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CLASSIFICATION OF IDEMPOTENT STATES ON THE COMPACT QUANTUM GROUPS $U_q(2)$, $SU_q(2)$, AND $SO_q(3)$

UWE FRANZ, ADAM SKALSKI, AND REIJI TOMATSU

Abstract. We give a simple characterisation of those idempotent states on compact quantum groups which arise as Haar states on quantum subgroups, show that all idempotent states on quantum groups $U_q(2)$, $SU_q(2)$, and $SO_q(3)$ ($q \in (-1,0) \cup (0,1]$) arise in this manner and list the idempotent states on compact quantum semigroups $U_0(2)$, $SU_0(2)$, and $SO_0(3)$. In the Appendix we provide a simple proof of coamenability of the deformations of classical compact Lie groups.

1. Introduction

It is well known that if X is a locally compact topological semigroup, then the space of regular probability measures on X possesses a natural convolution product. Analogously if A is a compact quantum semigroup, i.e. a unital C^*-algebra together with a coproduct (i.e., coassociative unital $*$-homomorphism) $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ then one can consider a natural associative convolution product on the state space of A,

$$\lambda \star \mu = (\lambda \otimes \mu) \circ \Delta, \quad \lambda, \mu \in S(\mathsf{A}).$$

It is natural to ask whether one can characterise the states which satisfy the idempotent property

$$\mu \star \mu = \mu.$$

A particular and most important example of an idempotent state is the Haar state on a given compact quantum group in the sense of Woronowicz [Wor98]. More general idempotent states arise naturally in considerations of Césaro limits of convolution operators on compact quantum groups, cf. [FS08a]. They are also an important ingredient in the construction of quantum hypergroups [CV99] and occur as initial value φ_0 of convolution semigroups $(\varphi_t)_{t\geq 0}$ of states on quantum groups, if one relaxes the initial condition $\varphi_0 = \varepsilon$, cf. [FSc00].

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For classical compact groups, Kawada and Itô have proven that all idempotent measures are induced by Haar measures of compact subgroups, see [KI40, theorem 3]. Later this result was extended to arbitrary locally compact topological groups, see [Hey77] and references therein. In [Pal96] Pal showed that this characterisation does not extend to quantum groups by giving an example of an idempotent state on the Kac-Paljutkin quantum group which cannot arise as the Haar state on a quantum subgroup. In [FS08b] the first two authors began a systematic study of idempotent states on compact quantum groups. In particular we exhibited further examples of idempotent states on quantum groups that are not induced by Haar states of quantum subgroups and gave a characterisation of idempotent states on finite quantum groups in terms of sub quantum hypergroups.

In this work we continue the analysis began in [FS08b] and show a simple characterisation of those idempotent states which arise as Haar states on quantum subgroups, so-called Haar idempotents. The main result of the present paper is the classification of all idempotent states on the compact quantum groups $U_q(2)$, $SU_q(2)$, and $SO_q(3)$ for $q \in (-1,1] \setminus \{0\}$. It turns out that they are all induced by quantum subgroups, cf. Theorems 4.5, 5.1, and 5.2. As a byproduct we obtain the classification of quantum subgroups of the afore-mentioned quantum groups, giving a new proof of the known result of Podleś [Pod95] for $SU_q(2)$ and $SO_q(3)$.

For the value $q = 0$, the quantum cancellation properties fail and $U_0(2)$, $SU_0(2)$, and $SO_0(3)$ are no longer compact quantum groups. But they can still be considered as compact quantum semigroups so that as explained above their state spaces have natural convolution products. Using this we determine all idempotent states on $U_0(2)$, $SU_0(2)$, and $SO_0(3)$, see Theorems 6.3 and 6.5 and 6.6. It turns out that in the case $q = 0$ there exist additional families of idempotent states, which do not appear when $q \neq 0$.

The detailed plan of the paper is as follows: in Section 2 we list the background results and definitions we need in the rest of the paper. In particular we recall the definitions of the quantum groups being the subject of the paper, discuss their corepresentation theory and present explicit formulas for their Haar states. Section 3 introduces idempotent states on compact quantum groups, provides the characterisation of those idempotent states which arise as Haar states on quantum subgroups (extending the results for finite quantum groups given in [FS08b]) and briefly discusses the commutative and cocommutative situation under the coamenability assumption. Section 4 contains main technical arguments of the paper and ends with the characterisation of all idempotent states on $U_q(2)$ for $q \in (-1,1] \setminus \{0\}$. In Section 5 we show how one can deduce the corresponding statements for $SU_q(2)$ and $SO_q(3)$. Section 6 contains the classification of the idempotent states on compact quantum semigroups $U_0(2)$, $SU_0(2)$, and $SO_0(3)$. Finally in Section 7, we use the result of [FS08b] showing that idempotent states on compact quantum groups are group-like projections in the dual quantum group, giving rise to algebraic quantum hypergroups by the construction given in [LvD07], and discuss the quantum hypergroups associated to these group-like projections for the case of $SU_q(2)$. The appendix contains a short direct proof of coamenability of deformations of classical compact Lie groups based on the representation theory developed in [KS98].

2. Preliminaries

The symbol $\otimes$ will denote the spatial tensor product of C^*-algebras, $\odot$ will be reserved for the purely algebraic tensor product.

2.1. **Compact quantum groups.** The notion of compact quantum groups has been introduced in [Wor87a]. Here we adopt the definition from [Wor98] (Definition 2.1 of that paper).

Definition 2.1. A *C^*-bialgebra* (a compact quantum semigroup) is a pair (A, Δ), where A is a unital C^*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphic map which is coassociative

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta.$$

If the quantum cancellation properties

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A},$$

are satisfied, then the pair (A, Δ) is called a *compact quantum group.*

In quantum group theory it is quite common to write $\mathsf{A} = C(\mathbb{G})$, to emphasize that A is considered as the algebra of functions on a compact quantum group $\mathbb{G}$ — but note that the symbol $\mathbb{G}$ itself has no meaning.

The map Δ is called the coproduct of A, it induces the convolution product

$$\lambda \star \mu := (\lambda \otimes \mu) \circ \Delta, \quad \lambda, \mu \in \mathsf{A}^*.$$

A unitary $U \in M_n(\mathsf{A})$ is called a *(finite-dimensional) unitary corepresentation of* A if for all $i, j = 1, \ldots, n$ we have $\Delta(U_{ij}) = \sum_{k=1}^{n} U_{ik} \otimes U_{kj}$. It is said to be *irreducible*, if the only matrices $T \in M_n(\mathbb{C})$ with $TU = UT$ are multiples of the identity matrix.

Possibly the most important feature of compact quantum groups is the existence of the dense $*$-subalgebra $\mathcal{A}$ (the algebra of matrix coefficients of irreducible unitary corepresentations of A), which is in fact a Hopf $*$-algebra - so for example $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. This $*$-Hopf algebra is also denoted by $\mathcal{A} = \mathrm{Pol}\,\mathbb{G}$, and treated as the analog of polynomial functions of $\mathbb{G}$.

Another fact of the crucial importance is given in the following result, Theorem 2.3 of [Wor98].

Proposition 2.2. *Let A be a compact quantum group. There exists a unique state $h \in \mathsf{A}^*$ (called the* Haar state *of A) such that for all $a \in \mathsf{A}$*

$$(h \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta(a) = (\mathrm{id}_{\mathsf{A}} \otimes h) \circ \Delta(a) = h(a)1.$$

A compact quantum group is said to be *in reduced form* if the Haar state h is faithful. If it is not the case we can always quotient out the kernel of h. This procedure in particular does not influence the underlying Hopf $*$-algebra $\mathcal{A}$; in fact the reduced object may be viewed as the natural completion of $\mathcal{A}$ in the GNS representation with respect to h (as opposed for example to the universal completion of $\mathcal{A}$, for details see [BMT01]). In general the reduced and universal object need not coincide. This leads to certain technical complications which are not of essential importance in our context (for example if a discrete group Γ is not amenable, the reduced C^*-algebra of Γ is a proper quantum subgroup of the universal C^*-algebra of Γ, even though they have 'identical' Haar states). To avoid such difficulties we focus on the class of coamenable compact quantum groups ([BMT01], see the Appendix to this paper for more information), for which the reduced and universal C^*-algebraic completions of $\mathcal{A}$ are naturally isomorphic. All deformations of classical compact Lie groups, so in particular quantum groups $U_q(2)$, $SU_q(2)$ and $SO_q(3)$ we consider in Sections 4 and 5 are known to be coamenable ([Ban99]); we give a simple proof of this fact in the Appendix.

The following definition was introduced by Podleś in the context of compact matrix pseudogroups (Definition 1.3 of [Pod95]).

Definition 2.3. A compact quantum group B is said to be a quantum subgroup of a compact quantum group A if there exists a surjective compact quantum group morphism $j : \mathsf{A} \to \mathsf{B}$, i.e. a surjective unital $*$-homomorphism $j : \mathsf{A} \to \mathsf{B}$ such that

$$\Delta_{\mathsf{B}} \circ j = (j \otimes j) \circ \Delta_{\mathsf{A}}. \tag{2.1}$$

Strictly speaking, one should consider the pairs (B, j), since A can contain several copies of B with different morphisms. We will not distinguish between (B, j) and (B', j') if there exists an isomorphism of quantum groups $\Theta : \mathsf{B} \to \mathsf{B}'$ such that $\Theta \circ j = j'$. Note that such isomorphic pairs induce the same idempotent state $\phi = h_{\mathsf{B}} \circ j = h_{\mathsf{B}'} \circ j'$, since uniqueness of the Haar states implies $h_{\mathsf{B}} = h_{\mathsf{B}'} \circ \Theta$.

Any compact quantum group contains itself and the trivial compact quantum group $\mathbb{C}$ as quantum subgroups; further these two quantum subgroups will be called trivial. If G is a compact group and a compact quantum group A contains G as a quantum subgroup, via a morphism $j : A \to C(G)$, we will sometimes simply say that G is a subgroup of A.

2.2. **q-Numbers.** Let $q \neq 1$. We will use the following notation for q-numbers,

$$\begin{aligned} (x;q)_n &= (1-x)(1-qx)\cdots(1-q^{n-1}x), \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \end{aligned}$$

for $n \in \mathbb{Z}_+$, $0 \le k \le n$, $x \in \mathbb{R}$.

2.3. **The Woronowicz quantum group $SU_q(2)$.** [Wor87a, Wor87b] For $q \in \mathbb{R}$, we denote by $\mathrm{Pol}\, SU_q(2)$ the $*$-bialgebra generated by α and γ, with the relations

$$\alpha\gamma = q\gamma\alpha, \qquad \alpha\gamma^* = q\gamma^*\alpha, \qquad \gamma^*\gamma = \gamma\gamma^*,$$
$$\gamma^*\gamma + \alpha^*\alpha = 1, \qquad \alpha\alpha^* - \alpha^*\alpha = (1-q^2)\gamma^*\gamma,$$

and comultiplication and counit defined by setting

$$\Delta(\alpha) = \alpha\otimes\alpha - q\gamma^*\otimes\gamma, \quad \Delta(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma,$$

and $\varepsilon(\alpha) = 1$, $\varepsilon(\gamma) = 0$. For $q \neq 0$, $\mathrm{Pol}\, SU_q(2)$ admits an antipode. On the generators, it acts as

$$S(\alpha) = \alpha^* \qquad \text{and} \qquad S(\gamma) = -q\gamma.$$

Denote the universal enveloping C^*-algebra of $\mathrm{Pol}\, SU_q(2)$ by $C(SU_q(2))$, then Δ extends uniquely to a non-degenerate coassociative homomorphism $\Delta : C(SU_q(2)) \to C(SU_q(2))\otimes C(SU_q(2))$, and the pair $(C(SU_q(2)), \Delta)$ is a C^*-bialgebra. For $q \neq 0$, $C(SU_q(2))$ is even a compact quantum group.

Note that the mapping $\alpha \mapsto \alpha^*$ and $\gamma \mapsto q\gamma^*$ induces isomorphisms $\mathrm{Pol}\, SU_{1/q}(2) \to \mathrm{Pol}\, SU_q(2)$, $C(SU_{1/q}(2)) \to C(SU_q(2))$, therefore it is sufficient to consider $q \in [-1, 1]$.

2.3.1. *Representation theory of* $C(SU_q(2))$*.* [Wor87b, VS88] The C^*-algebra $C(SU_q(2))$ has two families of irreducible representations. The first family consists of the one-dimensional representations ρ_θ, $0 \le \theta < 2\pi$, given by

$$\begin{aligned} \rho_\theta(\alpha) &= e^{i\theta}, \\ \rho_\theta(\gamma) &= 0. \end{aligned}$$

The other family consist of infinite-dimensional representations π_θ, $0 \le \theta < 2\pi$, acting on a separable Hilbert space $\mathfrak{h}$ by

$$\begin{aligned} \pi_\theta(\alpha)e_n &= \begin{cases} \sqrt{1-q^{2n}}\, e_{n-1} & \text{if} \quad n > 0, \\ 0 & \text{if} \quad n = 0, \end{cases} \\ \pi_\theta(\gamma)e_n &= e^{i\theta}q^n e_n, \end{aligned}$$

where $\{e_n; n \in \mathbb{Z}_+\}$ is an orthonormal basis for $\mathfrak{h}$.

This list is complete, i.e. any irreducible representation of $C(SU_q(2))$ is unitarily equivalent to a representation in one of the two families above (Theorem 3.2 of [VS88]). It is known that the C^*-algebra $C(SU_q(2))$ is of type I, therefore any representation can be written as a direct integral over the irreducible representations given above.

2.3.2. *Quantum subgroups of $SU_q(2)$.* Let $\mathrm{Pol}\,U(1)$ denote the $*$-algebra generated by one unitary w, $ww^* = w^*w = 1$. With $\Delta(w) = w \otimes w$, $\varepsilon(w) = 1$, $S(w) = w^*$, this becomes a $*$-Hopf algebra. Its enveloping C^*-algebra $C(U(1))$ is a compact quantum group. Note that the $*$-algebra homomorphism $\mathrm{Pol}\,SU_q(2) \to \mathrm{Pol}\,U(1)$ defined by $\alpha \mapsto w$, $\gamma \mapsto 0$ extends to a surjective compact quantum group morphism $j : C(SU_q(2)) \to C(U(1))$, i.e. $U(1)$ is a quantum subgroup of $SU_q(2)$. Furthermore, Podleś in Theorem 2.1 of [Pod95] has shown that, for $q \in (-1,0) \cup (0,1)$, $U(1)$ and its closed subgroups are the only non-trivial quantum subgroups of $SU_q(2)$. This will also follow from the results in Section 4.

There exists a second morphism $j' : C(SU_q(2)) \to C(U(1))$, determined by $j' : \alpha \mapsto w^*, \gamma \mapsto 0$. But we do not need to distinguish the pairs $(C(U(1)), j)$ and $(C(U(1)), j')$, since they are related by the automorphism Θ of $C(U(1))$ with $\Theta(w^k) = (w^*)^k$, $\Theta\big((w^*)^k\big) = w^k$, $k \in \mathbb{N}$.

2.3.3. *Corepresentations of $SU_q(2)$.* Let $q \in (-1,0) \cup (0,1)$. We recall a few basic facts about the corepresentations of $SU_q(2)$, for more details see [Wor87a, Wor87b, VS88, MMN$^+$88, Koo89]. For each non-negative half-integer $s \in \frac{1}{2}\mathbb{Z}_+$ there exists a $2s+1$-dimensional irreducible unitary corepresentation $u^{(s)} = (u^{(s)}_{k\ell})_{-s \le k,\ell \le s}$ of $SU_q(2)$, which is unique up to unitary equivalence. Note that that indices k, ℓ run over the set $\{-s, -s+1, \ldots, s-1, s\}$, they are integers if $s \in \mathbb{Z}_+$ is integer, and half-integer if $s \in (\frac{1}{2}\mathbb{Z}_+)\backslash\mathbb{Z}_+$ is half-integer. This convention is also used further in the paper.

The matrix elements $u^{(s)}_{k\ell}$, $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \le k, \ell \le s$, span $\mathrm{Pol}\,SU_q(2)$ and they are linearly dense in $C(SU_q(2))$, therefore will be sufficient for our calculations. We have

$$u^{(0)} = (1), \qquad u^{(1/2)} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

$$u^{(1)} = \begin{pmatrix} \alpha^2 & -q\sqrt{1+q^2}\gamma^*\alpha & q^2(\gamma^*)^2 \\ \sqrt{1+q^2}\gamma\alpha & 1-(1+q^2)\gamma^*\gamma & -q\sqrt{1+q^2}\alpha^*\gamma^* \\ \gamma^2 & \sqrt{1+q^2}\alpha^*\gamma & (\alpha^*)^2 \end{pmatrix},$$

and the matrix elements of the higher-dimensional corepresentations are of the form

$$u^{(s)}_{k\ell} = \begin{cases} \alpha^{-k-\ell} p^{(s)}_{k\ell} \gamma^{k-\ell} & \text{for} \quad k+\ell \le 0,\ k \ge \ell, \\ \alpha^{-k-\ell} p^{(s)}_{k\ell} (\gamma^*)^{\ell-k} & \text{for} \quad k+\ell \le 0,\ k \le \ell, \\ (\alpha^*)^{k+\ell} p^{(s)}_{k\ell} \gamma^{k-\ell} & \text{for} \quad k+\ell \ge 0,\ k \ge \ell, \\ (\alpha^*)^{k+\ell} p^{(s)}_{k\ell} (\gamma^*)^{\ell-k} & \text{for} \quad k+\ell \ge 0,\ k \le \ell, \end{cases}$$

where $p^{(s)}_{k\ell}$ is a polynomial in $\gamma^*\gamma$.

In particular, for s an integer, $u^{(s)}_{00} = p_s(\gamma^*\gamma; 1, 1; q^2)$ is the little q-Legendre polynomial and $u^{(s)}_{s0} = \sqrt{\left[\begin{array}{c} 2s \\ s \end{array}\right]_{q^2}} (\alpha^*)^s \gamma^s$.

If we define a $\mathbb{Z}$-grading on $\mathrm{Pol}\,SU_q(2)$ by

$$\deg\alpha = \deg\alpha^* = 0,\quad \deg\gamma = 1,\quad \deg\gamma^* = -1,$$

then we have

$$\deg u^{(s)}_{k\ell} = k-\ell.$$

With this grading, it is straight-forward to verify the following formula for the square of the antipode on homogeneous elements,

$$S^2(a) = q^{2\deg a}a. \tag{2.2}$$

2.3.4. *The Haar state of $SU_q(2)$.* As stated in Proposition 2.2, there exists a unique invariant state h on the compact quantum group $SU_q(2)$, called the Haar state. The Haar state is the identity on the one-dimensional corepresentation, and vanishes on the matrix elements of all other irreducible corepresentations, i.e.

$$h\left(u^{(s)}_{k\ell}\right) = \delta_{0s}$$

for $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \le k,\ell \le s$. On polynomials $p(\gamma^*\gamma) \in \mathbb{C}[\gamma^*\gamma]$, it is equal to Jackson's q-integral ([Koo89]),

$$h\big(p(\gamma^*\gamma)\big) = (1-q^2)\sum_{k=0}^{\infty} q^{2k}p(q^{2k}) =: \int_0^1 p(x)\mathrm{d}_{q^2}x.$$

2.4. **The compact quantum group $SO_q(3)$.** A compact quantum group B is called a quotient group of (A,Δ), if there exists an injective morphism of quantum groups $j:\mathsf{B}\to\mathsf{A}$. The compact quantum group $SO_q(3)$ can be defined as the quotient of $SU_q(2)$ by the quantum subgroup $\mathbb{Z}_2$, cf. [Pod95]. $\mathrm{Pol}\,SO_q(3)$ is the subalgebra of $\mathrm{Pol}\,SU_q(2)$ spanned by the matrix elements of the unitary irreducible corepresentations with integer label, and $C(SO_q(3))$ its norm closure. The Haar state on $SO_q(3)$ is simply the restriction of the Haar state on $SU_q(2)$.

Podleś [Pod95] has shown that $SO_q(3)$ and $SO_{-q}(3)$ are isomorphic.

2.4.1. *The semigroup case $q=0$.* We define $\mathrm{Pol}\,SO_0(3)$ as the unital $*$-subalgebra of $\mathrm{Pol}\,SU_0(2)$ generated by α^2, γ^2, $\gamma\alpha$, $\gamma^*\alpha$, and $\gamma^*\gamma$, i.e.

$$\mathrm{Pol}\,SO_0(3) = \mathrm{span}\left\{(\alpha^*)^r\gamma^k\alpha^s, (\alpha^*)^r(\gamma^*)^k\alpha^s : r,k,s\in\mathbb{Z}_+ \text{ s.t. } r+k+s \text{ even}\right\}.$$

Since

$$\begin{aligned}
\Delta\big((\alpha^*)^r\gamma^k\alpha^s\big) &= \sum_{\kappa=0}^{k}(\alpha^*)^{r+k-\kappa}\gamma^\kappa\alpha^s\otimes(\alpha^*)^r\gamma^{k-\kappa}\alpha^{s+\kappa},\\
\Delta\big((\alpha^*)^r(\gamma^*)^k\alpha^s\big) &= \sum_{\kappa=0}^{k}(\alpha^*)^{r}(\gamma^*)^\kappa\alpha^{s+k-\kappa}\otimes(\alpha^*)^{r+\kappa}(\gamma^*)^{k-\kappa}\alpha^{s},
\end{aligned}$$

this is a sub $*$-Hopf algebra in $\mathrm{Pol}\,SU_0(2)$. The C^*-bialgebra $C(SO_0(3))$ is then defined as the norm closure of $\mathrm{Pol}\,SO_0(3)$ in $C(SU_0(2))$.

2.4.2. *The conditional expectation $E : C(SU_q(2)) \to C(SO_q(3))$.* Looking at the defining relations of $SU_q(2)$, it is clear that $\vartheta : \alpha \to -\alpha, \gamma \to -\gamma$ extends to a unique $*$-algebra automorphism of $C(SU_q(2))$. Therefore $E = \frac{1}{2}(\mathrm{id} + \vartheta)$ defines a completely positive unital map from $C(SU_q(2))$ to itself. If $\phi_2 = h_{\mathbb{Z}_2} \circ j$ denotes the idempotent state on $SU_q(2)$ induced by the Haar measure of $\mathbb{Z}_2$ (with $j : SU_q(2) \to C(\mathbb{Z}_2)$ the corresponding surjective morphism), then we can write E also as

$$E = (\mathrm{id} \otimes \phi_2) \circ \Delta.$$

Checking

$$E(u^{(s)}_{k\ell}) = \begin{cases} u^{(s)}_{k\ell} & \text{if } s \in \mathbb{Z}_+, \\ 0 & \text{else.} \end{cases}$$

we can show that the range of E is equal to $C(SO_q(3))$. Furthermore, E satisfies

$$\Delta \circ E = (\mathrm{id} \otimes E) \circ \Delta = (E \otimes \mathrm{id}) \circ \Delta = (E \otimes E) \circ \Delta.$$

2.4.3. *Quantum subgroups of $SO_q(3)$.* The restriction of the morphism $j : C(SU_q(2)) \to C(U(1))$ to $C(SO_q(3))$ is no longer surjective, its range is equal to the subalgebra $\{f \in C(U(1)) : f(z) = f(-z) \forall z \in U(1)\} = C(U(1)/\mathbb{Z}_2)$. Since $U(1)/\mathbb{Z}_2 \cong U(1)$, we see that $SO_q(3)$ contains $U(1) \cong SO(2)$ and its closed subgroups as quantum subgroups. Podleś [Pod95] has shown that these are the only non-trivial quantum subgroups of $SO_q(3)$. Again this can be deduced from the results of Section 5.

2.5. **The compact quantum group $U_q(2)$.** [Koe91, Wys04, ZZ05] Let $q \in \mathbb{R}$. Then $\mathrm{Pol}\, U_q(2)$ is defined as the $*$-bialgebra generated by a, c, and v, with the relations

$$\begin{gathered} av = va, \qquad cv = vc, \qquad cc^* = c^*c, \\ ac = qca, \qquad ac^* = qc^*a, \qquad vv^* = v^*v = 1, \\ aa^* + q^2cc^* = 1 = a^*a + c^*c, \\ \Delta(a) = a \otimes a - qc^*v^* \otimes c, \quad \Delta(c) = c \otimes a + a \otimes c, \quad \Delta(v) = v \otimes v, \\ \varepsilon(a) = \varepsilon(v) = 1, \qquad \varepsilon(c) = 0. \end{gathered}$$

For $q \neq 0$, $\mathrm{Pol}\, U_q(2)$ admits an antipode, given by

$$S(a) = a^*, \qquad S(v) = v^*, \qquad S(c) = -qcv,$$

on the generators.

Denote the universal enveloping C^*-algebra of $\mathrm{Pol}\, U_q(2)$ by $C(U_q(2))$, then $\Delta : \mathrm{Pol}\, U_q(2) \to \mathrm{Pol}\, U_q(2) \odot \mathrm{Pol}\, U_q(2)$ extends uniquely to a non-degenerate coassociative homomorphism $\Delta : C(U_q(2)) \to C(U_q(2)) \otimes C(U_q(2))$, and the pair $(C(U_q(2)), \Delta)$ is a C^*-bialgebra. For $q \neq 0$, $C(U_q(2))$ is even a compact quantum group. It is again sufficient to consider $q \in [-1, 1]$, since $U_q(2)$ and $U_{1/q}(2)$ are isomorphic.

2.5.1. *Quantum subgroups of* $U_q(2)$. The mapping $a \mapsto \alpha$, $c \mapsto \gamma$, $v \mapsto 1$ extends to a surjective compact quantum group morphism $C(U_q(2)) \to C(SU_q(2))$ and shows that $SU_q(2)$ is a quantum subgroup of $U_q(2)$. Actually, as a C^*-algebra, $U_q(2)$ is isomorphic to the tensor product of $C(SU_q(2))$ and $C(U(1))$. As a compact quantum group, it is equal to a twisted product of $C(SU_q(2))$ and $C(U(1))$, cf. [Wys04], written as $U_q(2) = SU_q(2) \ltimes_\sigma U(1)$.

Another quantum subgroup of $U_q(2)$ is the two-dimensional torus. Denote by $\mathrm{Pol}\,\mathbb{T}^2$ the $*$-Hopf algebra generated by two commuting unitaries, i.e. by w_1, w_2 with the relations

$$w_1w_1^* = 1 = w_1^*w_1, \quad w_2w_2^* = 1 = w_2^*w_2, \quad w_1w_2 = w_2w_1, \quad w_1w_2^* = w_2^*w_1,$$
$$\Delta(w_1) = w_1 \otimes w_1, \quad \Delta(w_2) = w_2 \otimes w_2, \quad \varepsilon(w_1) = \varepsilon(w_2) = 1,$$

and $C(\mathbb{T}^2)$ the compact quantum group obtained as its C^*-enveloping algebra. Then the mapping $a \mapsto w_1$, $c \mapsto 0$, $v \mapsto w_2$ extends to a unique surjective compact quantum group morphism $C(U_q(2)) \to C(\mathbb{T}^2)$.

We will see that the twisted products $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$, $n \in \mathbb{N}$, the torus $\mathbb{T}^2$, and its closed subgroups are the only non-trivial quantum subgroups of $U_q(2)$, cf. Corollary 4.7.

2.5.2. *Corepresentations of* $U_q(2)$. Unitary irreducible corepresentations of $U_q(2)$ can be obtained as tensor product of unitary irreducible corepresentations of $SU_q(2)$ with corepresentations of $U(1)$, cf. [Wys04]. In this way one obtains the following family of unitary irreducible corepresentations of $U_q(2)$,

$$v^{(s,p)} = \left(u^{(s)}_{k\ell} v^{p+s+\ell}\right)_{-s\leq k,\ell\leq s}$$

for $p \in \mathbb{Z}$, $s \in \frac{1}{2}\mathbb{Z}_+$. The matrix elements of these corepresentations clearly span $\mathrm{Pol}\,U_q(2)$. Therefore they are dense in $C(U_q(2))$ and will be sufficient for the calculations in this paper.

Assume $q \neq 0$. We want to compute the action of the square of the antipode on the matrix elements of the unitary irreducible corepresentations defined above. Since we have $S^2(a) = a$, $S^2(c) = q^2c$, and $S^2(v) = v$, we get

$$S^2(u^{(s)}_{k\ell} v^{p+s+\ell}) = q^{2(k-\ell)} u^{(s)}_{k\ell} v^{p+s+\ell} \tag{2.3}$$

$p \in \mathbb{Z}$, $s \in \frac{1}{2}\mathbb{Z}_+$, and $-s \leq k, \ell \leq s$.

2.5.3. *The Haar state of* $U_q(2)$. The Haar state h on $U_q(2)$ can be written as a tensor product of the Haar state on $SU_q(2)$ and the Haar state on $U(1)$, it acts on the matrix elements of the unitary irreducible corepresentations given above as

$$h\left(u^{(s)}_{k\ell} v^{p+s+\ell}\right) = \delta_{0s}\delta_{0p}$$

for $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \leq k, \ell \leq s$, $p \in \mathbb{Z}$.

2.6. **Multiplicative domain of a completely positive unital map.** The following result by Choi on multiplicative domains will be useful for us.

Lemma 2.4. [Cho74, Theorem 3.1], [Pau02, Theorem 3.18] *Let $T : A \to B$ be a completely positive unital linear map between two C^*-algebras A and B. Set*

$$D_T = \{a \in A : T(aa^*) = T(a)T(a^*), T(a^*a) = T(a^*)T(a)\}.$$

Then we have

$$T(ab) = T(a)T(b) \qquad \text{and} \qquad T(ba) = T(b)T(a)$$

for all $a \in D_T$ and $b \in A$.

3. Idempotent states on compact quantum groups

In this section we formally introduce the notion of idempotent states on a C^*-bialgebra, provide a characterisation of those idempotent staets on compact quantum hroups which arise as Haar states on quantum subgroups and discuss commutative and cocommutative cases.

Definition 3.1. Let (A, Δ) be a C^*-bialgebra. A state $\phi \in \mathsf{A}^*$ is called an *idempotent state* if

$$(\phi \otimes \phi) \circ \Delta = \phi,$$

i.e. if it is idempotent for the convolution product.

Assume now that (A, Δ) is a compact quantum group.

Definition 3.2. A state $\phi \in \mathsf{A}^*$ is called a *Haar state on a quantum subgroup of* A (or a *Haar idempotent*) if there exists a quantum subgroup (B, j) of A and $\phi = h_{\mathsf{B}} \circ j$, where h_{B} denotes the Haar state on B.

It is easy to check that each Haar state on a quantum subgroup of A is idempotent. It follows from the example of Pal in [Pal96] and our work in [FS08b] that not every idempotent state is a Haar idempotent. We have the following simple characterisation, extending Theorem 4.5 of [FS08b].

Theorem 3.3. *Let A be a compact quantum group, let $\phi \in \mathsf{A}^*$ be an idempotent state and let $N_\phi = \{a \in \mathsf{A} : \phi(a^*a) = 0\}$ denote the null space of ϕ. Then ϕ is a Haar idempotent if and only if N_ϕ is a two-sided (equivalently, selfadjoint) ideal.*

Proof. It is an easy consequence of the Cauchy-Schwarz inequality that N_ϕ is a left ideal; thus it is a two-sided ideal if and only if it is selfadjoint.

Suppose first that ϕ is a Haar idempotent, i.e. there exists a compact quantum group B and a surjective compact quantum group morphism $j : \mathsf{A} \to \mathsf{B}$ such that $\phi = h_{\mathsf{B}} \circ j$. Recall that we assumed h_{B} to be faithful, so that $N_\phi = \{a \in \mathsf{A} : j(a^*a) = 0\} = \{a \in \mathsf{A} : j(a) = 0\}$, which is obviously self-adjoint.

Suppose then that N_ϕ is a two-sided selfadjoint ideal. Let $\mathsf{B} := \mathsf{A}/N_\phi$ and let $\pi_\phi : \mathsf{A} \to \mathsf{B}$ denote the canonical quotient map. We want to define the coproduct on B by the formula

$$\Delta_{\mathsf{B}} \circ \pi_\phi(a) = (\pi_\phi \otimes \pi_\phi) \circ \Delta(a) \in \mathsf{B} \otimes \mathsf{B}. \tag{3.1}$$

We need to check that it is well-defined - to this end we employ a slightly modified idea from the proof of Theorem 2.1 of [BMT01]. A standard use of Cauchy-Schwarz inequality implies that $\phi|_{N_\phi} = 0$, so that there exists a faithful state $\psi \in \mathsf{B}^*$ such that $\psi \circ \pi_\phi = \phi$. Faithfulness of ψ implies that also the map $\mathrm{id}_{\mathsf{B}} \otimes \psi : \mathsf{B} \otimes \mathsf{B} \to \mathsf{B}$ is faithful (note that here faithfulness of a positive map T is understood in the usual sense, namely $Ta = 0$ and $a \geq 0$ imply $a = 0$) and thus also $\psi \otimes \psi \in (\mathsf{B} \otimes \mathsf{B})^*$ is faithful. Suppose then that $a \in N_\phi$. We have then

$$0 = \phi(a^*a) = (\phi \otimes \phi) \circ \Delta(a^*a) = (\psi \otimes \psi) \circ (\pi_\phi \otimes \pi_\phi)(\Delta(a^*a)),$$

so also $(\pi_\phi \otimes \pi_\phi)\Delta(a^*a) = 0$. The last statement implies that $(\pi_\phi \otimes \pi_\phi)\Delta(a) = 0$ and validity of the definition given in the formula (3.1) is established. The fact that Δ_{B} is a coassociative unital $*$-homomorphism follows immediately from the analogous properties of Δ; similarly the cancellation properties of B follow from obvious equalities of the type

$$(\mathsf{B} \otimes 1_{\mathsf{B}})\Delta_{\mathsf{B}}(\mathsf{B}) = (\pi_\phi \otimes \pi_\phi)((\mathsf{A} \otimes 1_{\mathsf{A}})\Delta(\mathsf{A}))$$

and the cancellation properties of A. Thus $(\mathsf{B}, \Delta_{\mathsf{B}})$ is a compact quantum group and it remains to check that ψ defined above is actually the invariant state on B. This is however an immediate consequence of the following observation:

$$(\psi \otimes \psi) \circ \Delta_{\mathsf{B}} \circ \pi_\phi = (\phi \otimes \phi) \circ \Delta = \phi = \psi \circ \pi_\phi,$$

so that ψ is an idempotent state and, as it is faithful, it has to coincide with the Haar state of B ([Wor98]). □

Note that the first implication remains valid without the assumption of faithfulness of h_{B}; alternatively one could exploit the modular properties of Haar states on not-necessarily-coamenable compact quantum groups implying that their null spaces are always selfadjoint.

The following proposition will be useful for the classification of idempotent states on in the next two sections, cf. [FS08b, Section 3].

Proposition 3.4. *Let $\phi \in \mathsf{A}^*$ be an idempotent state. Then ϕ is invariant under the antipode, in the sense that $\phi(a) = \phi \circ S(a)$ for all a belonging to the $*$-Hopf algebra $\mathcal{A}$.*

3.1. **Idempotent states on cocommutative compact quantum groups.** Suppose now that A is cocommutative, i.e. $\Delta = \tau \circ \Delta$, where $\tau : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ denoted the usual tensor flip. It is easy to deduce from the general theory of duality for quantum groups ([KV00]) that A is isomorphic to the dual of $C_0(\Gamma)$, where Γ is a (classical) discrete group. For the reasons mentioned in the previous section in general we need to distinguish between the reduced and the universal dual; thus we restrict our attention to amenable Γ. The following generalises Theorem 6.2 of [FS08b] to the infinite-dimensional context.

Theorem 3.5. *Let Γ be an amenable discrete group and $\mathsf{A} = C^*(\Gamma)$. There is a one-to-one correspondence between idempotent states on A and subgroups of Γ. An idempotent state $\phi \in \mathsf{A}^*$ is a Haar idempotent if and only if the corresponding subgroup of Γ is normal.*

Proof. The dual of A may be identified with the Fourier-Stieltjes algebra $B(\Gamma)$. The convolution of functionals in A^* corresponds then to the pointwise multiplication of functions in $B(\Gamma)$ and $\phi \in B(\Gamma)$ corresponds to a positive (respectively, unital) functional on A if and only if it is positive definite (respectively, $\phi(e) = 1$). This implies that $\phi \in B(\Gamma)$ corresponds to an idempotent state if and only if it is an indicator function (of a certain subset $S \subset \Gamma$) which is positive definite. It is a well known fact that this happens if and only if S is a subgroup of Γ ([HR70], Cor. (32.7) and Example (34.3 a)). It remains to prove that if S is a subgroup of Γ then $\chi_S \in B(\Gamma)$ is a Haar state on a quantum subgroup of A if and only if S is normal. For the 'if' direction assume that S is a normal subgroup and consider the compact quantum group $\mathsf{B} = C^*(\Gamma/S)$ (recall that quotients of amenable groups are amenable). Let $F(\Gamma)$ denote the dense $*$-subalgebra of A given by the functions $f = \sum_{\gamma\in\Gamma} \alpha_\gamma \lambda_\gamma$ ($\alpha_\gamma \in \mathbb{C}$, $\{\gamma : \alpha_\gamma \neq 0\}$ finite). Define $j : F(\Gamma) \to \mathsf{B}$ by

$$j(f) = \sum_{\gamma\in\Gamma} \alpha_\gamma \lambda_{[\gamma]},$$

where f is as above. So-defined j is bounded: note that it is a restriction of the transpose of the map $T : \mathsf{B}^* \to \mathsf{A}^*$ given by

$$T(\phi)(\gamma) = \phi([\gamma]),\ \phi \in B(\Gamma/S), \gamma \in \Gamma.$$

The map T is well defined as it maps positive definite functions into positive definite functions; these generate the relevant Fourier-Stieltjes algebras. Further the closed graph theorem allows to prove that T is bounded; therefore so is $T^* : \mathsf{A}^{**} \to \mathsf{B}^{**}$ and $j = T^*|_{F(\Gamma)}$. It is now easy to check that the extension of j to A is a surjective unital $*$-homomorphism (onto B). As the Haar state on B is given by

$$h_{\mathsf{B}}\left(\sum_{\kappa\in\Gamma/S} \alpha_\kappa \lambda_\kappa\right) = \alpha_{[e]},$$

there is

$$h_{\mathsf{B}}(j(f)) = \sum_{\gamma\in S} \alpha_\gamma,$$

so that $h_{\mathsf{B}} \circ j$ corresponds via the identification of A^* to $B(\Gamma)$ exactly to the characteristic function of S.

The other direction follows exactly as in [FS08b]; we reproduce the argument for the sake of completeness. Suppose that S is a subgroup of Γ which is not normal and let $\gamma_0 \in \Gamma$, $s_0 \in S$ be such that $\gamma_0 s_0 \gamma_0^{-1} \notin S$. Denote by ϕ_S the state on A corresponding to the indicator function of S. Define $f \in \mathsf{A}$ by $f = \lambda_{\gamma_0 s_0} - \lambda_{\gamma_0}$. Then

$$f^* f = 2\lambda_e - \lambda_{s_0^{-1}} - \lambda_{s_0}, \quad f f^* = 2\lambda_e - \lambda_{\gamma_0 s_0^{-1}\gamma_0^{-1}} - \lambda_{\gamma_0 s_0 \gamma_0^{-1}}.$$

This implies that

$$\phi_S(f^* f) = 0, \quad \phi_S(f f^*) = 2,$$

so that $\operatorname{Ker}\phi_S$ is not selfadjoint and ϕ_S cannot be a Haar idempotent. □

Corollary 3.6. *Let* A *be a coamenable cocommutative compact quantum group. The following are equivalent:*

(1) *all idempotent states on* A *are Haar idempotents;*
(2) $\mathsf{A} \cong C^*(\Gamma)$ *for an amenable hamiltonian (i.e. containing no non-normal subgroups) discrete group* Γ.

4. Idempotent states on $U_q(2)$ ($q \in (-1, 0) \cup (0, 1]$)

For $q = 1$, $C(U_q(2))$ is equal to the C^*-algebra of continuous functions on the unitary group $U(2)$, and by Kawada and Itô's classical theorem all idempotent states on $C(U(2))$ come from Haar measures of compact subgroups of $U(2)$. In this section we shall classify the idempotent states on $C(U_q(2))$ for $-1 < q < 1, q \neq 0$. It turns out that they all correspond to Haar states of quantum subgroups of $U_q(2)$.

We begin with some preparatory lemmas.

Lemma 4.1. *Let* $\phi : \operatorname{Pol} U_q(2) \to \mathbb{C}$ *be an idempotent state. Then we have*

$$\phi(u^{(s)}_{k\ell} v^r) = 0 \qquad \text{if} \qquad k \neq \ell,$$

and $\phi(u^{(s)}_{kk} v^r) \in \{0, 1\}$, *for all* $s \in \frac{1}{2}\mathbb{Z}_+$, $r \in \mathbb{Z}$, $-s \leq k, \ell \leq s$.

Proof. By Proposition 3.4, we have $\phi \circ S = \phi$ on $\operatorname{Pol} U_q(2)$. Therefore, by Equation (2.3),

$$\phi(u^{(s)}_{k\ell} v^r) = \phi \circ S^2(u^{(s)}_{k\ell} v^r) = q^{2(k-\ell)} \phi(u^{(s)}_{k\ell} v^r)$$

i.e. $\phi(u^{(s)}_{k\ell} v^r) = 0$ for $k \neq \ell$.

Define the matrices $M_{s,p}(\phi) \in \mathcal{M}_{2s+1}(\mathbb{C})$ by

$$M_{s,p}(\phi) = \Big(\phi(u^{(s)}_{k\ell} v^{p+s+\ell})\Big)_{-s \leq k, \ell \leq s}.$$

Then $\phi = \phi \star \phi$ is equivalent to

$$M_{s,p}(\phi) = \big(M_{s,p}(\phi)\big)^2$$

for all $s \in \frac{1}{2}\mathbb{Z}_+$, $p \in \mathbb{Z}$. As we have already seen that these matrices are diagonal, it follows that the diagonal entries can take only the values 0 and 1. □

Lemma 4.2. *If $\phi : \operatorname{Pol} U_q(2) \to \mathbb{C}$ is an idempotent state with $\phi(u_{00}^{(1)}) = 1$, then there exists an idempotent state $\tilde{\phi} : \operatorname{Pol} \mathbb{T}^2 \to \mathbb{C}$ such that*

$$\phi = \tilde{\phi} \circ \pi_{\mathbb{T}^2}.$$

Proof. By the previous lemma $\phi(c) = \phi(c^*) = 0$

We have $u_{00}^{(1)} = 1-(1+q^2)c^*c$, therefore $\phi(u_{00}^{(s)}) = 1$ is equivalent to $\phi(c^*c) = 0$. Then by Lemma 2.4, $c, c^* \in D_\phi$, and ϕ vanishes on expressions of the form uc, cu, uc^*, c^*u with $u \in \operatorname{Pol} U_q(2)$. But since $vc = cv$ and $u_{k\ell}^{(s)} c = q^{-(k-\ell)} c u_{k\ell}^{(s)}$, $u_{k\ell}^{(s)} c^* = q^{-(k-\ell)} c^* u_{k\ell}^{(s)}$ for $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \leq k, \ell \leq s$, we can deduce that ϕ vanishes on the ideal

$$\mathcal{I}_c = \{u_1 c u_2, u_1 c^* u_2; u_1, u_2 \in \operatorname{Pol} U_q(2)\}$$

generated by c and c^*. It follows that we can divide out $\mathcal{I}_c$, i.e. there exists a unique state $\tilde{\phi}$ on $\operatorname{Pol} U_q(2)/\mathcal{I}_c$ such that the diagram

$$\begin{array}{ccc} \operatorname{Pol} U_q(2) & \xrightarrow{\pi} & \operatorname{Pol} U_q(2)/\mathcal{I}_c \\ {\scriptstyle \phi}\downarrow & \swarrow {\scriptstyle \tilde{\phi}} & \\ \mathbb{C} & & \end{array}$$

commutes.

But $\varepsilon(\mathcal{I}_c) = 0$,

$$\Delta(\mathcal{I}_c) \subseteq \mathcal{I}_c \odot \operatorname{Pol} U_q(2) + \operatorname{Pol} U_q(2) \odot \mathcal{I}_c,$$

and $S(\mathcal{I}_c) \subseteq \mathcal{I}_c$ i.e. $\mathcal{I}_c$ is also a Hopf $*$-ideal and $\operatorname{Pol} U_q(2)/\mathcal{I}_c$ is a $*$-Hopf algebra. One easily verifies that actually $\operatorname{Pol} U_q(2)/\mathcal{I}_c \cong \operatorname{Pol} \mathbb{T}^2$. Since $\pi_{\mathbb{T}^2} : \operatorname{Pol} U_q(2) \to \operatorname{Pol} \mathbb{T}^2$ is surjective coalgebra morphism, its dual $\pi_{\mathbb{T}^2}^* : (\operatorname{Pol} \mathbb{T}^2)^* \ni f \mapsto \pi_{\mathbb{T}^2}^*(f) = f \circ \pi_{\mathbb{T}^2} \in (\operatorname{Pol} U_q(2))^*$ is an injective algebra homomorphism, and $\tilde{\phi} = (\pi_{\mathbb{T}^2}^*)^{-1}(\phi)$ is again idempotent. ☐

Lemma 4.3. *If $\phi : \operatorname{Pol} U_q(2) \to \mathbb{C}$ is an idempotent state with $\phi(u_{00}^{(1)}) = 0$, then $\phi(u_{00}^{(s)}) = 0$ for all integers $s \geq 1$, i.e. we have $\phi|_{\mathbb{C}[c^*c]} = h|_{\mathbb{C}[c^*c]}$.*

Proof. Recall $u_{00}^{(1)} = 1 - (1+q^2)c^*c$. Therefore $\phi(u_{00}^{(s)}) = 0$ is equivalent to $\phi(c^*c) = \frac{1}{1+q^2}$.

Assume there exists an integer $s > 1$ with $\phi(u_{00}^{(s)}) = 1$. Then the Cauchy-Schwarz inequality implies $\phi((u_{00}^{(s)})^* u_{00}^{(s)}) \geq 1$. The unitarity of the corepresentation $v^{(s,p)}$ gives

$$1 = \sum_{k=-s}^{s} \left(u_{k0}^{(s)} v^{p+s}\right)^* u_{k0}^{(s)} v^{p+s} = \sum_{k=-s}^{s} \left(u_{k0}^{(s)}\right)^* u_{k0}^{(s)},$$

therefore

$$\phi\left(\sum_{\substack{k\in\{-s,\dots,s\}\\ k\neq 0}}\left(u_{k0}^{(s)}\right)^* u_{k0}^{(s)}\right)\leq 0,$$

and in particular $\phi\big((u_{s0}^{(s)})^* u_{s0}^{(s)}\big)=0$. We have

$$\begin{aligned}\left(u_s^{(s)}\right)^* u_{s0}^{(s)} &= \left[\begin{array}{c}2s\\ s\end{array}\right]_{q^2}\big((a^*)^s c^s\big)^*(a^*)^s c^s = \left[\begin{array}{c}2s\\ s\end{array}\right]_{q^2}(c^*)^s c^s a^s (a^*)^s\\ &= (c^*c)^s(1-q^2c^*c)\cdots(1-q^{2s}c^*c)\end{aligned}$$

By the representation theory of $C(SU_q(2))$, c^*c is positive self-adjoint contraction, with the spectrum $\sigma(c^*c)\subseteq\{q^{2n};n\in\mathbb{Z}_+\}\cup\{0\}$, and therefore the product $(1-q^2c^*c)\cdots(1-q^{2s}c^*c)$ defines a strictly positive operator. Therefore $\phi\left((u_{s,0}^{(s)})^* u_{s,0}^{(s)}\right)=0$ implies $\phi\big((c^*c)^s\big)=0$, which is impossible if $\phi(c^*c)=\frac{1}{1+q^2}>0$.

Therefore $\phi(u_{00}^{(s)})=0$ for all integers $s\geq 1$. □

Lemma 4.4. *Let*

$$\mathcal{A}_0=\mathrm{span}\{u_{k\ell}^{(s)}v^r; s\in\frac{1}{2}\mathbb{Z}_+, s>0, -s\leq k,\ell\leq s, r\in\mathbb{Z}\}$$

i.e. $\mathcal{A}_0$ is the subspace spanned by the matrix elements of the unitary irreducible corepresentations of dimension at least two.

Assume that $\phi|_{\mathbb{C}[c^*c]}=h|_{\mathbb{C}[c^*c]}$, i.e.

$$\phi(u_{00}^{(s)})=\delta_{0s}$$

for $s\in\mathbb{Z}_+$.

Then we have $\phi|_{\mathcal{A}_0}=h|_{\mathcal{A}_0}$, i.e.

$$\phi\left(u_{k\ell}^{(s)}v^r\right)=0$$

for all $r\in\mathbb{Z}$, $s\in\frac{1}{2}\mathbb{Z}_+$, $s>0$, and $-s\leq k,\ell\leq s$.

Proof. By Lemma 4.1, we already know that $\phi(u_{k\ell}^{(s)}v^r)\in\{0,1\}$, and $\phi(u_{k\ell}^{(s)}v^r)=0$ for $k\neq\ell$. Assume there exist $s\in\frac{1}{2}\mathbb{Z}_+$, $s>0$, $-s\leq k\leq s$ and $r\in\mathbb{Z}$ such that

$$\phi(u_{kk}^{(s)}v^r)=1.$$

We will show that this is impossible, if ϕ agrees with the Haar state h on the subalgebra generated by c^*c.

By the Cauchy-Schwarz inequality, we have

$$\phi\left((u_{kk}^{(s)})^* u_{kk}^{(s)}\right)\geq\left|\phi(u_{kk}^{(s)}v^r)\right|^2=1.$$

Applying ϕ to

$$\sum_{\ell=-s}^{s}(u_{\ell k}^{(s)})^* u_{\ell k}^{(s)}=1,$$

we can deduce

$$\sum_{\substack{\ell\in\{-s,\ldots,s\}\\ \ell\neq k}} \phi\left((u^{(s)}_{\ell k})^* u^{(s)}_{\ell k}\right) = 0.$$

But this contradicts $\phi|_{\mathbb{C}[c^*c]} = h|_{\mathbb{C}[c^*c]}$, because

$$\sum_{\substack{\ell\in\{-s,\ldots,s\}\\ \ell\neq k}} (u^{(s)}_{\ell k})^* u^{(s)}_{\ell k} = 1 - (u^{(s)}_{kk})^* u^{(s)}_{kk}$$

is a non-zero positive element in $\mathbb{C}[c^*c]$ and the Haar state is faithful. □

We can now give a description of all idempotent states on $U_q(2)$. It turns out that they are all induced by Haar states of quantum subgroups of $U_q(2)$.

Theorem 4.5. *Let $q \in (-1,0)\cup(0,1)$. Then the following is a complete list of the idempotent states on the compact quantum group $U_q(2)$.*

(1) *The Haar state h of $U_q(2)$.*
(2) *$\tilde{\phi}\circ\pi_{\mathbb{T}^2}$, where $\pi_{\mathbb{T}^2}$ denotes the surjective quantum group morphism $\pi_{\mathbb{T}^2}: C(U_q(2)) \to C(\mathbb{T}^2)$ and $\tilde{\phi}$ is an idempotent state on $C(\mathbb{T}^2)$. In particular if $\widetilde{\varepsilon}$ denotes the counit of $C(\mathbb{T}^2)$, then $\widetilde{\varepsilon}\circ\pi_{\mathbb{T}^2}$ is the counit of $U_q(2)$.*
(3) *The states induced by the Haar states of the compact quantum subgroups $SU_q(2)\ltimes_\sigma \mathbb{Z}_n$ of $U_q(2)\cong SU_q(2)\ltimes_\sigma U(1)$, for $n\in\mathbb{N}$. The case of $n=1$ corresponds to the Haar state on $SU_q(2)$ viewed as a quantum subgroup of $U_q(2)$.*

Remark 4.6. (1) The compact quantum group $C(\mathbb{T}^2)$ is commutative, by Kawada and Itô's theorem all idempotent states on $C(\mathbb{T}^2)$ are induced by Haar measures of compact subgroups of the two-dimensional torus $\mathbb{T}^2$.

(2) As a compact quantum group, $\mathbb{Z}_n$ is given by

$$\mathrm{Pol}\,\mathbb{Z}_n = C(\mathbb{Z}_n) = \mathrm{span}\,\{w_0,\ldots,w_{n-1}\},$$

with $w_k w_\ell = w_{k+\ell \bmod n}$, $S(w_k) = w_{n-k} = (w_k)^*$, $\Delta(w_k) = w_k\otimes w_k$, and $\varepsilon(w_k) = 1$ for $k = 0,\ldots,n-1$. The Haar state of $\mathbb{Z}_n$ is given by $h(w_k) = \delta_{0k}$. $C(\mathbb{Z}_n)$ can also be obtained from $\mathrm{Pol}\,U(1)$ by dividing out the Hopf $*$-ideal $\{u_1(w^n-1)u_2;\, u_1,u_2\in \mathrm{Pol}\,U(1)\}$.

Analogous to [Wys04, Section 4], one can define the twisted product $SU_q(2)\ltimes_\sigma\mathbb{Z}_n$. Alternatively, $\mathrm{Pol}(SU_q(2)\ltimes_\sigma\mathbb{Z}_n)$ can be obtained from $U_q(2)\cong SU_q(2)\ltimes_\sigma U(1)$ by dividing out the Hopf ideal $\{u_1(v^n-1)u_2;\, u_1,u_2\in\mathrm{Pol}\,U_q(2)\}$, and $C(SU_q(2)\ltimes_\sigma\mathbb{Z}_n)$ as its C^*-completion. This construction shows that $SU_q(2)\ltimes_\sigma\mathbb{Z}_n$ is a quantum subgroup of $U_q(2)\cong SU_q(2)\ltimes_\sigma U(1)$. As in the case of $U_q(2)$, $C(SU_q(2)\ltimes_\sigma\mathbb{Z}_n) = C(SU_q(2))\otimes C(\mathbb{Z}_n)$ as a C^*-algebra, and the Haar state of $SU_q(2)\ltimes_\sigma\mathbb{Z}_n$ is equal to the tensor product of the Haar states of $SU_q(2)$ and $\mathbb{Z}_n$.

Proof. (of Theorem 4.5) Let $\phi: C(U_q(2))\to\mathbb{C}$ be an idempotent state on $U_q(2)$. Clearly ϕ is uniquely determined by its restriction to $\mathrm{Pol}\,U_q(2)$.

We distinguish two cases.
Case (i) $\phi(u^{(1)}_{00}) = 1$. In this case Lemma 4.2 shows that ϕ is induced by an idempotent state on the quantum subgroup $\mathbb{T}^2$ of $U_q(2)$, i.e. $\phi = \tilde{\phi} \circ \pi_{\mathbb{T}^2}$ for some idempotent state $\tilde{\phi} : C(\mathbb{T}^2) \to \mathbb{C}$. This case includes the counit ε of $U_q(2)$, it corresponds to the trivial subgroup $\{1\}$ of $\mathbb{T}^2$.
Case (ii) $\phi(u^{(1)}_{00}) = 0$. In this case Lemma 4.3 and Lemma 4.4 imply that ϕ agrees with the Haar state h on the subspace $\mathcal{A}_0$, i.e.

$$\phi(u^{(s)}_{k\ell} v^r) = 0$$

for all $s \in \frac{1}{2}\mathbb{Z}_+$, $s > 0$, $-s \leq k, \ell \leq s$, and $r \in \mathbb{Z}$. It remains to determine ϕ on the $*$-subalgebra $\mathrm{alg}\{v, v^*\}$ generated by v, since $\mathrm{Pol}\, U_q(2) = \mathcal{A}_0 \oplus \mathrm{alg}\{v, v^*\}$ as a vector space. But this subalgebra is isomorphic to the $*$-Hopf algebra $\mathrm{Pol}\, U(1)$ of polynomials on the unit circle, and therefore $\phi|_{\mathrm{alg}\{v,v^*\}}$ has to be induced by the Haar measure of a compact subgroup of $U(1)$. We have the following possibilities.

(1) $\phi|_{\mathrm{alg}\{v,v^*\}} = \varepsilon_{U(1)}$, i.e. the restriction of ϕ to $\mathrm{alg}\{v, v^*\}$ is equal to the counit of $\mathrm{Pol}\, U(1)$. In this case we have

$$\phi(u^{(s)}_{k\ell} v^r) = \begin{cases} 1 & \text{if} \quad s = k = \ell = 0, \text{ and } r \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

This formula shows that $\phi = h_{SU_q(2)} \circ \pi_{SU_q(2)}$, where $\pi_{SU_q(2)}$ is the quantum groups morphism from $U_q(2)$ onto $SU_q(2)$ and $h_{SU_q(2)}$ denotes the Haar state of $SU_q(2)$.

(2) $\phi|_{\mathrm{alg}\{v,v^*\}} = h_{U(1)}$, i.e. the restriction of ϕ to $\mathrm{alg}\{v, v^*\}$ is equal to the Haar state of $\mathrm{Pol}\, U(1)$. In this case we have

$$\phi(u^{(s)}_{k\ell} v^r) = \begin{cases} 1 & \text{if} \quad s = k = \ell = 0 \text{ and } r = 0, \\ 0 & \text{else.} \end{cases}$$

We see that in this case ϕ is the Haar state h of $U_q(2)$.

(3) $\phi|_{\mathrm{alg}\{v,v^*\}}$ is the idempotent state on $U(1)$ induced by the Haar measure of the subgroup $\mathbb{Z}_n \subseteq U(1)$ for some $n \in \mathbb{N}$, $n \geq 2$. In this case we have

$$\phi(u^{(s)}_{k\ell} v^r) = \begin{cases} 1 & \text{if} \quad s = k = \ell = 0 \text{ and } r \equiv 0 \mod n, \\ 0 & \text{else.} \end{cases}$$

It follows that ϕ is induced by the Haar state of the quantum subgroup $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$ of $U_q(2) \cong SU_q(2) \ltimes_\sigma U(1)$.

Conversely, all the states we have found are induced by Haar states on quantum subgroups of $U_q(2)$, therefore they are clearly idempotent. It can be also checked directly. □

We see that all idempotent states on $U_q(2)$ are induced from Haar states of quantum subgroups of $U_q(2)$. We can also deduce the complete list of quantum subgroups of $U_q(2)$:

Corollary 4.7. *Let $q \in (-1,0) \cup (0,1)$. Then the following is a complete list of the non-trivial quantum subgroups of $U_q(2)$.*

1. *The two-dimensional torus and its closed subgroups.*
2. *The compact quantum groups of the form $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$, with $n \in \mathbb{N}$ (here the twisting is identical to that appearing in the identification $U_q(2) \cong SU_q(2) \ltimes_\sigma \mathbb{T}$).*

5. Idempotent states on compact quantum groups $SU_q(2)$ and $SO_q(3)$ $(q \in (-1,0) \cup (0,1])$

Let us first discuss the case $q = 1$. $C(SU_1(2))$ and $C(SO_1(3))$ are the algebras of continuous functions on the groups $SU(2)$ and $SO(3)$. All idempotent states correspond to Haar measures on compact subgroups. The list of these subgroups can be found, e.g., in [Pod95].

Consider now the generic case $q \in (-1,0) \cup (0,1)$. Every idempotent state on $SU_q(2)$ induces an idempotent state on $U_q(2)$, since $SU_q(2)$ is a quantum subgroup of $U_q(2)$. This observation allows us to deduce all idempotent states on $SU_q(2)$ from Theorem 4.5. We omit the details and just state the result.

Theorem 5.1. *Let $q \in (-1,0) \cup (0,1)$. The Haar state, the counit, and the idempotent states induced by the quantum subgroups $U(1)$ and $\mathbb{Z}_n$, $2 \leq n \leq \infty$, are the only idempotent states on $SU_q(2)$.*

Since the morphism $j : C(SU_q(2)) \to C(U(1))$ gives the diagonal matrices

$$\left(j(u^{(s)}_{k\ell})\right)_{-s\leq k,\ell\leq s} = \begin{pmatrix} z^{2s} & & & \\ & z^{2s-2} & & \\ & & \ddots & \\ & & & z^{-2s} \end{pmatrix},$$

we get

$$\left(h_{U(1)} \circ j\right)(u^{(s)}_{k\ell}) = \begin{cases} 1 & \text{if} \quad s \in \mathbb{Z}_+,\ k = \ell = 0, \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

and

$$\left(h_{\mathbb{Z}_{2n}} \circ j\right)(u^{(s)}_{k\ell}) = \begin{cases} 1 & \text{if} \quad s \in \mathbb{Z}_+,\ k = \ell,\ 2k \equiv 0 \mod 2n, \\ 0 & \text{else.} \end{cases} \tag{5.2}$$

$$\left(h_{\mathbb{Z}_{2n+1}} \circ j\right)(u^{(s)}_{k\ell}) = \begin{cases} 1 & \text{if} \quad k = \ell,\ 2k \equiv 0 \mod 2n+1, \\ 0 & \text{else.} \end{cases} \tag{5.3}$$

for $n \in \mathbb{N}$.

Consider now the idempotent states on $SO_q(3)$. Since $C(SO_q(3))$ is a subalgebra of $C(SU_q(2))$ and since the inclusion map is a quantum group morphism, every idempotent state on $SU_q(2)$ gives an idempotent state on $SO_q(3)$ by restriction. We will show that all idempotent states on $SO_q(3)$ arise in this way.

It follows that all idempotent states on $SO_q(3)$ are induced from Haar states of quantum subgroups.

Theorem 5.2. *Let $q \in (-1,0)\cup(0,1)$ and n an odd integer. Then the restrictions to $C(SO_q(3))$ of the idempotent states $h_{\mathbb{Z}_n} \circ j$ and $h_{\mathbb{Z}_{2n}} \circ j$ coincide.*

Furthermore, the Haar state, the counit, and the states induced from the Haar states on the quantum subgroups $U(1) \cong SO(2)$ and its closed subgroups are the only idempotent states on $SO_q(3)$.

Proof. The first statement follows from Equations (5.2) and (5.3).

Let now ϕ be an idempotent state on $SO_q(3)$. Denote by E the conditional expectation from $C(SU_q(2))$ onto $C(SO_q(3))$ introduced in Paragraph 2.4.2. Then $\hat{\phi} = \phi \circ E$ defines an idempotent state on $SU_q(2)$ such that $\phi = \hat{\phi}|_{C(SO_q(3))}$. With this observation Theorem 5.2 follows immediately from Theorem 5.1. □

Remark 5.3. This method applies to quotient quantum groups in general. Let A be a compact quantum group. A quantum subgroup (B, j) is called normal, if the images of the conditional expectations

$$\begin{aligned} E_{A/B} &= \big(\mathrm{id} \otimes (h_B \circ j)\big) \circ \Delta, \\ E_{B\backslash A} &= \big((h_B \circ i) \otimes \mathrm{id}\big) \circ \Delta, \end{aligned}$$

coincide, cf. [Wan08, Proposition 2.1 and Definition 2.2]. In this case the quotient $\mathsf{A}/\mathsf{B} = E_{A/B}(\mathsf{A})$ has a natural compact quantum group structure and all idempotent states on A/B arise as restrictions of idempotent states on A.

As a corollary to Theorems 5.1 and 5.2, we recover Podleś' classification [Pod95] of the quantum subgroups of $SU_q(2)$ and $SO_q(3)$.

Corollary 5.4. *Let $q \in (-1,0) \cup (0,1)$. Then $U(1) \cong SO(2)$ and its closed subgroups are the only non-trivial quantum subgroups of both $SU_q(2)$ and $SO_q(3)$.*

6. Idempotent states on compact quantum semigroups $U_0(2)$, $SU_0(2)$ and $SO_0(3)$

In this section we compute all idempotent states on $U_0(2)$, $SU_0(2)$ and $SO_0(3)$. As in the cases $q \neq 0$ considered earlier we begin with the C^*-bialgebra $C(U_0(2))$. Again we first need some preparatory observations and lemmas.

Note that $\pi_{\mathbb{T}^2} : C(U_q(2)) \to C(\mathbb{T}^2)$ is a well-defined $*$-algebra and coalgebra morphism also for $q = 0$, so $\mathbb{T}^2$ and its compact subgroups induce idempotent states on $U_0(2)$.

For $q = 0$ the algebraic relations of a and c become

$$cc^* = c^*c, \qquad aa^* = 1, \qquad ac = ac^* = 0, \qquad a^*a = 1 - c^*c.$$

As a is a coisometry, we have a decreasing family of orthogonal projections $(a^*)^n a^n$, $n \in \mathbb{N}$, which are group-like, i.e. $\Delta((a^*)^n a^n) = (a^*)^n a^n \otimes (a^*)^n a^n$, and $c^*c = 1 - a^*a$ is also an orthogonal projection.

Denote by M the unital semigroup $U(1) \times (\mathbb{Z}_+ \cup \{\infty\})$ with the operation

$$(z_1, n_1) \cdot (z_2, n_2) = (z_1 z_2, \min(n_1, n_2))$$

for $z_1, z_2 \in U(1)$, $n_1, n_2 \in \mathbb{Z}_+ \cup \{\infty\}$. This is an abelian semigroup with unit element $e_M = (1, \infty)$. Equip $\mathbb{Z}_+ \cup \{\infty\}$ with the topology in which a subset of $\mathbb{Z}_+ \cup \{\infty\}$ is open if and only if it is either an arbitrary subset of $\mathbb{Z}_+$ or the complement of a finite subset of $\mathbb{Z}_+$ (i.e. $\mathbb{Z}_+ \cup \{\infty\}$ is the one-point-compactification of $\mathbb{Z}_+$), and equip $U(1) \times (\mathbb{Z}_+ \cup \{\infty\})$ with the product topology.

The C^*-bialgebra $C(M)$ will play an important role in this section.

Lemma 6.1. *A probability measure μ on M is idempotent if and only if there exists an $n \in \mathbb{Z}_+ \cup \{\infty\}$ and an idempotent probability ρ on $U(1)$ such that $\mu = \rho \otimes \delta_n$.*

Proof. Any probability on M can be expressed as a sum $\mu = \sum_{n=0}^{\infty} \rho_n \otimes \delta_n + \rho_\infty \otimes \delta_\infty$, where ρ_n, $n \in \mathbb{Z}_+ \cup \{\infty\}$ are uniquely determined positive measures on $U(1)$ with total mass $\sum_{n=0}^{\infty} \rho_n(U(1)) + \rho_\infty(U(1)) = 1$, and δ_n denotes the Dirac measure on $\mathbb{Z}_+ \cup \{\infty\}$, i.e.

$$\delta_n(Q) = \begin{cases} 1 & \text{if} \quad n \in Q, \\ 0 & \text{if} \quad n \notin Q, \end{cases}$$

for $Q \subseteq \mathbb{Z}_+ \cup \{\infty\}$. We have

$$\begin{aligned} (\delta_n \star \delta_m)(Q) &= (\delta_n \otimes \delta_m)\Big(\big\{(k,\ell) \in (\mathbb{Z}_+ \cup \{\infty\})^2; \min(k,\ell) \in Q\big\}\Big) \\ &= \begin{cases} 1 & \text{if} \quad \min(n,m) \in Q, \\ 0 & \text{if} \quad \min(n,m) \notin Q, \end{cases} \end{aligned}$$

i.e. $\delta_n \star \delta_m = \delta_{\min(n,m)}$. Therefore

$$\mu^{\star 2} = \rho_\infty^{\star 2} \otimes \delta_\infty + \sum_{n=0}^{\infty} \left(\rho_n \star \left(\rho_n + 2 \sum_{m=n+1}^{\infty} \rho_m + 2\,\rho_\infty \right) \right) \otimes \delta_n.$$

Clearly, if ρ_∞ is an idempotent probability on $U(1)$ and $\rho_n = 0$ for $n \in \mathbb{Z}_+$, then $\mu = \rho_\infty \otimes \delta_\infty$ is idempotent.

Assume now that $\rho_\infty(U(1)) < 1$. Then there exists a unique $n \in \mathbb{Z}_+$ such that $\sum_{m=n+1}^{\infty} \rho_m(U(1)) < 1$, $\sum_{m=n}^{\infty} \rho_m(U(1)) = 1$ (i.e. n is the biggest integer m for which $\rho_{\geq m} = \rho_\infty + \sum_{k=m}^{\infty} \rho_k$ is a probability). Let $p = \rho_n(U(1))$. If μ is idempotent, then we have

$$p = \mu(U(1) \times \{n\}) = \mu^{\star 2}(U(1) \times \{n\}) = \left(\rho_n \star (2\rho_{\geq n} - \rho_n)\right)(U(1)) = 2p - p^2.$$

Since $p = \rho_n(U(1)) > 0$ by the choice of n, we get $p = 1$, i.e. $\rho_m = 0$ for $m \neq n$ and ρ_n is a probability. Then ρ_n has to be idempotent, and $\mu = \rho_n \otimes \delta_n$ is of the desired form.

Conversely, any probability of the form $\mu = \rho_n \otimes \delta_n$ with $n \in \mathbb{Z}_+ \cup \{\infty\}$ and ρ_n idempotent is idempotent. $\square$

For $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, define functions $\Theta_n^k : M \to \mathbb{C}$ by

$$\Theta_n^k(z,m) = \begin{cases} z^k & \text{if} \quad m \geq n, \\ 0 & \text{if} \quad m < n. \end{cases}$$

The span of these functions is dense in $C(M)$, and they satisfy

$$\Theta_n^k \Theta_m^\ell = \Theta_{\max(n,m)}^{k+\ell}, \qquad (\Theta_n^k)^* = \Theta_n^{-k}, \qquad \text{and} \qquad \varepsilon(\Theta_n^k) = \Theta_n^k(e_M) = 1.$$

For their coproduct, we have

$$\begin{aligned} \Delta\Theta_n^k\big((z_1,m_1),(z_2,m_2)\big) &= \Theta_n^k\big(z_1z_2, \min(m_1,m_2)\big) \\ &= \begin{cases} (z_1z_2)^k & \text{if} \quad m_1, m_2 \geq n, \\ 0 & \text{else,} \end{cases} \end{aligned}$$

i.e. $\Delta\Theta_n^k = \Theta_n^k \otimes \Theta_n^k$.

Proposition 6.2. *The semigroup M is a quantum quotient semigroup of $U_0(2)$, in the sense that there exists an injective $*$-algebra homomorphism j from $\mathrm{Pol}\, M := \mathrm{span}\{\Theta_n^k; n \in \mathbb{Z}_+, k \in \mathbb{Z}\}$ to $\mathrm{Pol}\, U_0(2)$ such that*

$$\Delta_M \circ j = (j \otimes j) \circ \Delta.$$

Proof. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, define $E_n^k = (\alpha^*)^n \alpha^n v^k \in \mathrm{Pol}\, U_q(2)$. From the defining relations of $U_0(2)$, one can check that the E_n^k satisfy the same $*$-algebraic and coalgebraic relations as the Θ_n^k, i.e. $j(\Theta_n^k) = E_n^k$ defines a $*$-bialgebra homomorphism $j : \mathrm{Pol}\, M \to \mathrm{Pol}\, U_0(2)$.

Let us show that j is injective. Assume there exists a non-zero function $f = \sum_{k,n} \lambda_{k,n} \Theta_n^k$ such that $j(f) = 0$. Let n_0 be the smallest integer for which there exists a $k \in \mathbb{Z}$ such that $\lambda_{k,n} \neq 0$. Take the representation $\pi = \pi_0 \otimes \mathrm{id}_{L^2(\mathbb{T})}$ of $C(U_0(2)) \cong C(SU_0(2)) \otimes C(U(1))$ (recall that π_0 was defined in Section 2.3.1). Since $j(f) = 0$, the operator $\chi = \pi(j(f)) = \sum \lambda_{k,n} \pi_0\left((\alpha^*)^n \alpha^n\right) v^k$ has to vanish. Apply χ to $e_{n_0} \otimes 1$. Since $\pi_0\left((\alpha^*)^n \alpha^n\right) e_{n_0} = 0$ for $n > n_0$, we get $\chi(e_{n_0} \otimes 1) = \sum_k \lambda_{n_0,k} e_0 \otimes v^k \in \mathfrak{h} \otimes L^2(\mathbb{T})$, which implies $\lambda_{n_0,k} = 0$ for all k, in contradiction to the choice of n_0. □

We can now give a description of all idempotent states on $U_0(2)$.

Theorem 6.3. *The following gives a complete list of the idempotents states on $U_0(2)$.*

(1) *The idempotent states induced by the Haar measures on the two-dimensional torus $\mathbb{T}^2$ and its closed subgroups. If ρ denotes the Haar measure of $\mathbb{T}^2$ or one of its closed subgroups, then the corresponding idempotent ϕ_ρ state is given by*

$$\phi_\rho\left((a^*)^r c^k a^s v^\ell\right) = \delta_{0k} \int_{\mathbb{T}^2} w_1^{s-r} w_2^\ell \mathrm{d}\rho(w_1,w_2)$$

for $n, m \in \mathbb{Z}_+$, $k, \ell \in \mathbb{Z}$. This includes the counit of $U_0(2)$, for the trivial subgroup $\{1\}$ of $\mathbb{T}^2$.

(2) *The family* $\Psi_{n,m} = \psi_n \otimes \phi_m$, $n \in \mathbb{Z}_+$, $m \in \mathbb{N} \cup \{\infty\}$. *Here* ϕ_m *is an idempotent state on* $C(U(1))$, *namely the Haar measure for* $m = \infty$ *and the idempotent state induced by the Haar measure of* $\mathbb{Z}_m$ *for* $m \in \mathbb{N}$. *And* ψ_n, $n \in \mathbb{Z}_+$ *is the idempotent state on* $C(SU_0(2))$ *defined by*

$$\psi_n\left((\alpha^*)^r \gamma^k \alpha^s\right) = \begin{cases} 1 & \text{if} \quad r = s \leq n \text{ and } k = 0 \\ 0 & \text{else.} \end{cases}$$

Remark 6.4. The state $\Psi_{0,\infty}$ can be considered as the Haar state on $U_0(2)$ since it is invariant, i.e.

$$\Psi_{0,\infty} \star f = f \star \Psi_{0,\infty} = f(1)\Psi_{0,\infty}$$

for any $f \in C(U_0(2))^*$. But $\Psi_{0,\infty}$ is not faithful. Its null space

$$\begin{aligned} N_{\Psi_{0,\infty}} &= \{u \in \operatorname{Pol} U_0(2); \Psi_{0,\infty}(u^*u) = 0\} \\ &= \operatorname{span}\{(a^*)^k c^m a^\ell v^n; k \in \mathbb{Z}_+, m, n \in \mathbb{Z}, \ell \geq 1\} \end{aligned}$$

is a left ideal, but not self-adjoint or two-sided. It is a subcoalgebra, i.e. we have

$$\Delta N_{\Psi_{0,\infty}} \subseteq N_{\Psi_{0,\infty}} \odot N_{\Psi_{0,\infty}},$$

but it is not a coideal, since the counit does not vanish on $\mathcal{N}_{\Psi_{0,\infty}}$.

Proof. (of Theorem 6.3) Let $\phi : C(U_0(2)) \to \mathbb{C}$ be an idempotent state on $U_0(2)$. Then ϕ induces an idempotent state $\phi \circ j$ on $C(M)$. By Lemma 6.1, $\phi \circ j$ is integration against a probability measure of the form $\rho \otimes \delta_n$ with $n \in \mathbb{Z}_+ \cup \{\infty\}$ and ρ an idempotent measure on $U(1)$. This determines ϕ on the subalgebra generated by v and $(a^*)^r a^r$, $r \in \mathbb{N}$, we have

$$\phi\left((a^*)^r a^r v^k\right) = \begin{cases} \rho(v^k) & \text{if} \quad r \leq n, \\ 0 & \text{else,} \end{cases}$$

for $k \in \mathbb{Z}$, $r \in \mathbb{Z}_+$.
Case (i): $n = \infty$. For $k > 0$ and any $r, s \in \mathbb{Z}_+$, $\ell \in \mathbb{Z}$, we have

$$\begin{aligned} \left|\phi\left((a^*)^r c^k a^s v^\ell\right)\right|^2 &\leq \phi\left((a^*)^r a^r\right) \phi\left((a^*)^s (c^*)^k c^k a^s\right) \\ &= \phi\left((a^*)^s a^s - (a^*)^{s+1} a^{s+1}\right) = 0, \end{aligned}$$

and therefore ϕ vanishes on the $*$-ideal $\mathcal{I}_c$ generated by c. As in the proof of Lemma 4.2, it follows that ϕ is induced by an idempotent state on $\operatorname{Pol} U_0(2)/\mathcal{I}_c \cong \operatorname{Pol} \mathbb{T}^2$, i.e. ϕ is of the form given in (1).
Case (ii): $n = 0$. Using again the Cauchy-Schwarz inequality, we get

$$\left|\phi\left((a^*)^r c^k a^0 v^\ell\right)\right|^2 \leq \phi\left((a^*)^r a^r\right) \phi\left((a^*)^s a^s - (a^*)^{s+1} a^{s+1}\right), \tag{6.1}$$

$$\left|\phi\left((a^*)^r c^k a^s v^\ell\right)\right|^2 \leq \phi\left((a^*)^r a^r - (a^*)^{r+1} a^{r+1}\right) \phi\left((a^*)^s a^s\right), \tag{6.2}$$

$$\left|\phi\left((a^*)^r a^s v^\ell\right)\right|^2 \leq \phi\left((a^*)^r a^r\right) \phi\left((a^*)^s a^s\right), \tag{6.3}$$

for $k, r, s \in \mathbb{Z}_+$, $\ell \in \mathbb{Z}$. This shows that $\phi\left((a^*)^r c^k a^r v^\ell\right)$ vanishes, unless $r = s = 0$. different from 0. But then $\phi = \phi \star \phi$ implies

$$\phi(c^k v^\ell) = (\phi \otimes \phi)\big(\Delta(c^k v^\ell)\big) = \sum_{\kappa=0}^{k} \phi\left((a^*)^{k-\kappa} c^\kappa v^\ell\right) \phi\left(c^{k-\kappa} a^\kappa v^\ell\right) = 0$$

for $k > 0$, $s \in \mathbb{Z}$. By hermitianity $\phi\left((c^*)^k v^\ell\right) = 0$ for $k > 0$, and ϕ has the form given in (2) with $n = 0$.
Case (iii): $n \in \mathbb{N}$. We use once more the Cauchy-Schwarz inequality. For $k \neq 0$, (6.1) and (6.2) imply that $\phi\left((a^*)^r c^k a^s v^\ell\right)$ vanishes unless $r = s = n$. But then we can show that $\phi = \phi \star \phi$ implies $\phi\left((a^*)^n c^k a^n v^\ell\right) = 0$ in the same way as in the previous case.

For $k = 0$, we see from (6.3) that $\phi\left((a^*)^r a^s v^\ell\right)$ vanishes unless $r, s \leq n$. The elements $(a^*)^r a^s v^\ell$ are group-like, therefore $\phi\left((a^*)^r a^s v^\ell\right) \in \{0, 1\}$. If we can show $\phi\left((a^*)^r a^s v^\ell\right) \neq 1$ for $r \neq s$, we are done, since then $\phi\left((a^*)^r c^k a^s v^\ell\right)$ is non-zero only if $r = s \leq n$. We get $\phi\left((a^*)^r a^s v^\ell\right) = \delta_{rs}\rho(v^\ell)$ for $r, s \leq n$, i.e. ϕ has the form given in (2).

We show $\phi\left((a^*)^r a^s v^\ell\right) \neq 1$ for $r \neq s$ by contradiction. Assume there exists a triple (r_0, s_0, ℓ_0) such that $\phi\left((a^*)^r a^s v^\ell\right) = 1$ and choose such a triple with maximal r_0. Set

$$b = (a^*)^{r_0} a^{s_0} v^{\ell_0} + (a^*)^{s_0} a^{r_0} v^{-\ell_0} - 1.$$

Maximality of r_0 implies $r_0 > s_0$ and

$$\phi\left((a^*)^{2r_0 - s_0} a^{s_0} v^{2\ell_0}\right) = \phi\left((a^*)^{s_0} a^{2r_0 - s_0} v^{-2\ell_0}\right) = 0,$$

therefore we get

$$\begin{aligned}
\phi(b^* b) &= \phi\left((a^*)^{r_0} a^{r_0}\right) + \phi\left((a^*)^{s_0} a^{s_0}\right) + 1 \\
&\quad + \phi\left((a^*)^{2r_0 - s_0} a^{s_0} v^{2\ell}\right) + \phi\left((a^*)^{s_0} a^{2r_0 - s_0} v^{-2\ell}\right) \\
&\quad - 2\phi\left((a^*)^{r_0} a^{s_0} v^\ell\right) - 2\phi\left((a^*)^{s_0} a^{r_0} v^{-\ell}\right) \\
&= -1,
\end{aligned}$$

which is clearly a contradiction to the positivity of ϕ.

Conversely, using the formulas

$$\begin{aligned}
\Delta\big((a^*)^r c^k a^s v^\ell\big) &= \sum_{\kappa=0}^{k} (a^*)^{r+k-\kappa} c^\kappa a^s v^\ell \otimes (a^*)^r c^{k-\kappa} a^{s+\kappa} v^\ell, \\
\Delta\big((a^*)^r (c^*)^k a^s v^\ell\big) &= \sum_{\kappa=0}^{k} (a^*)^r (c^*)^\kappa a^{s+k-\kappa} v^\ell \otimes (a^*)^{r+\kappa} (c^*)^{k-\kappa} a^s v^\ell,
\end{aligned}$$

for $r, s, k \in \mathbb{Z}_+$, $\ell \in \mathbb{Z}$, for the coproduct in $\mathrm{Pol}\, U_0(2)$, one can check that all states given in the theorem are indeed idempotent. □

The complete description of the idempotent states on $SU_0(2)$ follows now directly from Theorem 6.3 and the comments before Theorem 4.5.

Theorem 6.5. *The following gives a complete list of the idempotents states on* $SU_0(2)$.

(1) *The family* ϕ_n, $n \in \mathbb{N} \cup \{\infty\}$ *where* ϕ_1 *is the counit,* ϕ_∞ *the idempotent induced by the Haar state of quantum subgroup* $U(1)$, *and* ϕ_n, $2 \leq n < \infty$, *denotes the idempotent state induced by the Haar state on the quantum subgroup* $\mathbb{Z}_n$.

(2) *The family* ψ_n, $n \in \mathbb{Z}_+$ *defined by*

$$\psi_n\big((\alpha^*)^r \gamma^k \alpha^s\big) = \begin{cases} 1 & \text{if} \quad k = 0, r = s \leq n, \\ 0 & \text{else.} \end{cases}$$

for $r, s \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, *with the convention* $\gamma^{-k} = (\gamma^*)^k$.

Similarly using the conditional expectation introduced in Paragraph 2.4.2, we can derive a complete classification of the idempotent states on $SO_0(3)$. The proof is identical to the proof of Theorem 5.2 and therefore omitted.

Theorem 6.6. *All idempotent states on* $SO_0(3)$ *arise as restriction of idempotent states on* $SU_0(2)$. *Note also that we have again* $\phi_n|_{C(SO_0(3))} = \phi_{2n}|_{C(SO_0(3))}$ *for* n *an odd integer.*

7. The idempotent states on $SU_q(2)$ as elements of the dual and associated quantum hypergroups

Let A be a compact quantum group and $\mathcal{A}$ the corresponding Hopf $*$-algebra dense in A. Then $\mathcal{A}$ is an algebraic quantum group in the sense of van Daele, and so is its dual $\hat{\mathcal{A}}$, given by the functionals of the form $h(\cdot\, a)$ with $a \in \mathcal{A}$. By [FS08b, Lemma 3.1], an idempotent state ϕ on A defines a group-like projection p_ϕ in the multiplier algebra $M(\hat{\mathcal{A}})$ of the dual, and therefore, by [LvD07, Theorem 2.7] and [FS08b, Theorem 2.4] an algebraic quantum hypergroup $\hat{\mathcal{A}}_{p_\phi}$. As an algebra, $\hat{\mathcal{A}}_{p_\phi} = p_\phi \hat{\mathcal{A}} p_\phi$, and the coproduct of $\hat{\mathcal{A}}_{p_\phi}$ is given by

$$\hat{\Delta}_{p_\phi} = (p_\phi \otimes p_\phi)\hat{\Delta}(a)(p_\phi \otimes p_\phi)$$

for $a \in \hat{\mathcal{A}}_{p_\phi}$, where $\hat{\Delta}$ denotes the coproduct of $\hat{\mathcal{A}}$.

Let $q \in (-1, 0) \cup (0, 1)$. In this section we will consider the case of the compact quantum group $SU_q(2)$ and describe the algebraic quantum hypergroups associated to its idempotent states. Note that in this case the dense Hopf $*$-algebra is $\mathcal{A} = \mathrm{Pol}\, SU_q(2) = \mathrm{span}\,\{u^{(s)}_{k\ell} : s \in \frac{1}{2}\mathbb{Z}_+, -s \leq k, \ell \leq s\}$. We will use the basis

$$e^{(s)}_{kl} = \frac{1 - q^{2(2s+1)}}{q^{2(s-k)}(1 - q^2)} h\big((u^{(s)}_{k\ell})^* \ \cdot\ \big)$$

for $\hat{\mathcal{A}} = \mathrm{Pol}\,\widehat{SU_q(2)}$. Using the orthogonality relation

$$h\left((u^{(s)}_{k\ell})^* u^{(s')}_{k'\ell'}\right) = \delta_{ss'}\delta_{kk'}\delta_{\ell\ell'}\frac{q^{2(s-k)}(1-q^2)}{1-q^{2(2s+1)}},$$

for $s, s' \in \frac{1}{2}\mathbb{Z}_+$, $-s \leq k, \ell \leq s$, $-s' \leq k', \ell' \leq s'$, cf. [Koo89, Eq. (5.12)], we can check that this basis is dual to the basis $\{u^{(s)}_{k\ell} : s \in \frac{1}{2}\mathbb{Z}_+, -s \leq k, \ell \leq s\}$ of $\mathrm{Pol}\,SU_q(2)$. The algebraic quantum group $\hat{\mathcal{A}} = \mathrm{Pol}\,\widehat{SU_q(2)}$ is of discrete type and equal to the algebraic direct sum

$$\hat{\mathcal{A}} = \bigoplus_{s\in\frac{1}{2}\mathbb{Z}_+} M_{2s+1}.$$

The $e^{(s)}_{k\ell}$ form a basis of matrix units for $M_{2s+1} = \mathrm{span}\,\{e^{(s)}_{kl} : -s \leq k, \ell \leq s\}$.

The Haar state h and the counit ε give the elements $p_h = 1$ and $p_\varepsilon = e^{(0)}_{00}$ in $M(\hat{\mathcal{A}})$, and the associated algebraic quantum hypergroups are $\hat{\mathcal{A}}_{p_h} = \hat{\mathcal{A}}$ and $\hat{\mathcal{A}}_{p_\varepsilon} = \mathbb{C}$.

The remaining cases are more interesting.

7.1. **The idempotent state $\phi_2 = h_{\mathbb{Z}_2}\circ j$ induced by the quantum subgroup $\mathbb{Z}_2$.** We have

$$p_{\phi_2} = \sum_{s=0}^{\infty}\sum_{k=-s}^{s} e^{(s)}_{kk} = \sum_{s=0}^{\infty} 1_{2s+1},$$

i.e. p_{ϕ_2} is the sum of the identity matrices from the odd-dimensional matrix algebras M_{2s+1}, $s \in \mathbb{Z}_+$. This projection is in the center of $M(\hat{\mathcal{A}})$, therefore $\hat{\mathcal{A}}_2 = p_{\phi_2}\hat{\mathcal{A}}p_{\phi_2}$ will be an algebraic quantum group. We get

$$\hat{\mathcal{A}}_2 = \bigoplus_{s\in\mathbb{Z}_+} M_{2s+1} = \mathrm{Pol}\,\widehat{SO_q(3)},$$

i.e. $\hat{\mathcal{A}}_{p_{\phi_2}}$ is discrete algebraic quantum group dual to $SO_q(3)$. This is to be expected as $\mathbb{Z}_2$ is the only nontrivial normal quantum subgroup of $SU_q(2)$ and $SO_q(3)$ is the corresponding quotient quantum group.

7.2. **The idempotent state $\phi_\infty = h_{U(1)}\circ j$ induced by the quantum subgroup $U(1)$.** Here

$$p_{\phi_\infty} = \sum_{s=0}^{\infty} e^{(s)}_{00}$$

and this projection is not central. We get

$$\hat{\mathcal{A}}_{p_\infty} = \bigoplus_{s=0}^{\infty}\mathbb{C}$$

which is a commutative algebraic quantum hypergroup of discrete type. This is the dual of the hypergroup introduced in [Koo91, Section 7].

7.3. **The idempotent states $\phi_n = h_{\mathbb{Z}_n} \circ j$, $3 \le n < \infty$.** The remaining cases also give non-central projections,

$$\begin{aligned} p_{\phi_{2n}} &= \sum_{s=0}^{\infty} \sum_{k=-\lceil \frac{s}{n} \rceil}^{\lceil \frac{s}{n} \rceil} e^{(s)}_{nk,nk}, \\ p_{\phi_{2n+1}} &= \sum_{s\in\frac{1}{2}\mathbb{Z}_+} \sum_{\substack{-s\le k\le s \\ 2k\equiv 0 \bmod 2n+1}} e^{(s)}_{kk}, \end{aligned}$$

for $1 \le n < \infty$, cf. Equations (5.2) and (5.3). They lead to non-commutative algebraic quantum hypergroups $\hat{\mathcal{A}}_n = p_{\phi_n} \hat{\mathcal{A}} p_{\phi_n}$,

$$\begin{aligned} \hat{\mathcal{A}}_{2n} &= \bigoplus_{k=0}^{\infty} n M_{2k+1}, \\ \hat{\mathcal{A}}_{2n+1} &= \bigoplus_{k=0}^{\infty} (2n+1) M_k, \end{aligned}$$

of discrete type (in the formulas above nM_{2k+1} denotes n direct copies of the matrix algebra M_{2k+1} and similarly $(2n+1)M_k$ denotes $2n+1$ direct copies of the matrix algebra M_k).

Note that as the quantum subgroups consider in the last two paragraphs are not normal, the objects we obtain have only the quantum hypergroup structure (and can be informally thought of as duals of quantum hypergroups obtained via the double coset construction, [CV99]).

Appendix

The goal of the appendix is to provide a short proof of coamenability of the deformations of classical compact Lie groups. To facilitate the discussion we use here the symbol $\mathbb{G}$ to denote a compact quantum group, $\widehat{\mathbb{G}}$ to denote the dual of $\mathbb{G}$, $C(\mathbb{G})_{\text{red}}$ and $C(\mathbb{G})$ to denote respectively the reduced and universal C^*-algebras associated with $\mathbb{G}$ and $L^\infty(\mathbb{G})$ to denote the corresponding von Neumann algebra (we refer for example to [Tom07] for precise definitions). Note that contrary to the main body of the paper we do not assume that the Haar state on $\mathbb{G}$ is faithful, so that $\mathbb{G}$ need not be in the reduced form. We adopt the following definition ([BMT02], [Tom06]).

Definition A.1. A compact quantum group $\mathbb{G}$ is said to be *coamenable* if the dual quantum group $\widehat{\mathbb{G}}$ is amenable, that is, $L^\infty(\widehat{\mathbb{G}})$ has an invariant mean.

The following result gives a useful criterion to check coamenability:

Theorem A.2. [BMT02, Theorem 4.7], [Tom06, Corollary 3.7, Theorem 3.8] *A compact quantum group $\mathbb{G}$ is coamenable if and only if there exists a counit on $C(\mathbb{G})_{\text{red}}$ if and only if there exists a $*$-homorphism from $C(\mathbb{G})_{\text{red}}$ onto $\mathbb{C}$.*

The second equivalence is fairly easy to show, in the first the forward implication was established in [BMT02] and the backward implication in [Tom06].

Let $\mathbb{G}$ be a classical compact Lie group and $\mathbb{G}_q$ the q-deformation with the parameter $-1 < q < 1$, $q \neq 0$ (see [KS98]). The function algebra $C(\mathbb{G}_q)$ is the universal C^*-algebra generated by certain polynomial elements. The Haar state is denoted by h.

The following theorem was proved by T. Banica ([Ban99, Corollary 6.2]). We present another direct proof using L. I. Korogodski-Y. S. Soibelman's results on the representation theory of $C(\mathbb{G}_q)$.

Theorem A.3. *The quantum group $\mathbb{G}_q$ is coamenable.*

Proof. Let us introduce the left ideal $N_h := \{a \in C(\mathbb{G}_q) \mid h(a^*a) = 0\}$, which is in fact an ideal of $C(\mathbb{G}_q)$. The reduced compact quantum group $C(\mathbb{G}_q)_{\mathrm{red}}$ is defined as the quotient $C(\mathbb{G}_q)/N_h$. By Theorem A.2 to show that $\mathbb{G}_q$ is coamenable it suffices to show that the C^*-algebra $C(\mathbb{G}_q)_{\mathrm{red}}$ has a character.

Consider an irreducible representation $\pi\colon C(\mathbb{G}_q)_{\mathrm{red}} \to B(H_\pi)$. Composing this map with the canonical surjection $\rho\colon C(\mathbb{G}_q) \to C(\mathbb{G}_q)_{\mathrm{red}}$, we get an irreducible representation $\pi \circ \rho$ of $C(\mathbb{G}_q)$. Thanks to [KS98, Theorem 6.2.7 (3), §3], we may assume that $\pi \circ \rho$ is of the following form:

$$\pi \circ \rho = (\pi_{s_{i_1}} \otimes \cdots \otimes \pi_{s_{i_k}} \otimes \pi_t) \circ \delta^{(k)}$$

or $\pi \circ \rho = \pi_t$, where $s_{i_1} \cdots s_{i_k}$ is the reduced decomposition in the Weyl group of $\mathbb{G}$, and $t \in T$, the maximal torus of $\mathbb{G}$. In the latter case π is a one-dimensional representation. In the former case, we remark that the counit of $C(\mathbb{G}_q)$ factors through $\operatorname{Im} \pi_{s_i}$ for every i, that is, there exists $\eta_i\colon \operatorname{Im} \pi_{s_i} \to \mathbb{C}$ such that $\eta_i \circ \pi_{s_i} = \varepsilon$ (See the argument in [Tom07, p. 294]).

Then we introduce a representation $\tilde{\pi} := (\eta_{i_1} \otimes \cdots \otimes \eta_{i_k} \otimes \mathrm{id}) \circ \pi$ of $C(\mathbb{G}_q)_{\mathrm{red}}$, which is well-defined and one-dimensional. Indeed,

$$\tilde{\pi} \circ \rho = (\varepsilon \otimes \cdots \otimes \varepsilon \otimes \pi_t) \circ \delta^{(k)} = \pi_t.$$

Thus we have proved in each case the existence of a one-dimensional representation of the C^*-algebra $C(\mathbb{G})_{\mathrm{red}}$, and $\mathbb{G}_q$ is co-amenable. □

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