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Discrete Non-commutative Integrability: the Proof of a Conjecture by M. Kontsevich

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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DISCRETE NON-COMMUTATIVE INTEGRABILITY: THE PROOF OF A CONJECTURE BY M. KONTSEVICH

PHILIPPE DI FRANCESCO AND RINAT KEDEM

Abstract. We prove a conjecture of Kontsevich regarding the solutions of rank two recursion relations for non-commutative variables which, in the commutative case, reduce to rank two cluster algebras of affine type. The conjecture states that solutions are positive Laurent polynomials in the initial cluster variables. We prove this by use of a non-commutative version of the path models which we used for the commutative case.

1. Introduction

Let $\mathbb{F} = \mathbb{C}(x, y)$ denote the skew field of rational functions in the non-commutative variables x and y. Given any $a \in \mathbb{Z}$, Kontsevich introduced the following transformation on $\mathbb{F}^2$:

$$T_a \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \;\mapsto\; \begin{pmatrix} xyx^{-1} \\ (1+y^a)x^{-1}. \end{pmatrix} \tag{1.1}$$

which preserves the commutator $C = xyx^{-1}y^{-1}$.

Let $\mathcal{A} \subset \mathbb{F}$ be the algebra generated by the entries of all vectors obtained from iterations of the map T_cT_b (with $c, b \in \mathbb{Z}$), acting on the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Kontsevich made the following conjecture:

Conjecture 1.1. [8] *For any $b, c \in \mathbb{Z}_{>0}$, the entries of the vector $(T_cT_b)^m \begin{pmatrix} x \\ y \end{pmatrix}$, for all $m \geq 0$, are non-commutative Laurent polynomials in x and y with non-negative integer coefficients.*

That is, the generators of $\mathcal{A}$ are positive Laurent polynomials.

This conjecture is analogous to the similar conjecture [6] for rank 2 cluster algebras, in the commutative limit, $C = 1$. Although it is not quite clear what a "good" definition of a noncommutative cluster algebra should be in general, the equation (1.1) can be thought of an example of the mutations of the cluster variables in a rank 2 non-commutative cluster algebra.

In the commutative case [9], we introduced a method [4] which guarantees Laurent positivity, valid for the integrable cases of rank 2 cluster algebras, corresponding to a rank

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2 affine Cartan matrix (that is, $bc = 4$). This was done by writing explicit expressions for the cluster variables in terms of path models on weighted graphs. Equivalently, the generating function for cluster variables is a finite continued fraction with a manifestly positive expansion.

The case of affine A_1 is also the simplest example of a Q-system cluster algebra [7]. The path formulation can be generalized to the higher rank Q-systems [3, 4]. A non-commutative version of the Q-system cluster algebra is given by the so-called T-system. In [5], we found the solutions of the A_r T-system using the same path models, but with non-commutative weights. In this case, the non-commutative mutation relations are such that their matrix elements are the T-system equations, and the matrix elements of the non-commutative cluster variables are the T-system solutions. This is therefore another candidate for a non-commutative cluster algebra, of higher rank.

At rank 2, the Kontsevich evolution and the non-commutative Q-system relations are candidates for non-commutative cluster algebra mutations. Both can actually be obtained as different specializations of a more general evolution equation. We remark that the non-commutative Q-system is distinct from the quantum cluster algebras defined by [1], which is obtained as a specialization of the Kontsevich evolution, by setting $C = q$ to be a central element.

In this paper, we prove the conjecture of Kontsevich in the cases where it generalizes the integrable rank 2 cluster algebras, that is, the values of b, c are obtained from an affine Cartan matrix. We again use the path models introduced in [4] with non-commutative weights.

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2. Preliminaries

2.1. **Path models for the affine rank 2 cluster algebras.** Let us review briefly our solution for the rank 2 affine cluster algebras in the commutative case [6]. For the full details we refer to [4].

Let $\mathcal{F}$ be the field of rational functions in the two commuting formal variables x, y with rational coefficients. We consider the subring of $\mathcal{F}$ generated by the variables R_n, where R_n satisfy the recursion relations

$$R_{n+1}R_{n-1} = \begin{cases} 1 + R_n^b, & n \text{ odd;} \\ 1 + R_n^c, & n \text{ even,} \end{cases} \tag{2.1}$$

with initial conditions $(R_0, R_1) = (x, y)$. This is the cluster algebra of rank 2 corresponding to the exchange matrix

$$B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}. \tag{2.2}$$

with $b, c > 0$.

Some of these cluster algebras were studied in [9, 2], where, in particular, Laurent positivity was proven for the cases $bc \leq 4$ and $b = c$ in general. We are interested here in the *integrable* cases of the discrete evolution equations (2.1), that is, the following cases: (i) $(b, c) = (2, 2)$, (ii) $(b, c) = (1, 4)$ and (iii) $(b, c) = (4, 1)$. Each of these cases corresponds to an affine Dynkin diagram of rank 2, and each is integrable. These cases were studied in [9].

We may write a map $T_a : \mathcal{F}^2 \to \mathcal{F}^2$ as

$$T_a \begin{pmatrix} R_n \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} R_{n+1} \\ R_{n+2} \end{pmatrix}.$$

Then for th compound transformation $\mu = T_c T_b$ we have $\mu^m(R_0, R_1) = (R_{2m}, R_{2m+1})$, so there is a translational property, $\mu^m(R_{2n}, R_{2n+1}) = (R_{2(m+n)}, R_{2(m+n)+1})$. Therefore to get an expression for R_n in terms of any initial data (R_m, R_{m+1}), we need only find it in terms of two sets of initial data, say, (R_0, R_1) and (R_1, R_2). If $b = c$ we need only consider the first set of data because of the additional symmetry.

The path model solution of the system goes as follows. Define the generating function for the cluster variables as follows:

$$F(t) = \begin{cases} \sum_{n\geq 0} t^n R_n & \text{case (i);} \\ \sum_{n\geq 0} t^n R_{2n} & \text{case (ii);} \\ \sum_{n\geq 0} t^n R_{2n+1} & \text{case (iii).} \end{cases} \tag{2.3}$$

We write explicit expressions for $F(t)$ in each case, which make manifest the property that the coefficients of t^n are positive Laurent polynomials in any seed cluster data. One can then use the symmetries of these systems to express all other cluster variables (for negative values of n and for odd/even values of n in the cases (ii) and (iii)) in terms of the coefficients of $F(t)$. The expressions are always such that if the coefficients of $F(t)$ are positive Laurent polynomials, so are the remaining cluster variables.

One way of computing the generating function $F(t)$ is by expressing it as the partition function of weighted paths on a graph. If the weights are positive Laurent polynomials in some initial seed data, then so are the coefficients of t^n in $F(t)$, and hence the cluster variables.

For a given graph with vertices connected by edges, we assign a weight $w_{i,j}$ (i, j two vertices) associated to the edge connecting vertex i to vertex j. In general, we do not require that w_{ij} is equal to w_{ji}. The weight of a path from vertex i to vertex j is the product of all the weights along the edges traversed by the path, and its length is the number of steps traversed. The partition function of paths from vertex a to vertex b is the sum over all paths from vertex a to vertex b of the weights of the paths.

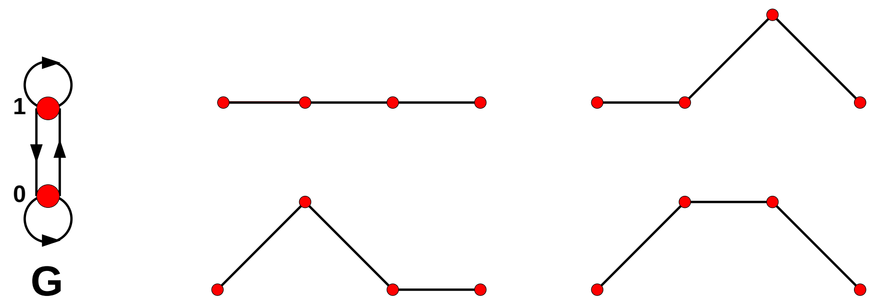

FIGURE 1. Weighted paths on the barbell graph G generate R_n for the cluster algebras $(1,4)$ and $(4,1)$. We have represented the 4 paths of length 3 on G, from the vertex 0 to itself, with respective weights y_1^3, y_1y_2, y_2y_1 and y_3y_2.

One can use the following presentations for the generating functions $F(t)$ in terms of path partition functions.

- $(b,c)=(2,2)$: Consider the graph composed of 4 vertices, labeled $0,1,2,3$, with edges connecting vertices i and $i+1$. We assign a weight w_{ij} to a step from i to j along each edge, when such an edge exists. The weights are $w_{i,i+1}=1,\ w_{i,i-1}=ty_i$, where
$$y_1=R_1R_0^{-1},\ y_2=R_1^{-1}R_0^{-1},\ y_3=R_1^{-1}R_0.$$
Note that these weights are positive Laurent monomials in the initial data R_0, R_1.
Then one can show that $F(t)R_0^{-1}$ is equal to the partition function of paths from vertex 0 to itself on this weighted graph. Since each weight is a positive Laurent monomial in the initial data, so is each coefficient of t^n in the partition function, which represents paths of length $2n$.
An equivalent statement is that the generating function $F(t)$ is the following finite continued fraction:
$$F(t)R_0^{-1}=\frac{1}{1-t\frac{y_1}{1-t\frac{y_2}{1-ty_3}}}.$$
We remark that a key fact which enables us to prove these formulas for the solution is the existence of an integral of motion, $K=y_1+y_2+y_3$, and also that $y_3y_1=1$. K is invariant under T_2, namely under $R_n\mapsto R_{n+1}$ for all n.
- $(b,c)=(1,4)$: Let G be the barbell graph, with vertices 0 and 1 connected by an edge, and each vertex connected to itself by a loop. See Figure 1. We consider paths on this graph from node 0 to itself of length n. We place a weight on each oriented edge. A path step from i to j contributes a weight $w_{i,j}$, so that a step from 1 to 1 around the loop connected to it contributes weight $w_{1,1}$. We choose $w_{0,0}=ty_1,\ w_{0,1}=t,\ w_{1,1}=ty_3,\ w_{1,0}=ty_2$, where
$$y_1=\frac{1+R_1}{R_0},\ y_2=\frac{R_0^4+(1+R_1)^2}{R_0^4R_1},\ y_3=\frac{R_0^4+1+R_1}{R_0^2R_1}.$$

This time, the weights are not monomials, but they are positive Laurent polynomials. Paths on this graph therefore all have weights which are positive Laurent polynomials. Explicitly, the generating function for R_{2n} in terms of the cluster seed (R_0, R_1) is obtained from the expansion in t of the finite continued fraction

$$F(t) = \frac{R_0}{1 - ty_1 - t^2 y_2 \frac{1}{1-ty_3}}.$$

The odd cluster variables are obtained from the relation $R_{2n}R_{2n-2} - 1 = R_{2n-1}$. One can show that 1 is a term in the Laurent polynomial $R_{2n}R_{2n-2}$ so that R_{2n-1} is positive. We note here that we have a conserved quantity, $K = y_1 + y_3$, and that $y_1y_3 - y_2 = 1$. K is left invariant under the compound mutation μ, i.e. under $R_n \mapsto R_{n+2}$ for all n.

Positivity with respect to the initial data (R_1, R_2) follows from the solution to the problem with $(b, c) = (4, 1)$, due to the symmetry between the two systems.

- $(b, c) = (4, 1)$: Here, we use the same graph, but replace the weights y_i with $y'_\alpha(R_0, R_1) = y_{4-\alpha}(R_1, R_0)$. The generating function for odd cluster variables is the partition function on this graph of paths from node 1 to itself. Even cluster variables are obtained from the equation $R_{2n+1}R_{2n-1} - 1 = R_{2n}$, and can again be shown to be positive. The conserved quantity is $K' = y'_1 + y'_3$, and we also have $y'_3y'_1 - y'_2 = 1$.

We will show below that each of these results generalizes to the non-commutative case. We will have more conserved quantities in the non-commutative case, due to the fact that the commutator $C \neq 1$, and is not, in fact, central. Thus, C itself will be a conserved quantity.

2.2. **A rank 2 non commutative cluster algebra.** We consider now the evolution (1.1). Define $C = xyx^{-1}y^{-1}$ and let $R_0 = Cx$ and $R_1 = y$. Given a pair $b, c \geq 0$, define $\{R_n\}_{n\in\mathbb{Z}_{\geq 0}}$ by

$$T_a \begin{pmatrix} CR_n \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} CR_{n+1} \\ R_{n+2} \end{pmatrix}$$

where $a = b$ if n is even and $a = c$ if n is odd. That is,

$$R_{n+1}CR_{n-1} = \begin{cases} 1 + R_n^b, & n \text{odd}; \\ 1 + R_n^c, & n \text{ even}. \end{cases} \tag{2.4}$$

Clearly, this defines R_n for negative values of n as well.

The expression for the commutator,

$$C = R_{n+1}^{-1} R_n R_{n+1} R_n^{-1}, \tag{2.5}$$

allows us to interpret it as a conserved quantity of the discrete evolution T_a, because its value is independent of n. In fact, one can check that, generally, any recursion relation of the form

$$R_{n+1}R_n^{-1}R_{n-1} = f_n(R_n) \tag{2.6}$$

has C as a conserved quantity. Equation (2.4) is a special case of this.

Note also that Eq. (2.5) implies a quasi-commutation relation

$$R_{n+1} C R_n = R_n R_{n+1}. \tag{2.7}$$

If $C = 1$, that is, if x and y commute, we recover the rank 2 cluster algebra of type (b, c).

Similarly, if we write $C = q$, a central element, then Eq. (2.7) turns into the quantum commutation relation $R_n R_{n+1} = q R_{n+1} R_n$, and we recover the rank 2 quantum cluster algebra of [1].

Therefore we call the transformation (1.1) a mutation and the ring $\mathcal{A}$ a non-commutative cluster algebra of rank 2. In general, the Laurent property is not proven for this algebra except in special cases. For example, in the case where B is obtained from the Dynkin diagram of finite type A_2, B_2 or G_2, one can check that the cluster algebra is finite, up to conjugation by C, with the same period as in the commutative case. In those cases, the cluster variables are positive Laurent polynomials with coefficients which are either 0 or 1. In the case where B is obtained from the Dynkin diagram of affine A_1, the Laurent property has been proved by Usnich, but not the positivity [8].

In this paper we do not attempt a general proof of Conjecture 1.1, but we generalize our proof for the commutative integrable (affine Dynkin diagram) cluster algebras of rank 2. We show that the path models of Ref. [4] have a simple non-commutative analogue. We therefore get an explicit expression for all cluster variables R_n. This allows us to prove the Laurent property and positivity for those cases.

What distinguishes these cases is that the transformation (2.4) is *integrable*. That is, in each case, there exist two Laurent polynomials (one of them being the commutator C) in the variables R_n, R_{n+1}, which are invariant under (2.4). The generating function for the cluster variables can be expressed in all cases as a finite continued fraction in any cluster seed variables. This can be interpreted as a generating function for paths with non-commutative weights.

2.3. **Symmetries.** The evolution equations (2.4) determine all R_n for $n \in \mathbb{Z}$ uniquely in terms of the initial data $CR_0 = x$ and $R_1 = y$.

One may relate the solutions of the (c, b) system to those of the (b, c) system by use of the translational symmetry. Let us denote by $f_n^{(b,c)}(x, y)$ the solution R_n of (2.4) expressed in terms of its initial data $CR_0 = x$ and $R_1 = y$. Let us also denote by $g_n^{(b,c)}(X, Y)$ the solution R_n of (2.4) expressed in terms the data $CR_1 = X$, $R_2 = Y$.

Lemma 2.1. *For all $n \in \mathbb{Z}$, we have:*

$$f_n^{(c,b)}(x, y) = g_{n+1}^{(b,c)}(x, y), \ \ n \in \mathbb{Z}.$$

Thus, solutions of the (c, b) system are given by those of the (b, c) system.

Moreover, one can relate the solution for $n < 0$ to that for $n \geq 0$. Define an anti-automorphism $*$ on $\mathbb{F}$ by

$$x \mapsto x^* = yC = yxyx^{-1}y^{-1}, \quad y \mapsto y^* = C^{-1}x = yxy^{-1}. \tag{2.8}$$

This is clearly an involution. In particular, we have $C^* = C$, $R_1^* = R_0$ and $R_0^* = R_1$.

Let $R_n = f_n^{(b,c)}(x,y)$ be a solution of the (b,c)-system, and $S_n = R_{1-n}^*$. Changing $n \to 1-n$ in (2.4) and applying $*$, we see that S_n satisfies the (c,b)-system. Since $S_0 = R_1^* = R_0$, so that $CS_0 = x$, and $S_1 = R_0^* = R_1 = y$, we have that $S_n = f_n^{(c,b)}(x,y)$. Therefore,

$$f_{-n}^{(c,b)}(x,y) = \left(f_{n+1}^{(b,c)}(x,y)\right)^* . \tag{2.9}$$

Note that the anti-automorphism $*$ (2.8) sends positive Laurent monomials of x, y to positive Laurent monomials of x, y.

To summarize, the symmetries assure us that, in the case $b = c = 2$, it is sufficient to prove that R_n is a positive Laurent polynomial of x, y for $n \geq 0$. For $(b,c) = (1,4)$, we may restrict our attention to $n \geq 0$ but we must find R_n as a function of both $(x,y) = (CR_0, R_1)$ and $(X,Y) = (CR_1, R_2)$. For $(b,c) = (4,1)$, the solutions will be expressed in terms of those of $(1,4)$. In all cases, the expressions for $n < 0$ follow from equation (2.9).

3. The non-commutative cluster algebra in the case $b = c = 2$

3.1. **Conserved quantities and linear recursions.**

The non-commutative $(2,2)$-system

$$R_{n+1} C R_{n-1} = (R_n)^2 + 1 \tag{3.1}$$

is a discrete integrable equation in the sense that it has a conserved quantity in addition to the commutator C (2.5).

Lemma 3.1. *The polynomial in the solutions R_n of the $(2,2)$ system (2.4) $K = R_{n+1}^{-1}R_n + R_{n+1}^{-1}R_n^{-1} + R_{n+1}R_n^{-1}$ is independent of n.*

Proof. Define

$$K_n = R_n^{-1}(R_{n+1}C + R_{n-1}), \qquad L_n = (R_{n+1} + CR_{n-1})R_n^{-1} . \tag{3.2}$$

Then $K_n = L_n$ as a consequence of the first conservation law, upon substituting $C = R_{n+1}^{-1}R_nR_{n+1}R_n^{-1}$ into the expression for K_n and $C = R_n^{-1}R_{n-1}R_nR_{n-1}^{-1}$ into that for L_n. Subtracting Equation (3.1) for n from that for $n+1$,

$$0 = (R_{n+2}CR_n - R_{n+1}^2) - (R_{n+1}CR_{n-1} - R_n^2) = R_{n+1}(K_{n+1} - L_n)R_n$$

So we deduce that $K_n = K$ is independent of n. □

Substituting $L_n = K_n = K$ into (3.2), we have

Lemma 3.2. *There exist two linear recursion relations with constant coefficients satisfied by the solutions of (2.4):*

$$R_{n+1}C + R_{n-1} = R_nK \tag{3.3}$$

$$R_{n+1} + CR_{n-1} = KR_n \tag{3.4}$$

These two recursion relations are equivalent modulo the first conserved quantity.

3.2. **Paths with noncommutative weights and positivity.** Define a generating function for the variables R_n with $n \geq 0$,

$$F(t) = \sum_{n\geq 0} t^n R_n.$$

Theorem 3.3.

$$F(t) = \left(1 - t\left(1 - t(1-ty_3)^{-1}y_2\right)^{-1} y_1\right)^{-1} R_0, \tag{3.5}$$

where the "weights" y_i are defined as

$$y_1 = R_1R_0^{-1},\ y_2 = R_1^{-1}R_0^{-1},\ y_3 = R_1^{-1}R_0. \tag{3.6}$$

Proof. Using Equation (3.4),

$$F(t) = (1 - tK + t^2C)^{-1}(R_0 - t(KR_0 - R_1))$$

Noting that $K = R_1R_0^{-1} + R_1^{-1}R_0^{-1} + R_1^{-1}R_0 = y_1 + y_2 + y_3$, $K - R_1R_0^{-1} = y_2 + y_3$, and $C = y_3y_1$, we have

$$\begin{aligned} F(t) &= (1 - t(y_1+y_2+y_3) + t^2y_3y_1)^{-1}(1 - t(y_2+y_3))R_0 \\ &= \left(1 - t(1 - t(y_2+y_3))^{-1}(1-ty_3)y_1\right)^{-1} R_0 \\ &= \left(1 - t\left(1 - t(1-ty_3)^{-1}y_2\right)^{-1} y_1\right)^{-1} R_0 \end{aligned}$$

and the Theorem follows. □

This expression for $F(t)$ is to be considered as a power series in t with coefficients which are words in the non-commutative variables y_1, y_2, y_3. Substituting

$$y_1 = y^2x^{-1}y^{-1},\ y_2 = x^{-1}y^{-1},\ y_3 = xy^{-1},\ R_0 = yxy^{-1} \tag{3.7}$$

into (3.5), we deduce

Corollary 3.4. *For all $n \geq 0$, the solution R_n of Equation* (3.1) *is a Laurent polynomial of x, y (with $x = CR_0$ and $y = R_1$) with only non-negative integer coefficients.*

From the discussion of the previous section, the same is true for R_n with $n < 0$.

We can use Equation (3.5) to interpret R_n as a partition function for paths with non-commuting weights.

Theorem 3.5. *For all $n \geq 0$, the quantity $R_nR_0^{-1}$, where R_n is the solution 3.1, is the partition function for paths along the segment $[0,3]$ starting and ending at 0 with $2n$ steps, with a weight 1 per step $i \to i+1$ and y_i per step $i \to i-1$ given by* (3.7)*, the total (non-commutative) weight of each path being the product from left to right of the step weights in the order in which they are visited.*

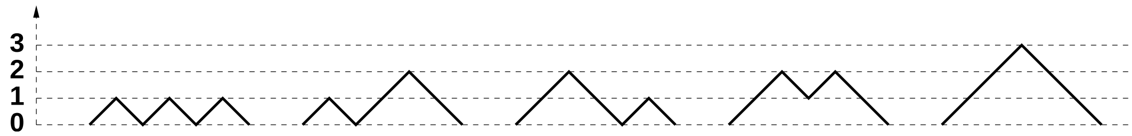

FIGURE 2. The five paths on $[0,3]$ of length 6, from $0 \to 0$.

Proof. The continued fraction $F(t)R_0^{-1}$ of Theorem 3.3 may be computed by the following recursion:

$$\begin{aligned} F_k &= (1 - tF_{k+1}y_k)^{-1} \quad (k = 1, 2, 3) \\ F_4 &= 1 \\ F(t)R_0^{-1} &= F_1 \end{aligned}$$

To get the series in t, we have to expand each intermediate step as:

$$F_k = \sum_{n \geq 0} t^n (F_{k+1}y_k)^n$$

Using this as an induction step, it allows to interpret F_k as the partition for paths on $[k-1, 3]$, from and to $(k-1)$, with weight 1 per step $i \to i+1$ and ty_i per step $i \to i-1$. This is clearly true for $F_4 = 1$, the partition function for the trivial path from $3 \to 3$ on the set $\{3\}$, with zero step. For intermediate k's, we simply decompose paths on $[k-1, 3]$ from $k-1 \to k-1$ into segments delimited by the ascending steps $k-1 \to k$ and the next descending step $k \to k-1$, the former receiving the weight 1 the latter the weight ty_k. In-between any two such steps, the path only explores the segment $[k, 3]$, with the partition function F_{k+1}. Finally, the weights are multiplied in the same order in which the steps are taken, and the Theorem follows. □

Example 3.6. *For $n = 3$, the five paths on $[0, 3]$ with 6 steps from $0 \to 0$ are depicted in Figure 2, and contribute respectively to R_3 (with weight y_i per descending step $i \to i-1$):*

$$\begin{aligned} R_3 &= (y_1^3 + y_1y_2y_1 + y_2y_1^2 + y_2^2y_1 + y_3y_2y_1)R_0 \\ &= y^2x^{-1}yx^{-1} + y^2x^{-1}y^{-1}x^{-1} + x^{-1}yx^{-1} + x^{-1}y^{-1}x^{-1} + xy^{-1}x^{-1} \\ &= \Big(\big((1 + y^2)x^{-1}\big)^2 + 1\Big)xy^{-1}x^{-1} = (R_2^2 + 1)R_1^{-1}C^{-1} \end{aligned}$$

An alternative formulation uses the following transfer matrix, with non-commutative entries indexed by $0, 1, 2, 3$:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ ty_1 & 0 & 1 & 0 \\ 0 & ty_2 & 0 & 1 \\ 0 & 0 & ty_3 & 0 \end{pmatrix} \tag{3.8}$$

The matrix element $T_{i,j}$ is nothing but the non-commutative weight of the step $i \to j$ of the above paths. Consequently, $(T^n)_{i,j}$ is the partition function for paths of n steps, from i to j. The partition function for paths from $0 \to 0$ is therefore

$$F(t)R_0^{-1} = \Big(\sum_{n=0}^{\infty} T^n\Big)_{0,0} = \Big((I-T)^{-1}\Big)_{0,0}.$$

There is a direct link between this formulation and the continued fraction expression (3.5). The latter is obtained by Gaussian elimination of the matrix $I - T$, by allowing only multiplication from the left and addition of rows (left multiplication by an upper triangular matrix), resulting in a lower triangular matrix. $F(t)R_0^{-1}$ is then computed as the inverse of the first diagonal element in the resulting matrix, leading to equation (3.5).

Both formulations display explicitly the positivity of R_n as a Laurent polynomial of x, y. We have

Theorem 3.7. *For any $n \in \mathbb{Z}$, the solution R_n of* (3.1) *is a Laurent polynomial of x, y with non-negative integer coefficients $\in \{0, 1\}$. Each Laurent monomial in the expression for R_n corresponds to the weight of a single path on the segment* $[0, 3]$ *with weights as in Theorem 3.5.*

Proof. The expression for R_n, $n \geq 0$, as the path partition function of Theorem 3.5 times R_0 is a manifestly positive Laurent polynomial of x, y. From (2.9), we deduce the positive Laurent polynomiality for all $n \in \mathbb{Z}$.

Moreover, as the weights y_i all have the form $R_1^{\pm 1} R_0^{\pm 1}$, they do not commute with each-other. Each path contributes an ordered product of such weights from left to right, which is encoded in the succession of down steps (taken towards the origin) along the path. This gives a bijection between the paths contributing to R_n and their total weights, which therefore occur exactly once in the expression of $R_n R_0^{-1}$. The last part of the Theorem follows. □

4. The cases $(b, c) = (1, 4)$ and $(4, 1)$.

The non-commutative $(1, 4)$ recursion relations (2.4) can be written as:

$$\begin{aligned} R_{2n} C R_{2n-2} &= 1 + R_{2n-1} \\ R_{2n+1} C R_{2n-1} &= 1 + (R_{2n})^4 \end{aligned} \tag{4.1}$$

As explained in Section 2.3, we consider two different sets of initial conditions: $(x, y) = (CR_0, R_1)$, where

$$R_0 = yxy^{-1}, \qquad R_1 = y \tag{4.2}$$

and $(X, Y) = (CR_1, R_2)$, where

$$R_1 = YXY^{-1}, \qquad R_2 = Y \tag{4.3}$$

We proceed as in [4]. Define $u_n = R_{2n}$, then odd index variables can be eliminated:

$$R_{2n+1} = u_{n+1} C u_n - 1$$

The variables u_n satisfy

$$(u_{n+2}Cu_{n+1}-1)C(u_{n+1}Cu_n-1)=1+u_{n+1}^4. \tag{4.4}$$

The initial data corresponding to (4.2) becomes

$$u_0=yxy^{-1}, \qquad u_1=(1+y)x^{-1}, \tag{4.5}$$

whereas that corresponding to (4.3) becomes

$$u_0=XY^{-1}X^{-1}Y(1+X)Y^{-1}, \qquad u_1=Y. \tag{4.6}$$

4.1. **Conserved quantities and linear recursions.** We use the following expressions for the commutator C, expressed as a function of u_n :

$$C = u_{n+1}^{-1}(u_{n+1}Cu_n-1)u_{n+1}(u_{n+1}Cu_n-1)^{-1} \tag{4.7}$$

$$C = (u_{n+1}Cu_n-1)^{-1}u_n(u_{n+1}Cu_n-1)u_n^{-1} \tag{4.8}$$

The first expression is obtained from $C=R_{2n+2}^{-1}R_{2n+1}R_{2n+2}R_{2n+1}^{-1}$, and the second from $C=R_{2n+1}^{-1}R_{2n}R_{2n+1}R_{2n}^{-1}$.

Starting from equation (4.4), let us substitute 4.7 for the term C in the center of the left hand side,

$$(u_{n+2}Cu_{n+1}-1)u_{n+1}^{-1}(u_{n+1}Cu_n-1)u_{n+1}=1+u_{n+1}^4,$$

or

$$u_{n+2}C(u_{n+1}Cu_n-1)=u_{n+1}^3+Cu_n.$$

We conclude that

$$u_{n+2}C=(u_{n+1}^3+Cu_n)(u_{n+1}Cu_n-1)^{-1} \tag{4.9}$$

By Eq.(4.8), we also have

$$(u_{n+1}Cu_n-1)C=u_nu_{n+1}C-1 \tag{4.10}$$

which is a quasi-commutation relation between u_n and u_{n+1}:

$$u_{n+1}Cu_n=u_nu_{n+1}+1-C^{-1}. \tag{4.11}$$

This is to be compared with equation (2.7) of the case $b=c=2$ above.

We can now prove the integrability of the evolution (4.4), by finding its conserved quantity.

Lemma 4.1. *The function*

$$K=\big((Cu_n)^2+(u_{n+1})^2\big)(u_{n+1}Cu_n-1)^{-1} \tag{4.12}$$

is independent of n*.*

Proof. Let

$$K_n = u_{n+1}^{-1}(u_{n+2}C + u_n)$$

Using (4.9),

$$K_n = \big((Cu_n)^2 + (u_{n+1})^2\big)(u_{n+1}Cu_n - 1)^{-1} \tag{4.13}$$

To prove that K_n is independent of n, we compute

$$K_n - K_{n-1} = \big((Cu_n)^2 + (u_{n+1})^2\big)(u_{n+1}Cu_n - 1)^{-1} - u_n^{-1}(u_{n+1}C + u_{n-1}) \tag{4.14}$$

We first need to move the factor $(u_{n+1}Cu_n - 1)^{-1}$ to the left of the first term. To do so, note that (4.8), implies that for all $m \geq 0$

$$(Cu_n)^m(u_{n+1}Cu_n - 1)^{-1} = (u_{n+1}Cu_n - 1)^{-1}(u_n)^m,$$

and (4.7), implies that for all $m \geq 0$

$$(u_{n+1})^m(u_{n+1}Cu_n - 1)^{-1} = (u_{n+1}Cu_n - 1)^{-1}(u_{n+1}C)^m.$$

Applying this to the first term in (4.14):

$$\begin{aligned} K_n - K_{n-1} &= (u_{n+1}Cu_n - 1)^{-1}\big((u_n)^2 + (u_{n+1}C)^2 - (u_{n+1}Cu_n - 1)u_n^{-1}(u_{n+1}C + u_{n-1})\big) \\ &= (u_{n+1}Cu_n - 1)^{-1}u_n^{-1}\big((u_n)^3 + u_{n+1}C - (u_nu_{n+1}C - 1)u_{n-1}\big) \end{aligned}$$

Substituting Eq. (4.10) in the last term,

$$\begin{aligned} K_n - K_{n-1} &= (u_{n+1}Cu_n - 1)^{-1}u_n^{-1}\big((u_n)^3 + u_{n+1}C - (u_{n+1}Cu_n - 1)Cu_{n-1}\big) \\ &= (u_{n+1}Cu_n - 1)^{-1}u_n^{-1}\big(u_n^3 + Cu_{n-1} - u_{n+1}C(u_nCu_{n-1} - 1)\big) \\ &= 0 \end{aligned}$$

as a consequence of 4.9. The lemma follows. □

By the definition (4.13) of K_n, we deduce:

Theorem 4.2. *The solution u_n to the system (4.4) satisfies the following linear recursion relations with constant coefficients:*

$$u_{n+2}C - u_{n+1}K + u_n = 0, \tag{4.15}$$

$$u_{n+2} - Ku_{n+1} + Cu_n = 0, \tag{4.16}$$

where

$$C = xyx^{-1}y^{-1}, \qquad K = (x^2 + \big((1+y)x^{-1}\big)^2)y^{-1} \tag{4.17}$$

in the case of the initial data (x, y) as in (4.5), or

$$C = XYX^{-1}Y^{-1}, \qquad K = (Y^2 + \big((1+X)Y^{-1}\big)^2)YX^{-1}Y^{-1} \tag{4.18}$$

in the case of the initial data (X, Y) as in (4.6).

Proof. The first relation follows from the definition of $K = K_n$ (4.13). The second follows from the first and from the quasi-commutation relation (4.11),

$$\begin{aligned} u_{n+1}Ku_{n+1} - u_{n+1}(u_{n+2} + Cu_n) &= u_{n+2}Cu_{n+1} + u_n u_{n+1} - u_{n+1}(u_{n+2} + Cu_n) \\ &= (1 - C^{-1}) - (1 - C^{-1}) = 0 \end{aligned}$$

Substituting the initial values (4.5)-(4.6) into the expressions for the conserved quantities C and K leads to (4.17)-(4.18). □

4.2. **(x,y) initial data: Paths with noncommutative weights and positivity.** As in Section 3.2, we introduce the generating function $F(t) = \sum_{n\geq 0} t^n u_n$, and use the second linear recursion relation of Theorem 4.2 to compute F explicitly.

Theorem 4.3. *The generating function $F(t)$ has the following non-commutative (finite) continued fraction expression:*

$$F(t) = \left(1 - ty_1 - t^2(1 - ty_3)^{-1}y_2\right)^{-1} u_0, \tag{4.19}$$

where

$$\begin{aligned} y_1 &= u_1u_0^{-1} = (1+y)x^{-1}yx^{-1}y^{-1}, \\ y_2 &= (K - y_1)y_1 - C = \Big(x^2 + (1+y)x^{-2}(1+y)\Big)y^{-1}x^{-1}yx^{-1}y^{-1}, \\ y_3 &= K - y_1 = \Big(x^3 + (1+y)x^{-1}\Big)x^{-1}y^{-1}, \end{aligned}$$

and $u_0 = yxy^{-1}$.

Proof. The recursion relation (4.16) implies

$$\begin{aligned} F(t) &= (1 - tK + t^2C)^{-1}\big(1 - t(K - u_1u_0^{-1})\big)u_0 \\ &= (1 - t(y_1 + y_3) + t^2(y_3y_1 - y_2))^{-1}\big(1 - ty_3\big)u_0 \\ &= (1 - ty_1 - t^2(1 - ty_3)^{-1}y_2))^{-1}u_0 \end{aligned}$$

where we have substituted $y_1 = u_1u_0^{-1}$, $y_3 = K - y_1$, and $y_2 = y_3y_1 - C$. □

Corollary 4.4. *For all $n \geq 0$, the solution u_n of the system (4.4-4.5) is a Laurent polynomial of x, y with only non-negative integer coefficients.*

As in Section 3.2, the continued fraction expression (4.19) allows to interpret R_n as a path partition function for all $n \geq 0$. The new feature is that the paths involved here will be Motzkin paths of height 1. That is, they are paths on the strip of $\mathbb{Z}^2$ delimited by $y = 0$ and $y = 1$, with steps $(x, y) \to (x + 1, 1 - y)$ or $(x, y) \to (x + 1, y)$. These are in bijection with paths on the graph G of Figure 1, if we consider horizontal steps at height y to be steps around the loop connecting vertex y to itself, and diagonal steps to up/down steps along the edge connecting vertex 0 and 1. The Motzkin paths of length 3 are represented in Figure 1 for illustration.

Theorem 4.5. *For all $n \in \mathbb{Z}_{\geq 0}$, $u_n u_0^{-1}$ is the partition function for Motzkin paths of height 1, from $(0,0)$ to $(n,0)$, with weights y_1 per horizontal step at height 0, weight 1 for an upward step, weightyy_2 per downward step and weight y_3 per horizontal step at height 1, with y_i as in equation (4.19).*

Notice that formally, these are precisely the paths of length n on the weighted graph G in Figure 1.

Proof. The expansion of the continued fraction F (4.19) in powers of t may be again decomposed into two steps:

$$\begin{aligned} F_1 &= (1-ty_3)^{-1} \\ F &= (1-ty_1-t(F_1)(ty_2))^{-1}u_0 \end{aligned}$$

We may now interpret F_1 as the partition function for paths on $\{1\}$ made of consecutive steps $1 \to 1$ (each receiving the weight ty_3). As to F, it generates paths on $[0,1]$ from 0 to 0, made of any shuffle of steps $0 \to 0$ (with weight ty_1) and segments made of one step $0 \to 1$ (weight t) followed by any number of steps $1 \to 1$ (weight F_1) and then one step $1 \to 0$ (weight ty_2). The Theorem follows. □

Alternatively, we may express the partition function $u_n u_0^{-1}$ by means of the path transfer matrix $T = \begin{pmatrix} y_1 & 1 \\ y_2 & y_3 \end{pmatrix}$ with entries indexed $0,1$, resulting in:

$$u_n u_0^{-1} = (T^n)_{0,0} \tag{4.20}$$

Example 4.6. *For $n=2$ we have*

$$\begin{aligned} u_2 u_0^{-1} &= (u_1^3 + Cu_0)(u_1 C u_0 - 1)^{-1} C^{-1} u_0^{-1} = \big(x + ((1+y)x^{-1})^3\big) x y^{-1} x^{-1} y x^{-1} y^{-1} \\ &= y_1^2 + y_2 \end{aligned}$$

by using the explicit values of y_1, y_2 from equation (4.19). This is the contribution of the two Motzkin paths of length 2 on $[0,1]$ starting and ending at 0, namely $0 \to 0 \to 0$ (weight y_1^2) and $0 \to 1 \to 0$ (weight $1 \times y_2$).

Let us now turn to the remaining variables $R_{2n+1} = u_{n+1} C u_n - 1 = u_n u_{n+1} - C^{-1}$. From the Laurent positivity result for u_n, it is easy to deduce that of R_{2n+1}. We simply have to show that the term C^{-1} occurs at least once in the product $u_n u_{n+1}$. We have

Lemma 4.7. *For $n \geq 1$, the expression of $u_n u_{n+1}$ as a Laurent polynomial of x, y contains the term C^{-1}.*

Proof. Using the path interpretation above, let us show that the contribution to $u_n u_{n+1}$ of a particular pair (m_1, m_2) of Motzkin paths on $[0,1]$ of lengths n and $n+1$ respectively also contains the term C^{-1}. For m_1 we take the flat motzkin path of length n: $0 \to 0 \to 0 \to \cdots \to 0$ (with weight y_1^n) and for m_2 the "maximal" Motzkin path of length $n+1$: $0 \to 1 \to 1 \to \cdots \to 1 \to 0$ (with weight $y_3^n y_2$). We are left with the task of proving that $y_1^n u_0 y_3^n y_2 u_0$, when expressed as a Laurent polynomial of x, y, contains the term $C^{-1} = yxy^{-1}x^{-1}$. In view of the explicit values of y_1, y_2, y_3 (4.19), let us only retain one term in each weight, namely $\tilde{y}_i$ instead of y_i, with $\tilde{y}_1 = yx^{-1}yx^{-1}y^{-1}$, $\tilde{y}_2 = x^2y^{-1}x^{-1}yx^{-1}y^{-1}$ and $\tilde{y}_3 = x^2y^{-1}$. We find easily that

$$\begin{aligned}(\tilde{y}_1)^n u_0 (\tilde{y}_3)^{n-1}\tilde{y}_2 u_0 &= yx^{-1}(yx^{-2})^{n-1}yx^{-1}y^{-1}\ yxy^{-1}\ (x^2y^{-1})^{n-1}x^2y^{-1}x^{-1}yx^{-1}y^{-1}\ yxy^{-1}\\ &= yxy^{-1}x^{-1} = C^{-1}\end{aligned}$$

and the Lemma follows. □

We summarize the results of this section with the following

Theorem 4.8. *For all $n \geq 0$, the solution R_n to the system (4.1-4.2) is a Laurent polynomial of x, y with only non-negative integer coefficients.*

4.3. **(X,Y) initial data: Paths with noncommutative weights and positivity.** Let us now re-express the generating function for u_n in terms of X, Y. More precisely, let us compute the new generating function $G(t) = \sum_{n\geq 0} t^n u_{n+1}$. We have the following

Theorem 4.9. *The generating function G has the following continued fraction form:*

$$G(t) = \Big(1 - ty_1' - t^2(1-ty_3')^{-1}y_2'\Big)^{-1} u_1,$$

where

$$\begin{aligned}y_1' &= K - y_3' = K - Cu_0u_1^{-1} = (Y^3 + (1+X)Y^{-1})X^{-1}Y^{-1},\\ y_2' &= y_3'y_1' - C = \big(Y + (1+X)Y^{-2}(1+X)Y^{-1}\big)X^{-1}Y^{-1},\\ y_3' &= Cu_0u_1^{-1} = (1+X)Y^{-2},\end{aligned}$$

where $u_1 = Y$.

Proof. Starting from the expression for $F(t)$ of Theorem 4.3, we compute $G(t) = (F(t) - u_0)/t$:

$$\begin{aligned}G(t) &= \frac{1}{t}\big((1-tK+t^2C)^{-1}(1-t(K-u_1u_0^{-1}))-1\big)\,u_0\\ &= (1-tK+t^2C)^{-1}(u_1u_0^{-1}-tC)u_0 = (1-tK+t^2C)^{-1}(1-tCu_0u_1^{-1})u_1\\ &= (1-t(y_1'+y_3')+t^2(y_3'y_1'-y_2'))^{-1}(1-ty_3')u_1\\ &= \Big(1-ty_1'-t^2(1-ty_3')^{-1}y_2'\Big)^{-1}u_1 \qquad (4.21)\end{aligned}$$

and the Theorem follows. □

Note that the path interpretation of the Theorem 4.5 still holds for G, but with the new weights y_1', y_2', y_3' and u_1 instead of u_0. Noting moreover that the weights y_1', y_2', y_3', u_1 of equation (4.21) are all positive Laurent polynomials of X, Y, we deduce:

Corollary 4.10. *For all $n \geq 1$, the solution u_n to the system (4.4-4.6) is a Laurent polynomial of X, Y with only non-negative integer coefficients.*

We now turn to the remaining variables $R_{2n+1} = u_n u_{n+1} - C^{-1}$. We may repeat the analysis of Lemma 4.7 in terms of the variables X, Y, by use of Theorem 4.9. The result is

Lemma 4.11. *The solution R_{2n+1} to the system (4.1-4.3) has a positive Laurent polynomial expression in terms of X, Y.*

Proof. We must show that $u_n u_{n+1}$ contains at least once the term C^{-1}. We repeat the analysis in the proof of Lemma 4.7, using the continued fraction of Theorem 4.9. We pick the contribution of the same two paths to $u_n u_{n+1}$, but we now retain in the weights only the terms $\tilde{y}_1' = Y^3 X^{-1} Y^{-1}$, $\tilde{y}_2' = XY^{-2}XY^{-1}X^{-1}Y^{-1}$ and $\tilde{y}_3' = XY^{-2}$. The contribution is then easily computed to be

$$\begin{aligned} (\tilde{y}_1')^n u_1 (\tilde{y}_3')^{n-1} \tilde{y}_2' u_1 &= Y(Y^2X^{-1})^n Y^{-1} Y (XY^{-2})^{n-1} XY^{-2}XY^{-1}X^{-1}Y^{-1}Y \\ &= YXY^{-1}X^{-1} = C^{-1} \end{aligned}$$

So the subtracted expression $R_{2n+1} = u_n u_{n+1} - C^{-1}$ is a positive Laurent polynomial of X, Y. □

We summarize the results of this section with the following

Theorem 4.12. *For all $n \geq 0$, the solution R_n to the system (4.1-4.3) is a Laurent polynomial of X, Y with only non-negative integer coefficients.*

4.4. **Main theorem and the case $(b, c) = (4, 1)$.** We conclude with our main theorem:

Theorem 4.13. *For all $n \in \mathbb{Z}$, the solution R_n of the system (4.1) for $(b, c) = (1, 4)$, with respectively initial data (x, y) (4.2) and initial data (X, Y) (4.3) is a positive Laurent polynomial of respectively x, y and X, Y, with only non-negative integer coefficients. The same holds for the system with $(b, c) = (4, 1)$ as well.*

Proof. By Theorems 4.8 and 4.12, we deduce that both $f_n^{(1,4)}(x, y)$ and $g_n^{(1,4)}(X, Y)$ (defined in Section 2.3) are positive Laurent polynomials for all $n \geq 0$. By Theorem 2.1, we deduce that $f_{n-1}^{(4,1)}(x, y) = g_n^{(1,4)}(x, y)$ and $g_{n+1}^{(4,1)}(X, Y) = f_n^{(1,4)}(X, Y)$ are also positive Laurent polynomials for all $n \geq 0$. Finally, by Equation (2.9), we deduce that both $f_{-n}^{(1,4)}(x, y) = \left(f_{n+1}^{(4,1)}(x, y)\right)^*$ and $f_{-n}^{(4,1)}(x, y) = \left(f_{n+1}^{(1,4)}(x, y)\right)^*$ are positive Laurent polynomials for all $n \geq 0$. We then apply again Theorem 2.1 to conclude that both $g_{-n}^{(1,4)}(X, Y) = f_{-n-1}^{(4,1)}(X, Y)$ and $g_{-n}^{(4,1)}(X, Y) = f_{-n-1}^{(1,4)}(X, Y)$ are positive Laurent polynomials for all $n \geq 0$. The Theorem follows. □

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PDF: Institut de Physique Théorique du Commissariat à l'Energie Atomique, Unité de Recherche associée du CNRS, CEA Saclay/IPhT/Bat 774, F-91191 Gif sur Yvette Cedex, FRANCE. e-mail: philippe.di-francesco@cea.fr

RK: Department of Mathematics, University of Illinois Urbana, IL 61801, U.S.A. e-mail: rinat@illinois.edu