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SEMI-INVERTIBLE EXTENSIONS OF C^*-ALGEBRAS

VLADIMIR MANUILOV AND KLAUS THOMSEN

Abstract. We prolonge the list of C^*-algebras for which all extensions by any stable separable C^*-algebra are semi-invertible. In particular, we handle certain amalgamations, both of C^*-algebras and of groups. Concerning groups we consider both reduced and full group C^*-algebras.

1. Introduction and statements of results

The number of examples of C^*-algebras for which the semi-group of extensions by the compact operators is not a group was only slowly increasing during the first decades following the first example of J. Anderson,[A], but recently the pace has picked up, cf. [HT], [HS], [HLSW] and [Se], and there are now whole series of C^*-algebras A for which it is known that there are non-invertible extensions of A by the C^*-algebra of compact operators $\mathbb{K}$. Furthermore, by considering extensions by general stable C^*-algebras the stock of examples of non-invertible extensions grows considerably. Indeed, a non-invertible extension of a C^*-algebra A by $\mathbb{K}$ gives rise to a non-invertible extension of A by $B \otimes \mathbb{K}$ for any unital C^*-algebra B.[1]

In a different direction the authors have shown that many of the non-invertible extensions are invertible in a slightly weaker sense, called *semi-invertibility*. Recall that an extension of a C^*-algebra A by a stable C^*-algebra B is invertible when there is another extension, the inverse, with the property that the direct sum extension of the two is a split extension. Semi-invertibility requires only that the sum is *asymptotically split*, in the sense that there is an asymptotic homomorphism as defined by Connes and Higson, [CH], consisting of right-inverses of the quotient map. It turns out that extensions of a suspended or a contractible C^*-algebra are always semi-invertible, [MT3], [MT1], and in [ST] it was shown that the extensions of the reduced group C^*-algebra of a free product of amenable groups are all semi-invertible. The main purpose of the present paper is to prolonge this list of C^*-algebras for which all the extensions by a separable stable C^*-algebra are semi-invertible.

To explain why semi-invertibility is a natural notion which can be considered as the best alternative when invertibility fails, we recall first the central definitions. Let A and B be separable C^*-algebras. The multiplier algebra of B will be denoted by $M(B)$, the generalized Calkin algebra of B by $Q(B)$ and $q_B : M(B) \to Q(B)$ is then the canonical surjection. We let $\mathrm{Ext}(A, B)$ denote the semi-group of unitary equivalence classes of extensions of A by B. Thus elements of $\mathrm{Ext}(A, B)$ are represented by $*$-homomorphisms $\varphi : A \to Q(B)$ and two extensions $\varphi, \psi : A \to Q(B)$ are unitarily

[1] Tensor the non-invertible extension with B using the maximal tensor-product, and pull back along the unital inclusion $A \subseteq A \otimes_{max} B$. It is easy to see that the resulting extension of A by $B \otimes \mathbb{K}$ does not have a completely positive section for the quotient map because the original extension does not.

equivalent when there is a unitary $u \in M(B)$ such that $\operatorname{Ad} q_B(u) \circ \varphi = \psi$. The addition $\varphi \oplus \psi$ of two extensions is defined from a choice of isometries $V_1, V_2 \in M(B)$ such that $V_1^* V_2 = 0$ to be the extension

$$(\varphi \oplus \psi)(a) = q_B(V_1)\varphi(a)q_B(V_1)^* + q_B(V_2)\psi(a)q_B(V_2)^* .$$

An extension $\varphi : A \to Q(B)$ is *split* when there is a $*$-homomorphism $\pi : A \to M(B)$ such that $\varphi = q_B \circ \pi$ and *asymptotically split* when there is an asymptotic homomorphism $\pi_t : A \to M(B), t \in [1, \infty)$, such that $q_B \circ \pi_t = \varphi$ for all t. We say that $\operatorname{Ext}(A, B)$ is a group when every extension $\varphi : A \to Q(B)$ has an inverse, meaning that there is another extension $\varphi' : A \to Q(B)$, *the inverse of* φ, such that $\varphi \oplus \varphi'$ is split. An extension $\varphi : A \to Q(B)$ is *semi-invertible* when there is another extension $\varphi' : A \to Q(B)$ such that $\varphi \oplus \varphi'$ is asymptotically split.

When the theory of C^*-extensions was first introduced, in the work of Brown, Douglas and Fillmore, [BDF1], [BDF1], the authors had very good (operator theoretic) reasons for wanting to trivialize the split extensions. [2] However, there are other reasons why split extensions must be trivialized in order to get a group from the semi-group $\operatorname{Ext}(A, B)$. For a split extension x it makes sense to define the direct sum x^∞ of a countably infinite collection of copies of x. Since $x \oplus x^\infty \oplus 0 = x^\infty \oplus 0$ in $\operatorname{Ext}(A, B)$ this shows that split extensions are trivial in any group-quotient of $\operatorname{Ext}(A, B)$. It is not difficult to show that x^∞ can also be defined when the extension x is asymptotically split. In fact, this is possible as soon as the extension splits via a discrete asymptotic homomorphism, e.g when it is quasi-diagonal. But by using the real parameter for the asymptotic section it can also be arranged that $x \oplus x^\infty \oplus 0$ becomes unitarily equivalent to $x^\infty \oplus 0$. It follows that also asymptotically split extensions must vanish in a group-quotient of $\operatorname{Ext}(A, B)$. In fact, any group-quotient of $\operatorname{Ext}(A, B)$ must factor through the cancellation semi-group of $\operatorname{Ext}(A, B)$. In retrospect it seems therefore not particularly surprising that it is not generally enough to trivialize only the split extensions to get a group, or even the asymptotically split extensions, as demonstrated in [MT4]. In fact, seen through the right looking-glasses it seems more surprising that $\operatorname{Ext}(A, B)$ actually *is* a group in so many cases, and that semi-invertibility prevails in many cases where invertibility fails.

Complementing on the cases covered by the results in [MT3], [MT1], [M], [Th4] and [ST] we shall show in this paper that all extensions in $\operatorname{Ext}(A, B)$ are semi-invertible when

a) A is the reduced group C^*-algebra $C^*_r(G)$ when the group G is an amalgamated free product $G = G_1 *_F G_2$ with F finite, G_2 is amenable and G_1 abelian, and when
b) A is the amalgamated free product of C^*-algebras, $A = A_1 *_D A_2$, when D is nuclear and all extensions of A_i by B are semi-invertible, $i = 1, 2$.

The result concerning a) is actually slightly more general and involves a KK-theory condition which is automatically fullfilled when G_1 is abelian. Furthermore we establish a few permanence properties for semi-invertibility: If all extensions of A and A' by B are semi-invertible then so are all extensions of $A \oplus A'$ by B, all extensions

[2] They also had good reasons for restricting the attention to essential extensions, but that's another story.

of $C(\mathbb{T}) \otimes A$ by B and all extensions of $\mathbb{K} \otimes A$ by B. It follows from this that all extensions of A by B are semi-invertible when

a') $A = C_r^*(G')$ provided $G' = \mathbb{Z}^k \times H \times G$ where H is a finite group and G is an amalgamated free product as in a) above, and when

b') A is the full group C^*-algebra $C^*(\mathbb{Z}^k \times H \times G'')$ where H is a finite group and G'' is obtained through successive amalgamations

$$G'' = (\cdots((G_1 *_{H_1} G_2) *_{H_2} G_3) *_{H_3} \cdots\cdots) *_{H_{n-1}} G_n,$$

provided all the groups $H_1, H_2, \dots, H_{n-1}$ are amenable, and all extensions of $C^*(G_i)$ by B are semi-invertible, $i = 1, 2, \dots, n$.

While we know from [HS], [HLSW] and [Se] that there are non-invertible extensions of A by B in many of the cases dealt with in a), our ignorance concerning invertibility of the extensions handled by b') is complete: There is no known example of an extension of a full group C^*-algebra by a stable C^*-algebra which is not invertible.

The proof of a) above is an elaboration of the ideas developed in [M], [Th4] and [ST]. In particular, the argument uses the notion of strong homotopy of extensions and depends on Lemma 4.3 in [MT1]. In contrast the method of proof of b) is new and does not use strong homotopy of extensions. Instead a key step uses methods devised for the classification of C^*-algebras by Lin, Dadarlat and Eilers. This difference in the proofs has consequences for the conclusions we obtain; in case a) the inverse (for semi-invertibility) can be chosen to be invertible while we do not know if this is so in case b).

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2. The reduced group C^*-algebra of free products with amalgamation over a finite subgroup

Throughout A and B are separable C^*-algebras and B is stable. Two extensions $\varphi, \varphi' : A \to Q(B)$ are *strongly homotopic* when there is a path $\psi_t, t \in [0,1]$, of extensions $\psi_t : A \to Q(B)$ such that

1) $t \mapsto \psi_t(a)$ is continuous for all $a \in A$, and
2) $\psi_0 = \varphi$ and $\psi_1 = \varphi'$.

By Lemma 4.3 of [MT1] we have the following

Theorem 2.1. *Assume that two extensions $\varphi, \varphi' : A \to Q(B)$ are strongly homotopic. Then φ is asymptotically split if and only if φ' is asymptotically split.*

In some of the cases we deal with below we show that for any extension $\varphi : A \to Q(B)$ there is an extension $\psi : A \to Q(B)$ such that $\varphi \oplus \psi$ is strongly homotopic to a split extension. This will be expressed by saying that φ is *strongly homotopy invertible*. Thanks to Theorem 2.1 this implies in particular that φ is semi-invertible. In some cases it turns out that ψ can be taken to be invertible. We express this by saying that φ *is strongly homotopy invertible with an invertible inverse.*

Lemma 2.2. *Let $G_i, i = 1, 2,$ be discrete countable amenable groups with a common finite subgroup $H \subseteq G_i, i = 1, 2$. Let $G_1 *_H G_2$ be the amalgamated free product group. Let $\mu : C^*(G_1 *_H G_2) \to C_r^*(G_1 *_H G_2)$ be the canonical surjection and let*

$h_\tau : C^*(G_1 *_H G_2) \to \mathbb{C}$ be the character corresponding to the trivial one-dimensional representation of $G_1 *_H G_2$. There are then a separable infinite-dimensional Hilbert space $\mathbb{H}$, $*$-homomorphisms $\sigma, \sigma_0 : C_r^*(G_1 *_H G_2) \to B(\mathbb{H})$, and a path

$$\zeta_s : C^*(G_1 *_H G_2) \to B(\mathbb{H}),\ s \in [0,1],$$

of unital $*$-homomorphisms such that

a) $\zeta_0 = \sigma \circ \mu$;
b) $\zeta_1 = h_\tau \oplus \sigma_0 \circ \mu$;
c) $\zeta_s(a) - \zeta_0(a) \in \mathbb{K},\ s \in [0,1]$, *and*
d) $s \mapsto \zeta_s(a)$ *is continuous for all* $a \in C^*(G_1 *_H G_2)$.

Proof. Set $G = G_1 *_H G_2$. Being amenable G_i has the Haagerup Property. See the discussion in 1.2.6 of [CCJJV]. It follows then from Propositions 6.1.1 and 6.2.3 of [CCJJV] that also G has the Haagerup Property. Since the Haagerup Property implies K-amenability by [Tu] (or Theorem 1.2 in [HK]) we conclude that G is K-amenable. We can therefore find a separable infinite-dimensional Hilbert space $\mathbb{H}$ and $*$-homomorphisms $\sigma, \sigma_0 : C_r^*(G) \to B(\mathbb{H})$ such that σ and $h_\tau \oplus \sigma_0$ are both unital and

1) $\sigma \circ \mu(x) - (h_\tau \oplus \sigma_0 \circ \mu)(x) \in \mathbb{K},\ x \in C^*(G)$, and
2) $[\sigma \circ \mu, h_\tau \oplus \sigma_0 \circ \mu] = 0$ in $KK(C^*(G), \mathbb{K})$,

cf. [C]. By adding the same unital and injective $*$-homomorphism to σ and σ_0 we can arrange that both σ and σ_0 are injective and have no non-zero compact operator in their range. Since $\mu|_{C^*(G_i)} : C^*(G_i) \to C_r^*(G_i)$ is injective because G_i is amenable, it follows that $\sigma \circ \mu|_{C^*(G_i)}$ and $(h_\tau \oplus \sigma_0 \circ \mu)|_{C^*(G_i)}$ are admissible in the sense of Section 3 of [DE] for each i. Thus Theorem 3.12 of [DE] applies to show that there is a norm-continuous path $u_s^i, s \in [1,\infty)$, of unitaries in $1 + \mathbb{K}$ such that

$$\lim_{s\to\infty} \left\| \sigma \circ \mu|_{C^*(G_i)}(a) - u_s^i \left(h_\tau \oplus \sigma_0 \circ \mu\right)|_{C^*(G_i)}(a) {u_s^i}^* \right\| = 0 \tag{2.1}$$

for all $a \in C^*(G_i)$ and

$$\sigma \circ \mu|_{C^*(G_i)}(a) - u_s^i \left(h_\tau \oplus \sigma_0 \circ \mu\right)|_{C^*(G_i)}(a) {u_s^i}^* \in \mathbb{K} \tag{2.2}$$

for all $a \in C^*(G_i)$ and all $s \in [1,\infty)$. Set

$$F = \left(h_\tau \oplus \sigma_0 \circ \mu\right)\left(C^*(H)\right)$$

which is a finite dimensional unital C^*-subalgebra of $B(\mathbb{H})$, and let $P : B(\mathbb{H}) \to F' \cap B(\mathbb{H})$ be the conditional expectation given by

$$P(x) = \int_{U(F)} uxu^*\ du,$$

where we integrate with respect to the Haar-measure on the unitary group $U(F)$ of F. Note that $P(1 + \mathbb{K}) \subseteq 1 + \mathbb{K}$. It follows from (2.1) that ${u_s^2}^* u_s^1$ asymptotically commutes with elements of F and hence also that

$$\lim_{s\to\infty} \left\| P\left({u_s^2}^* u_s^1\right) - {u_s^2}^* u_s^1 \right\| = 0. \tag{2.3}$$

Standard C^*-algebra techniques provides us then with a norm-continuous path $v_t, t \in [1,\infty)$, of unitaries in $F' \cap (1 + \mathbb{K})$ such that $\lim_{s\to\infty} \left\| v_s - P\left({u_s^2}^* u_s^1\right)\right\| = 0$, which combined with (2.3) implies that

$$\lim_{s\to\infty} \left\| u_s^2 v_s - u_s^1 \right\| = 0.$$

It follows that we can work with $u_s^2 v_s$ in the place of u_s^1 to arrange that besides (2.1) and (2.2) we have also that

$$\operatorname{Ad} u_s^1 \circ (h_\tau \oplus \sigma_0 \circ \mu)\,|_{C^*(H)} = \operatorname{Ad} u_s^2 \circ (h_\tau \oplus \sigma_0 \circ \mu)\,|_{C^*(H)}$$

for all s. It follows that the $*$-homomorphisms

$$\psi_s' = \left(\operatorname{Ad} u_s^1 \circ (h_\tau \oplus \sigma_0 \circ \mu)\right) *_{C^*(H)} \left(\operatorname{Ad} u_s^2 \circ (h_\tau \oplus \sigma_0 \circ \mu)\right)$$

are all defined and give us a norm-continuous path of unital $*$-homomorphisms $\eta_s : C^*(G) \to B(\mathbb{H}),\ s \in [0,1]$, such that

a') $\eta_0 = (\operatorname{Ad} u_1^1 \circ (h_\tau \oplus \sigma_0 \circ \mu)) *_{C^*(H)} (\operatorname{Ad} u_1^2 \circ (h_\tau \oplus \sigma_0 \circ \mu))$;
b') $\eta_1 = \sigma \circ \mu$;
c') $\eta_s(a) - \eta_0(a) \in \mathbb{K},\ a \in C^*(G), s \in [0,1]$.

The unitary group of $F' \cap (\mathbb{C}1 + \mathbb{K})$ is norm-connected; a fact which can be seen either from the spectral theory of compact operators or by observing that the algebra is AF. By using first a continuous path of unitaries connecting $u_1^{2*} u_1^1$ to 1 in $F' \cap (1 + \mathbb{K})$ and then a continuous path of unitaries connecting u_1^2 to 1 in the unitary group of $1 + \mathbb{K}$, we obtain continuous paths w_s^1 and w_s^2, $s \in [0,1]$, of unitaries in $1 + \mathbb{K}$ such that $w_0^1 = w_0^2 = 1$, $w_1^1 = u_1^1$, $w_1^2 = u_1^2$ and $\operatorname{Ad} w_s^1 \circ (h_\tau \oplus \sigma_0 \circ \mu)\,|_{C^*(H)} = \operatorname{Ad} w_s^2 \circ (h_\tau \oplus \sigma_0 \circ \mu)\,|_{C^*(H)}$ for all $s \in [0,1]$. It follows that the $*$-homomorphisms

$$\eta_s' = \left(\operatorname{Ad} w_s^1 \circ (h_\tau \oplus \sigma_0 \circ \mu)\right) *_{C^*(H)} \left(\operatorname{Ad} w_s^2 \circ (h_\tau \oplus \sigma_0 \circ \mu)\right)$$

are all defined and give us a norm-continuous path of unital $*$-homomorphisms $\eta_s' : C^*(G) \to B(\mathbb{H}),\ s \in [0,1]$, such that

a”) $\eta_0' = h_\tau \oplus (\sigma_0 \circ \mu)$;
b”) $\eta_1' = (\operatorname{Ad} u_1^1 \circ (h_\tau \oplus \sigma_0 \circ \mu)) *_{C^*(H)} (\operatorname{Ad} u_1^2 \circ (h_\tau \oplus \sigma_0 \circ \mu))$;
c”) $\eta_s'(a) - \eta_0'(a) \in \mathbb{K},\ a \in C^*(G), s \in [0,1]$.

The desired path ζ is then obtained by concatenation of the paths, η and η'. □

Theorem 2.3. *Let $G_i, i = 1,2$, be discrete countable amenable groups with a common finite subgroup $H \subseteq G_i, i = 1,2$, and let B be a separable stable C^*-algebra. Let $G_1 *_H G_2$ be the amalgamated free product group. Assume that the map*

$$i_1^* - i_2^* : KK\left(C^*(G_1), B\right) \oplus KK\left(C^*(G_2), B\right) \to KK\left(C^*(H), B\right),$$

induced by the inclusions $i_j : C^*(H) \to C^*(G_j), j = 1,2$, is rationally surjective, i.e. for every $x \in KK\left(C^*(H), B\right)$ there is an $n \in \mathbb{N}\backslash\{0\}$ such that nx is in the range of $i_1^* - i_2^*$.

It follows that every extension of $C_r^*\left(G_1 *_H G_2\right)$ by B is strongly homotopy invertible with an invertible inverse.

Proof. Set $G = G_1 *_H G_2$ and consider an extension $\varphi : C_r^*\left(G_1 *_H G_2\right) \to Q(B)$. Since $C^*(G) \simeq C^*\left(G_1\right) *_{C^*(H)} C^*\left(G_2\right)$ it follows from Proposition 2.8 of [Th2] that every extension of $C^*(G)$ by B is invertible. As observed in the proof of Lemma 2.2, G is K-amenable and it follows therefore from [C] that $\mu^* : \operatorname{Ext}^{-1}\left(C_r^*(G), B\right) \to \operatorname{Ext}^{-1}\left(C^*(G), B\right)$ is an isomorphism. In particular the inverse of $\varphi \circ \mu$ is in the range of μ^*, which means that there is an invertible extension $\varphi'' : C_r^*(G) \to Q(B)$ such that

$$[\varphi \circ \mu \oplus \varphi'' \circ \mu] = 0 \tag{2.4}$$

in $\operatorname{Ext}^{-1}\left(C^*(G), B\right)$. Let $\beta_0 : C_r^*(G) \to M(B)$ be an absorbing homomorphism, whose existence is guaranteed by [Th1] and set $\varphi' = \varphi \oplus q_B \circ \beta_0$. By Lemma 2.2 of

[Th2] $\beta_0|_{C_r^*(G_i)} : C_r^*(G_i) \to M(B)$ is absorbing for each $i = 1, 2$. Since G_i is amenable $\mu|_{C^*(G_i)} : C^*(G_i) \to C_r^*(G_i)$ is a $*$-isomorphism and it follows therefore from (2.4) that $(\varphi' \circ \mu \oplus \varphi'' \circ \mu)|_{C^*(G_i)}$ is a split extension for each i. In other words, there are $*$-homomorphisms $\pi_i : C^*(G_i) \to M(B)$ such that $(\varphi' \circ \mu \oplus \varphi'' \circ \mu)|_{C^*(G_i)} = q_B \circ \pi_i, i = 1, 2$. Note that

$$\pi_1(x) - \pi_2(x) \in B$$

for all $x \in C^*(H)$ so that (π_1, π_2) represents an element of $KK(C^*(H), B)$. We need to change the situation to a case where this pair represents 0 in $KK(C^*(H), B)$. This is done as follows:

$\beta_0|_{C^*(G_i)}, i = 1, 2$, are both absorbing so after adding $q_B \circ \beta_0$ to φ'' we get a situation where there are unitaries $u_i \in M(B)$ such that $\mathrm{Ad}\, u_i \circ \pi_i(y) - \beta_0(y) \in B$ for all $y \in C^*(G_i), i = 1, 2$. Then

$$\varphi' \circ \mu \oplus \varphi'' \circ \mu = Adq_B(u_2^*) \circ \left(\left(q_B \circ \mathrm{Ad}\, u_2 u_1^* \circ \beta_0|_{C^*(G_1)}\right) *_{C^*(H)} \left(q_B \circ \beta_0|_{C^*(G_2)}\right)\right).$$

It follows that we can choose the lifts, π_1, π_2, above such that $\left[\pi_1|_{C^*(H)}, \pi_2|_{C^*(H)}\right] = \left[\mathrm{Ad}\, w \circ \beta_0|_{C^*(H)}, \beta_0|_{C^*(H)}\right]$ in $KK(C^*(H), B)$ where $w = u_2 u_1^*$. To proceed we need a description of the KK-groups obtained in [Th1] and [Th3]: When A is a separable C^*-algebra and $\alpha : A \to M(B)$ is an absorbing $*$-homomorphism, there is an isomorphism between $K_1(\mathcal{D}_\alpha(A))$ and $KK(A, B)$, where

$$\mathcal{D}_\alpha(A) = \{m \in M(B) : \ \alpha(a)m - m\alpha(a) \in B \ \forall a \in A\}. \qquad (2.5)$$

The isomorphism sends a unitary $u \in \mathcal{D}_\alpha(A)$ to $[\mathrm{Ad}\, u \circ \alpha, \alpha]$. Ignoring the passage to matrices in K_1 our assumption implies, in this picture of KK-theory, that there is an $n > 0$ and a norm-continuous path of unitaries in $\mathcal{D}_{\beta_0}(C^*(H))$ connecting w^n to a product $w_2^* w_1$, where $w_i \in \mathcal{D}_{\beta_0}(C^*(G_i)), i = 1, 2$. Then $\left[\mathrm{Ad}\, w^n \circ \beta_0|_{C^*(H)}, \beta_0|_{C^*(H)}\right] = \left[\mathrm{Ad}\, w_1 \circ \beta_0|_{C^*(H)}, \mathrm{Ad}\, w_2 \circ \beta_0|_{C^*(H)}\right]$ in $KK(C^*(H), B)$. Note that

$$q_B \circ \beta_0 \circ \mu = \left(q_B \circ \mathrm{Ad}\, w_1^* \circ \beta_0|_{C^*(G_1)}\right) *_{C^*(H)} \left(q_B \circ \mathrm{Ad}\, w_2^* \circ \beta_0|_{C^*(G_2)}\right).$$

After adding

$$\underbrace{(\varphi' \oplus \varphi'') \oplus (\varphi' \oplus \varphi'') \oplus \cdots \oplus (\varphi' \oplus \varphi'')}_{n-1 \text{ times}} \oplus q_B \circ \beta_0$$

to φ'' we come in a position where the pair (π_1, π_2) can be chosen such that $[\pi_1, \pi_2] = 0$ in $KK(C^*(H), B)$. (If we take the passage to matrices in K_1 into account in the previous argument, it may be necessary to add a finite direct sum of copies of $q_B \circ \beta_0$ instead of a single copy.)

We can then proceed as follows: Set $\beta = q_B \circ \beta_0^\infty$ where β_0^∞ is the direct sum of a sequence of copies of β_0. By adding β to φ'' we come then in a situation where Theorem 3.8 of [DE] applies to give us a continuous path $u_t, t \in [1, \infty)$, of unitaries in $1 + B$ such that

$$\lim_{t \to \infty} \mathrm{Ad}\, u_t \circ \pi_1(x) = \pi_2(x)$$

for all $x \in C^*(H)$. Since $C^*(H)$ is finite dimensional we have that for t large enough there is a unitary $v \in 1 + B$ such that $v u_t \pi_1(x) u_t^* v^* = \pi_2(x)$ for all $x \in C^*(H)$. Hence, by exchanging π_1 with $\mathrm{Ad}\, v u_t \circ \pi_1$ we conclude that $\varphi' \circ \mu \oplus \varphi'' \circ \mu$ is split. By a standard argument, based on Kasparov's stabilization theorem, we may add a split extension to arrange that $\varphi' \circ \mu \oplus \varphi'' \circ \mu = q_B \circ \chi \oplus 0$ where $\chi : C^*(G) \to M(B)$ is a unital $*$-homomorphism. Let $\gamma : G \to M(B)$ be the unitary representation of G defined by χ and let η_s be the continuous path of $*$-homomorphisms from

Lemma 2.2, and ν_s the corresponding unitary representations. Let $h_{\gamma\otimes\nu_s}$ be the $*$-homomorphism $C^*(G) \to M(B)$ defined from the tensor product representation $\gamma \otimes \nu_s$ by use of a spatial isomorphism $B \otimes \mathbb{K} \simeq B$. Then

$$q_B \circ h_{\gamma\otimes\nu_s}, \ s \in [0,1],$$

is a strong homotopy of extensions of $C^*(G)$ by B. By the argument used in the proof of Theorem 2.3 of [Th3] and again in the proof of Theorem 2.2 in [ST] the properties of $\{\nu_s\}$ ensure that this homotopy factors through $C_r^*(G)$ and give us a strong homotopy, as well as split extensions ψ, ψ', of $C_r^*(G)$ by B connecting $\varphi \oplus q_B \circ \beta_0 \oplus \varphi'' \oplus \psi = \varphi' \oplus \varphi'' \oplus \psi$ to ψ'. Since $q_B \circ \beta_0 \oplus \varphi'' \oplus \psi$ is invertible, this completes the proof. □

As in [ST] the fact that the strong homotopy inverse is invertible implies that the group $\mathrm{Ext}^{-1/2}(C_r^*(G_1 *_H G_2), B)$ of extensions modulo asymptotically split extensions agrees with the corresponding KK-theory group and can be calculated from the universal coefficient theorem. The proof is the same as in [ST] and we omit it here.

The KK-condition of Theorem 2.3 is satisfied when G_1 is abelian since is this case already the map

$$i_1^* : KK\left(C^*\left(G_1\right), B\right) \to KK(C^*(H), B)$$

is surjective. This follows because there is in this case a $*$-homomorphism $p : C^*\left(G_1\right) \to C^*(H)$ which is a left-inverse for i_1. We get in this way the following corollary.

Corollary 2.4. *Let G_1 and G_2 be countable discrete amenable groups with a common finite subgroup $H \subseteq G_i, i = 1, 2$, and B a separable stable C^*-algebra. Let $G_1 *_H G_2$ be the amalgamated free product group. Assume that G_1 is abelian. It follows that every extension of $C_r^*\left(G_1 *_H G_2\right)$ by B is strongly homotopy invertible with an invertible inverse.*

Example 2.5. It is known that

$$Sl_2(\mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6,$$

cf. p. 11 in [S]. Hence Corollary 2.4 applies. (As the generator of $\mathbb{Z}_4$ one can use $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ can serve as the generator of $\mathbb{Z}_6$. The amalgamation is over the subgroup ± 1.) It has been shown by Hadwin and Shen in Corollary 4.4 of [HS] that one can get an example of an non-invertible extension of $C_r^*\left(Sl_2(\mathbb{Z})\right)$ by $\mathbb{K}$, starting from the non-invertible extension of $C_r^*\left(\mathbb{F}_2\right)$ found by Haagerup and Thorbjørnsen in [HT]. This means that concerning invertibility of extensions of $C_r^*\left(Sl_2(\mathbb{Z})\right)$ the situation is as for $C_r^*\left(\mathbb{F}_2\right)$: For every stabilization B of a unital separable C^*-algebra there are non-invertible extensions of $C_r^*\left(Sl_2(\mathbb{Z})\right)$ by B, but all are semi-invertible. And the inverse (for semi-invertibility) can be taken to be invertible.

For the full group C^*-algebra $C^*\left(Sl_2(\mathbb{Z})\right)$ the situation is also as for $\mathbb{F}_2$, namely that all extensions by $C^*\left(Sl_2(\mathbb{Z})\right)$ are invertible. This follows from [Br] when the ideal is $\mathbb{K}$ and from [Th2] when it is an arbitrary separable stable C^*-algebra.

Remark 2.6. The KK-condition of Theorem 2.3 can fail even when G_1 and G_2 are finite and equal, and H is abelian. Here is the simplest example. Let α be the unique non-trivial automorphism of $\mathbb{Z}_3$ which has order 2 and let $G_1 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$ be the semidirect product by this automorphism. Thus G_1 is a copy of the symmetric group S_3. Set $H = \mathbb{Z}_3 \subset G_1$. Let $B = \mathbb{K}$. Then $KK(C^*(G), B) \cong R(G)$ for any finite group G, where $R(G)$ denotes the Grothendieck group of the semigroup generated by irreducible (necessarily finite dimensional) representations of G. The functorial map $KK(C^*(G_1), B) \to KK(C^*(H), B)$ becomes the restriction map $R(G_1) \to R(H)$ after the above identification. The abelian group $R(H)$ is freely generated by the three one-dimensional representations, ρ_0, ρ_1 and ρ_2, that send a fixed generator a of H to 1, $e^{2\pi i/3}$ and $e^{-2\pi i/3}$, respectively. As the number of irreducible representations equals the number of conjugacy classes by the Burnside theorem, and as the group order equals the sum of squares of the dimensions of these representations, it follows that G_1 has three irreducible representations, two, σ_0 and σ_1, of dimension 1 and one, τ, of dimension 2. Thus, $R(G_1)$ is freely generated by three representations, σ_0, σ_1 and τ. One of the one-dimensional representations, σ_0, is the identity one, and the other, σ_1, maps H to 1 and $G_1 \setminus H$ to -1. Restrictions of both to H equal the trivial representation ρ_0 of H. The two-dimensional representation τ is the orthogonal complement to the constant functions in the obvious representation of G_1 on $l^2(H) \cong \mathbb{C}^3$. Then it is easy to see that $\tau|_H = \rho_1 \oplus \rho_2$. Thus, the restriction map $R(G_1) \to R(H)$ is not surjective.

This example shows only that the KK-condition of Theorem 2.3 is not vacuous. For all we know the conclusion of Theorem 2.3 may very well be true without this condition.

3. Amalgamated free product C^*-algebras

In this section we consider free products of C^*-algebras with amalgamation. The first result is an application of the relative K-homology developed by the authors in [MT2].

Theorem 3.1. *Let A_1, A_2 and B be separable C^*-algebras, B stable. Let D be a common C^*-subalgebra of A_1 and A_2, i.e. $D \subseteq A_1$ and $D \subseteq A_2$. Assume that*

1) *there is a $*$-homomorphism $\alpha_0 : A_1 *_D A_2 \to M(B)$ such that also $\alpha_0|_{A_1}, \alpha_0|_{A_2}$ and $\alpha_0|_D$ are absorbing, and assume that*
2) $\mathrm{Ext}(A_1, B)$ *and* $\mathrm{Ext}(A_2, B)$ *are both groups.*

It follows that every extension of $A_1 *_D A_2$ by B is strongly homotopy invertible.

Proof. Set $\alpha = q_B \circ \alpha_0$ and consider an extension $\varphi : A_1 *_D A_2 \to Q(B)$. By assumption 2) there is an extension $\psi_i : A_i \to Q(B)$ representing the inverse of $\varphi|_{A_i}$ in $\mathrm{Ext}(A_i, B)$ both for $i = 1$ and $i = 2$. Then $\psi_1|_D$ and $\psi_2|_D$ represent the same element in $\mathrm{Ext}(D, B)$, namely the inverse of the element represented by $\varphi|_D$. After addition of $\alpha_0|_{A_i}$ to $\varphi|_{A_i}$ we therefore assume that $\psi_1|_D$ and $\psi_2|_D$ are unitarily equivalent. Thus, after conjugating ψ_2 by a unitary, we can arrange that $\psi_1|_D = \psi_2|_D$. Then $\psi = \psi_1 *_D \psi_2 : A_1 *_D A_2 \to Q(B)$ is defined. Set $\Phi = \varphi \oplus \psi$. By adding a copy of α to Φ both extensions $\Phi|_{A_i} : A_i \to Q(B), i = 1, 2$, become split, i.e. there are $*$-homomorphisms $\Phi_i : A_i \to M(B)$ such that $q_B \circ \Phi_i = \Phi|_{A_i}, i = 1, 2$. By passing to a unitarily equivalent extension, i.e. by conjugating Φ by a unitary of the form $q_B(u)$, we can arrange that in addition $q_B \circ \Phi_2 = \alpha|_{A_2}$ and that $\Phi_2 = \alpha_0|_{A_2}$. Then $q_B \circ \Phi_1$ represents an element of the relative extension semi-group $\mathrm{Ext}_{D,\alpha|_{A_1}}(A_1, B)$, cf. [MT2]. In fact, it follows from Lemma 3.2 of [MT2] and assumption 2) that $q_B \circ \Phi_1$ is invertible in this semi-group, i.e. $q_B \circ \Phi_1 \in \mathrm{Ext}^{-1}_{D,\alpha|_{A_1}}(A_1, B)$. Let $\Phi_1' : A_1 \to Q(B)$ represent the inverse of $q_B \circ \Phi_1$ in $\mathrm{Ext}^{-1}_{D,\alpha|_{A_1}}(A_1, B)$ and note that $\Phi_1' *_D \alpha|_{A_2} : A_1 *_D A_2 \to Q(B)$ is then defined. After addition by this extension to Φ we can assume that Φ_1 represents 0 in $\operatorname{Ext}^{-1}_{D,\alpha|_{A_1}}(A_1,B)$. By definition of $\operatorname{Ext}_{D,\alpha|_{A_1}}(A_1,B)$ this means that there is a unitary u in the connected component of 1 in the relative commutant of $\alpha(D)$ in $Q(B)$ such that $\operatorname{Ad} u \circ q_B \circ \Phi_1 = \alpha|_{A_1}$. Let $u_t, t \in [0,1]$, be a continuous path of unitaries in $\alpha(D)' \cap Q(B)$ such that $u_0 = 1$ and $u_1 = u$. Then

$$\psi_t = (\operatorname{Ad} u_t \circ q_B \circ \Phi_1) *_D (q_B \circ \Phi_2)$$

is defined for every $t \in [0,1]$, and $\psi_t, t \in [0,1]$, is a strong homotopy of extensions connecting $\Phi = \psi_0$ to $\psi_1 = q_B \circ \alpha$. This completes the proof. □

Condition 1) of Theorem 3.1 is always satisfied when D is nuclear or is the range of a conditional expectation $A_i \to D$ for both $i = 1$ and $i = 2$, but it can fail in general. See [Th2]. Condition 2) is satisfied when A_1 and A_2 are nuclear so Theorem 3.1 has the following corollary.

Corollary 3.2. *Let A_1, A_2 and B be separable C^*-algebras, B stable. Let D be a common C^*-subalgebra of A_1 and A_2, i.e. $D \subseteq A_1$ and $D \subseteq A_2$. If A_1, A_2 and D are all nuclear it follows that every extension of $A_1 *_D A_2$ by B is strongly homotopy invertible.*

The next theorem shows that condition 2) of Theorem 3.1 can be weakened when D is nuclear, at the price of a slightly weaker conclusion.

Theorem 3.3. *Let A_1, A_2 and B be separable C^*-algebras, B stable. Let D be a common C^*-subalgebra of A_1 and A_2, i.e. $D \subseteq A_1$ and $D \subseteq A_2$. Assume that*

1) *there is a $*$-homomorphism $\beta : A_1 *_D A_2 \to M(B)$ such that $\beta|_D : D \to M(B)$ is absorbing,*
2) *that $\operatorname{Ext}(D,B)$ and $\operatorname{Ext}(D, C_0([1,\infty),B))$ are both groups, and*
3) *that all extensions of A_1 by B and all extensions of A_2 by B are semi-invertible.*

It follows that all extensions of $A_1 *_D A_2$ by B are semi-invertible.

Proof. By adding units to A_1, A_2 and D if necessary, we may assume that D is unital.

1. step: (Finding the first candidate for the inverse.)

Let $\varphi : A_1 *_D A_2 \to Q(B)$ be an extension. By assumption 2) there are extensions $\psi_i : A_i \to Q(B)$ such that $\varphi|_{A_i} \oplus \psi_i : A_i \to Q(B)$ are asymptotically split, $i = 1,2$. By assumption 2) $\operatorname{Ext}(D,B)$ is a group and hence $[\psi_1|_D] = [\psi_2|_D] = -[\varphi|_D]$ in $\operatorname{Ext}(D,B)$. (There are various ways to see this; it follows for example from Lemma 4.7 of [MT1].) Furthermore, by assumption 1) there is a $*$-homomorphism $\beta : A_1 *_D A_2 \to M(B)$ such that $\beta|_D$ is absorbing. So after adding by $q_B \circ \beta|_{A_1}$ to ψ_1 and $q_B \circ \beta|_{A_2}$ to ψ_2 we may assume that $\psi_1|_D$ and $\psi_2|_D$ are unitarily equivalent, and hence without loss of generality that $\psi_1|_D = \psi_2|_D$. Then we have a candidate for a semi-inverse to φ, namely $\psi_1 *_D \psi_2$. We will show that after addition by additional extensions (some of which may be non-trivial), $\varphi \oplus (\psi_1 *_D \psi_2)$ becomes asymptotically split.

2. step: (Removing a KK-obstruction.)

First note that $\varphi \oplus (\psi_1 *_D \psi_2)$ is split over D. Hence, by adding a copy of $q_B \circ \beta$ to φ and conjugating by a unitary we can arrange that

$$\varphi \oplus (\psi_1 *_D \psi_2)|_D = q_B \circ \beta|_D. \tag{3.1}$$

Let $\xi^i : A_i \to M(B)$, be equi-continuous asymptotic homomorphisms such that $q_B \circ \xi^i_t = \varphi|_{A_i} \oplus \psi_i$ for all t, $i = 1,2$. Note that by (3.1) we have that

$$\xi^i_t(d) - \beta(d) \in B \tag{3.2}$$

for all $t \in [1,\infty), d \in D, i = 1,2$. Let β^∞ denote the direct sum of a countable infinite number of copies of β and set $\pi = 1_{C_0[1,\infty)} \otimes \beta^\infty$; i.e. $1_{C_0[1,\infty)}$ is the unit in the multiplier algebra $M(C_0[1,\infty))$ and $\pi(x) = 1_{C_0[1,\infty)} \otimes \beta^\infty(x) \in M(C_0[1,\infty), B)$. Then $\pi : D \to M(C_0[1,\infty), B)$ is absorbing by Lemma 2.3 of [Th3]. Since $\operatorname{Ext}(D, C_0[1,\infty), B))$ is the trivial group by assumption 2), this implies that there is a strictly continuous path $U_t, t \in [1,\infty)$, of unitaries in $M(B)$ such that

$$t \mapsto U_t\left(\xi^1_t(d) \oplus \beta^\infty(d)\right)U_t^* - \left(\xi^2_t(d) \oplus \beta^\infty(d)\right) \tag{3.3}$$

is in $C_0[1,\infty), B)$ for all $d \in D$. For each $n \in \mathbb{N}$, $U_t, t \in [1,n]$, defines a unitary W_n in $M(C[1,n] \otimes B))$ in the natural way. Set $\pi_n = 1_{C[1,n]} \otimes \beta^\infty|_D$ and $\beta_n = 1_{C[1,n]} \otimes \beta|_D$. Then (3.3) and (3.2) imply that

$$W_n(\beta_n \oplus \pi_n)(d)W_n^* - (\beta_n \oplus \pi_n)(d) \in C[1,n] \otimes B \tag{3.4}$$

for all $d \in D$, i.e. W_n is a unitary in the C^*-algebra $\mathcal{D}_{\beta_n \oplus \pi_n}(D)$, cf. (2.5). Note that $\beta_n \oplus \pi_n$ is absorbing, again by Lemma 2.3 of [Th3], so that $K_1(\mathcal{D}_{\beta_n \oplus \pi_n}(D)) = KK(D, C[1,n] \otimes B)$ by (3.2) of [Th3]. Identifying $KK(D, C[1,n] \otimes B)$ and $KK(D,B)$ we can say that

$$[\operatorname{Ad} W_n \circ (\beta_n \oplus \pi_n), (\beta_n \oplus \pi_n)] = [\operatorname{Ad} U_1 \circ (\beta|_D \oplus \beta^\infty|_D), (\beta|_D \oplus \beta^\infty|_D)]. \tag{3.5}$$

in $KK(D, C[1,n] \otimes B)$. Add then the extension

$$(q_B \circ \operatorname{Ad} U_1 \circ (\beta \oplus \beta^\infty)|_{A_1}) *_D (q_B \circ (\beta \oplus \beta^\infty)|_{A_2})$$

to $\varphi \oplus (\psi_1 *_D \psi_2)$. We can then exchange ξ^1_t by $\xi^1_t \oplus \operatorname{Ad} U_1 \circ (\beta \oplus \beta^\infty)|_{A_1}$, ξ^2_t by $\xi^2_t \oplus (\beta \oplus \beta^\infty)|_{A_2}$, and U_t by $U_t \oplus U_1^*$. We may therefore return to the previous notation and conclude from (3.5) that

$$[\operatorname{Ad} W_n \circ (\beta_n \oplus \pi_n), (\beta_n \oplus \pi_n)] = 0$$

in $KK(D, C[1,n] \otimes B)$ for all n. It follows therefore that $\operatorname{diag}(W_n, 1, 1, \dots, 1)$ is in the connected component of 1 in the unitary group of $M_{k_n}(\mathcal{D}_{\beta_n \oplus \pi_n}(D))$ for some $k_n \in \mathbb{N}$, $k_n \geq 2$. Since $\beta_n \oplus \pi_n$ is absorbing, there is an isomorphism from $M_{k_n}(\mathcal{D}_{\beta_n \oplus \pi_n}(D))$ onto $M_2(\mathcal{D}_{\beta_n \oplus \pi_n}(D))$ which takes $\operatorname{diag}(W_n, 1, 1, \dots, 1)$ to $\operatorname{diag}(W_n, 1)$. It follows that $\operatorname{diag}(W_n, 1)$ is in the connected component of 1 in the unitary group of $M_2(\mathcal{D}_{\beta_n \oplus \pi_n}(D))$ for each n. After addition by the split extension β^∞ so that we can substitute $W_n \oplus 1$ for W_n, we may therefore assume that W_n is in the connected component of 1 in the unitary group of $\mathcal{D}_{\beta_n \oplus \pi_n}(D)$ for each $n \in \mathbb{N}$.

3. step: (The tricky part. This is an elaboration on ideas developed by Lin, Dadarlat and Eilers, in [L], [DE], and a very similar argument was used to prove Theorem 4.1 in [Th3].)

Let E_n denote the C^*-subalgebra of $M(C[1,n] \otimes B))$ generated by the unit $1_{C[0,1] \otimes B}$, $C[1,n] \otimes B$ and $(\beta_n \oplus \pi_n)(D)$. It follows from (3.4) that $\operatorname{Ad} W_n$ defines an automorphism α_n of E_n, and the path of unitaries in $\mathcal{D}_{\beta_n \oplus \pi_n}(D)$ connecting W_n to 1 gives us a uniform norm-continuous path of automorphisms in $\operatorname{Aut} E_n$ connecting α_n to the identity in $\operatorname{Aut} E_n$. Since E_n is separable, it follows from 8.7.8 and 8.6.12 in [P], cf. Proposition 2.15 of [DE], that α_n is asymptotically inner, i.e. there is a continuous path $V_s^n, s \in [1,\infty)$, of unitaries in E_n such that $\alpha_n(x) = \lim_{s\to\infty} V_s^n x V_s^{n*}$ for all $x \in E_n$.

Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite subsets with dense union in D. Since

$$\lim_{s\to\infty} \sup_{t\in[1,n]} \|V_s^n(t)\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)V_s^n(t)^* - U_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d){U_t}^*\| = 0$$

for all $d \in D$, we can find an $s_n \in [1,\infty)$ so big that

$$\|V_s^n(t)\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)V_s^n(t)^* - U_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d){U_t}^*\| \leq \frac{1}{n} \tag{3.6}$$

for all $s \geq s_n$, all $t \in [1,n]$ and all $d \in F_n$. Note that

$$\lim_{s\to\infty} V_s^{n+1}(n)^* V_s^n(n) x V_s^n(n)^* V_s^{n+1}(n) = x \tag{3.7}$$

for all $x \in B \cup (\xi_t^1 \oplus \beta^\infty)(D), t \in [1,n]$. To simplify notation, set $\Delta_s^k = V_s^{k+1}(k)^* V_s^k(k)$. It follows from (3.7) that if we increase s_n we can arrange that

$$\|\Delta_s^k\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)\Delta_s^{k*} - \left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)\| \leq \frac{1}{n^2} \tag{3.8}$$

for all $d \in F_n, t \in [1,n]$, and all $k = 2,3,\cdots,n$, when $s \geq s_n$. Proceeding inductively we can arrange that $s_n < s_{n+1}$ for all n. Let $s : [1,\infty) \to [1,\infty)$ be a continuous increasing function such that $s(n) = s_{n+1}, n = 1,2,3,\cdots$. Define a norm-continuous path $W_t, t \in [1,\infty)$, in

$$E = C^*\left(1_B, \left(\xi_1^1 \oplus \beta^\infty|_D\right)(D), B\right) = C^*\left(1_B, \left(\beta \oplus \beta^\infty|_D\right)(D), B\right)$$

such that $W_t = V_{s(t)}^2(t), t \in [1,2]$, and $W_t = V_{s(t)}^{k+1}(t)\Delta_{s(t)}^k \cdots \Delta_{s(t)}^3 \Delta_{s(t)}^2,\ t \in [k,k+1],\ k \geq 2$. Let $d \in F_n$ and consider $t \in [k,k+1]$, where $k \geq n$. Since $s(t) \geq s_{k+1}$ and $d \in F_{k+1}$, it follows from (3.8) that

$$W_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)W_t^* \ \sim_{k\cdot\frac{1}{k^2}}\ V_{s(t)}^{k+1}(t)\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)V_{s(t)}^{k+1}(t)^*, \tag{3.9}$$

where $\sim_\delta$ means that the distance between the two elements is at most δ. Furthermore, it follows from (3.6) that

$$V_{s(t)}^{k+1}(t)\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)V_{s(t)}^{k+1}(t)^* \ \sim_{\frac{1}{k}}\ U_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)U_t^*. \tag{3.10}$$

It follows from (3.10), (3.9) and (3.3) that

$$\lim_{t\to\infty} W_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)W_t^* - \left(\xi_t^2 \oplus \beta^\infty|_D\right)(d) = 0, \tag{3.11}$$

first when $d \in F_n$, and then for all $d \in D$ since n was arbitrary and $\{\xi_t^i\}_{i,t}$ equicontinuous.

Recall that D is unital. For each t there are unique elements $x_t \in D, \lambda_t \in \mathbb{C}$ and $b_t \in B$ such that

$$W_t = \left(\xi_t^1 \oplus \beta^\infty|_D\right)(x_t) + \lambda_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(1)^\perp + b_t.$$

Since $q_B \circ (\xi_t^1 \oplus \beta^\infty|_D) = q_B \circ (\xi_1^1 \oplus \beta^\infty|_D)$ is injective we find that $\{x_t\}$ must be a continuous path of unitaries in D such that $\lim_{t\to\infty} x_t d x_t^* = d$ for all $d \in D$. Set

$$U_t = W_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(x_t)^* + W_t\overline{\lambda_t}\left(\xi_t^1 \oplus \beta^\infty|_D\right)(1)^\perp.$$

Then $U_t, t \in [1,\infty)$, is a continuous path of unitaries $1 + B$ such that

$$\lim_{t\to\infty} U_t\left(\xi_t^1 \oplus \beta^\infty|_D\right)(d)U_t^* - \left(\xi_t^2 \oplus \beta^\infty|_D\right)(d) = 0$$

for all $d \in D$.

4. step: (Conclusion.)

By adding the split extension $q_B \circ \beta^\infty$ we can now return to the notation in the 1. step and assume that $U_t, t \in [1, \infty)$, is a continuous path of unitaries $1 + B$ such that

$$\lim_{t \to \infty} U_t \xi_t^1(d) U_t^* - \xi_t^2(d) = 0 \tag{3.12}$$

for all $d \in D$. Set

$$\mathcal{A} = \{ f \in C_b([1, \infty), M(B)) : \ f(1) - f(t) \in B \ \forall t \in [1, \infty) \}$$

and note that $C_0([1, \infty), B)$ is an ideal in $\mathcal{A}$. Let

$$p : \mathcal{A} \to \mathcal{A} / C_0([1, \infty), B)$$

be the quotient map. Define $*$-homomorphisms $\kappa_1 : A_1 \to \mathcal{A}$ and $\kappa_2 : A_2 \to \mathcal{A}$ such that $\kappa_1(a)(t) = U_t \xi_t^1(a) U_t^*$ and $\kappa_2(a) = \xi_t^2(a)$, respectively. Since $U_t \xi_t^1(d) U_t^* - \xi_t^2(d) \in D$ for all t and $d \in D$, it follows from (3.12) that

$$(p \circ \kappa_1) *_D (p \circ \kappa_2) : \ A_1 *_D A_2 \to \mathcal{A} / C_0([1, \infty), B)$$

is defined. By composing this $*$-homomorphism with a continuous right-inverse for p, whose existence follows from the Bartle-Graves selection theorem, we get an asymptotic homomorphism $\Phi : A_1 *_D A_2 \to M(B)$ such that $q_B \circ \Phi_t = \varphi \oplus (\psi_1 *_D \psi_2)$ for all t. ☐

Corollary 3.4. *Let A_1, A_2 and B be separable C^*-algebras, B stable. Let D be a common C^*-subalgebra of A_1 and A_2, i.e. $D \subseteq A_1$ and $D \subseteq A_2$. Assume that*

1) *D is nuclear, and*
2) *that all extensions of A_1 by B and all extensions of A_2 by B are semi-invertible.*

It follows that all extensions of $A_1 *_D A_2$ by B are semi-invertible.

Proof. It is well-known that condition 2) of Theorem 3.3 is fullfilled when D is nuclear. That condition 1) also holds follows from Lemma 2.2 of [Th2]. ☐

One important virtue of Theorem 3.3 and Corollary 3.4 when compared with Theorem 3.1 is the improved symmetry between assumptions and conclusions which allows to use it iteratively, for example to reach the following conclusion: Let A_1, A_2, A_3, A_4 be separable C^*-algebras, $D_1 \subseteq A_1, D_1 \subseteq A_2$, and $D_2 \subseteq A_3, D_2 \subseteq A_4$ common C^*-algebras. Assume that the A_i's and D_i's are all nuclear, and let E be a common nuclear C^*-subalgebra of $A_1 *_{D_1} A_2$ and $A_3 *_{D_2} A_4$. It follows that all extensions of

$$(A_1 *_{D_1} A_2) *_E (A_3 *_{D_2} A_4)$$

by a separable stable C^*-algebra B are semi-invertible.

4. Full group C^*-algebras

In this section we collect some consequence of Theorem 3.1 and Theorem 3.3 for the semi-invertibility of extensions by full group C^*-algebras.

Proposition 4.1. *Let G_1, G_2 be countable discrete groups and $H \subseteq G_i, i = 1, 2$, a common subgroup. Set $G = G_1 *_H G_2$ and let B be a separable stable C^*-algebra. Assume that $\operatorname{Ext}(C^*(G_i), B), i = 1, 2$, are both groups. It follows that every extension of $C^*(G)$ by B is strongly homotopy invertible.*

Proof. We can apply Theorem 3.1 because $C^*(G) = C^*(G_1) *_{C^*(H)} C^*(G_2)$. Indeed, there are canonical conditional expectations $C^*(G) \to C^*(H)$ and $C^*(G) \to C^*(G_i), i = 1,2$, so any absorbing $*$-homomorpshism $\alpha_0 : C^*(G) \to M(B)$, whose existence is guaranteed by [Th1], will meet the requirements in 1) of Theorem 3.1 by Lemma 2.1 of [Th2]. The conclusion of the corollary follows therefore from Theorem 3.1.

□

Similarly, Theorem 3.3 implies the following

Proposition 4.2. *Let $G_i, i = 1,2$, be discrete countable groups with a common subgroup $H \subseteq G_i, i = 1,2$, and B a separable stable C^*-algebra. Let $G_1 *_H G_2$ be the amalgamated free product group and let B be a separable stable C^*-algebra. Assume that*

1) $\mathrm{Ext}(C^*(H), B)$ *and* $\mathrm{Ext}\,(C^*(H), C_0[1,\infty) \otimes B)$ *are both group, and*
2) *for both $i = 1$ and $i = 2$ every extension of $C^*(G_i)$ by B is semi-invertible.*

It follows that every extension of $C^*(G_1 *_H G_2)$ by B is semi-invertible.

As is wellknown, condition 1) in Proposition 4.2 is satisfied when H is amenable, but it is also satisfied for certain non-amenable groups, e.g. free groups or an amalgamated free product of amenable groups over a finite subgroup.

We shall finish this paper by showing that the conclusions of Propositions 4.1 and 4.2, and partly also the conclusion of Theorem 2.3, are preserved by taking the product of the group with a group of the form $\mathbb{Z}^k \oplus H$, with H finite.

Lemma 4.3. *Let A and B be separable C^*-algebras, B stable. There are semi-group homomorphisms $\mu : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A \otimes \mathbb{K}, B)$ and $\nu : \mathrm{Ext}(A \otimes \mathbb{K}, B) \to \mathrm{Ext}(A, B)$ such that $\mu \circ \nu(x) \oplus 0 = x \oplus 0$ for all $x \in \mathrm{Ext}(A \otimes \mathbb{K}, B)$ and $\nu \circ \mu(y) \oplus 0 = y \oplus 0$ for all* $\mathrm{Ext}(A, B)$.

Proof. Since B is stable we can identify B and $\mathbb{K} \otimes B$. Let e be a minimal projection in $\mathbb{K}$ and let $V \in M(\mathbb{K} \otimes \mathbb{K} \otimes B)$ be an isometry such that $VV^* = e \otimes 1_{\mathbb{K}\otimes B}$. Then $\alpha(x) = V^*(e \otimes x)V$ is an isomorphism $\alpha : \mathbb{K} \otimes B \to \mathbb{K} \otimes \mathbb{K} \otimes B$, giving us isomorphisms $M(\mathbb{K} \otimes B) \to M(\mathbb{K} \otimes \mathbb{K} \otimes B)$ and $Q(\mathbb{K} \otimes B) \to Q(\mathbb{K} \otimes \mathbb{K} \otimes B)$ which we also denote by α. Let $s : A \to \mathbb{K} \otimes A$ be the $*$-homomorphism $s(a) = e \otimes a$. We can then define a map

$$\mathrm{Ext}\,(\mathbb{K} \otimes A, \mathbb{K} \otimes \mathbb{K} \otimes B) \to \mathrm{Ext}\,(A, \mathbb{K} \otimes B) \tag{4.1}$$

by $\varphi \mapsto \alpha^{-1} \circ \varphi \otimes s$. To get a map in the other direction note that the canonical embedding $\mathbb{K} \otimes M(\mathbb{K} \otimes B) \subseteq M(\mathbb{K} \otimes \mathbb{K} \otimes B)$ induce a $*$-homomorphism $L : \mathbb{K} \otimes Q(\mathbb{K} \otimes B) \to Q(\mathbb{K} \otimes \mathbb{K} \otimes B)$ which we can use to define a map

$$\mathrm{Ext}\,(A, \mathbb{K} \otimes B) \to \mathrm{Ext}\,(\mathbb{K} \otimes A, \mathbb{K} \otimes \mathbb{K} \otimes B) \tag{4.2}$$

by $\varphi \mapsto L \circ (\mathrm{id}_{\mathbb{K}} \otimes \varphi)$. Then $\alpha^{-1} \circ (L \circ (\mathrm{id}_{\mathbb{K}} \otimes \varphi)) \circ s = \mathrm{Ad}\, q_{\mathbb{K}\otimes B}(W) \circ \varphi$ for some isometry $W \in M(\mathbb{K} \otimes B)$, showing that

$$\left[\left(\alpha^{-1} \circ (L \circ (\mathrm{id}_{\mathbb{K}} \otimes \varphi)) \circ s\right) \oplus 0\right] = [\varphi \oplus 0]$$

in $\mathrm{Ext}\,(A, \mathbb{K} \otimes B)$.

Consider next an extension $\varphi : \mathbb{K} \otimes A \to Q\,(\mathbb{K} \otimes \mathbb{K} \otimes B)$. Note that

$$L \circ \left(\mathrm{id}_{\mathbb{K}} \otimes \left(\alpha^{-1} \circ \varphi \circ s\right)\right)(k \otimes a) = L\left(k \otimes \alpha^{-1}\left(\varphi(e \otimes a)\right)\right)$$

on simple tensors, $k \in \mathbb{K}, a \in A$. Since the automorphism of $Q(\mathbb{K} \otimes \mathbb{K} \otimes A)$ which interchange the two copies of $\mathbb{K}$ is given by a unitary in $M(\mathbb{K} \otimes \mathbb{K} \otimes B)$, the extension $L \circ (\mathrm{id}_{\mathbb{K}} \otimes (\alpha^{-1} \circ \varphi \circ s))$ is unitarily equivalent to an extension $\psi : \mathbb{K} \otimes A \to Q(\mathbb{K} \otimes \mathbb{K} \otimes B)$ such that

$$\psi(k \otimes a) = L\left(e \otimes \alpha^{-1}\left(\varphi(k \otimes a)\right)\right)$$

on simple tensors. Since $L\left(e \otimes \alpha^{-1}\left(\varphi(k \otimes a)\right)\right) = \operatorname{Ad} q_{\mathbb{K}\otimes\mathbb{K}\otimes B}(V)\left(\varphi(k \otimes a)\right)$, we see that the two maps, (4.1) and (4.2) are inverses of each other, up to addition by 0. Since both maps clearly are semi-group homomorphisms, the proof is complete. □

Corollary 4.4. *Let A and B be separable C^*-algebras, B stable. Then all extensions of A by B are semi-invertible or strongly homotopy invertible if and only if the same is true for all extensions of $M_n(A)$ by B, for any $n \in \mathbb{N}$.*

Lemma 4.5. *Let A_1, A_2 and B be separable C^*-algebras, B stable. Assume that all extensions of A_i by B are semi-invertible or are strongly homotopy invertible invertible (with an invertible inverse), $i = 1, 2$. It follows that all extensions of $A_1 \oplus A_2$ by B have the same property.*

Proof. Let $p_i : A_1 \oplus A_2 \to A_i \subseteq A_1 \oplus A_2, i = 1, 2$, be the canonical projections, and consider an extension $\varphi : A_1 \oplus A_2 \to Q(B)$. By a standard rotation argument $\varphi \oplus 0$ is strongly homotopic to the sum $(\varphi \circ p_1) \oplus (\varphi \circ p_2)$. The conclusion follows from this by use of Theorem 2.1. □

By combining Corollary 4.4 and Lemma 4.5 we get the following.

Corollary 4.6. *Let A, F and B be separable C^*-algebras, B stable, F finite dimensional. Assume that all extensions of A by B are semi-invertible or are strongly homotopy invertible (with an invertible inverse). It follows that all extensions of $F \otimes A$ by B have the same property.*

In particular, it follows that if G is a countable discrete group with the property that all extensions of $C^*_r(G)$ by B are semi-invertible or strongly homotopy invertible (with an invertible inverse), then the same is true for $C^*_r(H \times B)$ for any finite group H.

Lemma 4.7. *Let A and B be separable C^*-algebras, B stable. Assume that all extensions of A by B are semi-invertible or strongly homotopy invertible. It follows that all extensions of $C(\mathbb{T}) \otimes A$ by B have the same property.*

Proof. Let χ be the automorphism of $C(\mathbb{T}) \otimes A$ such that $\chi(f)(z) = f(\overline{z})$ and let $\mathrm{ev} : C(\mathbb{T}) \otimes A \to A$ be evaluation at $1 \in \mathbb{T}$. As is wellknown the $*$-homomorphism $C(\mathbb{T}) \otimes A \to M_2(C(\mathbb{T}) \otimes A)$ defined such that

$$f \mapsto \begin{pmatrix} f & \\ & \chi(f) \end{pmatrix}$$

is homotopic to a $*$-homomorphism which factorizes through ev. It follows that for any extension $\varphi : C(\mathbb{T}) \otimes A \to Q(B)$ the extension $\varphi \oplus \varphi \circ \chi$ is strongly homotopic to an extension of the form $\psi \circ \mathrm{ev}$, where $\psi : A \to Q(B)$ is an extension of A by B. By assumption there is an extension ψ' of A by B such that $\psi \oplus \psi'$ is either asymptotically split or strongly homotopic to a split extension. It follows that $\varphi \oplus \varphi \circ \chi \oplus \psi' \circ \mathrm{ev}$ has the same property by Theorem 2.1. Hence φ is semi-invertible or strongly homotopy invertible, as the case may be. □

Proposition 4.8. *Let G be a countable discrete group, H a finite group and $k \in \mathbb{N}$. Let B be a separable stable C^*-algebra and assume that all extensions of $C_r^*(G)$, (resp. $C^*(G)$), by B are semi-invertible or strongly homotopy invertible. It follows that all extensions of $C_r^*\left(\mathbb{Z}^k \times H \times G\right)$, (resp. $C^*\left(\mathbb{Z}^k \times H \times G\right)$), by B have the same property.*

Proof. Note that $C_r^*\left(\mathbb{Z}^k \times H \times G\right) \simeq C\left(\mathbb{T}^k\right) \otimes C^*(H) \otimes C_r^*(G)$, and that $C^*(H)$ is finite dimensional. It follows then from Corollary 4.6 and Lemma 4.7 that all extensions of $C_r^*\left(\mathbb{Z}^k \times H \times G\right)$ by B are semi-invertible or strongly homotopy invertible if $C_r^*(G)$ has this property. The same argument works for the full group C^*-algebra. $\square$

Finally, we observe that it also possible to use Theorem 3.1 and Theorem 3.3 to prove semi-invertibility for extensions of the full group C^*-algebra of certain HNN-extensions by using the realization obtained by Ueda in [U] of such group C^*-algebras as amalgamated free products.

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