilde{Z}}, \mathcal{O}_{\widetilde{Z}})$$

is an isomorphism both for a nodal and a cuspidal cubic curve.

8.2. **Simple vector bundles on a nodal Weierstraß curve.** Let E be a nodal Weierstraß curve, e.g. $zy^2 = x^3 + x^2 z$, $s = (0 : 0 : 1)$ the singular point and $\pi : \mathbb{P}^1 \longrightarrow E$ its normalisation. Choose homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1$ in such a way that $\pi^{-1}(s) = \{0, \infty\}$. Then, in notations of the previous subsection, Z and $\widetilde{Z}$ are reduced complex spaces as follows

$$Z = \{s\} \quad \text{and} \quad \widetilde{Z} = \{0\} \cup \{\infty\}.$$

Hence, for a triple $(\tilde{\mathcal{V}}, \mathcal{N}, \mu)$ the map μ is just an isomorphism of $\mathbb{C} \times \mathbb{C}$–modules.

Let $p = z_1 - z_0 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$, this is a section, which does not vanish on $\widetilde{Z}$. We define the collection of isomorphisms $\tau_l : \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(l) \longrightarrow \mathcal{O}_{\widetilde{Z}}$ $(l \in \mathbb{Z})$ by saying that, for each open subset $V \subset \mathbb{P}^1$ not containing $(1 : 1)$, τ_l maps a local section $s \in \Gamma\bigl(V, \mathcal{O}_{\mathbb{P}^1}(l)\bigr)$ to $\frac{s}{p^l}|_{\widetilde{Z}}$. This implies the following evaluation rule for morphisms of vector bundles on $\mathbb{P}^1$: if $\varphi = a_0 z_0^{m-n} + a_1 z_0^{m-n-1} z_1 + \cdots + a_{m-n} z_1^{m-n} \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\bigr)$ then we have a commutative diagram

$$\begin{tikzcd}
\tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(n) \arrow[r, "\eta^*(\varphi)"] \arrow[d, "\tau_n"'] & \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(m) \arrow[d, "\tau_m"] \\
\mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[r, "\mathrm{diag}\left((-1)^{m-n}a_0, a_{m-n}\right)"] & \mathbb{C}_0 \oplus \mathbb{C}_\infty .
\end{tikzcd}$$

Let us now determine the triple which corresponds to the line bundle $\mathcal{O}_E(y)$ for $y \in E_{\mathrm{reg}}$. Because the normalisation restricts to an isomorphism $\pi : \mathbb{P}^1 \setminus \{0, \infty\} \longrightarrow E_{\mathrm{reg}}$ and the chosen coordinates provide us with an isomorphism $\mathbb{C}^* \cong U = \mathbb{P}^1 \setminus \{0, \infty\}$, we obtain an identification $E_{\mathrm{reg}} \cong \mathbb{C}^*$, such that $y \in \mathbb{C}^*$ corresponds to $\tilde{y} := \pi^{-1}(y) = (1 : y) \in \mathbb{P}^1$. Obviously, $\pi^*\bigl(\mathcal{O}_E(y)\bigr) = \mathcal{O}_{\mathbb{P}^1}(\tilde{y}) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $H^0\bigl(\mathcal{O}_E(y)\bigr)$ is the one-dimensional subspace of $H^0\bigl(\mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ generated by a section vanishing at $(1 : y)$.

Lemma 8.9. *For the given choice of homogeneous coordinates on $\mathbb{P}^1$ and the set of trivialisations $\{\tau_l\}_{l \in \mathbb{Z}}$ described above, we obtain*

$$\mathbb{F}(\mathcal{O}_E(y)) = \bigl(\mathcal{O}_{\mathbb{P}^1}, \mathbb{C}_s, \bigl((y), (1)\bigr)\bigr).$$

Proof. Assume $\mathbb{F}(\mathcal{O}_E(y)) = \bigl(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, ((\lambda), (1))\bigr)$. Since we have an identification $H^0\bigl(\mathcal{O}_E(y)\bigr) = \mathsf{Hom}_E\bigl(\mathcal{O}_E, \mathcal{O}_E(y)\bigr)$ and an embedding

$$\mathsf{Hom}_{\mathbb{E}}\bigl(\mathcal{O}_E, \mathcal{O}_E(y)\bigr) \xrightarrow{\cong} \mathsf{Hom}_{\mathsf{Tri}(E)}\bigl(\mathbb{F}(\mathcal{O}_E), \mathbb{F}(\mathcal{O}_E(y))\bigr) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\bigr),$$

a section $az_0 + bz_1 \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ belongs to the image of π^* if and only if there exists a constant $f \in \mathbb{C}^*$ making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] & \mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[d, "\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}"] \\
\mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[r, "\begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix}"] & \mathbb{C}_0 \oplus \mathbb{C}_\infty .
\end{tikzcd}$$

This implies the relation $a + \lambda b = 0$, so $H^0\bigl(\mathcal{O}_E(y)\bigr)$ is generated by the section $\lambda z_0 - z_1$ vanishing at $(1 : \lambda)$. From this we conclude that $\lambda = y$ and $\mathbb{F}\bigl(\mathcal{O}_E(y)\bigr) \cong \bigl(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, ((y), (1))\bigr)$. □

Lemma 8.10. *Let E be a nodal Weierstraß curve. Then there exists a unique indecomposable semi-stable vector bundle $\mathcal{A}_m$ of rank m and degree 0 such that all* *its Jordan-Hölder factors are isomorphic to $\mathcal{O}$. This vector bundle is given by the triple $\big(\mathcal{O}_{\mathbb{P}^1}^m, \mathbb{C}_s^m, \mu\big)$, where*

$$\mu(0) = J_m(1) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \mu(\infty) = I_m = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Proof. The category of semi-stable vector bundles with the Jordan-Hölder factor $\mathcal{O}$ is equivalent to the category of finite-dimensional modules over $\mathbb{C}[[t]]$, see for example [22, Theorem 1.1 and Lemma 1.7]. Therefore, there exists a unique indecomposable vector bundle $\mathcal{A}_m$ of rank m recursively defined by the non-split exact sequences

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A}_{m+1} \longrightarrow \mathcal{A}_m \longrightarrow 0 \quad \text{and} \quad \mathcal{A}_1 = \mathcal{O}.$$

In order to get a description of $\mathcal{A}_m$ in terms of triples, first observe that $\pi^*\mathcal{A}_m \cong \mathcal{O}_{\mathbb{P}^1}^m$, hence $\mathbb{F}(\mathcal{A}_m) = \big(\mathcal{O}_{\mathbb{P}^1}^m, \mathbb{C}_s^m, \mu\big)$. The morphism μ is given by two invertible matrices $\mu(0), \mu(\infty) \in \mathsf{GL}_m(\mathbb{C})$. If $\mu' = (\mu'(0), \mu'(\infty))$ is another pair such that

$$\mu'(0) = S^{-1}\mu(0)T, \quad \mu'(\infty) = S^{-1}\mu(\infty)T$$

with $S, T \in \mathsf{GL}_m(\mathbb{C})$, then $\big(\mathcal{O}_{\mathbb{P}^1}^m, \mathbb{C}_s^m, \mu'\big)$ and $\big(\mathcal{O}_{\mathbb{P}^1}^m, \mathbb{C}_s^m, \mu\big)$ define isomorphic vector bundles on E. We may, therefore, assume $\mu(\infty) = I_m$. Keeping $\mu(\infty) = I_m$, $\mu(0)$ can still be transformed to $S^{-1}\mu(0)S$. Hence $\mu(0)$ splits into a direct sum of Jordan blocks. Since the vector bundle $\mathcal{A}_m$ is indecomposable, $\mu(0) \sim J_m(\lambda)$ for some $\lambda \in \mathbb{C}^*$. From the condition $\mathsf{Hom}_E(\mathcal{A}_m, \mathcal{O}) = \mathbb{C}$ one can easily deduce $\lambda = 1$. □

Our next aim is to get a description of simple vector bundles on E in terms of triples.

Theorem 8.11. *Let E be a nodal Weierstraß cubic curve and $\mathcal{V}$ a simple vector bundle on E of rank n and degree d. Then $\gcd(n,d) = 1$ and $\mathcal{V}$ is uniquely determined by $(n,d) \in \mathbb{N} \times \mathbb{Z}$ and $\det(\mathcal{E}) \in \mathsf{Pic}^d(E) \cong \mathbb{C}^*$. Conversely, for any pair $(n,d) \in \mathbb{N} \times \mathbb{Z}$ such that $\gcd(n,d) = 1$ the set of simple vector bundles on E of rank n and degree d is non-empty and is parametrised by $\mathbb{C}^*$.*

This result can be proved by various methods, see for example [14, Theorem 3.6] for a description of simple vector bundles on E in terms of étale coverings. For the reader's convenience we shall outline another proof, which is parallel to the case of a cuspidal cubic curve [10].

Proof.[2] Let $\mathcal{V}$ be a simple vector bundle of rank n and degree d on the curve E, then, by Lemma 8.7, $\pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(c)^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(c+1)^{n_2}$ for some $c \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{N}$. From the equalities

$$n_1 + n_2 = n, \quad cn_1 + (c+1)n_2 = d$$

[2]This proof is due to Lesya Bodnarchuk.

we obtain $n_2 = d - cn$ and $n_1 = (1+c)n - d$. Without loss of generality we may assume $c = 0$. In this case we have $n_1 = n - d$ and $n_2 = d$.

The isomorphism classes of vector bundles $\mathcal{V}$ on E with a fixed pull-back

$$\widetilde{\mathcal{V}} := \pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2} \in \mathsf{add}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$$

correspond to the equivalence classes of matrices μ with respect to the transformation rule $\mu \mapsto \tilde{\eta}^*(F)^{-1}\mu\tilde{\pi}^*(f)$, where F is an automorphism of $\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$ and $f \in \mathsf{GL}_{n_1+n_2}(\mathbb{C})$. An automorphism F of $\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$ can be written in block matrix form as

$$F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix},$$

where $F_{kk} \in \mathsf{GL}_{n_k}(\mathbb{C})$ for $k = 1, 2$ and $F_{21} \in \mathsf{Mat}_{n_2 \times n_1}(\mathbb{C}[z_0, z_1]_1)$. Note that the map

$$\tilde{\eta}^* : \mathsf{End}_{\mathbb{P}^1}(\widetilde{\mathcal{V}}) \to \mathsf{End}_{\widetilde{Z}}(\mathcal{O}_{\widetilde{Z}})$$

is an isomorphism and with respect to the chosen set of trivialisations $\{\tau_k\}_{k\in\mathbb{Z}}$

$$\tilde{\eta}^*(F) = \left[\begin{pmatrix} F_{11} & 0 \\ -F_{21}(1:0) & F_{22} \end{pmatrix}, \begin{pmatrix} F_{11} & 0 \\ F_{21}(0:1) & F_{22} \end{pmatrix}\right].$$

Since the matrix $\mu(\infty)$ is invertible, we can reduce it to the identity form by taking $f = \mu(\infty)^{-1}, F = \mathrm{id}$. It remains to find a canonical form for the matrix $\mu(0)$.

Definition 8.12. For a nodal cubic curve E we define the category $\mathsf{MP}_{\mathrm{nd}}$ as follows.

- Its objects are invertible matrices

$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right),$$

  where M_{11} and M_{22} are square matrices with entries in $\mathbb{C}$.
- Morphisms are pairs of block matrices

$$\mathsf{Hom}_{\mathsf{MP}_{\mathrm{nd}}}(M, M') = \{(S,T) \mid SM = M'T\},$$

  such that

$$S = \begin{pmatrix} A & 0 \\ C' & B \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A & 0 \\ C'' & B \end{pmatrix}$$

The category of those vector bundles $\mathcal{V}$ on E for which $\pi^*\mathcal{V} \in \mathsf{add}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is equivalent to $\mathsf{MP}_{\mathrm{nd}}$. Under this equivalence, $M \in \mathsf{MP}_{\mathrm{nd}}$ corresponds to the vector bundle $\mathcal{V}$ if $\mathbb{F}(\mathcal{V}) \cong (\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}, \mathbb{C}_s^{n_1+n_2}, (M, \mathrm{id}))$. The matrices S, T correspond to the two components of $\tilde{\eta}^*(F)$. In particular, the subcategory of simple vector bundles with such normalisation is equivalent to the full subcategory $\mathsf{MP}_{\mathrm{nd}}^s$ of simple[3] objects of $\mathsf{MP}_{\mathrm{nd}}$.

[3]Simple objects of $\mathsf{MP}_{\mathrm{nd}}$ are by definition the objects having only scalar endomorphisms. They are sometimes called *Schurian* objects or *bricks*.

Lemma 8.13. *Let $M \in \mathsf{MP}^s_{\mathrm{nd}}$ be a simple object such that the matrices M_{11} and M_{22} have sizes $n_1 \times n_1$ and $n_2 \times n_2$ respectively. Then, the matrix M_{12} has full rank.*

Proof. If this were not the case, the matrix M could be reduced to the form

$$M = \left(\begin{array}{cc|cc} M_1 & M_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline M_3 & M_4 & 0 & M_5 \\ M_6 & M_7 & 0 & M_8 \end{array}\right),$$

where we split M_{11}, M_{21} and M_{22} into blocks such that M_1 and M_8 are square matrices. Since M is invertible, either M_1 or M_2 is a non-zero matrix. Then M, as an object of $\mathsf{MP}_{\mathrm{nd}}$, had a non-scalar endomorphism given by

$$S = \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline M_5 & 0 & I & 0 \\ M_8 & 0 & 0 & I \end{array}\right) \quad T = \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ M_1 & M_2 & 0 & I \end{array}\right).$$

□

If $M \in \mathsf{MP}^s_{\mathrm{nd}}$ is a simple object such that the matrices M_{11} and M_{22} have the same size, i.e. $n_1 = n_2$, then M_{12} is invertible by Lemma 8.13. Hence, M can be reduced to the form

$$M = \left(\begin{array}{c|c} 0 & I \\ \hline X & 0 \end{array}\right)$$

where X splits into a direct sum of Jordan blocks with non-zero eigenvalues. It is easy to see that M is decomposable in $\mathsf{MP}_{\mathrm{nd}}$ unless $n_1 = n_2 = 1$. Now assume that $n_1 \neq n_2$. In this case, if $n_2 > n_1$, we can reduce M to the form

$$M = \left(\begin{array}{c||cc} 0 & I & 0 \\ \hline\hline M'_{11} & 0 & M'_{12} \\ M'_{21} & 0 & M'_{22} \end{array}\right)$$

and if $n_1 > n_2$ we can reduce M to

$$M = \left(\begin{array}{cc||c} M'_{11} & M'_{12} & 0 \\ 0 & 0 & I \\ \hline\hline M'_{21} & M'_{22} & 0 \end{array}\right).$$

In both cases, the additional split of the blocks is made in such a way that M'_{11} and M'_{22} are square matrices. A straightforward calculation shows that in both cases

$$M' = \begin{pmatrix} M'_{11} & M'_{12} \\ M'_{21} & M'_{22} \end{pmatrix}$$

is an object of $\mathsf{MP}^s_{\mathrm{nd}}$ again. Let us denote by $\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$ the full subcategory of $\mathsf{MP}^s_{\mathrm{nd}}$ whose objects have diagonal blocks of sizes $n_1 \times n_1$ and $n_2 \times n_2$. Then, sending

M to M' is an equivalence of categories

$$\mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2) \longrightarrow \begin{cases} \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2-n_1) & \text{if} \quad n_2 > n_1, \\ \mathsf{MP}^s_{\mathrm{nd}}(n_1-n_2,n_2) & \text{if} \quad n_1 > n_2. \end{cases}$$

This means that the matrix problem describing the simple objects of the category $\mathsf{MP}_{\mathrm{nd}}$ is *self-reproducing.* The chain of these reductions terminates when we achieve $n_1 = n_2 = 1$. In this case

$$M = \left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda & 0 \end{array}\right).$$

This implies that for a simple vector bundle $\mathcal{V}$ of rank n and degree d it must hold

$$\gcd(n_1,n_2) = \gcd((1-c)n-d, d-cn) = \gcd(n,d) = 1.$$

Moreover, we obtain an explicit algorithm to construct a canonical form of the matrix describing the family of triples corresponding to the set of simple vector bundles with prescribed rank and degree. This finishes the proof of Theorem 8.11. □

Example 8.14. If we let $\widetilde{\mathcal{V}} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $\mathcal{N} = \mathbb{C}^2_s$ and for any $\lambda \in \mathbb{C}^*$

$$\mu(0) = \left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda & 0 \end{array}\right) \quad \text{and} \quad \mu(\infty) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right),$$

the triple $(\widetilde{\mathcal{V}}, \mathcal{N}, \mu)$ defines a simple vector bundle of rank 2 and degree 1 on E. The corresponding matrix in $\mathsf{MP}^s_{\mathrm{nd}}$ for this triple is $M_{1,1}(\lambda) = \left(\begin{smallmatrix} 0 & 1 \\ \lambda & 0 \end{smallmatrix}\right) \in \mathsf{MP}^s_{\mathrm{nd}}$. □

Algorithm 8.15. For any pair of coprime positive integers (n_1,n_2), the set of simple objects $M_{n_1,n_2}(\lambda) \in \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$ is described in the following way.

(1) First, we produce a sequence of pairs of coprime integers by replacing at each step a pair (n_1,n_2) by (n_1-n_2,n_2) if $n_1 > n_2$ and by (n_1,n_2-n_1) if $n_2 > n_1$. We continue until we arrive at $(1,1)$.

(2) Starting with the matrix $M_{1,1}(\lambda) \in \mathsf{MP}^s_{\mathrm{nd}}$ from Example 8.14 we recursively construct the matrix $M_{n_1,n_2}(\lambda)$ as follows. We follow the sequence constructed in part (1) in reverse order and

- if we go from (m_1,m_2) to (m_1+m_2,m_2) we proceed as follows

$$M_{m_1,m_2}(\lambda) = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \Rightarrow M_{m_1+m_2,m_2}(\lambda) = \left(\begin{array}{cc||c} X & Y & 0 \\ \hline 0 & 0 & I_{m_2} \\ \hline\hline Z & W & 0 \end{array}\right).$$

- and similarly, if we go from (m_1,m_2) to (m_1,m_1+m_2) we set

$$M_{m_1,m_2}(\lambda) = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \Rightarrow M_{m_1,m_1+m_2}(\lambda) = \left(\begin{array}{c||cc} 0 & I_{m_1} & 0 \\ \hline\hline X & 0 & Y \\ Z & 0 & W \end{array}\right).$$

Remark 8.16. Note that $M_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda')$ in $\mathsf{MP}_{\mathrm{nd}}$ if and only if $\lambda = \lambda'$. Note also that $M_{n-d,d}(\lambda)$ is an $n \times n$–matrix having exactly one non-zero entry in each column and row and that $\lambda = \pm\det(M_{n_1,n_2}(\lambda))$. In particular, two stable vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ on a nodal Weierstraß curve having the same rank are isomorphic if and only if $\det(\mathcal{V}_1) \cong \det(\mathcal{V}_2)$.

Example 8.17. Let us describe the triple corresponding to the set of simple vector bundles on E of rank 5 and degree 12. First of all, the normalisation sheaf is $\widetilde{\mathcal{V}} = \mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2$, in particular $(n_1, n_2) = (3, 2)$. The sequence of reductions for sizes of matrices from the category $\mathsf{MP}_{\mathrm{nd}}$ is:

$$(3,2) \longrightarrow (1,2) \longrightarrow (1,1).$$

This induces a reverse sequence of functors

$$\mathsf{MP}^s_{\mathrm{nd}}(1,1) \longrightarrow \mathsf{MP}^s_{\mathrm{nd}}(1,2) \longrightarrow \mathsf{MP}^s_{\mathrm{nd}}(3,2),$$

giving the following sequence of canonical forms:

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda & 0 \end{array}\right) \mapsto \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \lambda & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, the set of stable vector bundles of rank 5 and degree 12 is described by the family of triples $\big(\mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2, \mathbb{C}^5_s, \mu\big)$, where

$$\mu(0) = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{array}\right), \quad \mu(\infty) = I_5.$$

Example 8.18. The indecomposable semi-stable vector bundles $\mathcal{V}$ of rank two and degree zero, whose Jordan-Hölder factors are locally free, are of the form $\mathcal{L} \otimes \mathcal{A}_2$, where $\mathcal{L} \in \mathsf{Pic}^0(E)$. They are described by the triples $\big(\mathcal{O}^2_{\mathbb{P}^1}, \mathbb{C}^2_s, \mu\big)$, where

$$\mu(0) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \quad \text{and} \quad \mu(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 8.19. *Let $\lambda \in \mathbb{C}^*$ and $X = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{C}^*$, $i = 1, \dots, n$, then $XM_{n-d,d}(\lambda) \cong M_{n-d,d}(\lambda \cdot \alpha_1 \cdot \ldots \cdot \alpha_n)$ as objects of $\mathsf{MP}_{\mathrm{nd}}$.*

Proof. It is not hard to see that $XM_{n-d,d}(\lambda) \in \mathsf{MP}_{\mathrm{nd}}$ is again simple, hence it is isomorphic to a canonical form $M_{n-d,d}(\lambda')$. This means that there exist invertible matrices $S = \left(\begin{smallmatrix} A & 0 \\ C' & B \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} A & 0 \\ C'' & B \end{smallmatrix}\right)$ such that $M_{n-d,d}(\lambda') = S^{-1}XM_{n-d,d}(\lambda)T$. Because $\det(S) = \det(A)\det(B) = \det(T)$, we obtain $\lambda' = \lambda\det(X)$. $\square$

Remark 8.20. The last lemma implies that there exists an object $N_{n_1,n_2}(\lambda) \in \mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$ isomorphic to $M_{n_1,n_2}(\lambda)$ and such that the corresponding triple describing a family of stable vector bundles on E of rank n and degree d is compatible with the action of the Jacobian. To construct $N_{n_1,n_2}(\lambda)$, we simply replace all non-zero entries of $M_{n_1,n_2}(\lambda)$ by $\sqrt[n]{\lambda}$, some fixed n-th root of λ. Another choice of $\sqrt[n]{\lambda}$ gives an isomorphic vector bundle.

In particular, in the family of triples $(\widetilde{\mathcal{V}}, \mathcal{N}, \mu)$ from Example 8.14, which describes the simple vector bundles of rank 2 and degree 1, we need to replace the matrices by

$$\mu(0) = \left(\begin{array}{c|c} 0 & \sqrt{\lambda} \\ \hline \sqrt{\lambda} & 0 \end{array}\right), \lambda \in \mathbb{C}^* \quad \text{and} \quad \mu(\infty) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

in order to obtain compatibility with the action of $\mathsf{Pic}^0(E)$.

The description of stable vector bundles on a nodal Weierstraß curve in terms of matrices constructed in Algorithm 8.15 can be translated into the language of moduli problems. Namely, for a pair $(n,d) \in \mathbb{N} \times \mathbb{Z}$ of coprime integers such that $0 \le d < n$ we give now a description of a universal family $\mathcal{P} := \mathcal{P}(n,d) \in \mathsf{VB}(E \times M)$, where $M = \mathsf{M}_E(n,d)$ is the moduli space of stable vector bundles of rank n and degree d. Recall that we have a pull-back diagram:

$$\begin{tikzcd} \widetilde{Z} \times M \arrow[r, "\tilde{\eta}\times 1"] \arrow[d, "\tilde{\pi}\times 1"'] & \mathbb{P}^1 \times M \arrow[d, "\pi\times 1"] \\ Z \times M \arrow[r, "\eta\times 1"] & E \times M. \end{tikzcd}$$

Let us denote $\mathcal{O}_{\mathbb{P}^1\times M}(n) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(n)$, where $\pi_1 : \mathbb{P}^1 \times M \to \mathbb{P}^1$ is the projection map and define $\widetilde{\mathcal{P}} := \mathcal{O}_{\mathbb{P}^1\times M}^{n-d} \oplus \mathcal{O}_{\mathbb{P}^1\times M}(1)^d$. Recall that $\widetilde{Z} = \{0, \infty\}$. The isomorphism

$$\mu : \mathcal{O}^n_{\widetilde{Z}\times M} \longrightarrow (\tilde{\eta} \times 1)^*\widetilde{\mathcal{P}}$$

is defined with the aid of Theorem 8.11, which tells us that $M \cong \mathbb{C}^*$ and which gives for each $v \in \mathbb{C}^*$ a triple $\big(\mathcal{O}_{\mathbb{P}^1}^{n-d} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d, \mathcal{O}^n_Z, \mu_v\big)$ such that $\mu_v(\infty) = \mathrm{id}$ and $\mu_v(0) = M_{n-d,d}(v)$. Note that $M_{n-d,d}(v)$ depends holomorphically on $v \in \mathbb{C}^*$.

Now we proceed as follows. Let $\overline{\mathcal{P}} := (\tilde{\eta} \times 1)_*(\tilde{\eta} \times 1)^*\widetilde{\mathcal{P}}$. Then the canonical map $\mathcal{O}_{Z\times M} \to (\tilde{\pi} \times 1)_*\mathcal{O}_{\widetilde{Z}\times M}$ and the chosen map $\mu : \mathcal{O}^n_{\widetilde{Z}\times M} \to (\tilde{\eta} \times 1)^*\widetilde{\mathcal{P}}$ induce the map $\tilde{\mu}$ defined as the composition

$$(\eta \times 1)_*\mathcal{O}^n_{Z\times M} \longrightarrow (\eta \times 1)_*(\tilde{\pi} \times 1)_*\mathcal{O}^n_{\widetilde{Z}\times M} \xrightarrow{\cong} (\pi\tilde{\eta} \times 1)_*\mathcal{O}^n_{\widetilde{Z}\times M} \xrightarrow{(\pi\tilde{\eta}\times 1)_*(\mu)} (\pi \times 1)_*\overline{\mathcal{P}}$$

Consider the pull-back diagram in the category $\mathsf{Coh}(E \times M)$ of coherent sheaves

$$\begin{tikzcd}
0 \arrow[r] & \ker(g) \arrow[r] \arrow[d, "="'] & \mathcal{Q} \arrow[r] \arrow[d] & (\eta \times 1)_* \mathcal{O}^n_{Z \times M} \arrow[r] \arrow[d, "\tilde{\mu}"] & 0 \\
0 \arrow[r] & \ker(g) \arrow[r] & (\pi \times 1)_* \tilde{\mathcal{P}} \arrow[r, "g"] & (\pi \times 1)_* \overline{\mathcal{P}} \arrow[r] & 0,
\end{tikzcd}$$

where $g = (\pi \times 1)_*(\mathrm{can})$ and $\mathrm{can} : \tilde{\mathcal{P}} \to (\tilde{\eta} \times 1)_*(\tilde{\eta} \times 1)^* \tilde{\mathcal{P}}$ is the canonical morphism.

Proposition 8.21. *The sheaf $\mathcal{Q} \in \mathsf{Coh}(E \times M)$ is locally free and is a universal family of stable vector bundles on E of rank n and degree d.*

Proof. Note that the sheaf $(\pi \times 1)_* \overline{\mathcal{P}}$ is flat over M, hence for any point $v \in M$ the restriction of the above diagram to $E \times \{v\}$ preserve exactness of the horizontal rows and boils down to a new pull-back diagram in the category $\mathsf{Coh}(E)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}\big(\pi_*(\mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d)\big) \arrow[r] \arrow[d, "="'] & \mathcal{Q}|_{E \times \{v\}} \arrow[r] \arrow[d] & \eta_* \mathcal{O}^n_Z \arrow[r] \arrow[d, "\tilde{\mu}_v"] & 0 \\
0 \arrow[r] & \mathcal{I}\big(\pi_*(\mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d)\big) \arrow[r] & \pi_*\big(\mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d\big) \arrow[r] & \eta_*(\tilde{\pi}_*(\mathcal{O}^n_{\tilde{Z}})) \arrow[r] & 0,
\end{tikzcd}$$

where $\mathcal{I} = \mathcal{A}nn_{\mathcal{O}}(\pi_* \mathcal{O}_{\tilde{\mathbb{P}}^1} / \mathcal{O}_E)$ is the conductor sheaf. By [13, Theorem 1.3] the sheaf $\mathcal{Q}|_{E \times \{v\}}$ is a (stable) vector bundle on E of rank n and degree d, corresponding to the triple $\big(\mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d, \mathcal{O}^n_{\tilde{Z}}, \mu_v\big)$, in particular, $\mathcal{Q}$ itself is locally-free. Moreover, for $v \neq v'$ it holds $\mathcal{Q}|_{E \times \{v\}} \not\cong \mathcal{Q}|_{E \times \{v'\}}$.

Let $\mathcal{P} \in \mathsf{VB}(E \times M)$ be a universal family of stable vector bundles of rank n and degree d on E. Then, by the universal property, there exists a unique map $f : M \to M$ such that $\mathcal{Q} = (1 \times f)^* \mathcal{P}$. Assume f is not injective and $v, v' \in M$ are two different points such that $f(v) = f(v')$. Then

$$\mathcal{Q}|_{E \times \{v\}} \cong \mathcal{P}|_{E \times \{f(v)\}} = \mathcal{P}|_{E \times \{f(v')\}} \cong \mathcal{Q}|_{E \times \{v'\}}$$

yielding a contradiction. So, $f : M \to M$ is holomorphic and bijective, hence it is biholomorphic and $\mathcal{Q}$ is a universal family. □

8.3. **Vector bundles on a cuspidal Weierstraß curve.** Let E be the cuspidal cubic curve, given by the equation $zy^2 = x^3$. Its normalisation $\pi : \mathbb{P}^1 \longrightarrow E$ is given by $\pi(z_0 : z_1) = (z_0^2 z_1 : z_0^3 : z_1^3)$. With these coordinates on $\mathbb{P}^1$ the preimage of the singular point $s = (0 : 0 : 1) \in E$ is $\pi^{-1}(s) = (0 : 1) = \infty$. Then Z is the reduced point $s \in E$ with the structure sheaf $\mathbb{C}$. Moreover, $\widetilde{Z}$ is non-reduced with support at $\infty = (0 : 1) \in \mathbb{P}^1$ and structure sheaf $\mathsf{R} = \mathbb{C}[\varepsilon]/\varepsilon^2$. The morphism $\tilde{\pi} : \widetilde{Z} \to Z$ corresponds to the ring homomorphism $\mathbb{C} \to \mathsf{R}$. By sending $y \in \mathbb{C}$ to $\pi(1 : y) \in E_{\mathrm{reg}}$ we obtain an isomorphism $\mathbb{C} \xrightarrow{\cong} E_{\mathrm{reg}}$.

Recall that $w = z_0/z_1$ is a coordinate in the neighbourhood U_∞ of the point $(0:1)$. The morphism $\tilde{\eta} : \widetilde{Z} \to \mathbb{P}^1$ is given by the map $\mathrm{ev}_{U_\infty} : \mathcal{O}_{\mathbb{P}^1}(U_\infty) \to \mathsf{R}, \quad w \mapsto \varepsilon$.

It is convenient to take the collection of trivialisations of line bundles

$$\tau_l : \mathcal{O}_{\mathbb{P}^1}(l) \otimes \mathcal{O}_{\widetilde{Z}} \to \mathcal{O}_{\widetilde{Z}}$$

such that for each open set $V \subset U_\infty$

$$\tau_l : \big(\mathcal{O}_{\mathbb{P}^1}(l) \otimes \mathcal{O}_{\widetilde{Z}}\big)(V) \xrightarrow{f \mapsto \mathrm{ev}_V\left(\frac{f}{z_1^l}|_V\right)} \mathsf{R}.$$

A morphism

$$p(z_0, z_1) = a_0 z_0^{m-n} + a_1 z_0^{m-n-1} z_1 + \dots + a_{m-n} z_1^{m-n} \in \mathsf{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m))$$

is therefore evaluated according to the rule

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\widetilde{Z}} \arrow[r, "p\otimes 1"] \arrow[d, "\tau_n"'] & \mathcal{O}_{\mathbb{P}^1}(m) \otimes \mathcal{O}_{\widetilde{Z}} \arrow[d, "\tau_m"] \\ \mathsf{R} \arrow[r, "a_{m-n-1}\varepsilon + a_{m-n}"] & \mathsf{R}. \end{tikzcd}$$

Lemma 8.22. *With respect to the given choice of the set of trivialisations $\{\tau_l\}_{l\in\mathbb{Z}}$ we have:*

$$\mathbb{F}(\mathcal{O}_E(y)) \cong \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, 1 - \varepsilon y\big)$$

for $y \in E_{\mathrm{reg}}$.

Proof. As in the case of a nodal cubic curve, because $\mathcal{O}_E(y)$ is a line bundle of degree one, we know

$$\mathbb{F}(\mathcal{O}_E(y)) = \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, (1 + \lambda\varepsilon)\big).$$

for some $\lambda \in \mathbb{C}$. Since $H^0\big(\mathcal{O}_E(y)\big) = \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{O}_E, \mathcal{O}_E(y)\big)$ and we have an embedding

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{O}_E, \mathcal{O}_E(y)\big) \xrightarrow{\cong} \mathsf{Hom}_{\mathsf{Tri}(E)}\big(\mathbb{F}(\mathcal{O}_E), \mathbb{F}(\mathcal{O}_E(y))\big) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)),$$

the section $az_0 + bz_1 \in \mathsf{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$ belongs to the image of π^* if and only if there exists a constant $f \in \mathbb{C}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathsf{R} \arrow[r, "f"] \arrow[d, "1"'] & \mathsf{R} \arrow[d, "1+\lambda\varepsilon"] \\ \mathsf{R} \arrow[r, "a\varepsilon+b"] & \mathsf{R}. \end{tikzcd}$$

This implies $a = \lambda b$ and the section $p(z_0, z_1) = b(\lambda z_0 + z_1)$ vanishes at $(1 : -\lambda)$. Hence, $y = -\lambda$ and $\mathbb{F}\big(\mathcal{O}(y)\big) = \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, (1 - y\varepsilon)\big)$. $\square$

Theorem 8.23. *Let E be a cuspidal cubic curve, $\mathcal{V}$ a simple vector bundle on E of rank n and degree d. Then $\gcd(n, d) = 1$ and $\mathcal{V}$ is determined by $(n, d) \in \mathbb{N} \times \mathbb{Z}$ and $\det(\mathcal{V}) \in \mathsf{Pic}^n(E) \cong \mathbb{C}$.*

Our proof (given after Corollary 8.27) is basically the same as in [10]. However, we shall need a slightly different canonical form of triples describing the set of simple vector bundles with prescribed rank and degree.

If $\mathcal{V}$ is simple, then by Lemma 8.7 there exists an integer $c \in \mathbb{Z}$ and natural numbers $n_1, n_2 \in \mathbb{N}$ such that

$$\widetilde{\mathcal{V}} = \pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(c)^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(c+1)^{n_2}.$$

As in the case of a nodal curve, without loss of generality we may assume $c = 0$. Let $\mathbb{F}(\mathcal{V}) = \big(\widetilde{\mathcal{V}}, \mathbb{C}^n_s, \mu\big)$, then

$$\mathsf{End}_E(\mathcal{V}) \cong \mathsf{End}_{\mathsf{Tri}(E)}\big(\widetilde{\mathcal{V}}, \mathbb{C}^n_s, \mu\big).$$

One can write

$$\mu = \mu(0) + \varepsilon\mu(\varepsilon) = \left(\begin{array}{c|c} \mu_{11}(0) & \mu_{12}(0) \\ \hline \mu_{21}(0) & \mu_2(0) \end{array}\right) + \varepsilon \left(\begin{array}{c|c} \mu_{11}(\varepsilon) & \mu_{12}(\varepsilon) \\ \hline \mu_{21}(\varepsilon) & \mu_2(\varepsilon) \end{array}\right),$$

where $\mu(0), \mu(\varepsilon) \in \mathsf{Mat}_{n_1+n_2}(\mathbb{C})$ with square diagonal blocks of sizes $n_1 \times n_1$ and $n_2 \times n_2$ respectively. A matrix $\mu \in \mathsf{GL}_{n_1+n_2}(\mathsf{R})$ is invertible if and only if $\mu(0) \in \mathsf{GL}_{n_1+n_2}(\mathbb{C})$ is invertible. Let

$$F = \left(\begin{array}{c|c} F_{11} & 0 \\ \hline F_{21} & F_{22} \end{array}\right)$$

be an automorphism of $\widetilde{\mathcal{V}}$, then $F_{21} \in \mathsf{Mat}_{n_2 \times n_1}(\mathbb{C}[z_0, z_1]_1)$ and $F_{kk} \in \mathsf{GL}_{n_k}(\mathbb{C})$ for $k = 1, 2$. Then

$$\tilde{\eta}^*(F) = \left(\begin{array}{c|c} F_{11} & 0 \\ \hline F_{21}(0:1) & F_{22} \end{array}\right) + \varepsilon \left(\begin{array}{c|c} 0 & 0 \\ \hline F_{21}(1:0) & 0 \end{array}\right).$$

Let $T : \mathbb{C}^{n_1+n_2}_s \to \mathbb{C}^{n_1+n_2}_s$ be a linear map, then $\tilde{\pi}^*(T) = T$. This means that the gluing matrix μ can be transformed by the rule $\mu \mapsto S^{-1}\mu T$, where $T \in \mathsf{GL}_{n_1+n_2}(\mathbb{C})$ and S has the form

$$S = \left(\begin{array}{c|c} S_{11} & 0 \\ \hline S_{21} & S_{22} \end{array}\right)$$

with $S_{kk} \in \mathsf{GL}_{n_k}(\mathbb{C})$ and $S_{21} \in \mathsf{Mat}_{n_2 \times n_1}(\mathsf{R})$. It is easy to see that one can transform the matrix μ to the form

$$\mu = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) + \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline 0 & M_{22} \end{array}\right).$$

Definition 8.24. For a cuspidal cubic curve E we define the category $\mathsf{MP}_{\mathrm{cp}}$ as follows:

- Its objects are "matrices" of the form

$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline \times & M_{22} \end{array}\right),$$

where $\times$ is an "empty" or "non-existing" block.

- Let $M, M' \in \mathsf{MP}_{\mathrm{cp}}$, then

$$\mathsf{Hom}_{\mathsf{MP}_{\mathrm{cp}}}(M, M') = \left\{ S = \left(\begin{array}{c|c} S_{11} & \times \\ \hline S_{21} & S_{22} \end{array} \right) \middle| \, SM = M'S \right\}.$$

  This condition means that

$$\begin{array}{rcl} S_{11}M_{11} & = & M'_{11}S_{11} + M'_{12}S_{21} \\ S_{11}M_{12} & = & M'_{12}S_{22} \\ S_{21}M_{12} + S_{22}M_{22} & = & M'_{22}S_{22}, \end{array}$$

  in other words: we ignore the lower left block in SM and $M'S$.

As in the nodal case, we denote by $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ the full subcategory whose objects have blocks M_{ii} of size $n_i \times n_i$. The following lemma is straightforward.

Lemma 8.25. *In the notation introduced above it holds:*

- *The category of vector bundles on E whose normalisation is in the category $\mathsf{add}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$ is equivalent to $\mathsf{MP}_{\mathrm{cp}}$. The corresponding fully faithful functor*

$$\mathsf{MP}_{\mathrm{cp}} \longrightarrow \mathsf{Tri}(E)$$

  is given by

$$M \mapsto \big(\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}, \mathbb{C}^n_s, \mu\big)$$

  where $\mu = I_n + \varepsilon M$ and $M \in \mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$.
- *Let $M \cong M'$ in $\mathsf{MP}_{\mathrm{cp}}$ then*

$$\mathrm{tr}(M) := \mathrm{tr}(M_{11}) + \mathrm{tr}(M_{22}) = \mathrm{tr}(M').$$

Our aim now is to describe the category $\mathsf{MP}^s_{\mathrm{cp}}$ of simple objects of $\mathsf{MP}_{\mathrm{cp}}$.

Lemma 8.26. *Let*

$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline \times & M_{22} \end{array} \right)$$

be a simple object of $\mathsf{MP}_{\mathrm{cp}}$. Then M_{12} has a full rank.

Proof. Assume it is not the case. Then the matrix M can be reduced to the form

$$M = \left(\begin{array}{c|c||c|c} M_1 & M_2 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline\hline \times & \times & M_3 & M_4 \\ \hline \times & \times & 0 & M_5 \end{array} \right)$$

and we obtain a non-scalar endomorphism

$$S = \left(\begin{array}{c|c||c|c} I & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 \\ \hline\hline 0 & 0 & I & 0 \\ \hline X & 0 & 0 & I \end{array} \right),$$

where X is an arbitrary matrix of appropriate size. $\square$

Corollary 8.27. *Let $M \in \mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ be a simple object. If $n_1 = n_2$ then $n_1 = n_2 = 1$ and M can be written in the form*

$$M_{1,1}(\lambda) = \left(\begin{array}{c|c} \lambda & 1 \\ \hline \times & 0 \end{array}\right)$$

for some $\lambda \in \mathbb{C}^*$.

Proof. Indeed, since the block M_{12} is square and invertible, we can transform the matrix M to the form

$$M = \left(\begin{array}{c|c} X & I \\ \hline \times & 0 \end{array}\right).$$

We can further reduce the block $M_{11} = X$ to its Jordan normal form keeping the block $M_{12} = I$ unchanged. This implies that M splits into a direct sum of objects of the form

$$\left(\begin{array}{c|c} J_m(\lambda) & I_m \\ \hline \times & 0 \end{array}\right)$$

which are simple in $\mathsf{MP}_{\mathrm{cp}}$ if and only if $m = 1$. □

Proof of Theorem 8.23. As seen above, isomorphism classes of simple bundles are in bijection with isomorphism classes of simple objects in the category $\mathsf{MP}_{\mathrm{cp}}$. Let $M \in \mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ be a simple object. If $(n_1, n_2) \neq (1, 1)$, then $n_1 \neq n_2$. If $n_1 > n_2$, we can reduce the matrix M to the form

$$M = \left(\begin{array}{c|c||c} M'_{11} & M'_{12} & 0 \\ \hline 0 & M'_{22} & I \\ \hline\hline \times & \times & 0 \end{array}\right)$$

and if $n_2 > n_1$ to the form

$$M = \left(\begin{array}{c||c|c} 0 & I & 0 \\ \hline\hline \times & M'_{11} & M'_{12} \\ \hline \times & 0 & M'_{22} \end{array}\right).$$

In both cases, the matrix

$$M' = \left(\begin{array}{c|c} M'_{11} & M'_{12} \\ \hline \times & M'_{22} \end{array}\right)$$

is an object of $\mathsf{MP}_{\mathrm{cp}}$. Moreover, in this way we get a functor

$$\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2) \longrightarrow \begin{cases} \mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2 - n_1) & \text{if} \quad n_2 > n_1, \\ \mathsf{MP}^s_{\mathrm{cp}}(n_1 - n_2, n_2) & \text{if} \quad n_1 > n_2, \end{cases}$$

which is an equivalence of categories. This means that the matrix problem describing simple objects of the category $\mathsf{MP}_{\mathrm{cp}}$ for a cuspidal cubic curve E is self-reproducing.

As in the case of a nodal curve, these functors allow to construct in a unique way the matrix $M_{n_1,n_2}(\lambda)$ for a given pair of coprime positive integers n_1 and n_2 and $\lambda \in \mathbb{C}$, starting with a given matrix

$$M_{1,1}(\lambda) = \left(\begin{array}{c|c} \lambda & 1 \\ \hline \times & 0 \end{array}\right)$$

Moreover, $M_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda')$ in $\mathsf{MP}_{\mathrm{cp}}$ if and only if $\lambda = \lambda'$. □

Remark 8.28. Note that the only non-zero diagonal element of $M_{n_1,n_2}(\lambda)$ is the moduli parameter $\lambda \in \mathbb{C}$.

Lemma 8.29. *For any pair of positive coprime integers (n_1, n_2) with $n = n_1 + n_2$, there exists an isomorphism in the category $\mathsf{MP}_{\mathrm{cp}}$*

$$M_{n_1,n_2}(\lambda) + \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n) \cong M_{n_1,n_2}(\lambda + \alpha_1 + \alpha_2 + \dots + \alpha_n).$$

Proof. We proceed by induction on the size n of the matrix M. The case $n_1 = n_2 = 1$ is obvious. Assume the statement is true for all pairs (n_1, n_2) such that $n_1 + n_2 \le n$.

Let us look at the case where M was obtained by a reductions step from an object of $\mathsf{MP}_{\mathrm{cp}}(n_1 + n_2, n_2)$, the other case is completely analogous. Let

$$M = \left(\begin{array}{c|c} A & B \\ \hline \times & C \end{array}\right) \quad \text{and} \quad M' = \left(\begin{array}{c|c||c} A & B & 0 \\ \hline 0 & C & I \\ \hline\hline \times & \times & 0 \end{array}\right)$$

be the corresponding object of $\mathsf{MP}_{\mathrm{cp}}(n_1 + n_2, n_2)$. Let A', C' and D' be diagonal matrices of appropriate sizes, then

$$\begin{aligned} \left(\begin{array}{c|c||c} A + A' & B & 0 \\ \hline 0 & C + C' & I \\ \hline\hline \times & \times & D' \end{array}\right) &\cong \left(\begin{array}{c|c||c} A + A' & B & 0 \\ \hline 0 & C + C' + D' & I \\ \hline\hline \times & \times & 0 \end{array}\right) \\ &\cong \left(\begin{array}{c|c||c} A + \alpha I & B & 0 \\ \hline 0 & C & I \\ \hline\hline \times & \times & 0 \end{array}\right), \end{aligned}$$

where the second isomorphism follows by induction and $\alpha = \mathrm{tr}(A') + \mathrm{tr}(C') + \mathrm{tr}(D')$. This implies the claim. □

Remark 8.30. Similarly to the case of a nodal cubic curve, this lemma implies that for any pair of coprime positive integers n_1, n_2 there exists an object $N_{n_1,n_2}(\lambda)$ isomorphic to $M_{n_1,n_2}(\lambda)$ in $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ and such that the canonical form of the corresponding triple is compatible with the action of the Jacobian $\mathsf{Pic}^0(E)$.

Remark 8.31. The above consideration shows that the action of the Jacobian $\mathsf{Pic}^0(E)$ on the set $M_E(n, d)$ of stable vector bundles of rank n and degree d is transitive. Note that the group $\mathsf{Pic}^0(E)$ is torsion free. If $\mathcal{V}$ is a vector bundle of rank n and $\mathcal{L}$ is a line bundle, we have $\det(\mathcal{V} \otimes \mathcal{L}) \cong \det(\mathcal{V}) \otimes \mathcal{L}^{\otimes n}$. Hence, $\det : M_E(n,d) \longrightarrow \mathsf{Pic}^d(E)$ is a bijection.

Remark 8.32. One can describe a universal family $\mathcal{P}(n,d)$ of stable vector bundles of rank n and degree d on a cuspidal cubic curve E in a similar way as it was done for a nodal cubic curve in Proposition 8.21.

Example 8.33. The following family of triples $\big(\widetilde{\mathcal{V}}, \mathcal{N}, \mu\big)$ defines a universal family of stable vector bundles of rank 2 and degree 1 on a cuspidal cubic curve E: $\widetilde{\mathcal{V}} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $\mathcal{N} = \mathbb{C}_s^2$ and

$$\mu = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) + \varepsilon \left(\begin{array}{c|c} \frac{\lambda}{2} & 1 \\ \hline 0 & \frac{\lambda}{2} \end{array}\right).$$

9. Computations of r–matrices for singular Weierstrass curves

Let E be a singular Weierstraß cubic curve, $(\mathcal{V}_1, \mathcal{V}_2, y_1, y_2)$ be as in Section 4. Our aim is to derive an explicit procedure to calculate the map

$$\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})$$

defined through the commutative diagram

$$\begin{tikzcd} & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \arrow[dl, "\mathrm{res}_{y_1}"'] \arrow[dr, "\mathrm{ev}_{y_2}"] & \\ \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \arrow[rr, "\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}"] & & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}). \end{tikzcd}$$

9.1. Computation of the residue and evaluation maps on singular Weierstraß cubic curves.

Let X be an arbitrary Gorenstein projective curve, $x \in X$ a smooth point, $i_x : \mathsf{pt} \to X$ the embedding map, $\mathcal{V}_1, \mathcal{V}_2$ two vector bundles on X. For the sheaf $\Omega_X = \Omega_X^{1,R}$ of regular differential 1-forms we have an exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(x) \xrightarrow{\mathrm{res}_x} \mathbb{C}_x \longrightarrow 0$$

inducing a commutative diagram

$$\begin{tikzcd} \mathcal{H}om_X\big(\mathcal{V}_1, \mathcal{V}_2(x)\big) \arrow[r] \arrow[dr, "res_x"'] & \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2 \otimes \Omega_X^\vee \otimes \mathbb{C}_x) \arrow[d] \\ & \mathcal{H}om_X(\mathcal{V}_1 \otimes \Omega_X \otimes \mathbb{C}_x, \mathcal{V}_2 \otimes \mathbb{C}_x) \end{tikzcd}$$

of coherent sheaves on X. Taking global sections we obtain a commutative diagram of vector spaces

$$\begin{tikzcd} \mathsf{Hom}_X\big(\mathcal{V}_1, \mathcal{V}_2(x)\big) \arrow[r] \arrow[dr, "\mathrm{res}_x"'] & \mathsf{Hom}_X(\mathcal{V}_1, \mathcal{V}_2 \otimes \Omega_X^\vee \otimes \mathbb{C}_x) \arrow[d, "\cong"] \\ & \mathsf{Hom}_X(\mathcal{V}_1 \otimes \Omega_X \otimes \mathbb{C}_x, \mathcal{V}_2 \otimes \mathbb{C}_x) \end{tikzcd}$$

where $\mathrm{res}_x = res_x(X)$. Moreover, we have a canonical isomorphism of sheaves

$$\mathcal{H}om_X\big(\mathcal{V}_1, \mathcal{V}_2(x)\big) \cong \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2) \otimes \mathcal{O}_X(x)$$

and the proof of the next lemma is straightforward.

Lemma 9.1. *The following diagram is commutative:*

$$\begin{tikzcd} \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2(x)) \arrow[r] \arrow[dr, "res_x"'] & \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2) \otimes \mathcal{O}_X(x) \arrow[d, "res'_x"] \\ & \mathcal{H}om_X(\mathcal{V}_1 \otimes \Omega_X \otimes \mathbb{C}_x, \mathcal{V}_2 \otimes \mathbb{C}_x) \end{tikzcd}$$

where the map res'_x is defined on the level of presheaves by the formula

$$res'_x\left(\sum_i s_i \otimes f_i\right)(v \otimes w \otimes 1) = \sum_i s_i(v) \otimes \mathrm{res}_x(f_i w).$$

Let E be a singular Weierstraß cubic curve, $\pi : \mathbb{P}^1 \to E$ its normalisation, $s \in E$ the singular point of E. Choose coordinates $(z_0 : z_1)$ on $\mathbb{P}^1$ in such a way that

$$\begin{aligned} \pi^{-1}(s) &= \{0, \infty\} && \text{for } E \text{ nodal;} \\ \pi^{-1}(s) &= \infty && \text{for } E \text{ cuspidal.} \end{aligned}$$

For $U = \pi^{-1}(E_{\mathrm{reg}})$ we denote by π_U the composition map $U \xrightarrow{\pi} E_{\mathrm{reg}} \hookrightarrow E$ and choose coordinates on E_{reg} such that for $y \in E_{\mathrm{reg}}$ we have $\tilde{y} := \pi^{-1}(y) = (1 : y)$. Next, we identify the one-dimensional space $H^0(E, \Omega_E)$ with its image in $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1} \otimes \mathcal{M}_{\mathbb{P}^1})$, where $\mathcal{M}_{\mathbb{P}^1}$ denotes the sheaf of meromorphic functions on $\mathbb{P}^1$. In these terms $H^0(E, \Omega_E)$ is generated by

$$\begin{aligned} \omega &= \frac{dz}{z} = -\frac{dw}{w} && \text{if } E \text{ is nodal} \\ \omega &= dz = \frac{dw}{w^2} && \text{if } E \text{ is cuspidal.} \end{aligned}$$

Since we want to make calculations for vector bundles on E using their pull-backs on $\mathbb{P}^1$, we have to describe the map $\tilde{r}^{\mathcal{V}_1, \mathcal{V}_2}_{y_1, y_2}$ in terms of morphisms of sheaves on $\mathbb{P}^1$.

Note that the following diagram is commutative

$$
\begin{tikzcd}
 & \mathsf{Hom}_E\big(\mathcal{V}_1 \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y\big) \\
\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y)\big) \arrow[r, "\mathrm{res}_y"] \arrow[d] \arrow[ur, bend left] & \mathsf{Hom}_E\big(\mathcal{V}_1 \otimes \Omega_E \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y\big) \arrow[u, "\cdot\omega"'] \arrow[d, "\cong"] \\
\mathsf{Hom}_{E_{\mathrm{reg}}}\big(\mathcal{V}_1|_{E_{\mathrm{reg}}}, \mathcal{V}_2(y)|_{E_{\mathrm{reg}}}\big) \arrow[r, "\mathrm{res}_y"] \arrow[d, "\cong"'] & \mathsf{Hom}_{E_{\mathrm{reg}}}\big(\mathcal{V}_1|_{E_{\mathrm{reg}}} \otimes \Omega_{E_{\mathrm{reg}}} \otimes \mathbb{C}_y, \mathcal{V}_2|_{E_{\mathrm{reg}}} \otimes \mathbb{C}_y\big) \arrow[d, "\cong"] \\
\mathsf{Hom}_U\big(\pi_U^*\mathcal{V}_1, (\pi_U^*\mathcal{V}_2)(\tilde{y})\big) \arrow[r, "\mathrm{res}_{\tilde{y}}"] & \mathsf{Hom}_E\big(\mathcal{V}_1|_U \otimes \Omega^1_U \otimes \mathbb{C}_{\tilde{y}}, \mathcal{V}_2|_U \otimes \mathbb{C}_{\tilde{y}}\big) \\
\mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{V}_1, (\pi^*\mathcal{V}_2)(\tilde{y})\big) \arrow[r, "\mathrm{res}_{\tilde{y}}"] \arrow[u] & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{V}_1 \otimes \Omega^1_{\mathbb{P}^1} \otimes \mathbb{C}_{\tilde{y}}, \pi^*\mathcal{V}_2 \otimes \mathbb{C}_{\tilde{y}}\big) \arrow[u, "\cong"']
\end{tikzcd}
$$

This means, we can compute the residue map res_y with the aid of the following commutative diagram

$$
\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y)\big) \arrow[r] \arrow[ddd, "\mathrm{res}_y"'] & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{V}_1, \pi^*\mathcal{V}_2(\tilde{y})\big) \arrow[d, "\mathrm{res}_{\tilde{y}}"] \\
 & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{V}_1 \otimes \Omega_{\mathbb{P}^1} \otimes \mathbb{C}_{\tilde{y}}, \pi^*\mathcal{V}_2 \otimes \mathbb{C}_{\tilde{y}}\big) \arrow[d, "\cong"] \\
 & \mathsf{Hom}_U\big(\pi_U^*\mathcal{V}_1 \otimes \Omega_U \otimes \mathbb{C}_{\tilde{y}}, \pi_U^*\mathcal{V}_2 \otimes \mathbb{C}_{\tilde{y}}\big) \arrow[d, "\cdot\omega"] \\
\mathsf{Hom}_E\big(\mathcal{V}_1 \otimes \mathbb{C}_y, \mathcal{V}_2 \otimes \mathbb{C}_y\big) \arrow[r, "\cong"] & \mathsf{Hom}_U\big(\pi_U^*\mathcal{V}_1 \otimes \mathbb{C}_{\tilde{y}}, \pi_U^*\mathcal{V}_2 \otimes \mathbb{C}_{\tilde{y}}\big).
\end{tikzcd}
$$

In these two diagrams and the following one, the non-labelled arrows are either defined in the previous sections or are canonical.

Let $\widetilde{\mathcal{V}} \cong \pi^*\mathcal{V}_1 \cong \pi^*\mathcal{V}_2 \in \mathsf{VB}(\mathbb{P}^1)$, then we describe $\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y)\big)$ as a subspace of $\mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)\big)$, see Section 8. The following diagram is commutative

$$\begin{tikzcd}
 & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_y, \mathcal{V}_2|_y) \arrow[r, "\cong"] & \mathsf{Hom}_{\mathbb{C}}\left(\widetilde{\mathcal{V}}|_{\tilde{y}}, \widetilde{\mathcal{V}}|_{\tilde{y}}\right) \\
 & \mathsf{Hom}_U(\pi^*\mathcal{V}_1 \otimes \mathbb{C}_{\tilde{y}}, \pi^*\mathcal{V}_2 \otimes \mathbb{C}_{\tilde{y}}) \arrow[u] & \\
\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y)\big) \arrow[uur, bend left, "\mathrm{res}_y"] \arrow[ddr, bend right] & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{V}_1, \pi^*\mathcal{V}_2(\tilde{y})\big) \arrow[u] & \mathsf{Hom}_U(\widetilde{\mathcal{V}} \otimes \mathbb{C}_{\tilde{y}}, \widetilde{\mathcal{V}} \otimes \mathbb{C}_{\tilde{y}}) \arrow[uu] \\
 & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(\tilde{y})\big) \arrow[u] \arrow[ur] & \\
 & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)\big) \arrow[u] \arrow[uur, bend right, "\widetilde{\mathrm{res}}_y"'] &
\end{tikzcd}$$

Take a set of isomorphisms $\{\tau_l : \mathcal{O}_{\mathbb{P}^1}(l) \otimes \mathcal{O}_U \to \mathcal{O}_U\}_{l \in \mathbb{Z}}$ given locally by $\tau_l(p) = \frac{p}{z_0^l}$. Then the map $\widetilde{\mathrm{res}}_y$, which is defined as the composition

$$\mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(\tilde{y})) \longrightarrow \mathsf{Hom}_U(\widetilde{\mathcal{V}} \otimes \mathbb{C}_{\tilde{y}}, \widetilde{\mathcal{V}} \otimes \mathbb{C}_{\tilde{y}}) \longrightarrow \mathsf{Mat}_n(\mathbb{C}),$$

is given by the formula

$$F \mapsto \frac{F(z_0, z_1)}{z_1 - z_0 y} \mapsto \mathrm{res}_y\left(\frac{F(1:z)}{z-y}\omega\right),$$

where ω is a global section of Ω_E viewed as a meromorphic 1-form on $\mathbb{P}^1$. Therefore, the residue map $\mathrm{res}_y : \mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2(y)) \longrightarrow \mathsf{Mat}_n(\mathbb{C})$ sends F to $\mathrm{res}_y\left(\frac{F(z)}{(z-y)}\omega\right)$, where $F \in \mathsf{Im}\left\{\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)\big)\right\}$ and $\omega \in H^0(\Omega_E)$ is the chosen global regular differential 1-form on E. Thus, we end up with the formula

$$\mathrm{res}_y(F) = \begin{cases} \dfrac{F(1:y)}{y} & \text{if } E \text{ is nodal} \\ F(1:y) & \text{if } E \text{ is cuspidal.} \end{cases}$$

The rule to compute the evaluation map ev_y is similar. Indeed, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \arrow[r] \arrow[d, "\pi^*"'] & \mathsf{Hom}_{\mathbb{C}}\big(\mathcal{V}_1|_{y_2}, \mathcal{V}_2(y_1)|_{y_2}\big) & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \arrow[l] \arrow[d] \\
\mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{V}_1, \pi^*\mathcal{V}_2(\tilde{y}_1)\big) & & \mathsf{Hom}_{\mathbb{C}}\big(\pi^*\mathcal{V}_1|_{\tilde{y}_2}, \pi^*\mathcal{V}_2(\tilde{y}_1)|_{\tilde{y}_2}\big) \\
\mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(\tilde{y}_1)\big) \arrow[u] \arrow[rr, "\mathrm{ev}_{\tilde{y}_2}"] & & \mathsf{Hom}_{\mathbb{C}}\big(\widetilde{\mathcal{V}}|_{\tilde{y}_2}, \widetilde{\mathcal{V}}(\tilde{y}_1)|_{\tilde{y}_2}\big) \arrow[u] \\
\mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)\big) \arrow[u, "\frac{1}{z_1 - z_0 y_1}"] \arrow[r] & \mathsf{Hom}_{\mathbb{C}}\big(\widetilde{\mathcal{V}}|_{\tilde{y}_2}, \widetilde{\mathcal{V}}(1)|_{\tilde{y}_2}\big) & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \arrow[l, "z_0"'] \arrow[u, "\frac{1}{y_2 - y_1}"']
\end{tikzcd}$$

Taking the same set of isomorphisms $\{\tau_l : \mathcal{O}_{\mathbb{P}^1}(l) \otimes \mathcal{O}_U \to \mathcal{O}_U\}_{l \in \mathbb{Z}}$ as for the map res_{y_1}, we get the following rule to calculate the map ev_{y_2}:

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2(y_1)) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)) \xrightarrow{\widetilde{\mathrm{ev}}_{y_2}} \mathsf{Mat}_n(\mathbb{C}),$$

where

$$\widetilde{\mathrm{ev}}_{y_2}(F) = \frac{1}{y_2 - y_1} F(1 : y_2)$$

for a global section $F \in \mathsf{Im}\Big(\pi^* : \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)\big)\Big)$ in both cases, the nodal and the cuspidal.

Combining the above, we see that we can compute the map $\tilde{r}^{\mathcal{V}_1, \mathcal{V}_2}_{y_1, y_2}$ using the commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{V}^{\lambda_1}, \mathcal{V}^{\lambda_2}(y_1)\big) \arrow[dr, "\pi^*"] \arrow[drr, bend left, "\mathrm{res}_{y_1}"] \arrow[ddr, bend right, "\mathrm{ev}_{y_2}"'] & & \\
& \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}(1)\big) \arrow[r, "\widetilde{\mathrm{res}}_{y_1}"] \arrow[d, "\widetilde{\mathrm{ev}}_{y_2}"] & \mathsf{Mat}_n(\mathbb{C}) \arrow[dl, "\tilde{r}^{v_1, v_2}_{y_1, y_2}"] \\
& \mathsf{Mat}_n(\mathbb{C}) &
\end{tikzcd}$$

where $\mathcal{V}^\lambda := \mathcal{P}(n, d)|_{E \times \{\lambda\}}$ for $\lambda \in M = M_E(n, d)$ and $\mathcal{P}(n, d) \in \mathsf{VB}(E \times M)$ is a universal sheaf.

9.2. **Trigonometric solutions obtained from a nodal cubic curve.** Let E be a nodal Weierstraß cubic curve. In this subsection we calculate the associative r-matrices corresponding to the moduli spaces of rank two (semi-)stable vector bundles on a nodal Weierstraß curve. We use the notation from Subsection 8.2.

We start with the case of the moduli space of stable vector bundles of rank 2 and degree 1, $M = M_E(2,1) = \mathbb{C}^*$. It is convenient to use the local homeomorphism $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ given by $\sigma(z) = z^2$, because, according to Example 8.14 and Remark 8.20, the family of stable vector bundles $(1 \times \sigma)^*\mathcal{P}(2,1)$ is then given by the triple $\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^2_s, \mu\bigr)$, where

$$\mu(0) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \lambda \in \mathbb{C}^* \quad \text{and} \quad \mu(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our goal is to compute the map

$$r^{\lambda_1,\lambda_2}_{y_1,y_2} : \mathsf{Mat}_2(\mathbb{C}) \longrightarrow \mathsf{Mat}_2(\mathbb{C}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix}.$$

<u>Step 1.</u> In order to calculate the entries φ, ψ, η, ξ we first need to describe the image of the map

$$\pi^* : \mathsf{Hom}_E\bigl(\mathcal{V}^{\lambda_1}, \mathcal{V}^{\lambda_2}(y_1)\bigr) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\bigr).$$

By Subsection 8.2 a morphism

$$F = \begin{pmatrix} a'z_0 + a''z_1 & t \\ b'z_0^2 + b''z_0z_1 + b'''z_1^2 & d'z_0 + d''z_1 \end{pmatrix} \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\bigr)$$

has the following evaluation rule:

$$F(0) = \begin{pmatrix} -a' & t \\ b' & -d' \end{pmatrix}, \qquad F(\infty) = \begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix}.$$

¿From the definition of the category of triples we see that F belongs to the image of $\mathsf{Hom}_E(\mathcal{V}^{\lambda_1}, \mathcal{V}^{\lambda_2}(y_1))$ under the linear map π^* if and only if there exists a matrix

$\varphi \in \mathsf{Mat}_2(\mathbb{C})$ making the following diagram commutative:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{C}^2_0 \oplus \mathbb{C}^2_\infty \arrow[r, "\begin{pmatrix} F(0) & 0 \\ 0 & F(\infty)\end{pmatrix}"] & \mathbb{C}^2_0 \oplus \mathbb{C}^2_\infty \\
\mathbb{C}^2_0 \oplus \mathbb{C}^2_\infty \arrow[u, "\begin{pmatrix} \left(\begin{smallmatrix} 0 & \lambda_1 \\ \lambda_1 & 0\end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)\end{pmatrix}"] \arrow[r, "\begin{pmatrix} \varphi & 0 \\ 0 & \varphi\end{pmatrix}"] & \mathbb{C}^2_0 \oplus \mathbb{C}^2_\infty \arrow[u, swap, "\begin{pmatrix} \left(\begin{smallmatrix} 0 & \lambda_2 y_1 \\ \lambda_2 y_1 & 0\end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)\end{pmatrix}"]
\end{tikzcd}$$

This is equivalent to the equations:

$$F(\infty) = \varphi \quad \text{and} \quad F(0)\begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_2 y_1 \\ \lambda_2 y_1 & 0 \end{pmatrix}\varphi.$$

Taking a'', b'', b''', d'' as free variables and solving the above system we get

$$\begin{cases} a' &= & -\lambda y_1 d'' \\ d' &= & -\lambda y_1 a'' \\ t &= & \lambda y_1 b''' \\ b' &= & (\lambda y_1)^2 b''', \qquad \text{where } \lambda = \dfrac{\lambda_2}{\lambda_1}. \end{cases}$$

Step 2. Next, the equation $\operatorname{res}_{y_1}(F) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ reads as

$$\begin{pmatrix} a' + a''y_1 & t \\ b' + b''y_1 + b'''y_1^2 & d' + d''y_1 \end{pmatrix} = y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

¿From this we obtain

$$\begin{cases} a' &= -\dfrac{\lambda y_1}{1-\lambda^2}(d + \lambda a) \\ a'' &= \dfrac{1}{1-\lambda^2}(a + \lambda d) \\ b' &= \lambda y_1^2 b \\ b'' &= c - \dfrac{\lambda^2+1}{\lambda} y_1 b \\ b''' &= \dfrac{1}{\lambda} b \\ d' &= -\dfrac{\lambda y_1}{1-\lambda^2}(a + \lambda d) \\ d'' &= \dfrac{1}{1-\lambda^2}(d + \lambda a) \\ t &= y_1 b. \end{cases}$$

Step 3. By the formula for the evaluation map it holds:

$$\mathrm{ev}_{y_2}(F) = \frac{1}{y_2 - y_1}\begin{pmatrix} a' + a''y_2 & t \\ b' + b''y_2 + b'''y_2^2 & d' + d''y_2 \end{pmatrix} = \begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix}.$$

This implies

$$\begin{cases} \varphi &= \dfrac{y_2 - \lambda^2 y_1}{1-\lambda^2}a + \dfrac{\lambda(y_2-y_1)}{1-\lambda^2}d \\ \psi &= y_1 b \\ \xi &= \dfrac{\lambda(y_2-y_1)}{1-\lambda^2}a + \dfrac{y_2 - \lambda^2 y_1}{1-\lambda^2}d \\ \eta &= \dfrac{(y_2-y_1)(y_2-\lambda^2 y_1)}{\lambda}b + y_2 c. \end{cases}$$

In order to calculate the corresponding solution of the associative Yang-Baxter equation we use the inverse of the canonical isomorphism

$$\mathsf{Mat}_2(\mathbb{C}) \otimes \mathsf{Mat}_2(\mathbb{C}) \longrightarrow \mathsf{Lin}\big(\mathsf{Mat}_2(\mathbb{C}), \mathsf{Mat}_2(\mathbb{C})\big)$$

given by $X \otimes Y \mapsto \mathrm{tr}(XY)Y$. It is easy to see that under this inverse

$$\mathsf{Lin}\big(\mathsf{Mat}_2(\mathbb{C}), \mathsf{Mat}_2(\mathbb{C})\big) \longrightarrow \mathsf{Mat}_2(\mathbb{C}) \otimes \mathsf{Mat}_2(\mathbb{C})$$

a linear function $e_{ij} \mapsto \alpha_{ij}^{kl} e_{kl}, \alpha_{ij}^{kl} \in \mathbb{C}^*$ corresponds to the tensor $\alpha_{ij}^{kl} e_{ji} \otimes e_{kl}$. Having this rule in mind we obtain the desired associative r–matrix:

$$r(\lambda; y_1, y_2) = \frac{y_2 - \lambda^2 y_1}{(y_2-y_1)(1-\lambda^2)}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) + \frac{\lambda}{1-\lambda^2}(e_{11}\otimes e_{22} + e_{22}\otimes e_{11})$$
$$+ \frac{y_1}{y_2-y_1}e_{21}\otimes e_{12} + \frac{y_2}{y_2-y_1}e_{12}\otimes e_{21} + \frac{y_2-\lambda^2 y_1}{\lambda}e_{21}\otimes e_{21}.$$

The gauge transformation $\varphi(z) = \varphi(\mu; z) : (\mathbb{C}^2, 0) \longrightarrow \mathsf{Aut}\big(\mathsf{Mat}_n(\mathbb{C})\big)$ (see Definition 2.5) given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{z}} & 0 \\ 0 & 1 \end{pmatrix}$$

yields the transformation

$$\begin{cases} e_{ii}\otimes e_{jj} \mapsto e_{ii}\otimes e_{jj}, \quad i,j \in \{1,2\} \\ e_{21}\otimes e_{12} \mapsto \sqrt{\frac{y_2}{y_1}}e_{21}\otimes e_{12} \\ e_{12}\otimes e_{21} \mapsto \sqrt{\frac{y_1}{y_2}}e_{12}\otimes e_{21} \\ e_{21}\otimes e_{21} \mapsto \frac{1}{\sqrt{y_1 y_2}}e_{21}\otimes e_{21}. \end{cases}$$

Thus, we end end up with the solution

$$r(\lambda, y) = \frac{y-\lambda^2}{(y-1)(1-\lambda^2)}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) + \frac{\lambda}{1-\lambda^2}(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}) +$$
$$+ \frac{\sqrt{y}}{y-1}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \left(\frac{\sqrt{y}}{\lambda} - \frac{\lambda}{\sqrt{y}}\right)e_{21}\otimes e_{21},$$

where $y = \dfrac{y_2}{y_1}$. Using the notation $\mathbb{1} = e_{11} + e_{22}$, this can be rewritten as

$$r(\lambda, y) = \frac{\mathbb{1} \otimes \mathbb{1}}{1-\lambda^2} + \frac{1}{y-1}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) - \frac{1}{\lambda+1}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) \\ + \frac{\sqrt{y}}{y-1}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \left(\frac{\sqrt{y}}{\lambda} - \frac{\lambda}{\sqrt{y}}\right) e_{21} \otimes e_{21}.$$

This is a solution of the associative Yang-Baxter equation of type (8), and by Theorem 2.8 this tensor also satisfies the quantum Yang-Baxter equation.

In order to rewrite $r(\lambda; y)$ in the additive form, we make the change of variables $y = \exp(2iz), \lambda = \exp(iv)$. Making a gauge transformation we can multiply the tensor $e_{21} \otimes e_{12}$ with an arbitrary scalar without changing the coefficients of the other tensors. Therefore, we obtain

$$2r_{\mathrm{trg}}(v, z) = \frac{\sin(z+v)}{\sin(z)\sin(v)}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) - \frac{1}{\cos(v)}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) + \\ + \frac{1}{\sin(z)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(z+v)e_{21} \otimes e_{12}.$$

Up to a scalar, the corresponding solution $\bar{r}(z) := \lim_{v \to 0}(\mathrm{pr} \otimes \mathrm{pr})r(v; z)$ of the classical Yang-Baxter equation is the trigonometric solution of Cherednik:

$$\bar{r}_{\mathrm{trg}}(z) = \frac{1}{2}\cot(z) h \otimes h + \frac{1}{\sin(z)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(z) e_{21} \otimes e_{21}.$$

Our next goal is to construct a solution $r(v; y)$ of the associative Yang-Baxter equation (5) having a higher-order pole with respect to v.

The triple $\left(\mathcal{O}_{\mathbb{P}^1}^2, \mathbb{C}_s^2, \mu\right)$ with

$$\mu(0) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \quad \text{and} \quad \mu(\infty) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

describes a universal family of semi-stable indecomposable vector bundles of rank two and degree one, having locally free Jordan-Hölder factors.

Step 1. First we compute the image of the normalisation map

$$\pi^* : \mathsf{Hom}_E(\mathcal{V}^{\lambda_1}, \mathcal{V}^{\lambda_2}(y_1)) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big).$$

Recall that for a morphism

$$F = \begin{pmatrix} a'z_0 + a''z_1 & b'z_0 + b''z_1 \\ c'z_0 + c''z_1 & d'z_0 + d''z_1 \end{pmatrix} \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$$

we have chosen the following evaluation rules

$$F(0) = -F' := -\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad F(\infty) = F'' := \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

Thus, F belongs to the image of the map π^* if and only if it holds

$$F(0)\begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_2 y_1 & \lambda_2 y_1 \\ 0 & \lambda_2 y_1 \end{pmatrix} F(\infty).$$

This implies that

$$F' = -\lambda y_1 F'' + \lambda y_1 \begin{pmatrix} -c'' & a'' + c'' - d'' \\ 0 & c'' \end{pmatrix}.$$

Step 2. The equation $\operatorname{res}_{y_1}(F) = y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ reads $F' + y_1 F'' = y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Solving this equation we obtain

$$\begin{cases} a'' & = & \dfrac{1}{1-\lambda}a + \dfrac{\lambda}{(1-\lambda)^2}c \\ b'' & = & \dfrac{\lambda}{1-\lambda}a + \dfrac{1}{1-\lambda}b - \dfrac{\lambda(\lambda+1)}{(1-\lambda)^3}c + \dfrac{\lambda}{(1-\lambda)^2}d \\ c'' & = & \dfrac{1}{1-\lambda}c \\ d'' & = & -\dfrac{\lambda}{(1-\lambda)^2}c + \dfrac{1}{1-\lambda}d. \end{cases}$$

Step 3. From the formula $\operatorname{ev}_{y_2}(F) = \dfrac{1}{y_2 - y_1}(F' + y_2 F'')$ we obtain:

$$\tilde{r}^{\lambda_1,\lambda_2}_{y_1,y_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi & \psi \\ \eta & \zeta \end{pmatrix},$$

where

$$\begin{cases} \varphi = \dfrac{y-\lambda}{(y-1)(1-\lambda)}a + \dfrac{\lambda}{(1-\lambda)^2}c \\ \eta = \dfrac{y-\lambda}{(y-1)(1-\lambda)}c \\ \xi = -\dfrac{\lambda}{(1-\lambda)^2}c + \dfrac{y-\lambda}{(y-1)(1-\lambda)}d \\ \psi = -\dfrac{\lambda}{(1-\lambda)^2}a + \dfrac{y-\lambda}{(y-1)(1-\lambda)}b - \dfrac{\lambda(1+\lambda)}{(1-\lambda)^3}c + \dfrac{\lambda}{(1-\lambda)^2}d \end{cases}$$

and $y = \dfrac{y_2}{y_1}$, $\lambda = \dfrac{\lambda_2}{\lambda_1}$. Hence, we obtain the associative r–matrix

$$r(\lambda; y) = \frac{y-\lambda}{(y-1)(1-\lambda)}\big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{21}\otimes e_{12} + e_{12}\otimes e_{21}\big) +$$

$$+\frac{\lambda}{(1-\lambda)^2}\big(e_{12}\otimes h - h\otimes e_{12}\big) - \frac{\lambda(1+\lambda)}{(1-\lambda)^3}e_{12}\otimes e_{12}.$$

Denoting $y = \exp(2iz)$, $\lambda = \exp(-2iv)$ and making a gauge transformation

$$e_{11} \mapsto e_{11}, e_{22} \mapsto e_{22}, e_{12} \mapsto 2e_{12} \text{ and } e_{21} \mapsto \frac{1}{2}e_{21}$$

we finally end up with an associative r–matrix

$$\begin{aligned} r(v; z) = & \frac{\sin(z+v)}{2\sin(z)\sin(v)}\bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{21}\otimes e_{12} + e_{12}\otimes e_{21}\bigr) \\ & + \frac{1}{2\sin^2(v)}\bigl(e_{12}\otimes h - h\otimes e_{12}\bigr) - \frac{\cos(v)}{\sin^3(v)}e_{12}\otimes e_{12}. \end{aligned}$$

Remark 9.2. Since $\lim\limits_{v\to 0}(\mathrm{pr}\otimes\mathrm{pr})(r(v;z))$ does not exist, the family of indecomposable semi-stable vector bundles of rank two and degree zero on a nodal Weierstraß curve E, whose Jordan-Hölder factors are locally free, does not give a solution of the classical Yang-Baxter equation.

9.3. **A rational solution obtained from a cuspidal cubic curve.** In this subsection we shall calculate the rational solution of the classical Yang-Baxter equation, obtained from a universal family of stable vector bundles of rank 2 and degree 1 on a cuspidal cubic curve. In terms of Subsection 8.3 it is described by the family of triples $\bigl(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^2_s, \mu\bigr)$, where

$$\mu = \mu(0) + \varepsilon\mu(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{C}.$$

As in the previous subsection, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Mat}_2(\mathbb{C}) \quad \text{and} \quad \begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix} = \tilde{r}^{\lambda_1,\lambda_2}_{y_1,y_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Step 1. Again, we start by calculating the image of the linear map

$$\pi^* : \mathsf{Hom}_E\bigl(\mathcal{V}^{\lambda_1}, \mathcal{V}^{\lambda_2}(y_1)\bigr) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\bigr).$$

Recall that we have the following rules to evaluate a morphism

$$F = \begin{pmatrix} a'z_0 + a''z_1 & t \\ b'z_0^2 + b''z_0z_1 + b'''z_1^2 & d'z_0 + d''z_1 \end{pmatrix} \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\bigr)$$

on the analytic subspace $\widetilde{Z}$:

$$F \mapsto \begin{pmatrix} a'' + a'\varepsilon & t \\ b''' + b''\varepsilon & d'' + d'\varepsilon \end{pmatrix}.$$

¿From the definition of the category of triples we see that F belongs to the image of $\mathsf{Hom}_E(\mathcal{V}^{\lambda_1}, \mathcal{V}^{\lambda_2}(y_1))$ if and only if there exists a matrix $f \in \mathsf{Mat}_2(\mathbb{C})$ making the following diagram commutative

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{R}^2 \arrow[r, "\begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix} + \varepsilon \begin{pmatrix} a' & 0 \\ b'' & d' \end{pmatrix}"] & \mathsf{R}^2 \\
\mathsf{R}^2 \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}"] \arrow[r, "f"] & \mathsf{R}^2, \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \lambda_2 - y_1 & 1 \\ 0 & \lambda_2 - y_1 \end{pmatrix}"']
\end{tikzcd}$$

where $\mathsf{R} = \mathbb{C}[\varepsilon]/\varepsilon^2$. This leads to the equality

$$\begin{pmatrix} a' & 0 \\ b'' & d' \end{pmatrix} + \begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_2 - y_1 & 1 \\ 0 & \lambda_2 - y_1 \end{pmatrix} \begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix}.$$

Taking a'', b', b''' and t as free variables we obtain

$$\begin{cases} a' = (\lambda - y_1)a'' + b''' \\ b'' = (\lambda - y_1)b''' \\ d' = (\lambda - y_1)a'' - b''' - (\lambda - y_1)^2 t \\ d'' = a'' - (\lambda - y_1)t. \end{cases}$$

<u>Step 2.</u> By the formula for the residue map res_{y_1} it holds:

$$\mathrm{res}_{y_1}(F) = \begin{pmatrix} a' + a''y_1 & t \\ b' + b''y_1 + b'''y_1^2 & d' + d''y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from which we get:

$$\begin{cases} t &= b \\ a'' &= \dfrac{1}{2\lambda}a + \dfrac{\lambda - y_1}{2}b + \dfrac{1}{2\lambda}d \\ b' &= -\dfrac{\lambda y_1}{2}a + \dfrac{\lambda^2 y_1(\lambda - y_1)}{2}b + c + \dfrac{\lambda y_1}{2}d \\ b''' &= \dfrac{1}{2}a - \dfrac{\lambda(\lambda - y_1)}{2}b - \dfrac{1}{2}d \end{cases}$$

<u>Step 3.</u> Since the formula for the map ev_{y_2} is given by:

$$\mathrm{ev}_{y_2}(F) = \frac{1}{y_2 - y_1} \begin{pmatrix} a' + a''y_2 & t \\ b' + b''y_2 + b'''y_2^2 & d' + d''y_2 \end{pmatrix} = \begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix}$$

we obtain:

$$\begin{cases} \varphi &= (1+\dfrac{y_2-y_1}{2\lambda})a + \dfrac{(\lambda-y_1)(y_2-y_1)}{2}b + \dfrac{y_2-y_1}{2\lambda}d \\ \psi &= t \\ \eta &= \dfrac{(y_2-y_1)(y_2-y_1+\lambda)}{2}a - \dfrac{\lambda(\lambda-y_1)(\lambda+y_2)(y_2-y_1)}{2}b + \\ & + c - \dfrac{(y_2-y_1)(\lambda+y_2)}{2}d \\ \xi &= \dfrac{y_2-y_1}{2\lambda}a - \dfrac{(y_2-y_1)(y_1-\lambda)}{2}b + \left(1+\dfrac{y_2-y_1}{2\lambda}\right)d. \end{cases}$$

¿From this we get the following associative r-matrix:

$$r(\lambda, y_1, y_2) = \frac{1}{2\lambda}\mathbb{1}\otimes\mathbb{1} + \frac{1}{y_2-y_1}\big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\big) +$$
$$+ \frac{\lambda-y_1}{2}e_{21}\otimes h + \frac{\lambda+y_2}{2}h\otimes e_{21} + \frac{\lambda(\lambda-y_1)(\lambda+y_2)}{2}e_{21}\otimes e_{21}.$$

Projecting this matrix to $\mathfrak{sl}_2(\mathbb{C})\otimes\mathfrak{sl}_2(\mathbb{C})$ we obtain a rational solution of the classical Yang-Baxter equation

$$\bar{r}(y_1, y_2) = \frac{1}{y_2-y_1}\left(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\right) + \frac{y_2}{2}h\otimes e_{21} - \frac{y_1}{2}e_{21}\otimes h.$$

which was found for the first time by Stolin in [46]. It is easy to check that $\bar{r}(y_1, y_2)$ does not have infinitesimal symmetries, hence by Theorem 2.8 the tensor $r(\lambda, y_1, y_2)$ satisfy the Quantum Yang-Baxter equation. This solution was recently found by Khoroshkin, Stolin and Tolstoy [29].

10. Summary

Let us sum up the main analytical results obtained is this article. Consider the Weierstraß family of plane cubic curves $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where $g_2, g_3 \in \mathbb{C}$. Our main results shows that for any pair $(n, d) \in \mathbb{N}\times\mathbb{Z}$ of coprime integers and any $t = (g_2, g_3)$ in some small neighbourhood of $(0, 0)$ there exists a germ of a meromorphic function

$$r_t = r_t^{(n,d)} : (\mathbb{C}^2\times\mathbb{C}^2, 0) \longrightarrow \mathsf{Mat}_n(\mathbb{C})\otimes\mathsf{Mat}_n(\mathbb{C})$$

which satisfies the *associative Yang-Baxter equation*

$$r_t(v_3, v_2; y_1, y_2)^{12} r_t(v_1, v_3; y_1, y_3)^{13} - r_t(v_1, v_3; y_1, y_3)^{23} r_t(v_1, v_2; y_1, y_2)^{12} +$$
$$+ r_t(v_1, v_2; y_1, y_3)^{13} r_t(v_2, v_3; y_2, y_3)^{23} = 0.$$

Moreover, the function $r_t(v_1, v_2; y_1, y_2)$, which is called the *geometric associative r–matrix* in this paper, depends analytically on the parameter t.

For any family of germs holomorphic functions $\phi_t : (\mathbb{C}^2, 0) \to \mathsf{GL}_n(\mathbb{C})$, depending holomorphically on $t = (g_2, g_3)$, the function

$$\big(\phi_t(v_1, y_1) \otimes \phi_t(v_2, y_2)\big) r_t(v_1, v_2; y_1, y_2) \big(\phi_t(v_2, y_1)^{-1} \otimes \phi_t(v_1, y_2)^{-1}\big)$$

is again a solution of the associative Yang-Baxter equation. This defines an equivalence relation on the set of associative r-matrices.

Recall the following classical result.

Proposition 10.1 (see Section II.4 in [27]). *Let $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}^2$ be the corresponding lattice. Then the complex torus $\mathbb{C}/\Lambda_\tau$ is isomorphic to the projective cubic curve $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where*
(15)

$$g_2 = 60 \sum_{(m', m'') \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m' + m''\tau)^4}, \quad g_3 = 140 \sum_{(m', m'') \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m' + m''\tau)^6}.$$

Conversely, for any pair $(g_2, g_3) \in \mathbb{C}^2$ such that $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$ there exists a unique τ from the domain D given below, such that $(g_2, g_3) = \big(g_2(\tau), g_3(\tau)\big)$.

$$D = \left\{ \tau \in \mathbb{C} \,\middle|\, |\mathrm{Re}(\tau)| \le \frac{1}{2}, \quad |\tau| \ge 1 \ \textit{if} \ \mathrm{Re}(\tau) \le 0, \quad |\tau| > 1 \ \textit{if} \ \mathrm{Re}(\tau) > 0 \right\}$$

In the case $(n, d) = (2, 1)$, we obtained the following explicit results for the geometric associative r–matrix $r_t^{(2,1)}(v_1, v_2; y_1, y_2) \in \mathsf{Mat}_2(\mathbb{C}) \otimes \mathsf{Mat}_2(\mathbb{C})$ attached to the Weierstraß family.

• If $t = (g_2, g_3)$ satisfies $\Delta(t) \neq 0$ and $\tau \in D$ is defined by the Equations (15), then $E_t \cong \mathbb{C}/\Lambda_\tau$ is a smooth elliptic curve and the corresponding solution of the associative Yang-Baxter equation $r_t^{(2,1)}(v_1, v_2; y_1, y_2)$ is *equivalent* to the elliptic solution

$$r_t^{\mathrm{ell}}(v; y) = \frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)} \left[\frac{\theta_1(y + v|\tau)}{\theta_1(v|\tau)} \mathbb{1} \otimes \mathbb{1} + \frac{\theta_2(y + v|\tau)}{\theta_2(v|\tau)} h \otimes h + \right.$$
$$\left. + \frac{\theta_3(y + v|\tau)}{\theta_3(v|\tau)} \sigma \otimes \sigma + + \frac{\theta_4(y + v|\tau)}{\theta_4(v|\tau)} \tau \otimes \tau \right],$$

where $\mathbb{1} = e_{11} + e_{22}$, $h = e_{11} - e_{22}$, $\sigma = i(e_{21} - e_{12})$ and $\tau = e_{21} + e_{12}$.
• If $\Delta(t) = 0$ but $t \neq 0$ then E_t is nodal and $r_t^{(2,1)}(v_1, v_2; y_1, y_2)$ is equivalent to the trigonometric solution

$$r_t^{\mathrm{trg}}(v, y) = \frac{\sin(y + v)}{\sin(y)\sin(v)}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) - \frac{1}{\cos(v)}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) +$$
$$+ \frac{1}{\sin(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(y + v) e_{21} \otimes e_{12}.$$

• Finally, if $t = 0$, the curve E_t is cuspidal and $r_t^{(2,1)}(v_1, v_2; y_1, y_2)$ is equivalent to the rational solution

$$r_t^{\mathrm{rat}}(v, y_1, y_2) = \frac{1}{v}\mathbb{1} \otimes \mathbb{1} + \frac{2}{y_2 - y_1}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \\ + (v - y_1)e_{21} \otimes h + (v + y_2)h \otimes e_{21} + v(v - y_1)(v + y_2)e_{21} \otimes e_{21}.$$

Moreover, for all fixed values of $v \neq 0$ in a neighbourhood of 0 and all $t = (g_2, g_3)$ the tensors $r_t^{\mathrm{ell}}(v, y)$, $r_t^{\mathrm{trg}}(v, y)$ and $r_t^{\mathrm{rat}}(v, y_1, y_2)$ also satisfy the *quantum Yang-Baxter equation.*

11. Appendix: Relatively stable sheaves on genus one fibrations

The goal of this Appendix is to show that the functor $\underline{\mathsf{M}}_{E/T}^{(n,d)}$ of relatively stable vector bundles on a genus one fibration with a section is representable in the category of complex analytic spaces. This statement is crucial for our construction of the geometric associative r–matrices.

The exact assumptions are the following:

- Let $E \xrightarrow{p} T$ be a flat projective morphism of complex spaces of relative dimension one and denote by $\breve{E}$ the smooth locus of p.
- Assume there exists a section $i : T \longrightarrow \breve{E}$ of p.
- Suppose that for all points $t \in T$ the fibre E_t is a reduced and irreducible projective curve of arithmetic genus one.

For a pair of coprime integers $(n, d) \in \mathbb{N} \times \mathbb{Z}$ and $t \in T$ let $\mathsf{St}^{(n,d)}(E_t)$ be the set of stable vector bundles of rank n and degree d on the curve E_t. Denote by Ans_T the category of complex spaces over T. Recall that the functor $\underline{\mathsf{M}}_{E/T}^{(n,d)} : \mathsf{Ans}_T \longrightarrow \mathsf{Sets}$ is defined as follows:

$$\underline{\mathsf{M}}_{E/T}^{(n,d)}(S \xrightarrow{f} T) = \left\{ \mathcal{F} \in \mathsf{Coh}(E_S) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is } S- \text{ flat} \\ \mathcal{F}|_{q^{-1}(s)} \in \mathsf{St}^{(n,d)}(E_t) \quad \forall s \in S \end{array} \right\} / \sim$$

where $t = f(s)$, q and E_S are defined via the Cartesian diagram

$$\begin{tikzcd} E_S \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ S \arrow[r, "f"] & T, \end{tikzcd}$$

$q^{-1}(s)$ is identified with $p^{-1}(t)$ and $\mathcal{F}_1 \sim \mathcal{F}_2$ if and only if there exists $\mathcal{L} \in \mathsf{Pic}(S)$ such that $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes q^*(\mathcal{L})$.

The following lemma is useful.[4]

Lemma 11.1. *Let $Y \xrightarrow{q} S$ be a proper and flat morphism of complex spaces, $\mathcal{F}$ and $\mathcal{G}$ two holomorphic vector bundles on Y such that for all points $s \in S$ it holds: $\mathcal{F}|_{Y_s} \cong \mathcal{G}|_{Y_s}$ and $\mathsf{End}_{Y_s}(\mathcal{F}|_{Y_s}) = \mathbb{C}$. Then there exists a line bundle $\mathcal{L} \in \mathsf{Pic}(S)$ such that $\mathcal{F} \cong \mathcal{G} \otimes q^*\mathcal{L}$.*

Proof. Let $s \in S$ be any closed point and $\varphi : \mathcal{F}|_{Y_s} \longrightarrow \mathcal{G}|_{Y_s}$ some isomorphism. Then the following diagram is commutative:

$$\begin{tikzcd} \mathcal{F}|_{Y_s} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{F}|_{Y_s} \otimes \big(\mathsf{Hom}_{Y_s}(\mathcal{F}|_{Y_s}, \mathcal{G}|_{Y_s}) \otimes_{\mathbb{C}} \mathcal{O}_{Y_s}\big) \arrow[d, "\mathrm{id}\otimes\mathrm{ev}"] \\ \mathcal{G}|_{Y_s} & \mathcal{F}|_{Y_s} \otimes \mathcal{H}om_{Y_s}(\mathcal{F}|_{Y_s}, \mathcal{G}|_{Y_s}) \arrow[l, "\mathrm{ev}"'] \end{tikzcd}$$

[4]We would like to thank Manfred Lehn for a helpful discussion about this question.

where the upper horizontal arrow is the isomorphism induced by φ. This implies that the canonical morphism

$$\mathcal{F}|_{Y_s} \otimes \big(\mathsf{Hom}_{Y_s}(\mathcal{F}|_{Y_s}, \mathcal{G}|_{Y_s}) \otimes_{\mathbb{C}} \mathcal{O}_{Y_s}\big) \longrightarrow \mathcal{G}|_{Y_s}$$

is an isomorphism for all $s \in S$. Since the sheaf $\mathcal{H}om_Y(\mathcal{F}, \mathcal{G})$ is flat over S, by the base-change isomorphism we obtain that $\mathcal{L} := q_*\mathcal{H}om_Y(\mathcal{F}, \mathcal{G})$ is a line bundle on S. Thus, the following composition of canonical morphisms of vector bundles on Y

$$\mathcal{F} \otimes q^*q_*\mathcal{H}om_Y(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{F} \otimes \mathcal{H}om_Y(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}$$

is an isomorphism on all fibres Y_s. This implies that $\mathcal{F} \otimes q^*\mathcal{L} \longrightarrow \mathcal{G}$ is an isomorphism as wanted. □

Proposition 11.2. *The moduli functor $\underline{\mathsf{M}}_{E/T}^{(n,d)}$ is representable under the above assumptions.*

Proof. By a result of Grothendieck [23, Théorème 3.1] the functor $\underline{\mathsf{Pic}}_{E/T}$ is representable, hence for any $d \in \mathbb{Z}$ the functor $\underline{\mathsf{Pic}}_{E/T}^{d}$ is representable, too. Our aim is to construct an isomorphism of functors $\xi : \underline{\mathsf{M}}_{E/T}^{(n,d)} \longrightarrow \underline{\mathsf{Pic}}_{E/T}^{0}$.
Consider the commutative diagram

$$\begin{tikzcd} & E \times_T E \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & T & \end{tikzcd}$$

Using [15, Theorem 2.12] one can construct a coherent sheaf $\mathcal{P} \in \mathsf{Coh}(E \times_T E)$, flat over both components and such that for any point $t \in T$ and $\mathcal{G} \in \mathsf{St}^{(n,d)}(E_t)$ the Fourier-Mukai transform

$$\widehat{\mathcal{G}} := \mathsf{FM}^{\mathcal{P}_t}(\mathcal{G}) = \mathbb{R}\pi_{2*}(\pi_1^*\mathcal{G} \overset{\mathbb{L}}{\otimes} \mathcal{P}_t)$$

is isomorphic to a line bundle of degree zero on E_t, where $\mathcal{P}_t := \mathcal{P}|_{E_t \times E_t}$. Recall that such a kernel $\mathcal{P}$ is constructed using a suitable composition of the functor $\mathcal{O}(D) \otimes -$ and the Fourier-Mukai transform $\mathsf{FM}^{\mathcal{I}_\Delta}$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset E \times_T E$.

Consider the diagram

$$\begin{tikzcd}
& E_S \times_S E_S \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] \arrow[rrr, "g\times g"] & & & E \times_T E \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
E_S \arrow[dr, "q"'] & & E_S \arrow[dl, "q"] & E \arrow[dr, "p"] & & E \arrow[dl, "p"'] \\
& S \arrow[rrr, "f"] & & & T &
\end{tikzcd}$$

denote $\mathcal{P}_S := (g \times g)^*\mathcal{P}$ and define the map

$$\xi_S : \underline{\mathsf{M}}^{(n,d)}_{E/T}(S) \longrightarrow \underline{\mathsf{Pic}}^0_{E/T}(S)$$

to be given by the Fourier-Mukai transform $\mathcal{F} \mapsto \widehat{\mathcal{F}} = \mathbb{R}\pi'_{2*}(\pi_1'^{*}\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{P}_S)$.

A priori, $\widehat{\mathcal{F}}$ is just an object of the derived category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E_S)$. It turns out, however, that it is isomorphic to a coherent sheaf on E_S belonging to $\underline{\mathsf{Pic}}^0_{E/T}(S)$. To show this, take any point $s \in S$; let $i_s : E_s \longrightarrow E_S$ be the canonical inclusion, where $E_s = q^{-1}(s) \cong E_t$. Since the morphisms π'_2 and q are flat, the base-change formula implies:

$$\mathbb{L}i_s^*\widehat{\mathcal{F}} = \mathbb{L}i_s^*\mathbb{R}\pi'_{2*}(\pi_1'^{*}\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{P}_S) \cong \mathbb{R}\pi''_{2\,*}\big(\pi_1''^{*}(\mathcal{F}|_{E_t}) \overset{\mathbb{L}}{\otimes} \mathcal{P}|_{E_t\times E_t})\big),$$

where $\pi''_1, \pi''_2 : E_s \times E_s \longrightarrow E_s$ are the canonical projections. So, for any $s \in S$ the complex $\mathbb{L}i_s^*\widehat{\mathcal{F}}$ is isomorphic to a line bundle, hence by [12, Lemma 4.3] $\widehat{\mathcal{F}}$ is a line bundle on E_S.

Because the integral transforms $\mathsf{FM}^{\mathcal{P}}$ and $\mathsf{FM}^{\mathcal{P}_t}$ are equivalences of categories and $\mathbb{L}i_{t*} \circ \mathsf{FM}^{\mathcal{P}} \cong \mathsf{FM}^{\mathcal{P}_t} \circ \mathbb{L}i_{t*}$, Lemma 11.1 implies that the map ξ_S is injective. Moreover, it is easy to see that ξ indeed defines a natural transformation of functors. Let $\mathcal{Q} \in \mathsf{Coh}(E \times_S E)$ be the kernel giving an inverse functor to $\mathsf{FM}^{\mathcal{P}}$ (up to a shift). Then the corresponding natural transformation of functors $\eta : \underline{\mathsf{Pic}}^0_{E/T} \longrightarrow \underline{\mathsf{M}}^{(n,d)}_{E/T}$ gives the inverse of ξ. □

Corollary 11.3. *The map of complex spaces* $\det : \mathsf{M}^{(n,d)}_{E/T} \longrightarrow \mathsf{Pic}^d_{E/T}$ *induced by the natural transformation* $\det : \underline{\mathsf{M}}^{(n,d)}_{E/T} \longrightarrow \underline{\mathsf{Pic}}^d_{E/T}$ *is an isomorphism.*

Proof. Indeed, since both functors $\underline{\mathsf{M}}^{(n,d)}_{E/T}$ and $\underline{\mathsf{Pic}}^d_{E/T}$ are representable, we get a map of complex spaces $\det : \mathsf{M}^{(n,d)}_{E/T} \longrightarrow \mathsf{Pic}^d_{E/T}$. Moreover, we know that for any $t \in T$ the map $\det : \mathsf{St}^{(n,d)}(E_t) \longrightarrow \mathsf{Pic}^d(E_t)$ is bijective, see Theorem 7.1, Remark 8.16 and Remark 8.31. Hence, det is an isomorphism and by Yoneda's Lemma the natural transformation of functors $\det : \underline{\mathsf{M}}^{(n,d)}_{E/T} \longrightarrow \underline{\mathsf{Pic}}^d_{E/T}$ is an isomorphism, too. □

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