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# REPRESENTATION THEORY OF IMPRIMITIVE NON-COMMUTATIVE ASSOCIATION SCHEMES OF ORDER 6

AKIHIDE HANAKI AND PAUL-HERMANN ZIESCHANG

Abstract. In this note, we investigate association schemes of order 6. We prove that non-normal closed subsets of such schemes have order 2 and that normal closed subsets of non-commutative schemes have order 2 or 3. After that, we investigate more closely schemes of order 6 which possess non-normal closed subsets and non-commutative schemes of order 6 which possess a symmetric normal closed subset of order 3. (Non-commutative schemes of order 6 which possess a non-symmetric normal closed subset of order 3 or a normal closed subset of order 2 will be investigated in a forthcoming article.) In both cases, we explicitly give all irreducible (complex) representations of such schemes. Among other structural consequences we obtain that association schemes of order 6 are Coxeter schemes if they have two different non-normal closed subsets and that they are semidirect products if they possess a normal and a non-normal closed subset or a thin non-normal closed subset.

The concept of an association scheme is a far-reaching generalization of the notion of a group. Many group theoretic facts have found a natural generalization in scheme theory; cf., e.g., [8] and [7]. One of these facts is the observation that, similar to groups, association schemes are commutative if they have at most five elements and not necessarily commutative if they have six elements.[1] While any two non-commutative groups of order 6 are isomorphic to each other, there exist infinitely many isomorphism classes of non-commutative schemes of order 6.[2]

The present article is a first attempt to obtain insight into the structure of non-commutative schemes of order 6. We restrict ourselves to imprimitive such schemes, which means that we assume these schemes to have at least one non-trivial closed subset, and our analysis is devoted to the representation theory of these schemes.

All imprimitive non-commutative schemes of order 6 that are known to us belong to (at least) one of the following three classes.

(A) Coxeter schemes of order 6.[3]

(B) Semidirect products with kernel of order 3 and complement of order 2.[4]

(C) Schemes with a normal thin closed subset of order 3.[5]

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[1]In this article, all association schemes are assumed to have finite valency. (By the valency of a scheme $S$ on $X$ we mean the cardinality of $X$.) For a proof of the former statement, see [6; (4.1)].

[2]By the order of a finite scheme $S$ we mean the cardinality of $S$.

[3]Coxeter schemes are schemes which are defined on the flags of a building. In the case where a Coxeter schemes has six elements, the corresponding building is a projective plane. Coxeter schemes have been investigated in general in [10] and in the last two chapters of [11].

[4]Semidirect products of association schemes are defined in [11; Section 7.3]. They were also considered in [1]. Note that the complement in a semidirect product is always thin.

[5]Thin closed subsets of schemes can be considered as groups; cf. [11; Section 5.5].

The intersection of any two of these three classes is the class of the Coxeter groups of order 6, so that all three classes can be viewed as generalizations of the class of Coxeter groups of order 6.

At an early stage of our investigation, we see that, similar to group theory, non-normal closed subsets of non-commutative schemes of order 6 have order 2 and that normal closed subsets of such schemes have order 2 or 3. After that, we investigate more closely schemes of order 6 which possess a non-normal closed subset and non-commutative schemes of order 6 which possess a symmetric normal closed subset of order 3. We intend to investigate non-commutative schemes of order 6 which possess a non-symmetric normal closed subset of order 3 or a normal closed subset of order 2 in a forthcoming article.

Our investigation requires a series of general results on association schemes. We shall compile these results in the first chapter of this article. In Section 1.1, we exhibit combinatorial sufficient conditions for a closed subset of a scheme to be normal. Section 1.2 is devoted to the ordinary character theory of schemes (of finite valency), and Section 1.3 deals with involutions of association schemes.

The second chapter of this article deals with non-normal closed subsets of schemes $S$ of order 6. In Section 2.1, we collect combinatorial results on $S$. First we prove that non-normal closed subsets of $S$ have order 2. In Section 2.2, we compute the character table of $S$ in terms of a few structure constants of $S$. In Section 2.3, we strengthen the arithmetic conditions which the valencies of $S$ have to obey. In Section 2.4, we provide sufficient conditions for $S$ to belong to one of the first two classes of the above trichotomy. In particular, we show that $S$ is a Coxeter scheme if $S$ has two different non-normal closed subsets and that $S$ belongs to the second case of the above trichotomy if $S$ possesses a thin non-normal closed subset. In Section 2.5, we compute the non-linear representation of $S$ in terms of a few structure constants of $S$. This provides new conditions on the structure constants of $S$.

In the third chapter of this article, we investigate non-commutative schemes $S$ of order 6 which possess a normal closed subset. In Section 3.1, we collect general results on $S$. In particular, we show that normal closed subsets of $S$ must have order 2 or 3. We also show that $S$ is a semidirect product (with kernel of order 3 and complement of order 2) if $S$ possesses a normal and a non-normal closed subset. In Section 3.2, we provide useful equations for the remaining four sections, which deal with the case where $S$ possesses a symmetric normal closed subset of order 3. In Section 3.3, we give the character table of $S$ in terms of the valencies of the elements of $S$, and we explicitly compute the representation of degree 2 of $S$ in these terms. We also compute the kernel of this latter representation. The control over the kernel allows us to compute all structure constants of $S$, again in terms of the (three) valencies of the elements of $S$. This is the subject of the remaining three sections of this article.

As for terminology and notation we refer to [11].

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## 1. Preliminaries

This chapter contains a series of general results on association schemes which will be needed in our investigation. The letter $X$ stands for a finite set, the letter $S$ for an association scheme on $X$.

### 1.1 Preliminaries on cosets

In this section, the letter $T$ stands for a closed subset of $S$.

**Lemma 1.1.1** *Let $p$ be an element in $S$ with $Tp \subseteq pT$, and let $q$ be an element in $pT$ such that $Tq^* = Tq$. Then $q \in Tp$.*

*Proof.* From $q \in pT$ we obtain $q^* \in Tp^*$; cf. [11; Lemma 1.3.2(iii)]. Thus, by [11; Lemma 2.1.4], $p^* \in Tq^*$. Thus, as $Tq^* = Tq$ and $q \in pT$,

$$p^* \in Tq \subseteq TpT = pT = qT;$$

cf. [11; Lemma 2.1.4]. Thus, $p \in Tq^* = Tq$, so that $q \in Tp$. □

Following [11] we define

$$N_S(T) := \{s \in S \mid Ts = sT\}$$

for each closed subset $T$ of $S$. (The set $N_S(T)$ is called the *normalizer* of $T$ *in* $S$.)

**Lemma 1.1.2** *Let $s$ be an element in $S$ such that $Ts = \{s\}$ and $Ts^* = \{s^*\}$. Then $s \in N_S(T)$.*

*Proof.* From $Ts^* = \{s^*\}$ we obtain $sT = \{s\}$; cf. [11; Lemma 1.3.2(iii)]. Thus, as $Ts = \{s\}$, $Ts = sT$, and that means that $s \in N_S(T)$. □

We define

$$S/T := \{Ts \mid s \in S\}.$$

From [11; Lemma 2.1.4] one knows that $S/T$ is a partition of $S$. We shall use this fact at various instances without further mentioning.

**Lemma 1.1.3** *Let $s$ be an element in $S$ such that $(Ts)^* \in S/T$. Then $s \in N_S(T)$.*

*Proof.* We are assuming that $(Ts)^* \in S/T$. Thus, as $s^* \in (Ts)^*$, $(Ts)^* = Ts^*$; cf. [11; Lemma 2.1.4]. Thus, by [11; Lemma 1.3.2(iii)], $Ts = sT$. □

One says that $T$ is *normal* in $S$ if $N_S(T) = S$. The remaining three results give sufficient conditions for $T$ to be normal in $S$.

**Lemma 1.1.4** *Assume that $|S \setminus T| = |S/T|$ and that $S \setminus T$ possesses exactly $|S \setminus T| - 2$ symmetric elements. Then $T$ is normal in $S$.*

*Proof.* We are assuming that $|S \setminus T| = |S/T|$. Thus, $S/T \setminus \{T\}$ possesses exactly one element of cardinality 2, all other elements of $S/T \setminus \{T\}$ consist of a single element. Let $p$ and $q$ be the two elements in the uniquely determined element of $S/T \setminus \{T\}$ of cardinality 2.

If $\{p,q\}^* = \{p,q\}$, we are done by Lemma 1.1.3. Thus, we assume that $\{p,q\}^* \neq \{p,q\}$. It follows that $p^* \notin \{p,q\}$ or $q^* \notin \{p,q\}$. Without loss of generality, we assume that $p^* \notin \{p,q\}$.

Since $\{p,q\}$ is the only element in $S/T \setminus \{T\}$ that has more than one element, $p^* \notin \{p,q\}$ forces $Tp^* = \{p^*\}$. Thus, $Tp^* \subseteq p^*T$. Since $q^* = q$, we also have $Tq^* = Tq$. Thus, by Lemma 1.1.1, $q \in Tp^* = \{p^*\}$, contradiction. □

**Lemma 1.1.5** *Assume that $|S \setminus T| \leq 3$. Then $T$ is normal in $S$.*

*Proof.* If $|S/T| = 1$, $T = S$, and we are done. If $|S/T| = 2$, we have $Ts = S\setminus T = sT$ for each element $s$ in $S \setminus T$. Thus, $T$ is normal also in this case.

Assume that $3 \leq |S/T|$. Then as $|S/T| \leq |S \setminus T| + 1 \leq 4$, we must have $|S/T| = |S \setminus T| = 3$ or $|S/T| = |S \setminus T| + 1$. In the former case, we are done by Lemma 1.1.3 and by Lemma 1.1.4. In the latter case, we are done by Lemma 1.1.2. □

We finish this section by mentioning that most of the arguments of this section do not make use of the fact that $S$ is assumed to have finite valency.

## 1.2. Preliminaries on scheme characters

In this section, the letter $T$ stands for a closed subset of $S$. The scheme ring of $T$ over the complex number field will be denoted by $\mathbb{C}T$.

If we speak about characters of $T$, we shall always mean characters of $\mathbb{C}T$.

The set of all characters of $T$ will be denoted by $\mathrm{Char}(T)$, the set of all irreducible characters of $T$ by $\mathrm{Irr}(T)$.

We define $\beta_T$ to be the uniquely defined $\mathbb{Z}$-bilinear form from $\mathrm{Char}(T) \times \mathrm{Char}(T)$ to $\mathbb{Z}$ which satisfies $\beta_T(\chi,\chi) = 1$ for each element $\chi$ in $\mathrm{Irr}(T)$ and $\beta_T(\phi,\psi) = 0$ for any two different elements $\phi$ and $\psi$ in $\mathrm{Irr}(T)$. In the following, we shall write $(\phi,\psi)_T$ instead of $\beta_T(\phi,\psi)$ whenever $\phi$ and $\psi$ are elements in $\mathrm{Irr}(T)$.

For each character $\psi$ of $T$, we define $\psi^S$ to be the induced character of $T$ to $S$; cf. [3; Section 5]. For each character $\phi$ of $S$, the restriction of $\phi$ to $T$ will be denoted by $\phi|_T$.

The following lemma is known as Frobenius Reciprocity.

**Lemma 1.2.1** *Let $\phi$ be a character of $S$, and let $\psi$ be a character of $T$. Then $(\psi, \phi|_T)_T = (\phi, \psi^S)_S$.*

*Proof.* This is [3; Theorem 5.2]. □

Let $\chi_X$ denote the complex-valued standard character of $S$. (In [11], this character was denoted by $\chi_{\mathbb{C}X}$.)

For each element $\chi$ in $\mathrm{Char}(S)$, we set

$$m_\chi := (\chi, \chi_X)_S.$$

This definition extends (and, therefore, is compatible with) the notion of the multiplicity of irreducible characters of $S$ introduced in [11; Section 9.1].

The following result is [5; Theorem 5.1]. For the sake of the completeness of this article we include the proof which was given in [5].

**Lemma 1.2.2** *For each character $\phi$ of $T$, we have $m_{\phi^S} = n_{S//T} m_\phi$.*

*Proof.* Without loss of generality, we may assume that $\phi$ is irreducible. Then, referring to Lemma 1.2.1 and [11; Theorem 9.1.7(ii)] we obtain

$$\begin{aligned} m_{\phi^S} &= \sum_{\chi \in \mathrm{Irr}(S)} (\chi, \phi^S)_S \; m_\chi \\ &= \sum_{\chi \in \mathrm{Irr}(S)} (\phi, \chi|_T)_T \; m_\chi \\ &= \sum_{\chi \in \mathrm{Irr}(S)} m_\chi \frac{m_\phi}{n_T \phi(1)} \sum_{t \in T} \frac{1}{n_t} \phi(\sigma_t)\chi(\sigma_{t^*}) \\ &= \frac{m_\phi}{n_T \phi(1)} \sum_{t \in T} \frac{1}{n_t} \phi(\sigma_t) \sum_{\chi \in \mathrm{Irr}(S)} m_\chi \chi(\sigma_{t^*}) \\ &= \frac{m_\phi}{n_T \phi(1)} \sum_{t \in T} \frac{1}{n_t} \phi(\sigma_t) \chi_{\mathbb{C}S}(\sigma_{t^*}) \\ &= \frac{m_\phi}{n_T \phi(1)} \phi(1) n_S = n_{S//T} m_\phi. \end{aligned}$$

This finishes the proof of the lemma. □

From [11; Lemma 9.1.2(i)] we know that there exists a uniquely determined ring homomorphism from $\mathbb{C}S$ to $\mathbb{C}$ which maps $\sigma_s$ to $n_s$ for each element $s$ in $S$. In [11], this homomorphism was denoted by $1_{\mathbb{C}S}$. Here we just write $1_S$.

For each element $\phi$ in $\mathrm{Irr}(S)$, we define

$$\mathrm{Irr}(T \mid \phi) := \{\psi \in \mathrm{Irr}(T) \mid (\psi, \phi|_T) \neq 0\}.$$

For each element $\psi$ in $\mathrm{Irr}(T)$, we define

$$\mathrm{Irr}(S \mid \psi) := \{\phi \in \mathrm{Irr}(S) \mid (\phi, \psi^S) \neq 0\}.$$

From Lemma 1.2.1 one obtains

$$\psi \in \mathrm{Irr}(T \mid \phi) \quad \Leftrightarrow \quad \phi \in \mathrm{Irr}(S \mid \psi)$$

for any two elements $\phi$ in $\mathrm{Irr}(S)$ and $\psi$ in $\mathrm{Irr}(T)$.

**Corollary 1.2.3** *The following statements hold.*

(i) *We have $(1_S, 1_T^S)_S = 1$. (In particular, $1_S \in \mathrm{Irr}(S \mid 1_T)$.)*

(ii) *We have $m_{1_T^S} = n_{S//T}$.*

(iii) *Assume that $T \neq S$. Then $\mathrm{Irr}(S \mid 1_T) \setminus \{1_S\}$ is not empty.*

*Proof.* (i) From Lemma 1.2.1 we obtain

$$(1_S, 1_T^S)_S = (1_T, 1_S|_T)_T = (1_T, 1_T)_T = 1.$$

(ii) From Lemma 1.2.2 we know that $m_{1_T^S} = n_{S//T} m_{1_T}$, from [11; Lemma 9.1.8(ii)] that $m_{1_T} = 1$. Thus, $m_{1_T^S} = n_{S//T}$.

(iii) From [11; Lemma 9.1.8(ii)] we know that $m_{1_S} = 1$. Thus, the claim follows from (i) and (ii). □

**Corollary 1.2.4** *Let $\chi$ be an element in $\mathrm{Irr}(S)$, and let $\phi$ and $\psi$ be elements in $\mathrm{Irr}(T \mid \chi)$. Assume that $\psi^S$ is irreducible and that $m_\phi \leq m_\psi$. Then $\phi^S = \chi$.*

*Proof.* We are assuming that $\psi \in \mathrm{Irr}(T \mid \chi)$. Thus, by definition, $(\psi, \chi|_T) \neq 0$. Thus, by Lemma 1.2.1, $(\chi, \psi^S) \neq 0$. Thus, as we are assuming that $\psi^S$ is irreducible, $\psi^S = \chi$. Thus, as we are assuming $m_\phi \leq m_\psi$

$$m_{\phi^S} = n_{S//T} m_\phi \leq n_{S//T} m_\psi = m_{\psi^S} = m_\chi;$$

cf. Lemma 1.2.2.

On the other hand, as $\phi \in \mathrm{Irr}(T \mid \chi)$, $m_\chi \leq m_{\phi^S}$. Thus, $m_\chi = m_{\phi^S}$, and that is equivalent to $\phi^S = \chi$. □

**Corollary 1.2.5** *Assume that $T \neq S$, and let $\chi$ be an element in $\mathrm{Irr}(S \mid 1_T)$. Then there is no element $\phi$ in $\mathrm{Irr}(T)$ with $\phi^S = \chi$.*

*Proof.* From $\chi \in \mathrm{Irr}(S \mid 1_T)$ we obtain $1_T \in \mathrm{Irr}(T \mid \chi)$.

Assume, by way of contradiction, that $\mathrm{Irr}(T)$ possesses an element $\phi$ with $\phi^S = \chi$. Then, by definition, $\phi \in \mathrm{Irr}(T \mid \chi)$.

On the other hand, we know from [11; Lemma 9.1.8(ii)] that $m_{1_T} = 1$. Thus, $m_{1_T} \leq m_\phi$. Thus, by Lemma 1.2.4, $1_T^S$ is irreducible. Thus, by Lemma 1.2.3(i), $1_T^S = 1_S$. Thus, by Lemma 1.2.3(ii), $T = S$, contradiction. □

**Lemma 1.2.6** *Let $\phi$ be an element in $\mathrm{Irr}(S)$, let $\psi$ be an element in $\mathrm{Irr}(T)$, and let $e_\psi$ denote the central idempotent which belongs to $\psi$. Then $\phi(e_\psi) = (\psi, \phi|_T)_T \psi(1)$.*

*Proof.* Let $\tau_1, \ldots, \tau_n$ denote the irreducible characters of $T$. Then

$$\phi|_T = (\tau_1, \phi|_\tau)_T \tau_1 + \cdots + (\tau_n, \phi|_\tau)_T \tau_n.$$

Thus,

$$\phi(e_\psi) = (\tau_1, \phi|_\tau)_T \tau_1(e_\psi) + \cdots + (\tau_n, \phi|_\tau)_T \tau_n(e_\psi) = (\psi, \phi|_\tau)_T \psi(e_\psi),$$

so that the claim follows from $\psi(e_\psi) = \psi(1)$. □

**Lemma 1.2.7** *The closed subset $T$ of $S$ is normal in $S$ if and only if*

$$(\chi, 1_T^S)_S \in \{0, \chi(1)\}$$

*for each element $\chi$ in $\mathrm{Irr}(S)$.*

*Proof.* Define

$$e_T := \frac{1}{n_T} \sum_{t \in T} \sigma_t.$$

It is easy to see that $T$ is normal in $S$ if and only if $e_T \in Z(\mathbb{C}S)$. Moreover, $e_T \in Z(\mathbb{C}S)$ if and only if

$$\chi(e_T) \in \{0, \chi(1)\}$$

for each element $\chi$ in $\mathrm{Irr}(S)$.[6] Finally, as $1_T(1) = 1$,

$$\chi(e_T) = (1_T, \chi|_T)_T 1_T(1) = (\chi, 1_T^S)_S;$$

[6] In general, if $e$ is an idempotent element of $\mathbb{C}S$, then we have $e \in Z(\mathbb{C}S)$ if and only if $\chi(e) \in \{0, \chi(1)\}$ for all elements $\chi$ in $\mathrm{Irr}(S)$. This is easy to see. In fact, let $e_1, \ldots, e_n$ be the identity elements of the Wedderburn components of $\mathbb{C}S$ and assume first that $e \in Z(\mathbb{C}S)$. Then there exist elements $c_1, \ldots, c_n$ in $\mathbb{C}$ such that

$$e = c_1 e_1 + \cdots + c_n e_n.$$

cf. Lemma 1.2.6 and Lemma 1.2.1. □

For each character $\chi$ of $T$, we define

$$\ker(\chi) := \{t \in T \mid \chi(\sigma_t) = n_t\chi(1)\}.$$

(The set $\ker(\chi)$ is called the *kernel* of $\chi$.)

**Lemma 1.2.8** *Let $\chi$ be a character of $T$. Then* $\ker(\chi)$ *is closed.*

*Proof.* This is [4; Theorem 3.2]. □

## 1.3 Preliminaries on involutions

In this section, the letter $t$ stands for an involution of $S$. (Recall that this means that $\{1, t\}$ is closed; cf. [11; Section 2.3].)

**Lemma 1.3.1** *Let $s$ be an element in $S$. Then $a_{tss} + 1$ divides $n_s$.*

*Proof.* Set $T := \langle t\rangle$ and $n := n_{S//T}$. Then there exist elements $x_1, \ldots, x_n$ in $X$ such that $x_iT \cap x_jT = \emptyset$ for any two different elements $i$ and $j$ in $\{1, \ldots, n\}$ and

$$X = x_1T \cup \cdots \cup x_nT;$$

cf. [11; Lemma 2.1.4].

Now we fix an element in $X$ and call it $x$. Then

$$n_{s^*} = |xs^*| = |xs^* \cap x_1T| + \cdots + |xs^* \cap x_nT|.$$

Note also that

$$|xs^* \cap x_iT| \in \{0, a_{tss} + 1\}$$

for each element $i$ in $\{1, \ldots, n\}$. Thus, $a_{tss}+1$ divides $n_{s^*}$, so that the claim follows from [11; Lemma 1.1.2(iii)]. □

For the remainder of this section, we fix elements $p$ and $q$ in $S$ with $q \in tp$ and $p \neq q$.

**Lemma 1.3.2** *We have $a_{tpq} = a_{tpp} + 1$.*

*Proof.* Since $t$ is an involution, $\{1, t\}$ is closed. Thus, as $q \in tp$, we obtain from [11; Lemma 2.3.1(i)] that

$$a_{1pp} + a_{tpp} = a_{1pq} + a_{tpq}.$$

Since $p \neq q$, $a_{1pq} = 0$. Thus, as $a_{1pp} = 1$, $a_{tpq} = a_{tpp} + 1$. □

**Lemma 1.3.3** *Assume that $tp \subseteq \{p, q\}$. Then the following hold.*

(i) *We have $\sigma_t\sigma_p = a_{tpp}\sigma_p + (a_{tpp} + 1)\sigma_q$.*

(ii) *We have*

$$n_q = \frac{(n_t - a_{tpp})n_p}{a_{tpp} + 1}.$$

---

Since $e$ is assumed to be idempotent, $c_i \in \{0, 1\}$ for all elements $i$ in $\{1, \ldots, n\}$. Thus, $\chi(e) \in \{0, \chi(1)\}$ for all elements $\chi$ in $\mathrm{Irr}(S)$.

Conversely, assume that $\chi(e) \in \{0, \chi(1)\}$ for all elements $\chi$ in $\mathrm{Irr}(S)$. Then, as $e$ is assumed to be idempotent, there exist elements $c_1, \ldots, c_n$ in $\{0, 1\}$ such that the above equation is satisfied. This implies $e \in Z(\mathbb{C}S)$.

(iii) *We have*
$$a_{tpp} = \frac{n_t n_p - n_q}{n_p + n_q}.$$

(iv) *Set* $T := \langle t \rangle$, *and let* $\lambda$ *be a linear character of* $S$ *such that* $(\lambda, 1_T^S)_S = 0$. *Then* $\lambda(\sigma_q) = -\lambda(\sigma_p)$.

*Proof.* (i) From Lemma 1.3.2 we know that $a_{tpq} = a_{tpp} + 1$. Thus, by [11; Lemma 9.1.1(i)],
$$\sigma_t \sigma_p = a_{tpp}\sigma_p + (a_{tpp} + 1)\sigma_q;$$
recall that we are assuming $tp \subseteq \{p, q\}$.

(ii) Since $1_S$ is an algebra homomorphism, this follows from (i) together with the observation that $a_{tpp} \neq -1$.

(iii) This follows from (ii).

(iv) We are assuming that $(\lambda, 1_T^S)_S = 0$. Thus, by Lemma 1.2.1, $(1_T, \lambda|_T)_T = 0$. Thus, as $t$ is an involution, $\lambda(\sigma_t) = -1$. Thus, as $\lambda$ is an algebra homomorphism, we obtain from (i) that
$$\begin{aligned} -\lambda(\sigma_p) = \lambda(\sigma_t)\lambda(\sigma_p) &= \lambda(\sigma_t\sigma_p) \\ &= \lambda(a_{tpp}\sigma_p + (a_{tpp}+1)\sigma_q) \\ &= a_{tpp}\lambda(\sigma_p) + (a_{tpp}+1)\lambda(\sigma_q). \end{aligned}$$
It follows that
$$-(a_{tpp} + 1)\lambda(\sigma_p) = (a_{tpp} + 1)\lambda(\sigma_q).$$
Now the claim follows from the fact that $a_{tpp} \neq -1$. □

Setting $T := \langle t \rangle$ in Lemma 1.3.3 the hypothesis $tp \subseteq \{p, q\}$ there is equivalent to the equation $Tp = \{p, q\}$.

**Lemma 1.3.4** *Assume that* $tp \subseteq \{p, q\}$. *Then* $a_{tpp} + a_{tqq} = n_t - 1$.

*Proof.* We are assuming that $tp \subseteq \{p, q\}$. Thus, by the first equation of [11; Lemma 1.1.3(iii)], $a_{tpq} + a_{tqq} = n_t$. On the other hand, we know from Lemma 1.3.2 that $a_{tpq} = a_{tpp} + 1$. Thus, $a_{tpp} + a_{tqq} = n_t - 1$. □

**Lemma 1.3.5** *Assume that* $tp \subseteq \{p, q\}$, *that* $p^* = p$, *and that* $q^* \neq q$. *Then the following hold.*

(i) *Let* $y$ *be an element in* $X$, *let* $z$ *be an element in* $yt$, *and let* $x$ *be an element in* $yp \cap zq$. *Then* $yp \cap xt = zq \cap xt$.

(ii) *We have* $a_{tpp} = a_{tq^*q^*}$.

(iii) *The integer* $a_{tpp} + 1$ *divides* $a_{ppt}$.

*Proof.* (i) We first show that $yp \cap xt \subseteq zq \cap xt$. In order to do so we fix an element in $yp \cap xt$ and call it $w$. We have to show that $w \in zq$.

From $w \in yp$ and $y \in zt$ we obtain $w \in ztp$. Thus, as we are assuming $tp \subseteq \{p, q\}$, we must have $w \in zp$ or $w \in zq$.

Assume that $w \in zp$. Then, as $x \in wt$, $x \in zpt$. On the other hand, our assumption $tp \subseteq \{p, q\}$ implies $pt \subseteq \{p, q^*\}$; cf. [11; Lemma 1.3.2(iii)]. Thus, $x \in zp$ or $x \in zq^*$. However, $x \in zq$ and $q \notin \{p, q^*\}$, contradiction.

Now we prove that $zq \cap xt \subseteq yp \cap xt$. In order to do so we fix an element in $zq \cap xt$ and call it $w$. We have to show that $w \in yp$.

From $w \in xt$ and $x \in yp$ we obtain $w \in ypt$. Thus, as $pt \subseteq \{p, q^*\}$, we must have $w \in yp$ or $w \in yq^*$.

From $w \in zq$ and $z \in yt$ we obtain $w \in ytq$. Thus, as $tq \subseteq \{p, q\}$, we must have $w \in yp$ or $w \in yq$. Thus, as $q^* \neq q$, $w \in yp$.

(ii) Let $z$ be an element in $X$, and let $x$ be an element in $zq$. Then, as $q \in tp$, $x \in ztp$. Thus, $zt$ possesses an element $y$ with $x \in yp$. Thus, by (i),

$$yp \cap xt = zq \cap xt.$$

On the other hand, since $x \in yp$, we have $|yp \cap xt| = a_{ptp}$. Similarly, as $x \in zq$, $|zq \cap xt| = a_{qtq}$. Thus, as $yp \cap xt = zq \cap xt$, $a_{ptp} = a_{qtq}$. Thus, by [11; Lemma 1.1.1(ii)], $a_{tpp} = a_{tq^*q^*}$.

(iii) Let $y$ be an element in $X$, let $z$ be an element in $yt$, and let $x$ be an element in $yp \cap zp$. Since $q \in tp$, $p \in tq$. Thus, as $x \in yp$, there exists an element $v$ in $yt$ such that $x \in vq$. Now, by (i), $yp \cap xt = vq \cap xt$. But, as $z \in vt$ we obtain from (i) also $zp \cap xt = vq \cap xt$. Thus, $yp \cap xt = zp \cap xt$.

Set $T := \langle t \rangle$. Then $yp \cap xT = yp \cap zp \cap xT$. Therefore,

$$|yp \cap zp \cap xT| = |yp \cap xT| = a_{ptp} + 1 = a_{tpp} + 1.$$

Thus, as $X/T$ is a partition of $X$, $a_{tpp} + 1$ divides $a_{ppt}$. □

The following corollary will not be needed in the remainder of this article.

**Corollary 1.3.6** *Assume that $tp \subseteq \{p, q\}$, that $p^* = p$, and that $q^* \neq q$. Let $v$ be an element in $X$, let $w$ be an element in $vp$, let $y$ be an element in $vt$, and let $z$ be an element in $yp \cap wt$. Then $z \in vp$.*

*Proof.* From $z \in yp$ and $y \in vt$ we obtain $z \in vtp$. Thus, $tp$ possesses an element $s$ such that $z \in vs$.

From $z \in wt$ and $w \in vp$ we obtain $z \in vpt$. Thus, as $z \in vs$, $s \in pt$. Thus, $s \in tp \cap pt = \{p\}$. It follows that $s = p$, so that $z \in vp$. □

Set $T := \langle t \rangle$. Corollary 1.3.6 says in particular, that the (symmetric) graph induced by $p$ on $yT \cup zT$ is a complete bipartite graph of valency $a_{tpp} + 1$ if one disregards the isolated points.

Let $p$ be an element in $S$, and let $q$ be an element in $tp$. Assume that $tp \subseteq \{p, q\}$, that $p^* = p$, and that $q^* \neq q$. Fix an element $x$ in $X$ and define

$$B := \{yp \cap xT \mid y \in X \setminus xT\}.$$

Then $B$ is a 2-design (with possibly repeated blocks) on $xT$. The parameters are

$$v = n_t + 1, \quad r = \frac{n_p}{a_{tpp} + 1}, \quad k = a_{tpp} + 1, \quad \lambda = \frac{a_{ppt}}{a_{tpp} + 1}.$$

We add two more results on structure constants of $S$.

**Lemma 1.3.7** *Assume that $tp \subseteq \{p, q\}$, that $p^* = p$, and that $q^* \neq q$. Then we have*

$$a_{tqq} = \frac{n_t n_q - n_p}{n_p + n_q}, \quad a_{tpq} = \frac{(n_t + 1)n_p}{n_p + n_q}, \quad a_{tqp} = \frac{(n_t + 1)n_q}{n_p + n_q}.$$

*Proof.* From Lemma 1.3.4 we know that $a_{tpp}+a_{tqq}=n_t-1$, from Lemma 1.3.3(iii) that

$$a_{tpp}=\frac{n_t n_p-n_q}{n_p+n_q}.$$

Thus,

$$a_{tqq}=n_t-a_{tpp}-1=\frac{n_t n_p+n_t n_q-n_t n_p+n_q-n_p-n_q}{n_p+n_q}=\frac{n_t n_q-n_p}{n_p+n_q}.$$

From [11; Lemma 1.1.3(iii)], we know that $a_{tqq}+a_{tpq}=n_t$. Thus, the second equation follows from the first one.

From Lemma 1.3.2 we know that $a_{tqp}=a_{tqq}+1$. Thus, the third equation follows from the first one. □

**Lemma 1.3.8** *Assume that* $tp\subseteq\{p,q\}$, *that* $p^*=p$, *and that* $q^*\neq q$. *Then* $n_t$ *divides* $n_p n_q$ *and* $n_q^2$.

*Proof.* From Lemma 1.3.3(iii) together with [11; Lemma 1.1.3(ii)] we obtain

$$a_{ppt}=a_{tpp}\frac{n_p}{n_t}=\frac{n_t n_p-n_q}{n_p+n_q}\cdot\frac{n_p}{n_t}=\frac{n_t n_p^2-n_q n_p}{(n_p+n_q)n_t}.$$

From Lemma 1.3.5(ii) we know that $a_{tpp}=a_{tq^*q^*}$. Thus, as $n_{q^*}=n_q$,

$$a_{q^*qt}=a_{tq^*q^*}\frac{n_q}{n_t}=a_{tpp}\frac{n_q}{n_t}=\frac{n_t n_p-n_q}{n_p+n_q}\cdot\frac{n_q}{n_t}=\frac{n_t n_p n_q-n_q^2}{(n_p+n_q)n_t},$$

and this finishes the proof. □

## 2. Non-normal closed subsets

In this chapter, we investigate association schemes of order 6 which have a non-normal closed subset. In order to do so we fix such a scheme and call it $S$. The non-normal closed subset that $S$ is supposed to have will be denoted by $T$.

### 2.1 Combinatorics

Since $T$ is assumed to be not normal in $S$, $T$ is different from $\{1\}$ and from $S$.

**Theorem 2.1.1** *We have* $|T|=2$.

*Proof.* This is an immediate consequence of Lemma 1.1.5. □

**Lemma 2.1.2** *The scheme* $S$ *possesses elements* $t$, $s$, $m$, *and* $j$ *with* $s^*=s$, $j^*=j$,

$$T=\{1,t\},\quad Ts=\{s,m\},\quad Tm^*=\{m^*,j\},$$

*and* $S=\{1,t,s,m,m^*,j\}$.

*Proof.* From Theorem 2.1.1 we know that $|T|=2$. Thus, $T\setminus\{1\}$ possesses an element $t$ with $T=\{1,t\}$.

From [6; (7.4)] we know that $S$ has exactly four symmetric elements. Thus, as 1 and $t$ are symmetric, $S$ possesses elements $s$, $m$, and $j$ such that $s^*=s$, $j^*=j$, and

$$S=\{1,t,s,m,m^*,j\}.$$

We are assuming that $T$ is not normal in $S$. Thus, $|S/T| \neq 2$. Moreover, by Lemma 1.1.4, $|S/T| \neq 4$ and, by Lemma 1.1.2, $|S/T| \neq 5$. Thus, $|S/T| = 3$.

Assume that $Tm = \{m\}$. Then, as $|S/T| = 3$, $Tm^* = \{m^*, s, j\}$. It follows that $mT = \{m, s, j\}$. In particular, $Tm \subseteq mT$ and $j \in mT \setminus Tm$. Thus, by Lemma 1.1.1, $j^* \neq j$, contradiction. Thus, $Tm \neq \{m\}$.

Similarly, $Tm^* \neq \{m^*\}$. Also, by Lemma 1.1.3, $Tm \neq \{m, m^*\}$. Thus, without loss of generality, $Tm = \{s, m\}$ and $Tm^* = \{m^*, j\}$. □

Lemma 2.1.2 allows us to apply our results of Section 1.3. Here we list the consequences.

**Lemma 2.1.3** *The following hold.*

(i) *The integer $a_{tss} + 1$ divides $a_{sst}$.*

(ii) *The integer $a_{tss} + 1$ divides $n_s$.*

(iii) *We have $a_{tsm} = a_{tss} + 1$.*

(iv) *We have $a_{tss} + a_{tmm} = n_t - 1$.*

(v) *We have $a_{tss} = a_{tm^*m^*}$ and $a_{tjj} = a_{tmm}$.*

(vi) *We have $a_{tss} + a_{tjj} = n_t - 1$.*

(vii) *We have $n_m(a_{tss} + 1) = (n_t - a_{tss})n_s$.*

(viii) *We have $n_j(a_{tss} + 1)^2 = (n_t - a_{tss})^2 n_s$.*

(ix) *We have $n_s n_j = n_m^2$.*

*Proof.* (i) This follows from Lemma 2.1.2 together with Lemma 1.3.5(iii).

(ii) This is a repetition of Lemma 1.3.1.

(iii) This follows from Lemma 2.1.2 together with Lemma 1.3.2.

(iv) This follows from Lemma 2.1.2 together with Lemma 1.3.4.

(v) This follows from Lemma 2.1.2 together with Lemma 1.3.5(ii).

(vi) This follows from (iv) and the second equation of (v).

(vii) This follows from Lemma 2.1.2 together with Lemma 1.3.3(ii) applied to $s$ and $m$ in place of $p$ and $q$.

(viii) This follows from Lemma 2.1.2 together with (v), (vii), and Lemma 1.3.3(ii) applied to $m^*$ and $j$ in place of $p$ and $q$.

(ix) This follows from (vii) and (viii). □

Now we define

$$k := n_{S//T}.$$

Note that, by [11; Lemma 4.3.3(i)], $n_S = (n_t + 1)k$.

**Lemma 2.1.4** *We have*

$$k - 1 = \frac{(n_t + 1)n_s}{(a_{tss} + 1)^2}.$$

*Proof.* Setting

$$f := \frac{n_t - a_{tss}}{a_{tss} + 1}$$

we obtain from Lemma 2.1.3(vii), (viii) that

$$(n_t + 1)(k - 1) = n_s + 2n_m + n_j = (1 + 2f + f^2)n_s = (f + 1)^2 n_s.$$

Thus, the claim follows. □

Lemma 2.1.3(vii), (viii) show that all valencies of $S$ can be expressed in terms of $n_t$, $n_s$, and $a_{tss}$. In the following section, we shall see that all character values of $S$ can be expressed in terms of $n_t$, $n_s$, and $a_{tss}$, too.

## 2.2 Characters

Recall that $S$ stands for a non-commutative association scheme of order 6. Thus, by [11; Corollary 8.6.5], $S$ has exactly three irreducible characters, two linear characters (one of them being the principal character of $S$) and one irreducible character of degree 2. In the following, the principal character of $S$ will be denoted by $1_S$, the non-principal linear character by $\lambda$, the character of degree 2 by $\chi$. Thus,

$$\mathrm{Irr}(S) = \{1_S, \lambda, \chi\}.$$

The letters $t$, $s$, $m$, $j$, and $k$ will have the same meaning as in the previous section.

**Lemma 2.2.1** *The following hold.*

(i) *We have* $1_T^S = 1_S + \chi$.

(ii) *The closed subset* $T$ *possesses a linear character* $\tau$ *with* $\mathrm{Irr}(T) = \{1_T, \tau\}$ *and* $\tau^S = \lambda + \chi$.

*Proof.* From Corollary 1.2.3(i) we know that $(1_S, 1_T^S)_S = 1$. Thus,

$$1_T^S = 1_S + (\lambda, 1_T^S)_S\lambda + (\chi, 1_T^S)_S\chi.$$

Recall also from Corollary 1.2.3(iii) that $1 \leq (\lambda, 1_T^S)_S + (\chi, 1_T^S)_S$.

From Theorem 2.1.1 we know that $|T| = 2$. Thus, $\mathrm{Irr}(T)$ possesses an element $\tau$ such that $\mathrm{Irr}(T) = \{1_T, \tau\}$.

Since $1_S|_T = 1_T$, $\tau \notin \mathrm{Irr}(T \mid 1_S)$. Thus $1_S \notin \mathrm{Irr}(S \mid \tau)$. Thus,

$$\tau^S = (\lambda, \tau^S)_S\lambda + (\chi, \tau^S)_S\chi.$$

With the help of Lemma 1.2.1 we also obtain

$$1 = \lambda(1) = \lambda|_T(1) = (1_T, \lambda|_T)_T 1_T(1) + (\tau, \lambda|_T)_T\tau(1) = (\lambda, 1_T^S)_S + (\lambda, \tau^S)_S$$

and

$$2 = \chi(1) = \chi|_T(1) = (1_T, \chi|_T)_T 1_T(1) + (\tau, \chi|_T)_T\tau(1) = (\chi, 1_T^S)_S + (\chi, \tau^S)_S.$$

If $1_T^S = 1_S + \lambda + \chi$, $\tau^S = \chi$, contrary to Corollary 1.2.5. Thus, we must have

$$1_T^S = 1_S + \lambda \quad \text{or} \quad 1_T^S = 1_S + \chi.$$

If $1_T^S = 1_S + \lambda$, $T$ is normal in $S$; cf. Lemma 1.2.7. Thus, $1_T^S = 1_S + \chi$. It follows that $\tau^S = \lambda + \chi$. □

**Lemma 2.2.2** *We have* $m_\lambda = (n_t - 1)(k - 1) + n_t$ *and* $m_\chi = k - 1$.

*Proof.* From Lemma 2.2.1(i) we know that $1_T^S = 1_S + \chi$, from [11; Lemma 9.1.8(ii)] that $m_{1_T} = 1$ and that $m_{1_S} = 1$. Thus, by Lemma 1.2.2,

$$n_{S//T} = n_{S//T} m_{1_T} = m_{1_T^S} = m_{1_S} + m_\chi = 1 + m_\chi.$$

Thus, the second equation follows from the definition of $k$.

From $m_\chi = k - 1$ and $\chi_X(1) = n_S$ we obtain

$$n_S = m_{1_S} + m_\lambda \lambda(1) + m_\chi \chi(1) = 1 + m_\lambda + 2(k-1) = m_\lambda + 2k - 1.$$

Thus, as $n_S = (n_t + 1)k$,

$$n_t k + k = (n_t + 1)k = m_\lambda + 2k - 1.$$

It follows that $m_\lambda = n_t k - k + 1 = (n_t - 1)(k-1) + n_t$. □

**Lemma 2.2.3** *We have*

$$m_\lambda = \frac{(n_t - 1)(n_t + 1)n_s}{(a_{tss} + 1)^2} + n_t \quad \textit{and} \quad m_\chi = \frac{(n_t + 1)n_s}{(a_{tss} + 1)^2}.$$

*Proof.* Considering Lemma 2.1.4 this follows from Lemma 2.2.2. □

**Lemma 2.2.4** *The following hold.*

(i) *We have* $\lambda(\sigma_t) = -1$.

(ii) *We have* $\lambda(\sigma_m) = \lambda(\sigma_{m^*}) = -\lambda(\sigma_s)$ *and* $\lambda(\sigma_s) = \lambda(\sigma_j)$.

*Proof.* (i) From Lemma 2.2.1(ii) we know that $(\lambda, \tau^S)_S = 1$. Thus, by Lemma 1.2.1, $(\tau, \lambda|_T)_T = 1$. Thus, $\lambda|_T = \tau$. Thus, $\lambda(\sigma_t) = \tau(\sigma_t) = -1$.

(ii) From Lemma 2.2.1(i) we know that $(\lambda, 1_T^S)_S = 0$. Thus, applying Lemma 1.3.3(iv) to $s$ and $m$ in place of $p$ and $q$ we obtain $\lambda(\sigma_m) = -\lambda(\sigma_s)$. Similarly, applying Lemma 1.3.3(iv) to $j$ and $m^*$ in place of $p$ and $q$ we obtain $\lambda(\sigma_j) = -\lambda(\sigma_{m^*})$.

That $\lambda(\sigma_m) = \lambda(\sigma_{m^*})$ follows from the fact that $\lambda$ is the only linear character of $S$. □

Now we are able to express the values of $\lambda$ on $S$ in terms of $n_t$, $n_s$, and $a_{tss}$.

**Lemma 2.2.5** *We have*

$$\lambda(\sigma_s)^2 = \frac{(n_t - a_{tss})^2 n_s^2}{n_t[(n_t - 1)(n_t + 1)n_s + (a_{tss} + 1)^2 n_t]}.$$

*Proof.* From [11; Theorem 9.1.7(ii)] together with Lemma 2.2.4 we obtain

$$\frac{(n_t + 1)k}{m_\lambda} = 1 + \frac{1}{n_t} + (\frac{1}{n_s} + \frac{2}{n_m} + \frac{1}{n_j})\lambda(\sigma_s)^2.$$

Now recall from Lemma 2.1.4 that

$$k - 1 = \frac{(n_t + 1)n_s}{(a_{tss} + 1)^2}$$

and from Lemma 2.1.3(vii) that

$$n_m = \frac{n_t - a_{tss}}{a_{tss} + 1} n_s.$$

Thus,

$$\frac{k-1}{n_m^2} = \frac{n_t+1}{(n_t-a_{tss})^2 n_s}.$$

Recall also that, by Lemma 2.1.3(ix), $n_s n_j = n_m^2$. Thus,

$$\frac{1}{n_s} + \frac{2}{n_m} + \frac{1}{n_j} = \frac{n_m(n_j+2n_m+n_s)}{n_s n_m n_j} = \frac{(n_t+1)(k-1)}{n_m^2} = \frac{(n_t+1)^2}{(n_t-a_{tss})^2 n_s}.$$

On the other hand, by Lemma 2.1.4 and the first equation of Lemma 2.2.2,

$$\frac{k}{m_\lambda} = \frac{(n_t+1)n_s + (a_{tss}+1)^2}{(n_t-1)(n_t+1)n_s + (a_{tss}+1)^2 n_t}$$

Thus,

$$\begin{aligned}
\frac{(n_t+1)k}{m_\lambda} - \frac{n_t+1}{n_t} &= \frac{n_t(n_t+1)^2 n_s + n_t(n_t+1)(a_{tss}+1)^2}{n_t[(n_t-1)(n_t+1)n_s + (a_{tss}+1)^2 n_t]} \\
&\quad - \frac{(n_t+1)[(n_t-1)(n_t+1)n_s + (a_{tss}+1)^2 n_t]}{n_t[(n_t-1)(n_t+1)n_s + (a_{tss}+1)^2 n_t]} \\
&= \frac{(n_t+1)^2 n_s}{n_t[(n_t-1)(n_t+1)n_s + (a_{tss}+1)^2 n_t]}.
\end{aligned}$$

Thus, by the first equation of this proof,

$$\frac{n_s}{n_t[(n_t-1)(n_t+1)n_s + (a_{tss}+1)^2 n_t]} = \frac{\lambda(\sigma_s)^2}{(n_t-a_{tss})^2 n_s}.$$

This finishes the proof. □

Recall that $\lambda$ is the only non-principal linear character of $S$. Thus, as $\lambda(\sigma_s)$ is an algebraic integer, $\lambda(\sigma_s)$ must be an integer.

**Lemma 2.2.6** *We have*

$$\chi(\sigma_s) = -(n_t-1)\lambda(\sigma_s) - \frac{n_t\lambda(\sigma_s)+n_s}{k-1}.$$

*Proof.* From [11; Lemma 9.1.2(ii)] we know that $\chi_X(\sigma_s) = 0$, from Lemma 2.2.2 that

$$m_\lambda = (n_t-1)(k-1)+n_t \quad \text{and} \quad m_\chi = k-1.$$

Thus, as

$$\chi_X = 1_S + m_\lambda\lambda + m_\chi\chi$$

and $1_S(\sigma_s) = n_s$,

$$0 = n_s + [(n_t-1)(k-1)+n_t]\lambda(\sigma_s) + (k-1)\chi(\sigma_s).$$

It follows that

$$\begin{aligned}
\chi(\sigma_s) &= \frac{-[(n_t-1)(k-1)+n_t]\lambda(\sigma_s) - n_s}{k-1} \\
&= \frac{-(n_t-1)(k-1)\lambda(\sigma_s) - n_t\lambda(\sigma_s) - n_s}{k-1} \\
&= -(n_t-1)\lambda(\sigma_s) - \frac{n_t\lambda(\sigma_s)+n_s}{k-1},
\end{aligned}$$

and this proves the lemma. □

Now we compute the fraction in the statement of Lemma 2.2.6 in terms of $n_t$, $n_s$, and $a_{tss}$.

**Lemma 2.2.7** *We have*

$$\frac{n_t\lambda(\sigma_s)+n_s}{k-1} = \pm\frac{n_t(n_t-a_{tss})(a_{tss}+1)^2}{(n_t+1)\sqrt{n_t[(n_t-1)(n_t+1)n_s+(a_{tss}+1)^2n_t]}} + \frac{(a_{tss}+1)^2}{n_t+1}.$$

*Proof.* From Lemma 2.2.5 we know that

$$\lambda(\sigma_s) = \pm\frac{(n_t-a_{tss})n_s}{\sqrt{n_t[(n_t-1)(n_t+1)n_s+(a_{tss}+1)^2n_t]}}.$$

Thus,

$$n_t\lambda(\sigma_s)+n_s = \pm\frac{n_t(n_t-a_{tss})n_s}{\sqrt{n_t[(n_t-1)(n_t+1)n_s+(a_{tss}+1)^2n_t]}} + n_s.$$

Now recall from Lemma 2.1.4 that

$$k-1 = \frac{(n_t+1)n_s}{(a_{tss}+1)^2}.$$

Thus, the desired equation follows. □

Lemma 2.2.5, Lemma 2.2.6, and Lemma 2.2.7 allow us to express $\chi(\sigma_s)$ in terms of $n_t$, $n_s$, and $a_{tss}$. The algebraic integers $\chi(\sigma_m)$, $\chi(\sigma_{m^*})$, and $\chi(\sigma_j)$ can be computed similarly. Similar to Lemma 2.2.6 one obtains

$$\chi(\sigma_m) = -(n_t-1)\lambda(\sigma_m) - \frac{n_t\lambda(\sigma_m)+n_m}{k-1},$$

$$\chi(\sigma_{m^*}) = -(n_t-1)\lambda(\sigma_{m^*}) - \frac{n_t\lambda(\sigma_{m^*})+n_{m^*}}{k-1},$$

and

$$\chi(\sigma_j) = -(n_t-1)\lambda(\sigma_j) - \frac{n_t\lambda(\sigma_j)+n_j}{k-1}.$$

Setting $c := -\lambda(\sigma_s)$ we obtain that

| | 1 | $t$ | $s$ | $m$ | $m^*$ | $j$ | multiplicity |
|---|---|---|---|---|---|---|---|
| $1_S$ | 1 | $n_t$ | $n_s$ | $n_m$ | $n_{m^*}$ | $n_j$ | 1 |
| $\lambda$ | 1 | $-1$ | $-c$ | $c$ | $c$ | $-c$ | $k(n_t-1)+1$ |
| $\chi$ | 2 | $n_t-1$ | $\frac{cm_\lambda-n_s}{k-1}$ | $-\frac{cm_\lambda+n_m}{k-1}$ | $-\frac{cm_\lambda+n_m}{k-1}$ | $\frac{cm_\lambda-n_j}{k-1}$ | $k-1$ |

is the character table of $S$.

## 2.3 Arithmetic

In this section, the letters $t$, $s$, $m$, $j$, $k$, $\lambda$, and $\chi$ will have the same meaning as in the previous section. We derive arithmetic consequences of the fact that the entries of the character table of $S$ must be integral.

**Lemma 2.3.1** *Assume that* $n_t \neq 1$. *Then*

$$(n_t-1)(n_t+1)n_s+(a_{tss}+1)^2n_t$$

*divides* $n_t(n_t-a_{tss})^2(a_{tss}+1)^4$.

*Proof.* From Lemma 2.2.6 we know that

$$\chi(\sigma_s) = -(n_t - 1)\lambda(\sigma_s) - \frac{n_t\lambda(\sigma_s) + n_s}{k-1}.$$

On the other hand, as $\chi$ is the only non-linear character of $S$, $\chi(\sigma_s)$ is integral. Thus, as $(n_t - 1)\lambda(\sigma_s)$ is integral,

$$\frac{n_t\lambda(\sigma_s) + n_s}{k-1}$$

must be integral. Now the claim follows from Lemma 2.2.7. □

We now add two consequences of Lemma 2.2.5.

**Corollary 2.3.2** *Assume that* $1 \leq a_{tss}$. *Then* $n_t + 1 \leq n_s$.

*Proof.* From Lemma 2.2.5 we obtain that

$$n_t[(n_t - 1)(n_t + 1)n_s + (a_{tss} + 1)^2 n_t]$$

divides $(n_t - a_{tss})^2 n_s^2$. Since $st \neq \{s\}$, $n_t \neq a_{tss}$; cf. [11; Lemma 1.1.3(iii)]. Thus,

$$n_t(n_t - 1)(n_t + 1)n_s \leq (n_t - a_{tss})^2 n_s^2.$$

Since we are assuming $1 \leq a_{tss}$, this implies $n_t + 1 \leq n_s$. □

We do not know examples in which $n_t$ does not divide $n_s$. However, we cannot prove this statement. In the remainder of this section, we provide sufficient conditions for $n_t$ to divide $n_s$.

**Corollary 2.3.3** *Let* $p$ *be a prime number, and let* $\alpha$ *be the multiplicity of* $p$ *in* $n_t$. *Then the following hold.*

(i) *Assume that* $p$ *does not divide* $a_{tss}$. *Then* $p^\alpha$ *divides* $n_s$.
(ii) *If* $\alpha = 1$, $p$ *divides* $n_s$.

*Proof.* (i) From [11; Lemma 1.1.3(ii)] we know that $a_{sst}n_t = a_{tss}n_s$. The claim is an immediate consequence of this equation.

(ii) Assume that $\alpha = 1$. Then $p^2$ does not divide $n_t$. However, by Lemma 2.2.5,

$$n_t[(n_t - 1)(n_t + 1)n_s + (a_{tss} + 1)^2 n_t]$$

is a square. Thus, $p$ must divide $(n_t - 1)(n_t + 1)n_s + (a_{tss} + 1)^2 n_t$. Thus, as $p$ divides $(a_{tss} + 1)^2 n_t$, $p$ must divide $(n_t - 1)(n_t + 1)n_s$. Thus, as $p$ does not divide $(n_t - 1)(n_t + 1)$, $p$ must divide $n_s$. □

**Corollary 2.3.4** *Assume that* $(a_{tss}, n_t) = 1$. *Then* $n_t$ *divides* $n_s$.

*Proof.* This follows immediately from Corollary 2.3.3(i). □

**Lemma 2.3.5** *Assume that* $a_{tss}$ *is square-free. Then* $n_t$ *divides* $n_s$.

*Proof.* Let $p$ be a prime divisor of $a_{tss}$, let $\alpha$ be the multiplicity of $p$ in $n_t$, and let $\beta$ be the multiplicity of $p$ in $n_s$. By [11; Lemma 1.1.3(ii)],

$$a_{sst}n_t = a_{tss}n_s.$$

Thus, as $a_{tss}$ is assumed to be square-free $\alpha \leq \beta + 1$.

Assume that $\alpha = \beta + 1$. Then the multiplicity of $p$ in

$$n_t[(n_t - 1)(n_t + 1)n_s + (p + 1)^2 n_t]$$

is $2\alpha - 1$. On the other hand, by Lemma 2.2.5, this multiplicity must be even, contradiction.

Thus, $\alpha \leq \beta$. Since $p$ has been chosen arbitrarily among the prime divisors of $a_{tss}$, we now obtain from Lemma 2.3.3(i) that $n_t$ divides $n_s$. □

Before we state (and prove) the final result of this section, we recall that the main hypothesis there, namely that $2a_{tss} = n_t - 1$, is equivalent to $a_{tss} = a_{tjj}$; cf. Lemma 2.1.3(vi). By Lemma 2.1.3(vii), (viii) it is also equivalent to $n_s = n_m = n_{m^*} = n_j$.

**Lemma 2.3.6** *Assume that $2a_{tss} = n_t - 1$. Then the following hold.*

(i) *We have $(a_{tss} + 1, n_t) = 1$.*

(ii) *The integer $n_t$ divides $n_s$.*

(iii) *The integer $a_{tss} + 2$ is a multiple of 4.*

(iv) *We have*

$$n_s = (a_{tss} + 1)(2a_{tss} + 1)\frac{a_{tss} + 2}{4}.$$

(v) *There exists a positive odd integer $u$ such that $a_{tss} = 2u$, $n_t = 4u + 1$,*

$$n_s = \frac{1}{2}(u + 1)(2u + 1)(4u + 1),$$

*and $a_{sst} = u(u + 1)(2u + 1)$.*

*Proof.* (i) Let $z$ be an integer dividing $a_{tss} + 1$. Then $z$ divides $2a_{tss} + 2 = n_t + 1$. Thus, if $z$ divides $n_t$, $z \in \{-1, 1\}$.

(ii) Our assumption $2a_{tss} = n_t - 1$ forces $(a_{tss}, n_t) = 1$. Thus, the claim follows from Corollary 2.3.4.

(iii), (iv) Set $a := a_{tss}$. Then, by Lemma 2.1.3(ii), $a + 1$ divides $n_s$. Moreover, by (ii), $n_t$ divides $n_s$. On the other hand, by (i), $a + 1$ and $n_t$ are coprime. Thus, $(a + 1)n_t$ divides $n_s$. Thus, there exists a positive integer $d$ such that

$$(a + 1)n_t d = n_s.$$

From Lemma 2.2.5 we know that

$$\begin{aligned} \lambda(\sigma_s)^2 &= \frac{(n_t - a)^2 n_s^2}{n_t[(n_t - 1)(n_t + 1)n_s + (a + 1)^2 n_t]} \\ &= \frac{(a + 1)^4 n_t^2 d^2}{n_t[4a(a + 1)^2 n_t d + (a + 1)^2 n_t]} = \frac{(a + 1)^2 d^2}{4ad + 1}. \end{aligned}$$

Thus, as $\lambda(\sigma_s)^2$ is integral, $4ad + 1$ divides $(a + 1)^2 d^2$. Thus, as $4ad + 1$ and $d$ are coprime, $4ad + 1$ divides $(a + 1)^2$. Thus, there exists a positive integer $e$ such that

$$(4ad + 1)e = (a + 1)^2.$$

It follows that $a$ divides $e - 1$.

Assume that $e \neq 1$. Then there exists a positive integer $g$ such that $ag = e - 1$. Thus, $ag + 1 = e$. Thus,

$$(4ad + 1)(ag + 1) = (a + 1)^2,$$

contradiction. Thus, $e = 1$, and that means that

$$4ad + 1 = (a + 1)^2.$$

Thus, $4ad = a^2 + 2a$, so that $4d = a + 2$. This proves (iii), and the equation in (iv) follows from $n_s = (a+1)n_t d$.

(v) From (iii) we obtain a positive odd integer $u$ such that $a_{tss} = 2u$. The other three equations now follow from $2a_{tss} = n_t - 1$, from (iv), and from [11; Lemma 1.1.3(ii)]. □

## 2.4 Some special cases

In this section, the letters $t$, $s$, $m$, $j$, and $\lambda$ will have the same meaning as in the previous section.

**Theorem 2.4.1** *If $n_t = 1$, $S$ is a semidirect product with complement $T$.*

*Proof.* Assume that $n_t = 1$. Then, by Lemma 2.1.3(vii), $a_{tss} = 0$ and $n_m = n_s$.

Since $a_{tss} = 0$, we obtain from Lemma 2.1.3(viii) that $n_j = n_s$ and from Lemma 2.2.5 that $\lambda(\sigma_s)^2 = n_s^2$.

Assume that $\lambda(\sigma_s) = n_s$. Then, by Lemma 2.2.4, $\ker(\lambda) = \{1, s, j\}$. Now recall from Lemma 1.2.8 that $\ker(\lambda)$ is closed. Thus, by Theorem 2.1.1, $\ker(\lambda)$ is normal in $S$. It follows that $S$ is a semidirect product with kernel $\ker(\lambda)$ and complement $T$.

If $\lambda(\sigma_s) = -n_s$, $\ker(\lambda) = \{1, m, m^*\}$. Again, by Theorem 2.1.1, $\ker(\lambda)$ is normal in $S$, and $S$ is a semidirect product with kernel $\ker(\lambda)$ and complement $T$. □

**Theorem 2.4.2** *Assume that $s$ is an involution. Then $S$ is a Coxeter scheme (with respect to $\{s, t\}$.)*

*Proof.* From Lemma 2.1.2 we know that $Ts = \{s, m\}$. Thus, $m \in ts$ and $m^* \notin ts$. (Similarly, $m^* \in st$ and $m \notin st$.) Thus, $st \neq ts$.

From Lemma 2.1.2 we also obtain that $\langle t, s\rangle = S$. Thus, by [9; Theorem 3.3], $S$ is a Coxeter scheme with respect to $\{t, s\}$. □

**Theorem 2.4.3** *If $a_{tss} = 0$, $S$ is a semidirect product with complement $T$ or a Coxeter scheme (with respect to $\{s, t\}$.)*

*Proof.* Assume that $a_{tss} = 0$ and set $c := -\lambda(\sigma_s)$. Then, by Lemma 2.2.5,

$$c^2 = \frac{n_t n_s^2}{n_t^2 n_s - n_s + n_t}.$$

It follows that

$$n_t(n_t n_s + 1)c^2 = n_s(n_t n_s + c^2).$$

Thus, $n_t n_s + 1$ divides $n_t n_s + c^2$. Thus, there exist a positive integer $d$ with

$$(n_t n_s + 1)d = n_t n_s + c^2.$$

Thus,

$$n_t n_s(d-1) = c^2 - d.$$

Assume first that $2 \leq d$. Then $n_t n_s \leq c^2 - 1$. Thus,

$$n_t n_s c^2 \leq n_t(n_t n_s + 1)c^2 = n_s(n_t n_s + c^2) \leq n_s(2c^2 - 1) = 2n_s c^2 - n_s.$$

This forces $n_t = 1$. Thus, by Theorem 2.4.1, $S$ is a semidirect product with complement $T$.

Assume now that $d = 1$. Then, as $n_t n_s(d-1) = c^2 - d$, we must have $c^2 = 1$. Thus, as $n_t(n_t n_s + 1)c^2 = n_s(n_t n_s + c^2)$,

$$n_t = n_s.$$

Now recall from [11; Lemma 1.1.3(iv)] that

$$n_s^2 = n_s + a_{sst}n_t + a_{sss}n_s + a_{ssm}n_m + a_{ssm^*}n_{m^*} + a_{ssj}n_j.$$

Note also that, as $a_{tss} = 0$, $a_{sst} = 0$; cf. [11; Lemma 1.1.3(ii)]. Moreover, from $a_{tss} = 0$, $n_t = n_s$, and Lemma 2.1.3(vii) one obtains $n_m = n_s^2$. Thus, $a_{ssm} = 0$. Similarly one obtains $a_{ssm^*} = 0$ and $a_{ssj} = 0$. It follows that

$$n_s^2 = n_s + a_{sss}n_s = (1 + a_{sss})n_s.$$

Thus, $1 + a_{sss} = n_s$, and that means that $s$ is an involution. Now Theorem 2.4.2 forces $S$ to be a Coxeter scheme with respect to $\{s, t\}$. □

**Theorem 2.4.4** *Assume that $a_{tss} = 1$. Then $S$ is a Coxeter scheme with respect to $\{t, j\}$ and has valency* 21.

*Proof.* Assuming $a_{tss} = 1$ we obtain from Lemma 2.1.3(vii) that $n_t \neq 1$. Thus, by Lemma 2.3.1,

$$(n_t - 1)(n_t + 1)n_s + 4n_t$$

divides $16n_t(n_t - 1)^2$.

From $a_{tss} = 1$ we also obtain $a_{sst}n_t = n_s$; cf. [11; Lemma 1.1.3(ii)]. Thus,

$$(n_t - 1)(n_t + 1)a_{sst} + 4$$

divides $16(n_t - 1)^2$. Thus, there exists a positive integer $d$ such that

$$[(n_t - 1)(n_t + 1)a_{sst} + 4]d = 16(n_t - 1)^2.$$

It follows that $a_{sst}d \leq 15$.

Assume first that $n_t$ is odd. (This will lead to a contradiction.) Then $(n_t-1)(n_t+1)$ is a multiple of 8. Thus, $(n_t - 1)(n_t + 1)a_{sst} + 4$ is not divisible by 8. Thus, by the last equation, $d$ is a multiple of 4.

On the other hand, as $a_{tss} = 1$, $a_{sst}$ must be even; cf. Lemma 2.1.3(i). Thus, as $a_{sst}d \leq 15$, $a_{sst} = 2$ and $d = 4$. It follows that $n_t^2 + 1 = 2(n_t - 1)^2$, contradiction.

Thus, $n_t$ is even. If $a_{sst}$ is not divisible by 4, $d$ must be divisible by 8, contradiction. Thus, $a_{sst}$ is a multiple of 4. Thus, there exists a positive integer $e$ such that $a_{sst} = 4e$. From this we obtain $de \leq 3$ and

$$[(n_t - 1)(n_t + 1)e + 1]d = 4(n_t - 1)^2.$$

Thus, $n_t - 1$ divides $d$. If $d = 3$, $e = 1$ and $n_t = 4$. Thus, $3n_t^2 = 4(n_t - 1)^2$, contradiction. Thus, $d \leq 2$. Thus, as $n_t$ is even, $n_t = 2$.

From $n_t = 2$ and $a_{tss} = 1$ we obtain $a_{tjj} = 0$; cf. Lemma 2.1.3(vi). Thus, by Theorem 2.4.3, $S$ is a Coxeter scheme with respect to $\{t, j\}$. □

## 2.5 The representation of degree 2

In this section, the letters $t$, $s$, $m$, $j$, $k$, $\lambda$, and $\chi$ will have the same meaning as in Section 2.3. In addition, we define $\Phi$ to be the representation which affords $\chi$.

**Proposition 2.5.1** *Set* $a := a_{tss}$,

$$p := \sqrt{(n_t+1)n_s + (a+1)^2}, \quad \textit{and} \quad q := \sqrt{\frac{(n_t^2-1)n_s + n_t(a+1)^2}{n_t}}.$$

*Then* $\Phi$ *admits a basis with respect to which*

$$\Phi(\sigma_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \Phi(\sigma_t) = \begin{pmatrix} n_t & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Phi(\sigma_s) = \frac{1}{n_t+1}\begin{pmatrix} -(a+1)^2 & (a+1)p \\ (n_t-a)p & \pm(n_t-a)q \end{pmatrix},$$

$$\Phi(\sigma_m) = \frac{1}{n_t+1}\begin{pmatrix} -(n_t-a)(a+1) & (n_t-a)p \\ -(n_t-a)p & \mp(n_t-a)q \end{pmatrix},$$

$$\Phi(\sigma_{m^*}) = \frac{1}{n_t+1}\begin{pmatrix} -(n_t-a)(a+1) & -(a+1)p \\ \frac{(n_t-a)^2p}{a+1} & \mp(n_t-a)q \end{pmatrix},$$

$$\Phi(\sigma_j) = \frac{1}{n_t+1}\begin{pmatrix} -(n_t-a)^2 & -(n_t-a)p \\ -\frac{(n_t-a)^2p}{a+1} & \pm(n_t-a)q \end{pmatrix}.$$

*Proof.* From Lemma 2.2.1 we know that $\chi|_T = 1_T + \tau$. Thus, we may assume that

$$\Phi(\sigma_t) = \begin{pmatrix} n_t & 0 \\ 0 & -1 \end{pmatrix}.$$

Set

$$\Phi(\sigma_s) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad \Phi(\sigma_m) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad \Phi(\sigma_j) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}.$$

Then

$$\Phi(\sigma_t\sigma_s) = \begin{pmatrix} n_t & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} n_ts_{11} & n_ts_{12} \\ -s_{21} & -s_{22} \end{pmatrix}.$$

Now recall from Lemma 2.1.3(iii) that $\sigma_t\sigma_s = a\sigma_s + (a+1)\sigma_m$. Thus,

$$\Phi(\sigma_t\sigma_s) = a\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} + (a+1)\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

It follows that

$$\Phi(\sigma_m) = \begin{pmatrix} \frac{n_t-a}{a+1}s_{11} & \frac{n_t-a}{a+1}s_{12} \\ -s_{21} & -s_{22} \end{pmatrix}.$$

Similarly, as $\sigma_s\sigma_t = a\sigma_s + (a+1)\sigma_{m^*}$,

$$\Phi(\sigma_{m^*}) = \begin{pmatrix} \frac{n_t-a}{a+1}s_{11} & -s_{12} \\ \frac{n_t-a}{a+1}s_{21} & -s_{22} \end{pmatrix}.$$

Thus,

$$\Phi(\sigma_t\sigma_{m^*}) = \begin{pmatrix} \frac{(n_t-a)n_t}{a+1}s_{11} & -n_ts_{12} \\ -\frac{n_t-a}{a+1}s_{21} & s_{22} \end{pmatrix}.$$

Recall also that, by Lemma 2.1.3(v), $\sigma_t\sigma_{m^*} = a\sigma_{m^*} + (a+1)\sigma_j$. Thus,

$$\Phi(\sigma_t\sigma_{m^*}) = a\begin{pmatrix} \frac{n_t-a}{a+1}s_{11} & -s_{12} \\ \frac{n_t-a}{a+1}s_{21} & -s_{22} \end{pmatrix} + (a+1)\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}.$$

So we have

$$\Phi(\sigma_j) = \begin{pmatrix} \frac{(n_t-a)^2}{(a+1)^2}s_{11} & -\frac{n_t-a}{a+1}s_{12} \\ -\frac{n_t-a}{a+1}s_{21} & s_{22} \end{pmatrix}.$$

Since $\Phi(\sigma_1) + \Phi(\sigma_t) + \Phi(\sigma_s) + \Phi(\sigma_m) + \Phi(\sigma_{m^*}) + \Phi(\sigma_j) = 0$, we now have

$$s_{11} = -\frac{(a+1)^2}{n_t+1}.$$

We set

$$f := \frac{n_t - a}{a+1}.$$

Applying the Schur relations [11; Theorem 9.2.4(ii)] for association schemes to the $(1,2)$-entries of the 2-dimensional representation of $S$ we then obtain

$$\frac{1}{n_S}\left(\frac{s_{12}s_{21}}{n_s} + \frac{f^2 s_{12}s_{21}}{fn_s} + \frac{s_{12}s_{21}}{fn_s} + \frac{f^2 s_{12}s_{21}}{f^2 n_s}\right) = \frac{1}{k-1};$$

cf. Lemma 2.2.2. Thus, referring to Lemma 2.1.4 we obtain from $n_S = (n_t+1)k$ that

$$\begin{aligned} s_{12}s_{21} &= \frac{(n_t+1)kfn_s}{(f+1)^2(k-1)} \\ &= \frac{(n_t+1)k(n_t-a)n_s(a+1)^2}{(a+1)(n_t+1)^2(k-1)} \\ &= \frac{k(n_t-a)n_s(a+1)}{(n_t+1)(k-1)} \\ &= \frac{k(n_t-a)(a+1)^3}{(n_t+1)^2} \\ &= \frac{(n_t-a)(a+1)[(n_t+1)n_s + (a+1)^2]}{(n_t+1)^2}. \end{aligned}$$

Now we apply the Schur relations [11; Theorem 9.2.4(ii)] for association schemes to the $(2,2)$-entries of the 2-dimensional representation of $S$. Then we obtain

$$\frac{1}{n_S}\left(1 + \frac{1}{n_t} + \frac{s_{22}^2}{n_s} + \frac{2s_{22}^2}{fn_s} + \frac{s_{22}^2}{f^2 n_s}\right) = \frac{1}{k-1}.$$

Thus,

$$\frac{n_t+1}{n_t} + \frac{(f+1)^2 s_{22}^2}{f^2 n_s} = \frac{n_S}{k-1}.$$

Thus, as $n_S = (n_t+1)k$,

$$\frac{(f+1)^2 s_{22}^2}{f^2 n_s} = \frac{(n_t+1)k}{k-1} - \frac{n_t+1}{n_t} = \frac{(kn_t - k + 1)(n_t+1)}{(k-1)n_t}.$$

Thus, referring to Lemma 2.1.4, we obtain

$$\begin{aligned} s_{22}^2 &= \frac{(kn_t-k+1)(n_t+1)f^2 n_s}{(k-1)n_t(f+1)^2} \\ &= \frac{(kn_t-k+1)(n_t-a)^2 n_s}{(k-1)n_t(n_t+1)} \\ &= \frac{(kn_t-k+1)(n_t-a)^2(a+1)^2}{n_t(n_t+1)^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(n_t-a)^2}{n_t(n_t+1)^2}(kn_t-k+1)(a+1)^2 \\
&= \frac{(n_t-a)^2}{n_t(n_t+1)^2}[n_t(n_t+1)n_s+n_t(a+1)^2-(n_t+1)n_s] \\
&= \frac{(n_t-a)^2}{n_t(n_t+1)^2}[(n_t^2-1)n_s+n_t(a+1)^2].
\end{aligned}
$$

Thus,

$$s_{22} = \pm\frac{n_t-a}{n_t+1}\sqrt{\frac{(n_t^2-1)n_s+n_t(a+1)^2}{n_t}},$$

and that finishes the proof. □

Theoretically, Proposition 2.5.1 allows us to compute all intersection numbers of $S$. However, we shall not do that here. Instead, we focus on the specific case which we started considering in Lemma 2.3.6.

**Corollary 2.5.2** *Set* $a := a_{tss}$ *and*

$$p := \sqrt{a^2+\frac{5}{2}a+2}.$$

*Assume that* $2a = n_t - 1$. *Then* $\Phi$ *admits a basis with respect to which*

$$\Phi(\sigma_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi(\sigma_t) = \begin{pmatrix} 2a+1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Phi(\sigma_s) = \frac{a+1}{2}\begin{pmatrix} -1 & p \\ p & \pm(a+1) \end{pmatrix}, \quad \Phi(\sigma_m) = \frac{a+1}{2}\begin{pmatrix} -1 & p \\ -p & \mp(a+1) \end{pmatrix},$$

$$\Phi(\sigma_{m^*}) = \frac{a+1}{2}\begin{pmatrix} -1 & -p \\ p & \mp(a+1) \end{pmatrix}, \quad \Phi(\sigma_j) = \frac{a+1}{2}\begin{pmatrix} -1 & -p \\ -p & \pm(a+1) \end{pmatrix}.$$

*Proof.* Considering Lemma 2.3.6(v) this follows easily from Proposition 2.5.1. □

**Corollary 2.5.3** *Assume that* $2a_{tss} = n_t - 1$. *Then there exists a positive integer* $v$ *such that* $n_t = 16v+5$, $n_s = (2v+1)(8v+3)(16v+5)$, $a_{tss} = 2(4v+1)$,

$$a_{sst} = 2(2v+1)(4v+1)(8v+3), \quad a_{sss} = 2(2v+1)(16v^2+3v-1),$$

*and*

$$a_{ssm} = (4v+1)(16v^2+19v+5), \quad a_{ssj} = 2(2v+1)(16v^2+11v+2).$$

*Proof.* From Lemma 2.3.6(iii) we know that $a_{tss}+2$ is a multiple of 4. Thus, there exists a positive odd integer $u$ such that

$$a_{tss} = 2u.$$

From Corollary 2.5.2 we now obtain (after a longer computation)

$$a_{sss} = \frac{1}{4}(u+1)(4u^2-5u-3), \quad a_{ssm} = \frac{1}{4}u(4u^2+11u+5),$$

and

$$a_{ssj} = \frac{1}{4}(u+1)(4u^2+3u+1).$$

Thus, as $a_{ssm}$ is integral, $11u+5$ must be a multiple of 4. Thus, $u \equiv 1$ (4). Thus, there exists a positive integer $v$ such that

$$u = 4v+1.$$

Thus, as $a_{tss} = 2u$,

$$a_{tss} = 2(4v+1).$$

From $n_t = 2a_{tss}+1$ and $a_{tss} = 2(4v+1)$ we obtain $n_t = 16v+5$.

The value for $n_s$ follows from Lemma 2.3.6(iv) together with $a_{tss} = 2(4v+1)$.

The value for $a_{sst}$ follows from the values of $n_t$ and $n_s$ together with $a_{tss} = 2(4v+1)$; cf. [11; Lemma 1.1.3(ii)].

The values for $a_{sss}$, $a_{ssm}$, and $a_{ssj}$ follow from $u = 4v+1$ together with the above-mentioned values for these three structure constants. □

**Corollary 2.5.4** *Assume that $2a_{tss} = n_t - 1$. Then $n_s$ is odd.*

*Proof.* This follows from Corollary 2.5.3. □

We now list the parameters of the first three cases for $2a_{tss} = n_t - 1$.

| $n_S$ | $n_t$ | $n_s$ | $a_{tss}$ | $a_{sst}$ | $a_{sss}$ | $a_{ssm}$ | $a_{ssj}$ | $v$ | $k$ | $\lambda$ | $r$ | $b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 66 | 5 | 15 | 2 | 6 | $-2$ | 5 | 4 | 6 | 3 | 2 | 5 | 10 |
| 2794 | 21 | 693 | 10 | 330 | 108 | 200 | 174 | 22 | 11 | 30 | 63 | 126 |
| 182001 | 37 | 3515 | 18 | 1710 | 690 | 963 | 880 | 38 | 19 | 90 | 185 | 370 |

(In this table, as an exception, the letter $\lambda$ does, of course, not stand for the linear character of $S$. It has the meaning which it usually has in the theory of block designs.)

In the first case (where $v = 0$ in the notation of Corollary 2.5.3), we obtain $a_{sss} = -2$. Thus, there is no scheme with the parameter set of the first row.

Let us compute the character table of $S$ in the second case. From Corollary 2.5.2 we obtain

$$\Phi(\sigma_s) = \begin{pmatrix} -\frac{11}{2} & \frac{11}{2}\sqrt{127} \\ \frac{11}{2}\sqrt{127} & \frac{121}{2} \end{pmatrix} \quad \text{or} \quad \Phi(\sigma_s) = \begin{pmatrix} -\frac{11}{2} & \frac{11}{2}\sqrt{127} \\ \frac{11}{2}\sqrt{127} & -\frac{121}{2} \end{pmatrix}.$$

Thus, the characteristic polynomial of $\Phi(\sigma_s)$ is

$$x^2 - 55x - \frac{8349}{2} \quad \text{or} \quad x^2 + 66x - 3509.$$

It follows that (in the notation of Proposition 2.5.1) $s_{22} = -\frac{121}{2}$. Thus,

| | 1 | $t$ | $s$ | $m$ | $m^*$ | $j$ | multiplicity |
|---|---|---|---|---|---|---|---|
| $1_S$ | 1 | 21 | 693 | 693 | 693 | 693 | 1 |
| $\lambda$ | 1 | $-1$ | 3 | $-3$ | $-3$ | 3 | 2541 |
| $\chi$ | 2 | 20 | $-66$ | 55 | 55 | $-66$ | 126 |

is the character table of $S$ in this case. (From Lemma 2.2.5 we obtain $\lambda(\sigma_s)^2 = 9$.) However, we do not know if a scheme with the parameters in the second row exists at all.

We conclude this section by showing that up to $n_t = 7$ there is only one parameter set which is still open.

**Theorem 2.5.5** *Assume that $n_t \leq 7$, that $S$ is not a semidirect product with complement $T$ and not a Coxeter scheme. Then $n_S = 301$, $n_t = 6$, $a_{tss} = 2$, $n_s = 54$, $n_m = 72$, and $n_j = 96$.*

*Proof.* From Lemma 2.1.3(vi) we know that $a_{tss} + a_{tjj} = n_t - 1$. Without loss of generality, we may assume that $a_{tss} \leq a_{tjj}$. From Theorem 2.4.3 and Theorem 2.4.4 we also obtain that $2 \leq a_{tss}$. It follows that

$$2 \leq a_{tss} \leq \frac{1}{2}(n_t - 1).$$

In particular, $5 \leq n_t$.

Assume that $n_t = 5$. Then, by the above inequality, $a_{tss} = 2$. Thus, by Corollary 2.5.3, $a_{sss} = -2$, contradiction.

Assume that $n_t = 6$. Then, by the above inequality, $a_{tss} = 2$. Thus, by Lemma 2.1.4, $n_s$ is a multiple of 9. Thus, there exists a positive integer $d$ such that $n_s = 9d$. Thus, by Lemma 2.3.1, $35\cdot 9d + 9\cdot 6$ divides $6\cdot 16\cdot 81$. Thus, $35d + 6$ divides $32\cdot 27$. Thus, $d = 6$. Thus, $n_s = 54$. Thus, by Lemma 2.1.3(vii), $n_m = 72$ and, by Lemma 2.1.3(viii), $n_j = 96$.

Assume that $n_t = 7$. Then, by Corollary 2.5.3, $a_{tss} \neq 3$. Thus, by the above inequality, $a_{tss} = 2$. Thus, by Lemma 2.1.4, 9 divides $n_s$. Thus, there exists a positive integer $e$ such that $n_s = 9e$. Thus, by Lemma 2.3.1, $48\cdot 9e + 9\cdot 7$ divides $7\cdot 25\cdot 81$. Thus, $48e + 7$ divides $7\cdot 25$, contradiction. □

Let us compute the character table of $S$ in the exceptional case arising in Theorem 2.5.5. From Proposition 2.5.1 we obtain

$$\Phi(\sigma_s) = \begin{pmatrix} -\frac{9}{7} & \frac{9\sqrt{43}}{7} \\ \frac{12\sqrt{43}}{7} & \frac{72}{7} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\frac{9}{7} & \frac{9\sqrt{43}}{7} \\ \frac{12\sqrt{43}}{7} & -\frac{72}{7} \end{pmatrix}.$$

Thus, the characteristic polynomial of $\Phi(\sigma_s)$ is

$$x^2 - 9x - 108 \quad \text{or} \quad x^2 + \frac{81}{7}x - \frac{3996}{49}.$$

It follows that (in the notation of Proposition 2.5.1) $s_{22} = \frac{72}{7}$. Thus, one obtains that

| | $1$ | $t$ | $s$ | $m$ | $m^*$ | $j$ | multiplicity |
|---|---|---|---|---|---|---|---|
| $1_S$ | 1 | 6 | 54 | 72 | 72 | 96 | 1 |
| $\lambda$ | 1 | $-1$ | $-2$ | 2 | 2 | $-2$ | 216 |
| $\chi$ | 2 | 5 | 9 | $-12$ | $-12$ | 8 | 42 |

is the character table of $S$.

We conclude this section with a list of triples $(n_t, n_s, a_{tss})$ of positive integers which satisfy all of our conditions. We also include the parameters of the 2-design which we introduced right after Corollary 1.3.6. (Again, the meaning of $\lambda$ in this list does not stand for the linear character of $S$.)

| $n_S$ | $n_t$ | $n_s$ | $n_m$ | $n_j$ | $a_{tss}$ | $v$ | $k$ | $\lambda$ | $r$ | $b$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 301 | 6 | 54 | 72 | 96 | 2 | 7 | 3 | 6 | 18 | 42 |
| 1221 | 10 | 250 | 300 | 360 | 4 | 11 | 5 | 20 | 50 | 110 |
| 2041 | 12 | 192 | 432 | 972 | 3 | 13 | 4 | 12 | 48 | 156 |
| 3165 | 14 | 686 | 784 | 896 | 6 | 15 | 7 | 42 | 98 | 210 |
| 6517 | 18 | 1458 | 1620 | 1800 | 8 | 19 | 9 | 72 | 162 | 342 |
| 8841 | 20 | 500 | 1600 | 5120 | 4 | 21 | 5 | 20 | 100 | 420 |
| 10186 | 21 | 1029 | 2205 | 4725 | 6 | 22 | 7 | 42 | 147 | 462 |
| 28861 | 30 | 1080 | 4500 | 18750 | 5 | 31 | 6 | 30 | 180 | 930 |
| 77701 | 42 | 2058 | 10584 | 54432 | 6 | 43 | 7 | 42 | 294 | 1806 |
| 182001 | 56 | 3584 | 21952 | 134456 | 7 | 57 | 8 | 56 | 448 | 3192 |

For each of the parameter sets arising in the above table, there exists a design. Also the corresponding integral table algebras exist.

## 3. Normal closed subsets

In this chapter, we investigate non-commutative association schemes of order 6 which have a normal closed subset different from $\{1\}$ and from $S$. In order to do so we fix such a scheme and call it $S$.

### 3.1 General remarks

In this section, the letter $U$ stands for a normal closed subset of $S$ different from $\{1\}$ and from $S$.

**Lemma 3.1.1** *We have* $|S//U| = 2$.

*Proof.* Since $U \neq \{1\}$, $|S//U| \leq 5$. Thus, by [6; (4.1)], $S//U$ is commutative. Thus, all irreducible characters of $S//U$ are linear.

On the other hand, by [3; Theorem 3.4], $\mathrm{Irr}(S//U) \subseteq \mathrm{Irr}(S)$. Thus, as $S$ has only two linear characters, $|S//U| \leq 2$. Thus, as $U \neq S$, $|S//U| = 2$. □

We define

$$l := n_{S//U}.$$

The following lemma will be needed in Corollary 3.3.4 and in Corollary 3.4.6.

**Lemma 3.1.2** *Let $s$ be an element in $S \setminus U$. Then the following hold.*

(i) *We have* $n_{s^U} = l - 1$.

(ii) *The positive integer $l - 1$ divides $n_s$.*

*Proof.* (i) Since $s \in S \setminus U$, $S = U \cup UsU$; cf. Lemma 3.1.1. Thus, by [11; Lemma 2.1.3], $n_{UsU} = n_U(l - 1)$. Thus, by [11; Theorem 4.1.3(iii)], $n_{s^U} = l - 1$.

(ii) Since $U$ is normal in $S$, $n_{s^U}$ divides $n_s$; cf. [11; Lemma 4.3.1(ii)]. Thus, the claim follows from (i). □

Since $S$ is assumed to be a non-commutative association scheme of order 6, $S$ has exactly three irreducible characters, two linear characters (one of them being the principal character of $S$) and one irreducible character of degree 2; cf. [11; Corollary 8.6.5]. As before the principal character of $S$ will be denoted by $1_S$, the non-principal linear character by $\lambda$, and the character of degree 2 by $\chi$. Thus, as before, $\mathrm{Irr}(S) = \{1_S, \lambda, \chi\}$.

**Lemma 3.1.3** *We have* $1_U^S = 1_S + \lambda$.

*Proof.* From Corollary 1.2.3(i) we know that

$$(1_S, 1_U^S)_S = 1.$$

Since $U \neq \{1\}$, $|S//U| \leq 5$. Thus, by [6; (4.1)], $S//U$ is commutative. Thus, as $U \neq S$, $S//U$ has a non-principal linear character. By [3; Theorem 3.4], this irreducible character must be $\lambda$ and we must have $U \subseteq \ker(\lambda)$. Thus, $(1_U, \lambda|_U)_U = 1$. Thus, by Lemma 1.2.1,

$$(\lambda, 1_U^S)_S = 1.$$

Since $\chi(1) = 2$, $(1_U, \chi|_U)_U \leq 2$.

If $(1_U, \chi|_U)_U = 2$, $(\chi, 1_U^S)_S = 2$; cf. Lemma 1.2.1. Thus, $1_U^S = 1_S + \lambda + 2\chi$. Thus, $\mathrm{Irr}(U) = \{1_U\}$, contrary to $U \neq S$.

If $(1_U, \chi|_U)_U = 1$, $(\chi, 1_U^S)_S = 1$; cf. Lemma 1.2.1. Thus, $1_U^S = 1_S + \lambda + \chi$. Thus, $U$ possesses an irreducible character $\upsilon$ such that $\upsilon^S = \chi$, contrary to Corollary 1.2.5.

Thus, $(\chi, 1_U^S)_S = 0$, and that means that $1_U^S = 1_S + \lambda$. □

**Theorem 3.1.4** *The following holds.*

(i) *We have* $|U| = 2$ *or* $|U| = 3$.

(ii) *Assume that* $|U| = 2$. *Then* $U$ *possesses a linear character* $\upsilon$ *with* $\mathrm{Irr}(U) = \{1_U, \upsilon\}$ *and* $\upsilon^S = 2\chi$.

(iii) *Assume that* $|U| = 3$. *Then* $U$ *possesses linear characters* $\upsilon_1$ *and* $\upsilon_2$ *with* $\mathrm{Irr}(U) = \{1_U, \upsilon_1, \upsilon_2\}$, $\upsilon_1^S = \chi$, *and* $\upsilon_2^S = \chi$.

*Proof.* Considering Lemma 3.1.3 this follows from $\mathrm{Irr}(S) = \{1_S, \lambda, \chi\}$. □

**Lemma 3.1.5** *We have*

$$m_\lambda = l - 1 \quad \textit{and} \quad m_\chi = \frac{1}{2}(n_S - l).$$

*Proof.* From Lemma 3.1.3 we know that $1_U^S = 1_S + \lambda$, from [11; Lemma 9.1.8(ii)] that $m_{1_U} = 1$. Thus, by Lemma 1.2.2,

$$n_{S//U} = n_{S//U} m_{1_U} = m_{1_U^S} = m_{1_S} + m_\lambda = 1 + m_\lambda,$$

and this equation is equivalent to the first of the desired equations.

From $m_\lambda = l - 1$ we obtain

$$n_S = 1 + m_\lambda + 2m_\chi = 1 + l - 1 + 2m_\chi,$$

and this equation is equivalent to the second of the desired equations. □

**Lemma 3.1.6** *We have*

$$\lambda(\sigma_s) = -\frac{n_s}{l-1}$$

*for each element $s$ in $S \setminus U$.*

*Proof.* We have $\lambda(\sigma_{s^U}) = -1$. Thus, by [3; Theorem 3.4] and Lemma 3.1.2(i),

$$\lambda(\sigma_s) = \frac{n_s}{n_{s^U}} \lambda(\sigma_{s^U}) = -\frac{n_s}{l-1}.$$

This finishes the proof of the lemma. □

**Theorem 3.1.7** *Assume that $S$ possesses a non-normal closed subset $T$. Then $S$ is a semidirect product with kernel $U$ and complement $T$.*

*Proof.* From Theorem 2.1.1 we know that $|T| = 2$. Let $t$ be the element in $T \setminus \{1\}$. Then, by Lemma 2.2.4(i), $\lambda(\sigma_t) = -1$ and, by Lemma 3.1.6,

$$\lambda(\sigma_t) = -\frac{n_t}{n_{S//U} - 1}.$$

It follows that $n_t = n_{S//U} - 1$. Thus,

$$n_T = n_{S//U} \quad \text{and} \quad n_U = n_{S//T}.$$

From Lemma 2.2.2 we know that $m_\lambda = n_{S//T}(n_T - 2) + 1$, from Lemma 3.1.5 that $m_\lambda = n_{S//U} - 1$. Thus,

$$n_{S//T}(n_T - 2) = n_{S//U} - 2.$$

Thus, as $n_T = n_{S//U}$ and $n_U = n_{S//T}$,

$$n_U(n_T - 2) = n_T - 2.$$

Thus, as $n_U \neq 1$, $n_T = 2$. This means that $t$ is thin, so that the claim follows from Theorem 2.4.1. □

Note that $|U| = 3$ if $S$ is a semidirect product with kernel $U$.

For the remainder of this section, we assume that $|U| = 3$. Thus, $U \setminus \{1\}$ possesses elements $u_1$ and $u_2$ such that

$$U = \{1, u_1, u_2\}.$$

From $|U| = 3$ we obtain $|\mathrm{Irr}(U)| = 3$. The two elements in $\mathrm{Irr}(U) \setminus \{1_U\}$ will be denoted by $\upsilon_1$ and $\upsilon_2$.

**Lemma 3.1.8** *The following hold.*

(i) *We have $m_{\upsilon_1} = m_{\upsilon_2}$.*

(ii) *We have $n_{u_1} = n_{u_2}$.*

(iii) *The valencies $n_{u_1}$ and $n_{u_2}$ are even.*

*Proof.* (i) From Lemma 1.2.2 together with Theorem 3.1.4(iii) we obtain

$$n_{S//U} m_{\upsilon_1} = m_{\upsilon_1^S} = m_\chi = m_{\upsilon_2^S} = n_{S//U} m_{\upsilon_2}.$$

This proves (i).

(ii) Considering [2; Corollary 2] this follows from (i).

(iii) This follows from [11; Lemma 1.2.1]. □

We set

$$n := n_{u_1}.$$

Then, by Lemma 3.1.8(ii), $n = n_{u_2}$. Moreover, by Lemma 3.1.8(iii), $n$ is even.

Define

$$\omega := \sqrt{2n+1}, \quad \theta_1 := \frac{1}{2}(\omega - 1), \quad \text{and} \quad \theta_2 := -\frac{1}{2}(\omega + 1).$$

Then we can establish the character table of $U$.

**Lemma 3.1.9** *Let $\upsilon_1$, $\upsilon_2$ be the elements in* $\mathrm{Irr}(U) \setminus \{1_U\}$. *Then*

| | 1 | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $1_U$ | 1 | $n$ | $n$ | 1 |
| $\upsilon_1$ | 1 | $\theta_1$ | $\theta_2$ | $n$ |
| $\upsilon_2$ | 1 | $\theta_2$ | $\theta_1$ | $n$ |

*is the character table of $U$.*

*Proof.* This is obvious. □

## 3.2 Basic equations

In this short section, we establish several equations which we shall need in Section 3.3 and Section 3.4.

Let $\omega$ be a real number, and define

$$\theta_1 := \frac{1}{2}(\omega - 1) \quad \text{and} \quad \theta_2 := -\frac{1}{2}(\omega + 1).$$

Then we have $\theta_1 + \theta_2 = -1$ and $\theta_1 - \theta_2 = \omega$.

**Lemma 3.2.1** *The following hold.*

(i) *We have* $\theta_1^2 + \theta_2^2 = \frac{1}{2}(\omega^2 + 1)$.

(ii) *We have* $\theta_1^2 - \theta_2^2 = -\omega$.

*Proof.* (i) We have

$$\begin{aligned} \theta_1^2 + \theta_2^2 &= \frac{1}{4}(\omega - 1)^2 + \frac{1}{4}(\omega + 1)^2 \\ &= \frac{1}{4}(\omega^2 - 2\omega + 1 + \omega^2 + 2\omega + 1) = \frac{1}{2}(\omega^2 + 1). \end{aligned}$$

(ii) We have

$$\begin{aligned} \theta_1^2 - \theta_2^2 &= \frac{1}{4}(\omega - 1)^2 - \frac{1}{4}(\omega + 1)^2 \\ &= \frac{1}{4}(\omega^2 - 2\omega + 1 - \omega^2 - 2\omega - 1) = -\omega. \end{aligned}$$

This finishes the proof of the lemma. □

Let $\mu$ and $\sigma$ be real numbers, let $\delta$ and $\varepsilon$ be elements in $\{-1, 1\}$, and define

$$m_{12} := \frac{1}{2}\mu(\delta\sigma + \varepsilon\omega) \quad \text{and} \quad m_{21} := \frac{1}{2}\mu(\delta\sigma - \varepsilon\omega).$$

The following lemma will be needed in Lemma 3.4.5 and in Lemma 3.4.8.

**Lemma 3.2.2** *The following hold.*

(i) *We have*
$$(m_{12}+m_{21})(m_{12}\theta_1-m_{21}\theta_2)=\frac{1}{2}\mu^2(\sigma^2-\delta\varepsilon\sigma)\omega.$$

(ii) *We have*
$$(m_{12}+m_{21})(m_{21}\theta_1-m_{12}\theta_2)=\frac{1}{2}\mu^2(\sigma^2+\delta\varepsilon\sigma)\omega.$$

(iii) *We have*
$$m_{21}^2\theta_2-m_{12}^2\theta_1=\frac{1}{4}\mu^2(-\sigma^2+2\delta\varepsilon\sigma-\omega^2)\omega.$$

(iv) *We have*
$$m_{12}^2\theta_2-m_{21}^2\theta_1=-\frac{1}{4}\mu^2(\sigma^2+2\delta\varepsilon\sigma+\omega^2)\omega.$$

*Proof.* (i) We have
$$m_{12}\theta_1=\frac{1}{4}\mu(\delta\sigma+\varepsilon\omega)(\omega-1)=\frac{1}{4}\mu(\delta\sigma\omega-\delta\sigma+\varepsilon\omega^2-\varepsilon\omega)$$
and
$$m_{21}\theta_2=-\frac{1}{4}\mu(\delta\sigma-\varepsilon\omega)(\omega+1)=\frac{1}{4}\mu(-\delta\sigma\omega-\delta\sigma+\varepsilon\omega^2+\varepsilon\omega).$$
Thus,
$$m_{12}\theta_1-m_{21}\theta_2=\frac{1}{2}\mu(\delta\sigma-\varepsilon)\omega.$$
Thus, the claim follows from $m_{12}+m_{21}=\mu\delta\sigma$.
(ii) We have
$$m_{21}\theta_1=\frac{1}{4}\mu(\delta\sigma-\varepsilon\omega)(\omega-1)=\frac{1}{4}\mu(\delta\sigma\omega-\delta\sigma-\varepsilon\omega^2+\varepsilon\omega)$$
and
$$m_{12}\theta_2=-\frac{1}{4}\mu(\delta\sigma+\varepsilon\omega)(\omega+1)=\frac{1}{4}\mu(-\delta\sigma\omega-\delta\sigma-\varepsilon\omega^2-\varepsilon\omega).$$
Thus,
$$m_{21}\theta_1-m_{12}\theta_2=\frac{1}{2}\mu(\delta\sigma+\varepsilon)\omega.$$
Again, the claim follows from $m_{12}+m_{21}=\mu\delta\sigma$.
(iii) We have
$$m_{21}^2\theta_2=-\frac{1}{8}\mu^2(\sigma^2-2\delta\varepsilon\sigma\omega+\omega^2)(\omega+1)=\frac{1}{8}\mu^2(-\sigma^2\omega-\sigma^2+2\delta\varepsilon\sigma\omega^2+2\delta\varepsilon\sigma\omega-\omega^3-\omega^2)$$
and
$$m_{12}^2\theta_1=\frac{1}{8}\mu^2(\sigma^2+2\delta\varepsilon\sigma\omega+\omega^2)(\omega-1)=\frac{1}{8}\mu^2(\sigma^2\omega-\sigma^2+2\delta\varepsilon\sigma\omega^2-2\delta\varepsilon\sigma\omega+\omega^3-\omega^2).$$
Thus,
$$m_{21}^2\theta_2-m_{12}^2\theta_1=\frac{1}{4}\mu^2(-\sigma^2+2\delta\varepsilon\sigma-\omega^2)\omega,$$
and this finishes the proof.
(iv) We have
$$m_{12}^2\theta_2=-\frac{1}{8}\mu^2(\sigma^2+2\delta\varepsilon\sigma\omega+\omega^2)(\omega+1)=\frac{1}{8}\mu^2(-\sigma^2\omega-\sigma^2-2\delta\varepsilon\sigma\omega^2-2\delta\varepsilon\sigma\omega-\omega^3-\omega^2)$$

and

$$m_{21}^2\theta_1 = \frac{1}{8}\mu^2(\sigma^2-2\delta\varepsilon\sigma\omega+\omega^2)(\omega-1) = \frac{1}{8}\mu^2(\sigma^2\omega-\sigma^2-2\delta\varepsilon\sigma\omega^2+2\delta\varepsilon\sigma\omega+\omega^3-\omega^2).$$

Thus,

$$m_{12}^2\theta_2 - m_{21}^2\theta_1 = -\frac{1}{4}\mu^2(\sigma^2+2\delta\varepsilon\sigma+\omega^2)\omega,$$

and this finishes the proof. □

### 3.3 Symmetric normal closed subsets with three elements I

In this section, the letters $U$, $l$, $\lambda$, $\chi$, $\omega$, $\theta_1$, and $\theta_2$ have the same meaning as in Section 3.1. Additionally, we assume $U$ to be symmetric and to have three elements.

Since $|U| = 3$, $U$ possesses non-principal linear characters $\upsilon_1$ and $\upsilon_2$ such that

$$\mathrm{Irr}(U) = \{1_U, \upsilon_1, \upsilon_2\}, \quad \upsilon_1^S = \chi, \quad \text{and} \quad \upsilon_2^S = \chi;$$

cf. Theorem 3.1.4(iii). The two elements in $U \setminus \{1\}$ will be denoted by $u_1$ and $u_2$, so that

$$U = \{1, u_1, u_2\}.$$

Recall that $u_1$ and $u_2$ have the same valency. As before, we denote this valency by $n$.

We assume that $S$ is not commutative and that $U$ is symmetric and contains three elements. Thus, by [11; Lemma 1.2.5], $S$ possesses elements $s$ and $m$ such that

$$S \setminus U = \{s, m, m^*\}.$$

Recall that

$$l := n_{S//U}.$$

Now we are ready to give the character table of $S$.

**Lemma 3.3.1** *The table*

| | $1$ | $u_1$ | $u_2$ | $s$ | $m$ | $m^*$ | |
|---|---|---|---|---|---|---|---|
| $1_S$ | $1$ | $n$ | $n$ | $n_s$ | $n_m$ | $n_m$ | $1$ |
| $\lambda$ | $1$ | $n$ | $n$ | $-\frac{n_s}{l-1}$ | $-\frac{n_m}{l-1}$ | $-\frac{n_m}{l-1}$ | $l-1$ |
| $\chi$ | $2$ | $-1$ | $-1$ | $0$ | $0$ | $0$ | $nl$ |

*is the character table of $S$.*

*Proof.* The values for $\lambda$ follow from Lemma 3.1.6. The multiplicity of $\chi$ is obtained from the second equation of Lemma 3.1.5. □

Recall that $\omega := \sqrt{2n+1}$, and set

$$\mu := \sqrt{\frac{n_m}{n}}, \quad \sigma := \sqrt{\frac{n_s}{l-1}}.$$

Then we have the following.

**Lemma 3.3.2** *Fix elements $\delta$ and $\varepsilon$ in $\{-1, 1\}$, and define*

$$m_{12} := \frac{1}{2}\mu(\delta\sigma + \varepsilon\omega) \quad \text{and} \quad m_{21} := \frac{1}{2}\mu(\delta\sigma - \varepsilon\omega).$$

*Then*

$$(m_{12}+m_{21})^2 = \frac{n_s n_m}{n(l-1)} \quad \textit{and} \quad m_{12}m_{21} = -\frac{n_m^2}{2n(l-1)}.$$

*Proof.* We have

$$(m_{12}+m_{21})^2 = (\mu\sigma)^2 = \frac{n_s n_m}{n(l-1)}.$$

Since $(2n+1)(l-1) = n_s + 2n_m$, we have

$$\begin{aligned} m_{12}m_{21} &= \frac{n_m}{4n}(\delta\sigma+\varepsilon\omega)(\delta\sigma-\varepsilon\omega) \\ &= \frac{n_m}{4n}(\sigma^2-\omega^2) \\ &= \frac{n_m}{4n}[\frac{n_s}{l-1}-(2n+1)] \\ &= \frac{n_m}{4n}(\frac{n_s}{l-1}-\frac{n_s+2n_m}{l-1}) = -\frac{n_m^2}{2n(l-1)}. \end{aligned}$$

This finishes the proof. □

Recall that

$$\theta_1 := \frac{1}{2}(\omega-1) \quad \text{and} \quad \theta_2 := -\frac{1}{2}(\omega+1).$$

In the following, we denote by $\Phi$ the representation of $S$ affording $\chi$. Thus, $\Phi$ is a surjective ring homomorphism from $\mathbb{C}S$ to $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.

**Lemma 3.3.3** *There exist elements $\delta$ and $\varepsilon$ in $\{-1,1\}$ such that, for*

$$m_{12} := \frac{1}{2}\mu(\delta\sigma+\varepsilon\omega) \quad \textit{and} \quad m_{21} := \frac{1}{2}\mu(\delta\sigma-\varepsilon\omega),$$

*the map $\Phi$ admits a basis with respect to which*

$$\Phi(\sigma_{u_1}) = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \quad \textit{and} \quad \Phi(\sigma_{u_2}) = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_1 \end{pmatrix},$$

$$\Phi(\sigma_s) = \begin{pmatrix} 0 & -(m_{12}+m_{21}) \\ -(m_{12}+m_{21}) & 0 \end{pmatrix},$$

$$\Phi(\sigma_m) = \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}, \quad \Phi(\sigma_{m^*}) = \begin{pmatrix} 0 & m_{21} \\ m_{12} & 0 \end{pmatrix}.$$

*Proof.* From Theorem 3.1.4(iii) we obtain that $\chi|_U = \upsilon_1 + \upsilon_2$. Thus, we may assume that

$$\Phi(\sigma_{u_1}) = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \quad \text{and} \quad \Phi(\sigma_{u_2}) = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_1 \end{pmatrix}.$$

We also may assume that there exists real numbers $m_{11}$, $m_{12}$, $m_{21}$, and $m_{22}$ such that

$$\Phi(\sigma_m) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \text{and} \quad \Phi(\sigma_{m^*}) = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}.$$

Then, as $\theta_1 + \theta_2 = -1$,

$$\Phi(\sigma_s) = \begin{pmatrix} -2m_{11} & -(m_{12}+m_{21}) \\ -(m_{12}+m_{21}) & -2m_{22} \end{pmatrix}.$$

Recall from Lemma 3.3.1 that $m_\chi = nl$. Thus,

$$\frac{n_S}{m_\chi} = \frac{(2n+1)l}{m_\chi} = \frac{2n+1}{n}.$$

Applying the Schur relations [11; Theorem 9.2.4(ii)] for association schemes to the $(1,1)$-entries of the 2-dimensional representation of $S$ we now obtain

$$\frac{2n+1}{n} = 1 + \frac{\theta_1^2}{n} + \frac{\theta_2^2}{n} + \frac{4m_{11}^2}{n_s} + \frac{m_{11}^2}{n_m} + \frac{m_{11}^2}{n_m} = \frac{2n+1}{n} + m_{11}^2(\frac{4}{n_s} + \frac{2}{n_m});$$

cf. Lemma 3.2.1(i). It follows that $m_{11} = 0$. Thus, by Lemma 3.3.1, $m_{22} = 0$.

Now we apply [11; Theorem 9.2.4(ii)] to the $(1,2)$-entries of the 2-dimensional representation of $S$. Then we obtain

$$\frac{2n+1}{n} = \frac{1}{n_s}(m_{12} + m_{21})^2 + \frac{1}{n_m}(m_{12}^2 + m_{21}^2).$$

Applying [11; Theorem 9.2.4(ii)] to the $(1,2)$-entries and to the $(2,1)$-entries of the 2-dimensional representation of $S$ we obtain

$$\frac{1}{n_s}(m_{12} + m_{21})^2 + \frac{2}{n_m} m_{12} m_{21} = 0.$$

Adding these two equations one obtains

$$(m_{12} + m_{21})^2 = \frac{(2n+1)n_s n_m}{n(n_s + 2n_m)} = \frac{n_s n_m}{n(l-1)}.$$

Now, by the last two equations,

$$m_{12} m_{21} = -\frac{n_m^2}{2n(l-1)}.$$

Thus, the claim follows from Lemma 3.3.2. □

From Lemma 3.1.2(ii) we know that $l-1$ divides $n_s$ and $n_m$. Here is more.

**Corollary 3.3.4** *The integer $2n(l-1)$ divides $n_m^2$ and $n(l-1)$ divides $n_s n_m$.*

*Proof.* From Lemma 3.3.2 and Lemma 3.3.3 one obtains that the determinant of $\Phi(\sigma_m)$ is

$$-m_{12} m_{21} = \frac{n_m^2}{2n(l-1)}$$

and that the determinant of $\Phi(\sigma_s)$ is

$$-(m_{12} + m_{21})^2 = -\frac{n_s n_m}{n(l-1)}.$$

Since both of these numbers are algebraic integers. This proves the lemma. □

The following lemma is the key to the computation of the structure constants of $S$.

**Lemma 3.3.5** *The kernel of $\Phi$ is spanned by $\sigma_1 + \sigma_{u_1} + \sigma_{u_2}$ and $\sigma_s + \sigma_m + \sigma_{m^*}$.*

*Proof.* Recall that $\theta_1 + \theta_2 = -1$, we obtain from Lemma 3.3.3 that $\sigma_1 + \sigma_{u_1} + \sigma_{u_2}$ is in the kernel of $\Phi$. From Lemma 3.3.3 we also obtain that $\sigma_s + \sigma_m + \sigma_{m^*}$ is in the kernel of $\Phi$.

On the other hand, the domain of the surjective ring homomorphism $\Phi$ has dimension 6 and the codomain of $\Phi$ has dimension 4. Thus, the kernel of $\Phi$ has dimension 2. □

## 3.4 Symmetric normal closed subsets with three elements II

In this section, the letters $U$, $l$, $u_1$, $u_2$, $s$, $m$, $n$, $\omega$, $\theta_1$, $\theta_2$, and $\Phi$ have the same meaning as in the previous section. It is the purpose of this section to compute, for each element $r$ in $S$, the values

$$a_{ssr}, \quad a_{mmr}, \quad a_{m^*m^*r}, \quad a_{smr}, \quad a_{m^*sr}, \quad a_{mm^*r}$$

in terms of the valencies of $S$. Our main tool is Lemma 3.3.5.

**Lemma 3.4.1** *We have*

$$a_{ssu_1} = a_{ssu_2} = n_s - \frac{n_s n_m}{n(l-1)}$$

and

$$a_{sss} = a_{ssm} = a_{ssm^*} = \frac{n_s^2(l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* From Lemma 3.3.3 we obtain

$$\Phi(\sigma_s^2) = (m_{12} + m_{21})^2 \Phi(\sigma_1).$$

Thus, $\sigma_s^2 - (m_{12}+m_{21})^2\sigma_1$ is in the kernel of $\Phi$. Thus, by Lemma 3.3.5, there exist integers $\alpha$ and $\beta$ such that

$$\sigma_s^2 = (m_{12} + m_{21})^2\sigma_1 + \alpha(\sigma_1 + \sigma_{u_1} + \sigma_{u_2}) + \beta(\sigma_s + \sigma_m + \sigma_{m^*}).$$

Thus, as $a_{ss1} = n_s$, we obtain from the first equation of Lemma 3.3.2 that

$$\alpha = n_s - (m_{12} + m_{21})^2 = n_s - \frac{n_s n_m}{n(l-1)}.$$

Now we apply $1_S$ to $\sigma_s^2$ and obtain

$$n_s^2 = n_s + (n_s - \frac{n_s n_m}{n(l-1)})2n + \beta(l-1)(2n+1).$$

It follows that

$$\begin{aligned}\beta &= \frac{n_s[n_s(l-1) - (2n+1)(l-1) + 2n_m]}{(l-1)^2(2n+1)} \\ &= \frac{n_s[n_s(l-1) - (n_s + 2n_m) + 2n_m]}{(l-1)^2(2n+1)} \\ &= \frac{n_s^2(l-2)}{(l-1)^2(2n+1)}.\end{aligned}$$

This finishes the proof of the lemma. □

The last equation of Lemma 3.4.1 implies that $2n+1$ divides $l-2$ if $n_s = l-1$.

**Lemma 3.4.2** *We have*

$$a_{mmu_1} = a_{mmu_2} = \frac{n_m^2}{2n(l-1)}$$

and

$$a_{mms} = a_{mmm} = a_{mmm^*} = \frac{n_m^2(l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* From Lemma 3.3.3 we obtain

$$\Phi(\sigma_m^2) = m_{12}m_{21}\Phi(\sigma_1).$$

Thus, $\sigma_m^2 - m_{12}m_{21}\sigma_1$ is in the kernel of $\Phi$. Thus, by Lemma 3.3.5, there exist integers $\alpha$ and $\beta$ such that

$$\sigma_m^2 = m_{12}m_{21}\sigma_1 + \alpha(\sigma_1 + \sigma_{u_1} + \sigma_{u_2}) + \beta(\sigma_s + \sigma_m + \sigma_{m^*}).$$

Thus, as $a_{mm1} = 0$, $\alpha = -m_{12}m_{21}$. Thus, by the first equation of Lemma 3.3.2,

$$\alpha = \frac{n_m^2}{2n(l-1)}.$$

By applying $1_S$ to $\sigma_m^2$, we now obtain

$$n_m^2 = \frac{n_m^2}{2n(l-1)}2n + \beta(l-1)(2n+1).$$

It follows that

$$\beta = \frac{n_m^2(l-2)}{(l-1)^2(2n+1)},$$

and this finishes the proof. □

**Corollary 3.4.3** *We have*

$$a_{m^*m^*u_1} = a_{m^*m^*u_2} = \frac{n_m^2}{2n(l-1)}$$

and

$$a_{m^*m^*s} = a_{m^*m^*m} = a_{m^*m^*m^*} = \frac{n_m^2(l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* Considering [11; Lemma 1.1.1(ii)] this follows from Lemma 3.4.2. □

The following lemma will be convenient in the proofs of Lemma 3.4.5 and Lemma 3.4.8. Recall that $\omega$ is our abbreviation of $\sqrt{2n+1}$.

**Lemma 3.4.4** *We have*

$$\Phi(\frac{\theta_2\sigma_{u_2} - \theta_1\sigma_{u_1}}{\omega}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \textit{and} \quad \Phi(\frac{\theta_2\sigma_{u_1} - \theta_1\sigma_{u_2}}{\omega}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

*Proof.* From Lemma 3.2.1(ii) we know that $\theta_2^2 - \theta_1^2 = \omega$. Thus, by Lemma 3.3.3,

$$\Phi(\theta_2\sigma_{u_2} - \theta_1\sigma_{u_1}) = \theta_2\begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_1 \end{pmatrix} - \theta_1\begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}.$$

The second equation is proved similarly. □

**Lemma 3.4.5** *We have*

$$a_{smu_1} = \frac{n_m}{2n}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}}), \quad a_{smu_2} = \frac{n_m}{2n}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

*and*

$$a_{sms} = a_{smm} = a_{smm^*} = \frac{n_s n_m(l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* From Lemma 3.3.3 together with Lemma 3.4.4 we obtain

$$\begin{aligned}\Phi(\sigma_s\sigma_m) &= \begin{pmatrix} -m_{21}(m_{12}+m_{21}) & 0 \\ 0 & -m_{12}(m_{12}+m_{21}) \end{pmatrix} \\ &= \Phi(\frac{-m_{21}(m_{12}+m_{21})(\theta_2\sigma_{u_2}-\theta_1\sigma_{u_1}) - m_{12}(m_{12}+m_{21})(\theta_2\sigma_{u_1}-\theta_1\sigma_{u_2})}{\omega}).\end{aligned}$$

Thus,

$$\sigma_s\sigma_m - \frac{m_{21}(m_{12}+m_{21})(\theta_1\sigma_{u_1}-\theta_2\sigma_{u_2}) + m_{12}(m_{12}+m_{21})(\theta_1\sigma_{u_2}-\theta_2\sigma_{u_1})}{\omega}$$

is in the kernel of $\Phi$. Thus, by Lemma 3.3.5, there exist integers $\alpha$ and $\beta$ such that

$$\begin{aligned}\sigma_s\sigma_m &= \frac{m_{21}(m_{12}+m_{21})(\theta_1\sigma_{u_1}-\theta_2\sigma_{u_2}) + m_{12}(m_{12}+m_{21})(\theta_1\sigma_{u_2}-\theta_2\sigma_{u_1})}{\omega} \\ &\quad +\alpha(\sigma_1+\sigma_{u_1}+\sigma_{u_2}) + \beta(\sigma_s+\sigma_m+\sigma_{m^*}).\end{aligned}$$

Thus, as $a_{sm1} = 0$, $\alpha = 0$. Thus, by Lemma 3.2.2(ii),

$$a_{smu_1} = \frac{(m_{12}+m_{21})(m_{21}\theta_1 - m_{12}\theta_2)}{\omega} = \frac{n_m}{2n}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}})$$

and, by Lemma 3.2.2(i),

$$a_{smu_2} = \frac{(m_{12}+m_{21})(m_{12}\theta_1 - m_{21}\theta_2)}{\omega} = \frac{n_m}{2n}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}}).$$

By applying $1_S$ to $\sigma_s\sigma_m$ we obtain

$$n_s n_m = \frac{n_s n_m}{l-1} + \beta(l-1)(2n+1).$$

This finishes the proof of the lemma. □

Recall that

$$\frac{n_s}{l-1}$$

is an integer; cf. Lemma 3.1.2(ii). Here is more.

**Corollary 3.4.6** *The integer*

$$\frac{n_s}{l-1}$$

*is an odd square.*

*Proof.* This follows from the first equation of Lemma 3.4.5 together with the fact that $n_s + 2n_m = (2n+1)(l-1)$. □

**Corollary 3.4.7** *We have*

$$a_{m^*su_1} = \frac{n_m}{2n}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}}), \quad a_{m^*su_2} = \frac{n_m}{2n}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

*and*

$$a_{m^*ss} = a_{m^*sm} = a_{m^*sm^*} = \frac{n_s n_m(l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* Considering [11; Lemma 1.1.1(ii)] this follows from Lemma 3.4.5. □

**Lemma 3.4.8** *We have*

$$a_{mm^*u_1} = n_m - \frac{n_m}{4n}(\frac{n_s}{l-1} + 2n + 1 - 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

$$a_{mm^*u_2} = n_m - \frac{n_m}{4n}(\frac{n_s}{l-1} + 2n + 1 + 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

*and*

$$a_{mm^*s} = a_{mm^*m} = a_{mm^*m^*} = \frac{n_m^2(l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* From Lemma 3.3.3 together with Lemma 3.4.4 we obtain

$$\Phi(\sigma_m\sigma_{m^*}) = \begin{pmatrix} m_{12}^2 & 0 \\ 0 & m_{21}^2 \end{pmatrix} = \Phi(\frac{m_{12}^2(\theta_2\sigma_{u_2} - \theta_1\sigma_{u_1}) + m_{21}^2(\theta_2\sigma_{u_1} - \theta_1\sigma_{u_2})}{\omega}).$$

Thus,

$$\sigma_m\sigma_{m^*} - \frac{m_{21}^2\theta_2 - m_{12}^2\theta_1}{\omega}\sigma_{u_1} - \frac{m_{12}^2\theta_2 - m_{21}^2\theta_1}{\omega}\sigma_{u_2}$$

is in the kernel of $\Phi$. Thus, by Lemma 3.3.5, there exist integers $\alpha$ and $\beta$ such that

$$\sigma_m\sigma_{m^*} - \frac{m_{21}^2\theta_2 - m_{12}^2\theta_1}{\omega}\sigma_{u_1} - \frac{m_{12}^2\theta_2 - m_{21}^2\theta_1}{\omega}\sigma_{u_2} =$$
$$\alpha(\sigma_1 + \sigma_{u_1} + \sigma_{u_2}) + \beta(\sigma_s + \sigma_m + \sigma_{m^*}).$$

Thus, as $a_{mm^*1} = n_m$, $\alpha = n_m$. Thus, by Lemma 3.2.2(iii),

$$a_{mm^*u_1} = n_m + \frac{m_{21}^2\theta_2 - m_{12}^2\theta_1}{\omega} = n_m + \frac{n_m}{4n}[-\frac{n_s}{l-1} + 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}} - (2n+1)]$$

and, by Lemma 3.2.2(iv),

$$a_{mm^*u_2} = n_m + \frac{m_{12}^2\theta_2 - m_{21}^2\theta_1}{\omega} = n_m - \frac{n_m}{4n}(\frac{n_s}{l-1} + 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}} + 2n + 1).$$

By applying $1_S$ to $\sigma_m^2$, we now obtain (in a longer computation) that

$$a_{mm^*m} = \frac{n_m^2(l-2)}{(l-1)^2(2n+1)},$$

so that the lemma is proved. □

## 3.5 Symmetric normal closed subsets with three elements III

In this section, the letters $U$, $l$, $u_1$, $u_2$, $s$, $m$, and $n$ have the same meaning as in the previous section. It is the purpose of this section to compute, for each element $r$ in $S$, the values

$$a_{sm^*r}, \quad a_{msr}, \quad a_{m^*mr}$$

in terms of the valencies of $S$. Our main tool is the following proposition.

**Proposition 3.5.1** *The ring $\mathbb{C}S$ possesses a ring automorphism $\alpha$ satisfying*

$$\alpha(\sigma_{u_1}) = \sigma_{u_2}, \quad \alpha(\sigma_{u_2}) = \sigma_{u_1}, \quad \alpha(\sigma_m) = \sigma_{m^*}, \quad \alpha(\sigma_{m^*}) = \sigma_m$$

*and fixing $\sigma_s$.*

*Proof.* Considering Lemma 3.3.3 we obtain that conjugation by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is a ring automorphism of $\mathbb{C}S$ satisfying the required equations. □

**Corollary 3.5.2** *We have*

$$a_{sm^*u_1} = \frac{n_m}{2n}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}}), \quad a_{sm^*u_2} = \frac{n_m}{2n}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

*and*

$$a_{sm^*s} = a_{sm^*m} = a_{sm^*m^*} = \frac{n_s n_m (l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* Considering Proposition 3.5.1 this follows from Lemma 3.4.5. □

**Corollary 3.5.3** *We have*

$$a_{msu_1} = \frac{n_m}{2n}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}}), \quad a_{msu_2} = \frac{n_m}{2n}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

*and*

$$a_{mss} = a_{msm} = a_{msm^*} = \frac{n_s n_m (l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* Considering Proposition 3.5.1 this follows from Corollary 3.4.7. □

Corollary 3.5.3 also follows from [11; Lemma 1.1.1(ii)] together with Corollary 3.5.2.

**Corollary 3.5.4** *We have*

$$a_{m^*mu_1} = n_m - \frac{n_m}{4n}(\frac{n_s}{l-1} + 2n + 1 + 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

$$a_{m^*mu_2} = n_m - \frac{n_m}{4n}(\frac{n_s}{l-1} + 2n + 1 - 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}}),$$

*and*

$$a_{m^*ms} = a_{m^*mm} = a_{m^*mm^*} = \frac{n_m^2 (l-2)}{(l-1)^2(2n+1)}.$$

*Proof.* Considering Proposition 3.5.1 this follows from Lemma 3.4.8. □

Note that

$$a_{pqr} = \frac{n_p n_q (l-2)}{(l-1)^2(2n+1)}$$

for any three elements $p$, $q$, and $r$ in $\{s, m, m^*\}$.

## 3.6 Symmetric normal closed subsets with three elements IV

In this section, the letters $U$, $l$, $u_1$, $u_2$, $s$, $m$, and $n$ have the same meaning as in the previous section. We shall compute the remaining set of structure constants of $S$.

**Lemma 3.6.1** *We have*

$$a_{u_1ss} = a_{su_1s} = a_{u_2ss} = a_{su_2s} = n - \frac{n_m}{l-1}.$$

*Proof.* This follows from the first two equations of Lemma 3.4.1 together with [11; Lemma 1.1.3(ii)] and [11; Lemma 1.1.1(ii)]. □

**Lemma 3.6.2** *We have*

$$a_{u_1sm} = a_{su_1m^*} = a_{u_2sm^*} = a_{su_2m} = \frac{1}{2}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}})$$

*and*

$$a_{u_1sm^*} = a_{su_1m} = a_{u_2sm} = a_{su_2m^*} = \frac{1}{2}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}}).$$

*Proof.* The first four equations follow from the first equation of Corollary 3.5.3 together with [11; Lemma 1.1.3(ii)], [11; Lemma 1.1.1(ii)], and Proposition 3.5.1. The second four equations follow similarly from the second equation of Corollary 3.5.3 together with [11; Lemma 1.1.3(ii)], [11; Lemma 1.1.1(ii)], and Proposition 3.5.1. □

**Lemma 3.6.3** *We have*

$$a_{u_1ms} = a_{m^*u_1s} = a_{u_2m^*s} = a_{mu_2s} = \frac{n_m}{2n_s}(\frac{n_s}{l-1} - \delta\varepsilon\sqrt{\frac{n_s}{l-1}})$$

*and*

$$a_{u_2ms} = a_{m^*u_2s} = a_{u_1m^*s} = a_{mu_1s} = \frac{n_m}{2n_s}(\frac{n_s}{l-1} + \delta\varepsilon\sqrt{\frac{n_s}{l-1}})$$

*Proof.* The first four equations follow from the first equation of Corollary 3.5.2 together with [11; Lemma 1.1.3(ii)], [11; Lemma 1.1.1(ii)], and Proposition 3.5.1. The second four equations follow from the second equation of Corollary 3.5.2 together with [11; Lemma 1.1.3(ii)], [11; Lemma 1.1.1(ii)], and Proposition 3.5.1. □

**Lemma 3.6.4** *We have*

$$a_{u_1mm} = a_{m^*u_1m^*} = a_{u_2m^*m^*} = a_{mu_2m} = n - \frac{1}{4}(\frac{n_s}{l-1} + 2n + 1 - 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}})$$

*and*

$$a_{u_2mm} = a_{m^*u_2m^*} = a_{u_1m^*m^*} = a_{mu_1m} = n - \frac{1}{4}(\frac{n_s}{l-1} + 2n + 1 + 2\delta\varepsilon\sqrt{\frac{n_s}{l-1}})$$

*Proof.* The first four equations follow from the first equation of Lemma 3.4.8 together with [11; Lemma 1.1.3(ii)], [11; Lemma 1.1.1(ii)], and Proposition 3.5.1. The second four equations follow from the second equation of Lemma 3.4.8 together with [11; Lemma 1.1.3(ii)], [11; Lemma 1.1.1(ii)], and Proposition 3.5.1. □

**Lemma 3.6.5** *We have*

$$a_{u_1m^*m} = a_{mu_1m^*} = a_{u_2m^*m} = a_{mu_2m^*} = \frac{n_m}{2(l-1)}$$

*and*

$$a_{u_1mm^*} = a_{m^*u_1m} = a_{u_2mm^*} = a_{m^*u_2m} = \frac{n_m}{2(l-1)}.$$

*Proof.* The first four equations follow from the first two equations of Lemma 3.4.2 together with [11; Lemma 1.1.3(ii)] and [11; Lemma 1.1.1(ii)]. The second four equations follow from the first four equations together with Proposition 3.5.1. □

**Lemma 3.6.6** *Let $p$, $q$, and $r$ be elements in $S$ such that exactly two of the elements $p$, $q$, and $r$ are in $U$. Then $a_{pqr} = 0$.*

*Proof.* This follows from the definition of a closed subset. □

**Lemma 3.6.7** *We have*

$$a_{u_1u_1u_1} = a_{u_2u_2u_2} = \frac{n}{2} - 1$$

*and*

$$a_{u_1u_1u_2} = a_{u_1u_2u_1} = a_{u_2u_1u_1} = a_{u_2u_2u_1} = a_{u_2u_1u_2} = a_{u_1u_2u_2} = \frac{n}{2}.$$

*Proof.* This is easy to compute. □

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