Oberwolfach Preprints

OWP 2011 - 03

J. MATTHEW DOUGLASS, GÖTZ PFEIFFER, AND GERHARD RÖHRLE

Coxeter Arrangements and Solomon's Descent Algebra

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA

J. MATTHEW DOUGLASS, GÖTZ PFEIFFER, AND GERHARD RÖHRLE

1. Introduction

Suppose V is a finite-dimensional, complex vector space. A linear transformation t in $\mathrm{GL}(V)$ is called a *reflection* if the fixed point set of t is a hyperplane in V, or equivalently, the 1-eigenspace of t has codimension 1. Suppose that $W \subset \mathrm{GL}(V)$ is a finite Coxeter group with Coxeter generating set S. Then each s in S acts on V as a reflection with order two and W is generated by S subject to the relations $(st)^{m_{s,t}} = 1$, where $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} > 1$ for $s \neq t$ in S. Let T denote the set of all reflections in W. For each t in T, let $H_t = \mathrm{Fix}(t)$ denote the fixed point set of t and set $\mathcal{A} = \{ H_t \mid t \in T \}$. Then $(V, \mathcal{A})$ is a hyperplane arrangement and W acts on the complement $M = V \setminus \cup_{t\in T} H_t$.

The action of W on M determines a representation of W on the singular cohomology of M. For $p \geq 0$ let $H^p(M)$ denote the p^{th} singular cohomology space of M with complex coefficients and let $H^\bullet(M) = \oplus_{p\geq 0} H^p(M)$ denote the total cohomology of M. It follows from a result of Brieskorn [3] that $\dim H^\bullet(M) = |W|$ and so a naive guess would be that the representation of W on $H^\bullet(M)$ is the regular representation of W. A simple computation for the symmetric group S_3 shows that this is not the case.

In 1986, Lehrer [12] computed the character of the representation of W on $H^\bullet(M)$ when W is a symmetric group . Subsequently, the character of W on $H^\bullet(M)$ was computed case-by-case for other types of Coxeter groups by various authors. In 2001, Blair and Lehrer [2] gave a case-free computation of the character of $H^\bullet(M)$. Recently, Felder and Veselov [6] have found an elegant description of this character that describes precisely how it differs from the regular character of W. Specifically Felder and Veselov show that the character of $H^\bullet(M)$ is given as

$$\sum_\sigma (2\,\mathrm{Ind}_{\langle\sigma\rangle}^G(1) - \rho),$$

where the sum runs over a set of *special* involutions in W and ρ is the regular character of W.

In contrast, while the representation of W on $H^\bullet(M)$ is well-understood, very little is known about the representations of W on the individual graded pieces $H^p(M)$. Lehrer and Solomon [13] have described these representations as sums of induced representations from linear characters of centralizers of elements in W when W is the symmetric group. They conjecture that a similar decomposition exists in general. The first author [4] extended Lehrer and Solomon's construction to hyperoctahedral groups and expressed each $H^p(M)$ as a sum

2010 *Mathematics Subject Classification.* Primary 20F55; Secondary 05E10, 52C35.

The authors would like to thank their charming wives for unwavering support during the preparation of this paper.

of representations induced from linear characters of subgroups. However, the subgroups appearing are not always centralizers of elements of W.

In this paper we state a conjecture that refines the conjecture of Lehrer and Solomon [13, Conjecture 1.6] and directly relates the representation of W on $H^p(M)$ and a subrepresentation of the right regular representation of W. We prove the conjecture when W is a symmetric group. The conjecture is true for all rank two Coxeter groups [5] and has been checked using the computer algebra system GAP [18] and package CHEVIE [7] for some low rank Coxeter groups.

The subrepresentations of the right regular representation we consider arise from the decomposition of a subalgebra of the group algebra of W, known as "Solomon's descent algebra," into projective, indecomposable modules. Projective, indecomposable modules in an artinian $\mathbb{C}$-algebra are generated by idempotents. These idempotents were first described explicitly by Bergeron, Bergeron, Howlett, and Taylor [1]. The second author [17] has subsequently used these idempotents to give quiver presentations for Coxeter groups. In §5.2 we show that when W is a symmetric group the subrepresentations of the right regular representation generated by these idempotents are induced representations. Schocker [19] has proved a similar result using different methods.

In section §2 of this paper we state our conjecture, in section §3 we prove some preliminary general results, and in §4 and §5 we prove the conjecture when W is a symmetric group.

2. The Orlik-Solomon algebra and Solomon's descent algebra

We assume that W is a subgroup of the unitary group of V with respect to a fixed positive, definite, Hermitian form $\langle \cdot\,, \cdot \rangle$. It is known that V has a basis $\Pi = \{\, \alpha_s \mid s \in S \,\}$ indexed by S so that $\langle \alpha_s, \alpha_t \rangle = -\cos(\pi/m_{s,t})$ for s and t in S. Then s acts on V as the reflection through the hyperplane orthogonal to α_s and $\Phi = \{\, w(\alpha_s) \mid w \in W, s \in S \,\}$ is a root system in V.

2.1. **Shapes and conjugacy classes.**

We begin by recalling a parameterization of the conjugacy classes in W due to Geck and Pfeiffer (see [8, §3.2]) in a form compatible with the arrangement $(V, \mathcal{A})$ of W.

The *lattice of* $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of subspaces of V that arise as intersections of hyperplanes in $\mathcal{A}$:

$$L(\mathcal{A}) = \{\, H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_p} \mid t_1, t_2, \ldots, t_p \in T \,\}.$$

For X in $L(\mathcal{A})$ define

$$W_X = \{\, w \in W \mid X \subseteq \mathrm{Fix}(w) \,\}$$

to be the pointwise stabilizer of X in W. It follows from Steinberg's Theorem [23] that W_X is generated by $\{t \in T \mid X \subseteq H_t\,\}$. It then follows that $X = \mathrm{Fix}(W_X)$, and so the rule $X \mapsto W_X$ defines an injection from $L(\mathcal{A})$ to the set of subgroups of W. Notice that W_X is again a Coxeter group.

The action of W on $\mathcal{A}$ induces an action of W on $L(\mathcal{A})$. Obviously $wW_Xw^{-1} = W_{w(X)}$ and so for X and Y in $L(\mathcal{A})$, the subgroups W_X and W_Y are conjugate if and only if X and Y lie in the same W-orbit. Thus, the rule $X \mapsto W_X$ induces a bijection between the set of orbits of W on $L(\mathcal{A})$ and the set of conjugacy classes of subgroups W_X.

By a *shape of* W we mean a W-orbit in $L(\mathcal{A})$. We denote the set of shapes of W by Λ. For example, if W is the symmetric group S_n, then Λ is in bijection with the set of partitions of n and with the set of Young diagrams with n boxes.

It is shown in [16, §6.2] that the rule $w \mapsto \mathrm{Fix}(w)$ defines a surjection from W to $L(\mathcal{A})$. Composing with the map that sends an element X in $L(\mathcal{A})$ to its W-orbit, we get a map

$$\mathrm{sh}\colon W \to \Lambda.$$

We say $\mathrm{sh}(w)$ is the *shape of* w. Thus, $\mathrm{sh}(w)$ is the W-orbit of $\mathrm{Fix}(w)$ in $L(\mathcal{A})$. Clearly, sh is constant on conjugacy classes and so we can define the *shape* of a conjugacy class to be the shape of any of its elements.

An element w in W, or its conjugacy class, is called *cuspidal* if $\mathrm{Fix}(w) = \mathrm{Fix}(W)$. For example, if W is the symmetric group S_n, then the conjugacy class consisting of n-cycles is the only cuspidal class. In general, there is more than one cuspidal conjugacy class. Cuspidal elements and conjugacy classes are called *elliptic* by some authors.

Suppose that λ is a shape, X in $L(\mathcal{A})$ has shape λ, and C is a conjugacy class in W with shape λ. If w is in C, then $\mathrm{Fix}(w)$ is in the W-orbit of X and so $C \cap W_X$ is a non-empty union of cuspidal W_X-conjugacy classes. Geck and Pfeiffer [8, §3.2] have shown that in fact $C \cap W_X$ is a single W_X-conjugacy class. It follows that $C \mapsto C \cap W_X$ defines a bijection between the set of conjugacy classes in W with shape λ and the set of cuspidal conjugacy classes in W_X.

Fix a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of W-orbit representatives in $L(\mathcal{A})$. Summarizing the preceding discussion we see that conjugacy classes in W are parametrized by pairs (λ, C_λ) where λ is a shape and C_λ is a cuspidal conjugacy class in W_{X_λ}.

2.2. **The Orlik-Solomon algebra.** Next we consider the cohomology ring $H^\bullet(M)$. Arnold and Brieskorn (see [16, §1.1]) have computed $H^\bullet(M)$. In the following we use the presentation of this algebra given by Orlik and Solomon [14].

Recall that the set T of reflections in W parametrizes the hyperplanes in $\mathcal{A}$. The *Orlik-Solomon algebra of* W is the $\mathbb{C}$-algebra, $A = A(\mathcal{A})$, with generators $\{\, a_t \mid t \in T \,\}$ and relations

- $a_{t_1} a_{t_2} = -a_{t_2} a_{t_1}$ for t_1 and t_2 in T and
- $\sum_{i=1}^{p} (-1)^i a_{t_1} \cdots \widehat{a_{t_i}} \cdots a_{t_p} = 0$ for every linearly dependent subset $\{\, H_{t_1}, \dots, H_{t_p} \,\}$ of $\mathcal{A}$.

The algebra A is a skew-commutative, graded, connected $\mathbb{C}$-algebra that is isomorphic as a graded algebra to $H^\bullet(M)$. Let A^p denote the degree p subspace of A. Then

- $A^0 \cong \mathbb{C}$,
- for $1 \leq p \leq |S|$ the subspace A^p is spanned by the set of all $a_{t_1} \cdots a_{t_p}$ where $\operatorname{codim} H_{t_1} \cap \cdots \cap H_{t_p} = p$, and

- $A^p = 0$ for $p > |S|$.

See [16, §3.1] for details.

The action of W on $\mathcal{A}$ extends to an action of W on A as algebra automorphisms. An element w in W acts on a generator a_t of A by $wa_t = a_{wtw^{-1}}$. With this W-action A is isomorphic to $H^\bullet(M)$ as graded W-algebras.

Orlik and Solomon [15] have shown that the normalizer of W_X in W is the setwise stabilizer of X in W, that is

$$N_W(W_X) = \{\, w \in W \mid w(X) = X \,\}.$$

For X in $L(\mathcal{A})$, define A_X to be the span of $\{\, a_{t_1} \cdots a_{t_r} \mid H_{t_1} \cap \cdots \cap H_{t_r} = X \,\}$. Proofs of the following statements may be found in [16, Chapter 6].

- If $\operatorname{codim} X = p$, then $A_X \subseteq A^p$.
- There are vector space decompositions $A^p \cong \bigoplus_{\operatorname{codim} X = p} A_X$ and $A \cong \bigoplus_{X \in L(\mathcal{A})} A_X$.
- For w in W, $wA_X = A_{w(X)}$. Thus, A_X is an $N_W(W_X)$-stable subspace of A.

For a shape λ in Λ, set $A_\lambda = \bigoplus_{X \in \lambda} A_X$. Suppose X is a fixed subspace in λ and that $\operatorname{codim} X = p$. Then A_λ is a W-stable subspace of A^p and as representations of W we have

$$A_\lambda \cong \operatorname{Ind}_{N_W(W_X)}^W (A_X) \quad \text{and} \qquad A \cong \bigoplus_{\lambda \in \Lambda} A_\lambda$$

(see [13]).

2.3. **Solomon's descent algebra.** In contrast with the Orlik-Solomon algebra A, which is constructed from the set T of reflections in W, Solomon's descent algebra is constructed from the Coxeter generating set S of W.

Suppose that I is a subset of S. Define

$$X_I = \bigcap_{s \in I} H_s \qquad \text{and} \qquad W_I = \langle I \rangle.$$

Then X_I is in $L(\mathcal{A})$ and $\operatorname{codim} X_I = |I|$. It follows from Steinberg's Theorem [23] that $W_I = W_{X_I}$. Recall that $\Pi = \{\, \alpha_s \mid s \in S \,\}$ is a basis of V. For $I \subseteq S$ define

$$\Pi_I = \{\, \alpha_s \mid s \in I \,\}.$$

Then X_I is the orthogonal complement of the span of Π_I.

Orlik and Solomon (see [16, §6.2]) have shown that each orbit of W on $L(\mathcal{A})$ contains a subspace X_I for some subset I of S. For subsets I and J of S define $I \sim J$ if there is a w in W with $w(\Pi_I) = \Pi_J$. Then $\sim$ is an equivalence relation. It is well-known that W_I and W_J are conjugate if and only if $I \sim J$ (see [21]). It follows that the rule $I \mapsto X_I$ induces a bijection between $S/\sim$, the set of $\sim$-equivalence classes, and Λ, the set of shapes of W.

Next, let ℓ denote the length function of W determined by the generating set S and define

$$W^I = \{\, w \in W \mid \ell(ws) > \ell(w) \, \forall \, s \in I \,\}.$$

Then W^I is a set of left coset representatives of W_I in W. Also, define

$$x_I = \sum_{w \in W^I} w$$

in the group algebra $\mathbb{C}W$. Solomon [20] has shown that the span of $\{ x_I \mid I \subseteq S \}$ is in fact a subalgebra of $\mathbb{C}W$. We denote this subalgebra by $\Sigma(W)$ and call it the *descent algebra* of W. It is not hard to see that $\{ x_I \mid I \subseteq S \}$ is linearly independent and so $\dim \Sigma(W) = 2^{|S|}$. Notice that $x_S = 1$ is the identity in both $\mathbb{C}W$ and its subalgebra $\Sigma(W)$.

Bergeron, Bergeron, Howlett, and Taylor [1] have defined a basis $\{ e_I \mid I \subseteq S \}$ of $\Sigma(W)$ that consists of quasi-idempotents and is compatible with the set of shapes of W. For λ in Λ define

$$S_\lambda = \{ I \subseteq S \mid X_I \in \lambda \} \qquad \text{and} \qquad e_\lambda = \sum_{I \in S_\lambda} e_I.$$

Then each e_λ is idempotent and $\{ e_\lambda \mid \lambda \in \Lambda \}$ is a complete set of primitive, orthogonal idempotents in $\Sigma(W)$. (See §3 for more details.) In particular, $\sum_{\lambda \in \Lambda} e_\lambda = 1$ in $\mathbb{C}W$.

Define $E_\lambda = e_\lambda \mathbb{C}W$. In §3 we show that $e_I \mathbb{C}W_I$ affords an action of $N_W(W_I)$ and that E_λ is induced from $e_I \mathbb{C}W_I$ when I is in S_λ . Thus, in analogy with the decomposition in §2.2 of the Orlik-Solomon algebra A, as representations of W we have

$$E_\lambda \cong \operatorname{Ind}_{N_W(W_I)}^W (e_I \mathbb{C}W_I) \qquad \text{and} \qquad \mathbb{C}W = \bigoplus_{\lambda \in \Lambda} E_\lambda.$$

2.4. **Centralizers and complements.** The last ingredient we need in order to state the conjecture is a certain set of characters of centralizers of elements of W. These characters, together with the sign character, should quantify the difference between the representation of W on $H^p(M)$ and a subrepresentation of the regular representation. They naturally arise in work of Howlett and Lehrer [10] and in recent results of the second author [11] that describe the structure of the centralizer of an element in W.

Suppose that I is a subset of S and C is a conjugacy class in W such that $C \cap W_I$ is a cuspidal conjugacy class in W_I. Howlett [9] has shown that W_I has a complement, N_I, in $N_W(W_I)$. Moreover, it is shown in [11] that if c is in $C \cap W_I$, then $Z_W(c) \subseteq N_W(W_I)$ and $Z_{W_I}(c) W_I = N_W(W_I)$. It follows that

(1) for any X in $L(\mathcal{A})$, W_X has a complement, say N_X, in $N_W(W_X)$ such that

$$N_W(W_X) \cong W_X \rtimes N_X,$$

and

(2) for a cuspidal element c in W_X, $Z_W(c) \subseteq N_W(W_X)$ and $Z_W(c)/Z_{W_X}(c) \cong N_X$.

Recall that $N_W(W_X)$ is the setwise stabilizer of X in W. We may define a linear character $\alpha_X \colon N_W(W_X) \to \mathbb{C}$ by $\alpha_X(n) = \det n|_X$ for n in $N_W(W_X)$. For c in W, we define α_c to be the restriction of α_X to $Z_W(c)$ when $X = \operatorname{Fix}(c)$.

2.5. **Relating the Orlik-Solomon algebra and the descent algebra.** We now have all the concepts we need in order to state the conjecture.

Let ϵ denote the sign character of W. For c in W, define $X_c = \operatorname{Fix}(c)$, $\operatorname{rk}(c) = \operatorname{codim} X_c$, and $W_c = W_{X_c}$. Notice that c is a cuspidal element in W_c.

Associated with each λ in Λ we have the W-stable subspace A_λ of the Orlik-Solomon algebra A, a right ideal E_λ in $\mathbb{C}W$, and the set of conjugacy classes with shape λ. We conjecture that A_λ and E_λ are related to the set of conjugacy classes with shape λ as follows.

Conjecture 2.1. *Choose a set $\mathcal{C}$ of conjugacy class representatives in W. For λ in Λ set $\mathcal{C}_\lambda = \{\, c \in \mathcal{C} \mid \operatorname{sh}(c) = \lambda \,\}$. Then, for each conjugacy class representative c in $\mathcal{C}$, there is a linear character ϕ_c of $Z_W(c)$, such that for every λ in Λ,*

(a) the character of E_λ is

$$\bigoplus_{c\in\mathcal{C}_\lambda} \operatorname{Ind}_{Z_W(c)}^W(\phi_c) = \bigoplus_{c\in\mathcal{C}_\lambda} \operatorname{Ind}_{N_W(W_c)}^W(\operatorname{Ind}_{Z_W(c)}^{N_W(W_c)}(\phi_c))$$

and

(b) the character of A_λ is

$$\bigoplus_{c\in\mathcal{C}_\lambda} \operatorname{Ind}_{Z_W(c)}^W(\epsilon_c\alpha_c\phi_c) = \epsilon \bigoplus_{c\in\mathcal{C}_\lambda} \operatorname{Ind}_{N_W(W_c)}^W(\alpha_{X_c} \operatorname{Ind}_{Z_W(c)}^{N_W(W_c)}(\phi_c)),$$

where ϵ_c denotes the restriction of ϵ to $Z_W(c)$.

In particular,

$$H^p(M) \cong A^p \cong \bigoplus_{\operatorname{rk}(c)=p} \operatorname{Ind}_{Z_W(c)}^W(\epsilon_c\alpha_c\phi_c)$$

for $0 \leq p \leq |S|$, and

$$\mathbb{C}W \cong \bigoplus_{c\in\mathcal{C}} \operatorname{Ind}_{Z_W(c)}^W(\phi_c) \quad \textit{and} \quad H^\bullet(M) \cong \bigoplus_{c\in\mathcal{C}} \operatorname{Ind}_{Z_W(c)}^W(\epsilon_c\alpha_c\phi_c).$$

We show in Corollary 3.5 that $\dim A_\lambda$ is the number of elements in W with shape λ. Bergeron, Bergeron, Howlett, and Taylor [1] have shown that $\dim E_\lambda$ is also the number of elements in W with shape λ. Thus $\dim A_\lambda = \dim E_\lambda$.

As stated in the introduction, the main result in this paper is a proof of Conjecture 2.1 for symmetric groups. The conjecture is known to be true for all Coxeter groups with rank at most two [5].

We in fact prove a slightly stronger result than is stated in the conjecture. We show that the character ϕ_c of $Z_W(c)$ may be chosen to be a character of the normal subgroup $Z_{W_c}(c)$ of $Z_W(c)$ that is N_{X_c}-stable and extends to a character of $Z_W(c)$.

3. E_λ IS AN INDUCED REPRESENTATION

Suppose λ is in Λ. We have seen in §2.2 that $A_\lambda \cong \operatorname{Ind}_{N_W(W_X)}^W(A_X)$ for X in λ. In this section we show that E_λ has a similar description as an induced representation for I in S_λ. We begin by recalling some results of Bergeron, Bergeron, Howlett, and Taylor [1].

For subsets I, J, and K of S define

$$W^{IJ} = \big(W^I\big)^{-1} \cap W^J \qquad \text{and} \qquad W^{IJK} = \{\, w \in W^{IJ} \mid w^{-1}(\Pi_I) \cap \Pi_J = \Pi_K \,\}.$$

Then W^{IJ} is the set of minimal length (W_I, W_J)-double coset representatives in W. Solomon [20] has shown that $x_I x_J = \sum_K a_{IJK} x_K$ where $a_{IJK} = |W^{IJK}|$.

The quasi-idempotents e_I in §2.3 are defined as follows. For subsets J and K of S set

$$m_{JK} = \left| \coprod_{L \sim J} \{\, w \in W^{LK} \mid w^{-1}(\Pi_L) \cap \Pi_K = \Pi_J \,\} \right| .$$

Then $m_{JK} = 0$ when $J \not\subseteq K$ and $m_{JK} = |\{\, w \in W^K \mid w(\Pi_J) \subseteq \Pi \,\}|$ when $J \subseteq K$. Moreover, $m_{JJ} \neq 0$ for all J and so the system of equations

$$x_K = \sum_{J \subseteq S} m_{JK} e_J, \qquad K \subseteq S$$

can be solved for $\{\, e_J \mid J \subseteq S \,\}$. Define n_{JK} and e_K by

$$e_K = \sum_{J \subseteq S} n_{JK} x_J.$$

Notice that $n_{JJ} = m_{JJ}^{-1}$ and $n_{JK} = 0$ when $J \not\subseteq K$.

The next two lemmas give some translation properties for the quantities just defined.

Lemma 3.1. *Suppose that $K \subseteq S$ and d is in W with $d^{-1}(\Pi_K) \subseteq \Pi$.*

(a) $W^K d = W^{K^d}$.

(b) $x_K d = x_{K^d}$.

(c) $m_{I^d J^d} = m_{IJ}$ *for* $I \subseteq J \subseteq K$.

(d) $e_L d = e_{L^d}$ *for* $L \subseteq K$.

Proof. Statement (a) is proved in [1, Lemma 2.4]. Statement (b) follows immediately from (a).

Suppose that $I \subseteq J \subseteq K$. Clearly, $\Pi_{L^d} = d^{-1}(\Pi_L)$ for all $L \subseteq K$ and so

$$\begin{aligned} m_{I^d J^d} &= |\{\, w \in W^{J^d} \mid w(\Pi_{I^d}) \subseteq \Pi \,\}| \\ &= |\{\, w \in W^J d \mid w d^{-1}(\Pi_I) \subseteq \Pi \,\}| \\ &= |\{\, w d^{-1} \in W^J \mid w d^{-1}(\Pi_I) \subseteq \Pi \,\}| \\ &= |\{\, w \in W^J \mid w(\Pi_I) \subseteq \Pi \,\}| \\ &= m_{IJ}. \end{aligned}$$

This proves (c).

Using (b) and (c) we see that for $J \subseteq K$,

$$\sum_I m_{IJ}(e_I d) = x_J d = x_{J^d} = \sum_I m_{IJ^d} e_I = \sum_I m_{{}^dIJ} e_I = \sum_I m_{IJ} e_{I^d}.$$

Thus, $\sum_I m_{IJ}(e_I d) = \sum_I m_{IJ} e_{I^d}$. Now, fix a subset L of K, multiply both sides by n_{JL}, and sum over J, to get $e_L d = e_{L^d}$. (Note that $n_{JL} = 0$ unless $J \subseteq L$.) This proves (d). ☐

Lemma 3.2. *Suppose n is in $\mathrm{GL}(V)$ with $n(\Pi) = \Pi$. Then $n^{-1} e_I n = e_{I^n}$ for $I \subseteq S$. In particular, n centralizes e_S in $\mathbb{C}W$.*

Proof. It follows from the assumption that $n(\Pi) = \Pi$ that $\ell(nwn^{-1}) = \ell(w)$ for all w in W. Therefore, $n^{-1} W^I n = W^{I^n}$ and hence $n^{-1} x_I n = x_{I^n}$ for all $I \subseteq S$.

Suppose $I \subseteq J \subseteq S$. Then

$$\begin{aligned} m_{I^n J^n} &= |\{\, w \in W^{J^n} \mid w(\Pi_{I^n}) \subseteq \Pi \,\}| \\ &= |\{\, w \in n^{-1} W^J n \mid wn^{-1}(\Pi_I) \subseteq \Pi \,\}| \\ &= |\{\, nwn^{-1} \in W^J \mid nwn^{-1}(\Pi_I) \subseteq n(\Pi) \,\}| \\ &= |\{\, w \in W^J \mid w(\Pi_I) \subseteq \Pi \,\}| \\ &= m_{IJ}. \end{aligned}$$

Thus, we see that for $K \subseteq S$,

$$\sum_J m_{JK}(n^{-1} e_J n) = n^{-1} x_K n = x_{K^n} = \sum_J m_{JK^n} e_J = \sum_J m_{{}^nJK} e_J = \sum_J m_{JK} e_{J^n}.$$

Therefore, $\sum_J m_{JK}(n^{-1} e_J n) = \sum_J m_{JK} e_{J^n}$. Now, fix a subset I of S, multiply both sides by n_{KI}, and sum over K, to get $n^{-1} e_I n = e_{I^n}$. ☐

Suppose λ is in Λ. Recall that $e_\lambda = \sum_{I \in S_\lambda} e_I$ and $E_\lambda = e_\lambda \mathbb{C}W$. It is shown in [1, §7] that if I and J are in S_λ, then

$$e_I e_J = \tfrac{1}{|S_\lambda|} e_J.$$

It follows that e_λ is an idempotent. It is also shown in [1, §7] that $1 = \sum_{\lambda \in \Lambda} e_\lambda$ and that $\{\, e_\lambda \mid \lambda \in \Lambda \,\}$ is a complete set of pairwise orthogonal, primitive idempotents in $\Sigma(W)$. Since $\Sigma(W)$ is a subalgebra of $\mathbb{C}W$, it follows that $\{\, e_\lambda \mid \lambda \in \Lambda \,\}$ is a set of pairwise orthogonal idempotents in $\mathbb{C}W$.

Lemma 3.3. *Suppose that λ is in Λ and I is in S_λ. Then, $E_\lambda = e_I \mathbb{C}W$.*

Proof. Say $S_\lambda = \{J_1, \dots, J_r\}$. For $1 \le i \le r$, choose d_i in W with $J_i = I^{d_i}$. Then $e_{J_i} = e_{I^{d_i}} = e_I d_i$ by Lemma 3.1(d). Therefore, $e_\lambda = e_I(d_1 + \cdots + d_r)$ and so $E_\lambda \subseteq e_I \mathbb{C}W$. On the other hand,

$$e_\lambda e_I = (e_{J_1} + \cdots + e_{J_r}) e_I = e_{J_1} e_I + \cdots + e_{J_r} e_I = \tfrac{1}{r} e_I + \cdots + \tfrac{1}{r} e_I = e_I,$$

and so $e_I \mathbb{C}W = e_\lambda e_I \mathbb{C}W \subseteq e_\lambda \mathbb{C}W$. ☐

We next compute the dimensions of the various spaces we are studying in terms of shapes.

Lemma 3.4. *Suppose that λ is a shape in Λ, X is a subspace in λ, and C is a conjugacy class in W with shape λ. Then*

(a) $|C| = |W : N_W(W_X)|\,|C \cap W_X|$ *and*

(b) $|\mathrm{sh}^{-1}(\lambda)| = |W : N_W(W_X)|\,|\mathrm{sh}^{-1}(\lambda) \cap W_X|$.

Proof. Notice that $C \cap W_X$ is a cuspidal conjugacy class in W_X. Thus, it follows from (1) and (2) in §2.4 that $|N_W(W_X) : Z_W(c)| = |W_X : Z_{W_X}(c)|$ for c in C. Therefore

$$|C| = |W : N_W(W_X)|\,|N_W(W_X) : Z_W(c)| = |W : N_W(W_X)|\,|C \cap W_X|.$$

This proves (a). Statement (b) follows from (a) and the observation that $\mathrm{sh}^{-1}(\lambda)$ is the union of those conjugacy classes in W whose intersection with W_X is a cuspidal conjugacy class in W_X. □

The quasi-idempotents e_I are defined relative to the ambient set S. We use a superscript to indicate this ambient set when it is not equal S. Thus, for $I \subseteq J \subseteq S$, e_I^J denotes the quasi-idempotent in $\mathbb{C}W_J$ defined using J as the ambient set instead of S.

Corollary 3.5. *Suppose that λ is in Λ and I is in S_λ. Then*

(a) $\dim A_{X_I} = \dim e_I \mathbb{C}W_I = |\mathrm{sh}^{-1}(\lambda) \cap W_I|$ *and*

(b) $\dim A_\lambda = \dim E_\lambda = |\mathrm{sh}^{-1}(\lambda)|$.

Proof. It is clear that $\{\, w \in W \mid \mathrm{Fix}(w) = X_I \,\} = \mathrm{sh}^{-1}(\lambda) \cap W_I$ is the set of all cuspidal elements in W_I. It is shown in [4, Proposition 2.4] that $\dim A_{X_I} = |\{\, w \in W \mid \mathrm{Fix}(w) = X_I \,\}|$. Therefore, $\dim A_{X_I} = |\mathrm{sh}^{-1}(\lambda) \cap W_I|$, and it follows from Lemma 3.4(b) that $\dim A_\lambda = |\mathrm{sh}^{-1}(\lambda)|$. As remarked above, Bergeron, Bergeron, Howlett, and Taylor [1] have shown that $\dim E_\lambda = |\mathrm{sh}^{-1}(\lambda)|$. This proves (b).

To complete the proof of (a) it remains to show that $\dim e_I \mathbb{C}W_I = |\mathrm{sh}^{-1}(\lambda) \cap W_I|$. It is shown in [1, Proposition 7.3] that e_I factors as $e_I = x_I e_I^I$. Thus, $e_I \mathbb{C}W_I = x_i e_I^I \mathbb{C}W_I$. Also, it follows from [1, Theorem 7.15] that $\dim e_I^I \mathbb{C}W_I = |\mathrm{sh}^{-1}(\lambda) \cap W_I|$. Because W^I is a complete set of left coset representatives for W_I in W, it is clear that left multiplication by x_I defines an isomorphism of right $\mathbb{C}W_I$-modules $e_I^I \mathbb{C}W_I \cong e_I \mathbb{C}W_I$. Therefore, $\dim e_I \mathbb{C}W_I = |\mathrm{sh}^{-1}(\lambda) \cap W_I|$ as desired. □

We can now show that E_λ is an induced representation.

Proposition 3.6. *Suppose that λ is in Λ and I is in S_λ.*

(a) *$N_W(W_I)$ acts on $e_I \mathbb{C}W_I$ by right multiplication and $E_\lambda \cong \mathrm{Ind}_{N_W(W_I)}^W (e_I \mathbb{C}W_I)$.*

(b) *$N_W(W_X)$ acts on A_X and $A_\lambda \cong \mathrm{Ind}_{N_W(W_X)}^W (A_X)$.*

Proof. Statement (b) is proved in [13, §2]. We prove (a).

Recall that $N_W(W_I) = W_I N_I$. Obviously $e_I \mathbb{C}W_I$ is stable under right multiplication by W_I. We have seen in the proof of Corollary 3.5 that $e_I \mathbb{C}W_I = x_i e_I^I \mathbb{C}W_I$. If n is in N_I, then $n(\Pi_I) = \Pi_I$ and it follows from Lemmas 3.1 and 3.2 that

$$e_I \mathbb{C}W_I n = e_I n \mathbb{C}W_I = (x_I n)(n^{-1} e_I^I n)\mathbb{C}W_I = x_I e_I^I \mathbb{C}W_I = e_I \mathbb{C}W_I.$$

Therefore, $e_I \mathbb{C}W_I$ is stable under right multiplication by $N_W(W_I)$.

Since $E_\lambda = e_I \mathbb{C}W$, by Lemma 3.3, to prove that $E_\lambda \cong \operatorname{Ind}_{N_W(W_I)}^W (e_I \mathbb{C}W_I)$ it is enough to show that the multiplication map $e_I \mathbb{C}W_I \otimes_{\mathbb{C}N_W(W_I)} \mathbb{C}W \to E_\lambda$ is a bijection. The map is obviously a surjection. Moreover, using Lemma 3.4 and Corollary 3.5, we have

$$\begin{aligned} \dim E_\lambda &= |\mathrm{sh}^{-1}(\lambda)| \\ &= |W : N_W(W_I)|\, |\mathrm{sh}^{-1}(\lambda) \cap W_I| \\ &= |W : N_W(W_I)| \dim e_I \mathbb{C}W_I \\ &= \dim e_I \mathbb{C}W_I \otimes_{\mathbb{C}N_W(W_I)} \mathbb{C}W \end{aligned}$$

and so the multiplication map is a bijection. □

4. Symmetric groups: $\lambda = (n)$

The rest of this paper is devoted to the proof of Conjecture 2.1 for symmetric groups.

From now on, we take W to be the symmetric group on n letters with $n \geq 2$ and we identify W with the subgroup of $\mathrm{GL}_n(\mathbb{C})$ that acts on the basis $\{v_1, v_2, \ldots, v_n\}$ as permutations. Here, v_i is the column vector whose j^{th} entry is 0 for $j \neq i$ and 1 for $j = i$. For $1 \leq i \leq n-1$ let s_i denote the matrix in W that interchanges v_i and v_{i+1} and fixes v_j for $j \neq i, i+1$. Then $S = \{s_1, s_2, \ldots, s_{n-1}\}$ is a Coxeter generating set for W.

By a *partition of* n we mean a non-increasing finite sequence of positive integers whose sum is n. Say $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a partition of n. Then $\lambda_1 \geq \cdots \geq \lambda_p > 0$ and $\sum_{k=1}^p \lambda_k = n$. The integers λ_i are called the *parts of* λ.

It is well-known that for $W = S_n$ we may identify Λ with the set of partitions of n. We make this identification precise as follows. Suppose that λ is a partition of n with p parts. Define partial sums τ_i for $i = 0, 1, \ldots, p$ by $\tau_0 = 0$ and $\tau_i = \lambda_1 + \cdots + \lambda_i$ for $1 \leq i \leq p$. Define

$$I_\lambda = S \setminus \{s_{\tau_1}, s_{\tau_2}, \ldots, s_{\tau_{p-1}}\} \qquad \text{and} \qquad W_\lambda = \langle I_\lambda \rangle.$$

Then W_λ is isomorphic to the product of symmetric groups $S_{\lambda_1} \times \cdots \times S_{\lambda_p}$, where the factor S_{λ_i} acts on the subset $\{v_{\tau_{i-1}+1}, v_{\tau_{i-1}+2}, \ldots, v_{\tau_i}\}$ of $\{v_1, v_2, \ldots, v_n\}$. Next, define

$$X_\lambda = \mathrm{Fix}(W_\lambda).$$

Then X_λ is in $L(\mathcal{A})$ and $W_{X_\lambda} = W_\lambda$. We have seen in Proposition 3.6 that

$$E_\lambda \cong \operatorname{Ind}_{N_W(W_\lambda)}^W (e_{I_\lambda} \mathbb{C}W_\lambda) \quad \text{and} \quad A_\lambda \cong \operatorname{Ind}_{N_W(W_\lambda)}^W (A_{X_\lambda}).$$

It is well-known and straightforward to check that $\{\, X_\lambda \mid \lambda \text{ is a partition of } n \,\}$ is a complete set of orbit representatives for the action of W on $L(\mathcal{A})$ and that $\{\, I_\lambda \mid \lambda \text{ is a partition of } n \,\}$ is a complete set of representatives for $S/\sim$.

Notice that in the extreme case when all parts of λ are equal 1 we have $I_\lambda = \emptyset$ and $W_\lambda = W_\emptyset = \{1\}$. At the other extreme, when $\lambda = (n)$, we have $I_\lambda = S$ and $W_\lambda = W_S = W$. We first prove Conjecture 2.1 when $\lambda = (n)$.

For the rest of this section we take $\lambda = (n)$. Then $W_\lambda = N_W(W_\lambda) = W$ and so $E_{(n)} = e_{I_{(n)}}\mathbb{C}W_{(n)}$ and $A_{(n)} = A_{X_{(n)}}$. Moreover, $A_{X_{(n)}} = A^{n-1}$ is the top, non-zero graded piece of A. To simplify the notation, we denote $A_{(n)}$, $E_{(n)}$, and $e_{I_{(n)}}$ by A_n, E_n, and e_n respectively.

Define $c_1 = 1$ in W and for $1 \leq i \leq n$ define $c_i = s_{i-1}\cdots s_2 s_1$, so c_i acts on the basis $\{v_1, v_2, \dots, v_n\}$ as an i-cycle. Also, set $c = c_n$. Then,

- c is a cuspidal element in W,
- the set of cuspidal elements in W is precisely the conjugacy class of c, and
- $Z_W(c) = \langle c \rangle$ is the cyclic group of order n generated by c.

Set $\zeta = e^{2\pi i/n}$ in $\mathbb{C}$ and define $\phi\colon Z_W(c) \to \mathbb{C}$ by $\phi(c^{-1}) = \zeta$. The elements we have denoted by c_i are denoted by c_i^{-1} by Lehrer and Solomon [13]. However, the character ϕ of $Z_W(c)$ is the same as in [13].

Theorem 4.1. *With the preceding notation we have that*

(a) the character of W on E_n is $\operatorname{Ind}_{Z_W(c)}^W(\phi)$ and

(b) the character of W on A_n is $\epsilon\operatorname{Ind}_{Z_W(c)}^W(\phi)$.

Notice that with the notation of §2.1, we have $Z_c = Z_W(c)$ and so $\phi = \phi_c$.

Statement (b) has been proved by Stanley [22, Theorem 7.2] and by Lehrer and Solomon [13, Theorem 3.9]. Our proof below that the character of W on E_n is $\operatorname{Ind}_{Z_W(c)}^W(\phi)$ follows the Lehrer-Solomon argument. To emphasize and differentiate the parallel arguments, we use the convention that the superscript $+$ denotes quantities associated with E_n and the superscript $-$ denotes quantities associated with A_n.

Suppose t is an indeterminate. For $0 \leq k \leq n$, define elements $b^+(n,k)$ and $b^-(n,k)$ in $\mathbb{C}W$ by

$$(1 - c_1 t)(1 - c_2 t)\cdots(1 - c_n t) = \sum_{k=0}^{n} b^+(n,k)t^k$$

and

$$(1 + c_n t)(1 - c_{n-1} t)\cdots(1 + (-1)^{n-1} c_1 t) = \sum_{k=0}^{n} b^-(n,k)t^k$$

respectively (the k^{th} factor in the product on the left-hand side of the last equation is $(1 + (-1)^{k-1} c_{n-k+1} t)$).

Set $W_{n-1} = \langle s_1, s_2, \dots, s_{n-2} \rangle$. Then $W_{n-1} \cong S_{n-1}$. The analog of the idempotent e_n in E_n is the basis element $a_n = a_{s_1} a_{s_2} \cdots a_{s_{n-1}}$ in $A_n = A^{n-1}$. Lehrer and Solomon [13, §3] prove the following statements.

(i) $A_n = \mathbb{C}W a_n$.

(ii) $c^{-k}a_n = b^-(n-1,k)a_n$ for $0 \le k \le n-1$. In particular, $A_n = \mathbb{C}W_{n-1}a_n$.

(iii) Consider the homomorphism of left $\mathbb{C}W$-modules from $\mathbb{C}W$ to A_n given by right multiplication by a_n. The kernel of this mapping is the left $\mathbb{C}W_{n-1}$-module generated by $\{\, c^{-k} - b^-(n-1,k) \mid 0 \le k \le n-1 \,\}$.

(iv) $\{\, wa_n \mid w \in W_{n-1} \,\}$ is a $\mathbb{C}$-basis of A_n and A_n is the left regular $\mathbb{C}W_{n-1}$-module.

Next we show that the analogous statements hold with A_n replaced by E_n and $b^-(n,k)$ replaced by $b^+(n,k)$.

For $k = 1, 2, \dots, n-1$, define $x_k = x_{S\setminus\{s_k\}}$ and $w_k = c_1c_2\cdots c_k$. Then w_k is the longest element in $\langle s_1, s_2, \dots, s_{k-1}\rangle$.

Lemma 4.2. *Suppose* $1 \le k \le n-1$. *Then*

$$W^{S\setminus\{s_k\}}w_k = \{\, c_{i_1}\cdots c_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n \,\}.$$

Proof. It suffices to show that if $1 \le i_1 < \cdots < i_k \le n$, then $c_{i_1}\cdots c_{i_k}w_k$ is in $W^{S\setminus\{s_k\}}$. For this, we consider elements in W as acting on $\{1,\dots,n\}$. That is, we identify the vector v_j with j for $1 \le j \le n$. Then

$$W^{S\setminus\{s_k\}} = \{\, w \in W \mid w(j) < w(j+1)\,\forall\, j \in \{1,\dots,n-1\} \setminus \{k\} \,\}$$

and

$$w_k(j) = \begin{cases} k+1-j & 1 \le j \le k \\ j & k+1 \le j \le n. \end{cases}$$

Fix $i_1, \dots, i_k$ with $1 \le i_1 < \cdots < i_k \le n$ and set $w = c_{i_1}\cdots c_{i_k}w_k$. If $1 \le j \le k$, then $w(j) = i_j < i_{j+1} = w(j+1)$. If $j \ge i_k$, then $w(j) \le j < j+1 = w(j+1)$. Suppose that $k < j < i_k$. Choose r minimal such that

$$j+1 \le i_k,\ j+1-1 \le i_{k-1},\ \dots,\ j+1-r \le i_{k-r}, \text{ and } j+1-r-1 > i_{k-r-1}.$$

Then $w(j) \le j-r-1 < j-r = j+1-r-1 = w(j+1)$. □

Corollary 4.3. *For* $1 \le k \le n-1$, *we have* $b^+(n,k) = (-1)^k x_k w_k$.

Proof. Using the definition and Lemma 4.2 we have

$$b^+(n,k) = (-1)^k \sum_{1\le i_1<\cdots<i_k\le n} c_{i_1}\cdots c_{i_k} = (-1)^k x_k w_k.$$

□

Proposition 4.4. *The following analogs of (i)-(iv) above hold.*

(a) $E_n = e_n\mathbb{C}W$.

(b) $e_nc^k = e_nb^+(n-1,k)$ *for* $0 \le k \le n-1$. *In particular,* $E_n = e_n\mathbb{C}W_{n-1}$.

(c) Consider the endomorphism of $\mathbb{C}W$ *considered as a right* $\mathbb{C}W$*-module given by left multiplication by* e_n. *The kernel of this mapping is the free, right* $\mathbb{C}W_{n-1}$*-module with basis* $\{\, c^k - b^+(n-1,k) \mid 0 \le k \le n-1 \,\}$.

(d) $\{\, e_nw \mid w \in W_{n-1} \,\}$ *is a* $\mathbb{C}$*-basis of* E_n *and* E_n *is the right regular* $\mathbb{C}W_{n-1}$*-module.*

Proof. The first statement follows immediately from the definitions.

We prove (b) by recursion. It is clear that $e_n c^k = e_n b^+(n-1,k)$ for $k=0$, since $b(n-1,0) = 1 = c^0$. Suppose $e_n c^{k-1} = e_n b^+(n-1,k-1)$. It follows from [1, Theorem 7.8] that $e_n x_J = 0$ unless $J = S$. Thus, it follows from Corollary 4.3 that $e_n b^+(n,k) = (-1)^k e_n x_k w_k = 0$ for $1 \leq k \leq n-1$. On the other hand, it follows from the definition that

$$\sum_{k=0}^{n} b^+(n,k)t^k = \left(\sum_{k=0}^{n-1} b^+(n-1,k)t^k\right)(1-c_n t)$$

and hence $b^+(n,k) = b^+(n-1,k) - b^+(n-1,k-1)c$ for $1 \leq k \leq n-1$. Therefore,

$$e_n c^k = e_n c^{k-1} c = e_n b^+(n-1,k-1)c = e_n b^+(n-1,k).$$

Next, consider the endomorphism of $\mathbb{C}W$ given by $x \mapsto e_n x$. Let K denote the kernel of this mapping and let K_1 denote the $\mathbb{C}W_{n-1}$-submodule of $\mathbb{C}W$ generated by $\{\, c^k - b^+(n-1,k) \mid 0 \leq k \leq n-1 \,\}$. It follows from (b) that $K_1 \subseteq K$. Moreover, $\{\, c^k - b^+(n-1,k) \mid 0 \leq k \leq n-1 \,\}$ is a $\mathbb{C}W_{n-1}$ basis of K_1 because the cyclic subgroup generated by c is a left transversal of W_{n-1} in W. Therefore, $\dim_{\mathbb{C}} K_1 = (n-1)(n-1)!$. However,

$$\dim K = \dim \mathbb{C}W - \dim E_n = n! - |W : Z_W(c)| = (n-1)(n-1)! = \dim K_1.$$

Therefore $K_1 = K$. This proves (c).

Because $b^+(n-1,k)$ is in $\mathbb{C}W_{n-1}$ for $1 \leq k \leq n-1$, it follows from (b) that the image of the mapping $x \mapsto e_n x$ is $e_n \mathbb{C}W_{n-1}$. Therefore, $E_n = e_n \mathbb{C}W_{n-1}$. Since $\dim E_n = (n-1)!$, it follows that $\{\, e_n w \mid w \in W_{n-1} \,\}$ is a $\mathbb{C}$-basis of E_n. This proves (d). □

Finally, define idempotents f^+ and f^- in $\mathbb{C}Z_W(c)$ by

$$f^+ = \tfrac{1}{n} \sum_{k=0}^{n-1} \phi(c^k) c^{-k} = \tfrac{1}{n} \sum_{k=0}^{n-1} \zeta^{-k} c^{-k}$$

and

$$f^- = \tfrac{1}{n} \sum_{k=0}^{n-1} \epsilon(c)^k \phi(c^k) c^{-k} = \tfrac{1}{n} \sum_{k=0}^{n-1} \epsilon(c)^k \zeta^{-k} c^{-k}.$$

Obviously, the lines $\mathbb{C}f^+$ and $\mathbb{C}f^-$ in $\mathbb{C}W$ are stable under left and right multiplication by $Z_W(c)$ and afford the characters ϕ and $\epsilon\phi$ of $Z_W(c)$ respectively. Moreover, $\mathrm{Ind}_{Z_W(c)}^W(\phi)$ is afforded by the right $\mathbb{C}W$-module $f^+\mathbb{C}W$ and $\epsilon \mathrm{Ind}_{Z_W(c)}^W(\phi) = \mathrm{Ind}_{Z_W(c)}^W(\epsilon\phi)$ is afforded by the left $\mathbb{C}W$-module $\mathbb{C}Wf^-$. Thus, to prove Theorem 4.1 it is enough to find $\mathbb{C}W$-isomorphisms $E_n \cong f^+\mathbb{C}W$ and $A_n \cong \mathbb{C}Wf^-$.

Lemma 4.5. *The idempotent f^+ acts invertibly by right multiplication on e_n and the idempotent f^- acts invertibly by left multiplication on a_n.*

Proof. Lehrer and Solomon [13, §3] show that f^- acts invertibly on a_n. Their argument is easily modified to show that f^+ acts invertibly by right multiplication on e_n as follows.

We have $(1-c_1\zeta)\cdots(1-c_{n-1}\zeta)=\sum_{k=0}^{n-1} b^+(n-1,k)\zeta^k$. Multiply both sides on the left by $\frac{1}{n}e_n$ and use Proposition 4.4(b) to get

$$\tfrac{1}{n}e_n(1-\zeta c_1)\cdots(1-\zeta c_{n-1})=\tfrac{1}{n}\sum_{k=0}^{n-1}\zeta^k e_n b^+(n-1,k)=\tfrac{1}{n}\sum_{k=0}^{n-1}\zeta^k e_n c^k=e_n f^+.$$

If $1\le k\le n-1$, then

$$1-\zeta^k=1-\zeta^k c_k^k=(1-\zeta c_k)(1+\zeta c_k+\cdots+\zeta^{k-1}c_k^{k-1}).$$

Since ζ is a primitive n^{th} root of unity, $1-\zeta^k\neq 0$ in $\mathbb{C}$. Thus, $1-\zeta c_k$ acts invertibly on e_n for $1\le k\le n-1$ and so f^+ acts invertibly on e_n. □

Proof of Theorem 4.1. (See [13, §3].) Consider the mapping from $f^+\mathbb{C}W$ to E_n given by $x\mapsto e_nx$. It follows from Lemma 4.5 and the discussion preceding it that $e_nf^+\neq 0$, that $Z_W(c)$ acts on the line $\mathbb{C}e_nf^+$ in E_n as the character ϕ, and that the mapping is a surjection. Since $\dim f^+\mathbb{C}W=|W:Z_W(c)|=(n-1)!=\dim E_n$, the mapping is also an injection. Thus, we have an isomorphism of right $\mathbb{C}W$-modules, $E_n\cong f^+\mathbb{C}W$.

As in [13, §3], similar reasoning applies to the mapping from $\mathbb{C}Wf^-$ to A_n given by $x\mapsto xa_n$ and shows that $A_n\cong\mathbb{C}Wf^-$. □

5. Symmetric groups: arbitrary λ

In this section we consider the case of an arbitrary partition of n and complete the proof of Conjecture 2.1 for symmetric groups.

Suppose $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_p)$ is a partition of n. Recall that $I_\lambda=S\setminus\{s_{\tau_1},s_{\tau_2},\dots,s_{\tau_{p-1}}\}$ and that $W_\lambda=\langle I_\lambda\rangle$ is isomorphic to the product of symmetric groups $S_{\lambda_1}\times\cdots\times S_{\lambda_p}$, where the factor S_{λ_i} acts on $\{v_{\tau_{i-1}+1},v_{\tau_{i-1}+2},\dots,v_{\tau_i}\}$. For $1\le i\le p$ define $g_{\lambda_i}=s_{\tau_i-1}\cdots s_{\tau_{i-1}+2}s_{\tau_{i-1}+1}$. Then g_{λ_i} is the λ_i-cycle in S_{λ_i} that corresponds to the n-cycle $c=c_n$ in §4. Next, define $c_\lambda=g_{\lambda_1}g_{\lambda_2}\cdots g_{\lambda_p}$ and $Z_\lambda=Z_{W_\lambda}(c_\lambda)$. Then

- c_λ is a cuspidal element in W_λ,
- the set of cuspidal elements in W_λ is precisely the conjugacy class of c_λ, and
- $Z_\lambda\cong\langle g_{\lambda_1}\rangle\times\langle g_{\lambda_2}\rangle\times\cdots\times\langle g_{\lambda_p}\rangle$.

Notice that $\{\, c_\lambda \mid \lambda \text{ is a partition of } n\,\}$ is a complete set of conjugacy class representatives in W.

With λ as above, for $1\le i\le p$, define ϕ_{λ_i} to be the character of $\langle g_{\lambda_i}\rangle$ with $\phi_{\lambda_i}(g_{\lambda_i}^{-1})=e^{2\pi i/\lambda_i}$. Then ϕ_{λ_i} is the analog of the character ϕ in §4 for the factor S_{λ_i} of W_λ. Next, define the character ϕ_λ of $Z_\lambda\cong\langle g_{\lambda_1}\rangle\times\langle g_{\lambda_2}\rangle\times\cdots\times\langle g_{\lambda_p}\rangle$ to be

$$\phi_\lambda=\phi_{\lambda_1}\otimes\cdots\otimes\phi_{\lambda_p}.$$

Note that this notation is not consistent with that of Lehrer and Solomon; our character ϕ_λ corresponds to the character $\phi_\lambda\epsilon$ in [13]. Applying the special case $\lambda=(n)$ considered in §4 to each factor S_{λ_i} of W_λ, for $1 \le i \le p$ define

$$f^+_{\lambda_i} = \tfrac{1}{\lambda_i}\sum_{k=0}^{\lambda_i-1} \phi_{\lambda_i}(g^k_{\lambda_i})g^{-k}_{\lambda_i} \quad \text{and} \quad f^-_{\lambda_i} = \tfrac{1}{\lambda_i}\sum_{k=0}^{\lambda_i-1} \epsilon(g^k_{\lambda_i})\phi(g^k_{\lambda_i})g^{-k}_{\lambda_i}.$$

Finally, define idempotents f^+_λ and f^-_λ in $\mathbb{C}Z_\lambda$ by

$$f^+_\lambda = f^+_{\lambda_1}f^+_{\lambda_2}\cdots f^+_{\lambda_p} \quad \text{and} \quad f^-_\lambda = f^-_{\lambda_1}f^-_{\lambda_2}\cdots f^-_{\lambda_p}.$$

Obviously the lines $\mathbb{C}f^+_\lambda$ and $\mathbb{C}f^-_\lambda$ in $\mathbb{C}W$ are stable under left and right multiplication by Z_λ and afford the characters ϕ_λ and $\epsilon\phi_\lambda$ of Z_λ respectively.

Now consider the canonical complement N_{X_λ} of W_λ in $N_W(W_\lambda)$. Set $N_\lambda = N_{X_\lambda}$. If λ has m_j parts equal j, then N_λ is isomorphic to the product of symmetric groups $\prod_j S_{m_j}$ (see [9] or [13]). In particular, N_λ has one Coxeter generator, say r_i, for each i such that $\lambda_i = \lambda_{i+1}$. The generator r_i acts on the set $\{v_1, v_2, \dots, v_n\}$ by interchanging $v_{\tau_{i-1}+j}$ and v_{τ_i+j} for $1 \le j \le \lambda_i$, and fixing v_k for $k \le \tau_{i-1}$ and $k > \tau_{i+1}$.

It is well-known and easy to check ([13], [11]) that $N_\lambda \subseteq Z_W(c_\lambda)$, and so $Z_W(c_\lambda) \cong Z_\lambda \rtimes N_\lambda$.

Lemma 5.1. *The subgroup N_λ of $Z_W(c_\lambda)$ stabilizes the characters ϕ_λ and $\epsilon\phi_\lambda$ of Z_λ, and centralizes the idempotents f^+_λ and f^-_λ. In particular, ϕ_λ extends to a character, also denoted by ϕ_λ, of $Z_W(c_\lambda)$, with $\phi_\lambda(nz) = \phi_\lambda(z)$ for n in N_λ and z in Z_λ.*

Proof. Suppose that i is such that $\lambda_i = \lambda_{i+1}$ and consider the generator r_i of N_λ. Then r_i is an involution and it follows from the description of the action of r_i on the basis $\{v_1, \dots, v_n\}$ of V that

$$r_i g_{\lambda_j} r_i^{-1} = r_i g_{\lambda_j} r_i = \begin{cases} g_{\lambda_{i+1}} & j = i \\ g_{\lambda_i} & j = i+1 \\ g_{\lambda_j} & j \ne i, i+1. \end{cases}$$

Since $\phi_\lambda(g_{\lambda_i}) = \phi_\lambda(g_{\lambda_{i+1}})$, it follows that r_i stabilizes ϕ_λ and $\epsilon\phi_\lambda$.

The group N_λ is generated by $\{\, r_i \mid \lambda_i = \lambda_{i+1}, \}$ and so N_λ stabilizes the characters ϕ_λ and $\epsilon\phi_\lambda$ of Z_λ. Moreover, N_λ acts on $\{g_{\lambda_1}, \dots, g_{\lambda_p}\}$ by conjugation as a group of permutations. Thus, it follows from the definition of $f^+_{\lambda_i}$ and $f^-_{\lambda_i}$ that conjugation by N_λ permutes $\{f^+_{\lambda_1}, \dots, f^+_{\lambda_p}\}$ and $\{f^-_{\lambda_1}, \dots, f^-_{\lambda_p}\}$. Since the $f^+_{\lambda_i}$'s pairwise commute and the $f^-_{\lambda_i}$'s pairwise commute, we see that N_λ centralizes $f^+_{\lambda_1}\cdots f^+_{\lambda_p} = f^+_\lambda$ and $f^-_{\lambda_1}\cdots f^-_{\lambda_p} = f^-_\lambda$. ☐

Set $\alpha_\lambda = \alpha_{X_\lambda}$. Then α_λ is a character of $N_W(W_\lambda)$ and $\alpha_\lambda(r_i) = -1$. Note that this notation is not consistent with that of Lehrer and Solomon; our character α_λ corresponds to the character $\alpha_\lambda\epsilon$ in [13] as $\epsilon(r_i) = (-1)^{\lambda_i}$.

Theorem 5.2. *Suppose that λ is a partition of n. Then the $N_W(W_\lambda)$-modules $e_{I_\lambda}\mathbb{C}W_\lambda$ and A_{X_λ}, and the character ϕ_λ of $Z_W(C_\lambda)$, are related by*

(a) *the character of the right $N_W(W_\lambda)$-module $e_{I_\lambda}\mathbb{C}W_\lambda$ is $\operatorname{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\phi_\lambda)$ and*

(b) *the character of the left $N_W(W_\lambda)$-module A_{X_λ} is $\epsilon\,\alpha_\lambda \operatorname{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\phi_\lambda)$.*

Proof. Statement (b) has been proved by Lehrer and Solomon [13, Theorem 4.4]. Their argument may be rephrased as follows. Extending the definition of the element a_n in A_n when $\lambda = (n)$, Lehrer and Solomon define an element a_λ in A_{X_λ} on which f_λ^- acts invertibly. Then:

(i) $Z_W(c_\lambda)$ acts on the line $\mathbb{C}f_\lambda^- a_\lambda$ in A_{X_λ} via the character $\epsilon_\lambda\alpha_\lambda\phi_\lambda$.

(ii) $A_{X_\lambda} = \mathbb{C}N_W(W_\lambda)f_\lambda^- a_\lambda$.

(iii) The multiplication map $\mathbb{C}N_W(W_\lambda) \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}f_\lambda^- a_\lambda \to A_{X_\lambda}$ is an isomorphism.

Therefore, $A_{X_\lambda} \cong \mathrm{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\mathbb{C}f_\lambda^- a_\lambda)$ and the character of A_{X_λ} is indeed $\mathrm{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\epsilon_\lambda\alpha_\lambda\phi_\lambda)$. Our proof of (a) follows a similar line of reasoning, with e_{I_λ} in place of a_λ.

For the rest of this proof we fix a partition $\lambda = (\lambda_1, \dots, \lambda_p)$ of n. To simplify the notation, set $I = I_\lambda$. It suffices to show that the line $\mathbb{C}e_I f_\lambda^+$ in the right $N_W(W_\lambda)$-module $e_I\mathbb{C}W_\lambda$ satisfies properties analogous to (i), (ii), and (iii) above.

(i′) *$Z_W(c_\lambda)$ acts on the line $\mathbb{C}e_I f_\lambda^+$ via the character ϕ_λ:* We have $e_I = x_I e_I^I = x_I e_{\lambda_1}^{\lambda_1} \cdots e_{\lambda_p}^{\lambda_p}$, where $e_{\lambda_i}^{\lambda_i}$ in S_{λ_i} is defined using the partition (λ_i) of λ_i. By Lemma 4.5, the idempotent $f_{\lambda_i}^+$ acts invertibly by right multiplication on $e_{\lambda_i}^{\lambda_i}$ for $1 \le i \le p$. Since

$$e_I f_\lambda^+ = x_I\,(e_{\lambda_1}^{\lambda_1} \cdots e_{\lambda_p}^{\lambda_p})\,(f_{\lambda_1}^+ \cdots f_{\lambda_p}^+) = x_I\,(e_{\lambda_1}^{\lambda_1} f_{\lambda_1}^+) \cdots (e_{\lambda_p}^{\lambda_p} f_{\lambda_p}^+),$$

it follows that f_λ^+ acts invertibly on e_I. In particular, $e_I f_\lambda^+ \neq 0$. Moreover, it is clear that Z_λ acts on $\mathbb{C}e_I f_\lambda^+$ via the character ϕ_λ. We have seen in Lemma 3.1 that $e_I n = e_I$ for n in N_λ. Thus, to show that $Z_W(c_\lambda)$ acts on $\mathbb{C}e_I f_\lambda^+$ via the character ϕ_λ, it is enough to show that N_λ centralizes f_λ^+ in $\mathbb{C}W$, but this was shown in Lemma 5.1.

(ii′) *$e_I\mathbb{C}W_\lambda = e_I f_\lambda^+ \mathbb{C}N_W(W_\lambda)$:* Because $e_I N_\lambda = e_I$ and f_λ^+ acts invertibly on e_I, we have

$$e_I\mathbb{C}W_\lambda = e_I\mathbb{C}N_W(W_\lambda) = e_I f_\lambda^+ \mathbb{C}N_W(W_\lambda).$$

(iii′) *The multiplication map $\mathbb{C}e_I f_\lambda^+ \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}N_W(W_\lambda) \to e_I\mathbb{C}W_\lambda$ is an isomorphism:* It follows from (ii′) that the mapping is surjective. Moreover,

$$\begin{aligned}\dim e_I f_\lambda^+ \mathbb{C}N_W(W_\lambda) &= \dim e_I\mathbb{C}W_\lambda \\ &= \dim e_I^I \mathbb{C}W_\lambda \\ &= |W_\lambda : Z_\lambda| \\ &= |N_W(W_\lambda) : Z_W(c_\lambda)| \\ &= \dim \mathbb{C}e_I f_\lambda^+ \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}N_W(W_\lambda)\end{aligned}$$

and so the mapping is an isomorphism.

Therefore, $e_I\mathbb{C}W_\lambda \cong \mathrm{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\mathbb{C}e_I f_\lambda^+)$ and the character of $e_I\mathbb{C}W_\lambda$ is $\mathrm{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\phi_\lambda)$ as claimed. □

The proof of Conjecture 2.1 for symmetric groups now follows from Proposition 3.6, Theorem 5.2, and transitivity of induction.

Theorem 5.3. *For each partition* λ *of* n *there is a linear character* ϕ_λ *of* $Z_W(c_\lambda)$ *such that*

(a) the character of E_λ *is* $\mathrm{Ind}_{Z_W(c_\lambda)}^W(\phi_\lambda)$ *and*

(b) the character of A_λ *is* $\mathrm{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda\alpha_\lambda\phi_\lambda)$*, where* ϵ_λ *denotes the restriction of* ϵ *to* $Z_W(c_\lambda)$*.*

In particular,

$$H^p(M) \cong \bigoplus_{\lambda\vdash n,\, \mathrm{rk}(c_\lambda)=p} \mathrm{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda\alpha_\lambda\phi_\lambda)$$

for $0 \leq p \leq n-1$*, and*

$$\mathbb{C}W \cong \bigoplus_{\lambda\vdash n} \mathrm{Ind}_{Z_W(c_\lambda)}^W(\phi_\lambda) \quad \textit{and} \quad A \cong \bigoplus_{\lambda\vdash n} \mathrm{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda\alpha_\lambda\phi_\lambda).$$

Acknowledgments: The authors acknowledge the financial support of the DFG-priority programme SPP1489 "Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory". Part of the research for this paper was carried out while the authors were staying at the Mathematical Research Institute Oberwolfach supported by the "Research in Pairs" programme. The second author wishes to acknowledge support from Science Foundation Ireland.

References

[1] F. Bergeron, N. Bergeron, R.B. Howlett, and D.E. Taylor. A decomposition of the descent algebra of a finite Coxeter group. *J. Algebraic Combin.*, 1(1):23–44, 1992.

[2] J. Blair and G.I. Lehrer. Cohomology actions and centralisers in unitary reflection groups. *Proc. London Math. Soc. (3)*, 83(3):582–604, 2001.

[3] E. Brieskorn. Sur les groupes de tresses [d'après V. I. Arnold]. In *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401*, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973.

[4] J.M. Douglass. On the cohomology of an arrangement of type B_l. *J. Algebra*, 146:265–282, 1992.

[5] J.M. Douglass, G. Pfeiffer, and G. Röhrle. Coxeter arrangements and Solomon's descent algebra: dihedral groups. Preprint 2011.

[6] G. Felder and A. P. Veselov. Coxeter group actions on the complement of hyperplanes and special involutions. *J. Eur. Math. Soc. (JEMS)*, 7(1):101–116, 2005.

[7] M. Geck, G. Hiß, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE — A system for computing and processing generic character tables. *Appl. Algebra Engrg. Comm. Comput.*, 7:175–210, 1996.

[8] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, volume 21 of *London Mathematical Society Monographs. New Series.* The Clarendon Press,‘ Oxford University Press, New York, 2000.

[9] R. B. Howlett. Normalizers of parabolic subgroups of reflection groups. *J. London Math. Soc. (2)*, 21(1):62–80, 1980.

[10] R.B. Howlett and G.I. Lehrer. Duality in the normalizer of a parabolic subgroup of a finite Coxeter group. *Bull. London Math. Soc.*, 14(2):133–136, 1982.

[11] M. Konvalinka, G. Pfeiffer, and C. Röver. A note on element centralizers in finite Coxeter groups. `arXiv:1005.1186`.

[12] G. I. Lehrer. On hyperoctahedral hyperplane complements. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 219–234. Amer. Math. Soc., Providence, RI, 1987.

[13] G.I. Lehrer and L. Solomon. On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes. *J. Algebra*, 104(2):410–424, 1986.
[14] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Invent. Math.*, 56(2):167–189, 1980.
[15] P. Orlik and L. Solomon. Coxeter arrangements. In *Singularities*, volume 40 of *Proc. Symp. Pure Math.*, pages 269–292. Amer. Math. Soc., 1983.
[16] P. Orlik and H. Terao. *Arrangements of Hyperplanes*. Springer-Verlag, 1992.
[17] G. Pfeiffer. A quiver presentation for Solomon's descent algebra. *Adv. Math.*, 220(5):1428–1465, 2009.
[18] Martin Schönert et al. *GAP – Groups, Algorithms, and Programming – version 3 release 4*, RWTH Aachen, 1997.
[19] M. Schocker. Über die höheren Lie-Darstellungen der symmetrischen Gruppen. *Bayreuth. Math. Schr.*, (63):103–263, 2001.
[20] L. Solomon. A decomposition of the group algebra of a finite Coxeter group. *J. Algebra*, 9:220–239, 1968.
[21] L. Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.
[22] R.P. Stanley. Some aspects of groups acting on finite posets. *J. Combin. Theory Ser. A*, 32(2):132–161, 1982.
[23] R. Steinberg. Differential equations invariant under finite reflection groups. *Trans. Amer. Math. Soc.*, 112:392–400, 1964.

Department of Mathematics, University of North Texas, Denton TX, USA 76203

E-mail address: douglass@unt.edu

URL: http://hilbert.math.unt.edu

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, University Road, Galway, Ireland

E-mail address: goetz.pfeiffer@nuigalway.ie

URL: http://schmidt.nuigalway.ie/~goetz

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany

E-mail address: gerhard.roehrle@rub.de

URL: http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Lehrstuhl-VI