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ZETA FUNCTIONS OF 3-DIMENSIONAL p-ADIC LIE ALGEBRAS

BENJAMIN KLOPSCH AND CHRISTOPHER VOLL

ABSTRACT. We give an explicit formula for the subalgebra zeta function of a general 3-dimensional Lie algebra over the p-adic integers $\mathbb{Z}_p$. To this end, we associate to such a Lie algebra a ternary quadratic form over $\mathbb{Z}_p$. The formula for the zeta function is given in terms of Igusa's local zeta function associated to this form.

1. INTRODUCTION

For a prime p, let $\mathbb{Z}_p$ denote the ring of p-adic integers. The *(subalgebra) zeta function* of a $\mathbb{Z}_p$-algebra L, additively isomorphic to $\mathbb{Z}_p^n$, is the Dirichlet series

$$\zeta_L(s) = \sum_{H \leq L} |L : H|^{-s},$$

where the sum ranges over the subalgebras of finite index in L, and s is a complex variable.

Zeta functions of $\mathbb{Z}_p$-*Lie* algebras play an important role in the subject of subgroup growth. Indeed, to every saturable p-adic analytic pro-p group G there is an associated $\mathbb{Z}_p$-Lie algebra $L = L(G)$ and if $\dim(G) \leq p$ then

$$\zeta_{L(G)}(s) = \zeta_G(s) := \sum_{H \leq G} |G : H|^{-s},$$

the *(subgroup) zeta function* of the group G (cf. [13, 14] and references therein). Similarly, for a finitely generated nilpotent group G, there is a nilpotent $\mathbb{Z}$-Lie algebra $L(G)$ such that, for almost all primes p,

$$\zeta_{\mathbb{Z}_p \otimes_{\mathbb{Z}} L(G)}(s) = \zeta_{G,p}(s) := \sum_{H \leq_p G} |G : H|^{-s},$$

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the *local (subgroup) zeta function* of G at the prime p, enumerating subgroups of finite p-power index in G (cf. [8]). Thus, to some degree, the study of subgroup zeta functions reduces to the study of subalgebra zeta functions of $\mathbb{Z}_p$-Lie algebras. Zeta functions of groups and rings in general have attracted considerable interest over the last few decades; we refer to [5] for a recent survey.

By now, numerous examples of zeta functions of nilpotent and soluble $\mathbb{Z}_p$-Lie algebras have been calculated. One of the first examples is the zeta function of the Heisenberg Lie algebra $\mathfrak{h}(\mathbb{Z}_p)$ which was computed in [8]. For further explicit calculations see, for example, [18, 19, 7]. On the other hand, explicit examples of zeta functions of insoluble $\mathbb{Z}_p$-Lie algebras are thin on the ground. Only for zeta functions associated to the two $\mathbb{Q}_p$-forms of the simple Lie algebra of type A_1 have explicit formulae been found: using results of Ilani ([11]), du Sautoy gave a formula for the zeta function of $\mathfrak{sl}_2(\mathbb{Z}_p)$ ([3]; see also [6]). In [12], Klopsch computed the zeta function of $\mathfrak{sl}_1(\Delta_p)$, where Δ_p denotes the maximal $\mathbb{Z}_p$-order in a central $\mathbb{Q}_p$-division algebra of index 2. No explicit formula for the zeta function of any 'semi-simple' $\mathbb{Z}_p$-Lie algebra of dimension greater than 3 is known (cf. [15, Problem 9(c) on p. 431]).

In [20], Voll introduced a method for computing zeta functions of $\mathbb{Z}_p$-algebras in terms of certain p-adic integrals generalising Igusa's local zeta function. Given a polynomial $f(\mathbf{x}) \in \mathbb{Z}_p[x_1, \ldots, x_n]$, Igusa's local zeta function associated to f is defined as the p-adic integral

$$Z_f(s) = \int_{\mathbb{Z}_p^n} |f(\mathbf{x})|_p^s \mathrm{d}\mu.$$

Here $|\ |_p$ denotes the p-adic absolute value, s is a complex variable and $\mathrm{d}\mu$ stands for the normalised additive Haar measure on $\mathbb{Z}_p^n$. Igusa's local zeta function is closely connected to the Poincaré series enumerating the numbers of solutions of the congruences $f(\mathbf{x}) \equiv 0 \mod (p^m)$ for $m \in \mathbb{N}$ (see Section 2 for further details and references).

The purpose of the current paper is to demonstrate that this point of view may be used to unify and generalise the existing computations of zeta functions of 3-dimensional $\mathbb{Z}_p$-Lie algebras. These specific calculations draw on a variety of methods: they range from elementary counting arguments for the Heisenberg Lie algebra in [8] over a carefully chosen resolution of singularities for a high-dimensional hypersurface in [6] to a structural analysis of a division algebra in [12]. Our main result generalises these results and subsumes them under a unified description in terms of a rather tame p-adic integral: Igusa's local zeta function of a ternary quadratic form, naturally associated to a 3-dimensional $\mathbb{Z}_p$-Lie algebra. To formulate our result we recall that

the zeta function of the abelian 3-dimensional $\mathbb{Z}_p$-Lie algebra is

$$\zeta_{\mathbb{Z}_p^3}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2).$$

Here $\zeta_p(s) = (1-p^{-s})^{-1}$ denotes the p-th local Riemann zeta function.

Theorem 1.1. *Let L be a 3-dimensional $\mathbb{Z}_p$-Lie algebra. Then there is a ternary quadratic form $f(\mathbf{x}) \in \mathbb{Z}_p[x_1, x_2, x_3]$, unique up to equivalence, such that, for $i \geq 0$,*

$$\zeta_{p^iL}(s) = \zeta_{\mathbb{Z}_p^3}(s) - Z_f(s-2)\zeta_p(2s-2)\zeta_p(s-2)p^{(2-s)(i+1)}(1-p^{-1})^{-1},$$

where $Z_f(s)$ is Igusa's local zeta function associated to f.

In the course of the proof of Theorem 1.1 we define $f(\mathbf{x})$ in terms of the structure constants of L with respect to a given $\mathbb{Z}_p$-basis; different bases give rise to equivalent quadratic forms (see Section 3 for details).

The following table lists the ternary quadratic forms controlling the subalgebra growth in several special cases mentioned above.

Lie algebra	ternary quadratic form $f(\mathbf{x})$
$\mathbb{Z}_p^3$	0
$\mathfrak{h}(\mathbb{Z}_p)$	x_3^2
$\mathfrak{sl}_2(\mathbb{Z}_p)$	$x_3^2 + 4x_1x_2$
$\mathfrak{sl}_1(\Delta_p)$	$\begin{cases} -2(x_1^2 - 2(x_2^2 - x_2x_3 + x_3^2)) & \text{for } p = 2 \\ x_3^2 - \rho x_2^2 - p x_1^2 \ (\rho \text{ a non-square mod } p) & \text{for } p \text{ odd} \end{cases}$

It is comparatively easy to compute Igusa's local zeta functions associated to these forms. By Theorem 1.1 we immediately recover the known formulae for the zeta functions of the Lie algebras in this table; see Sections 4.1, 4.2 and 4.3. In Section 4.4 we use Theorem 1.1 to treat the soluble case, which has not been previously studied. We give a complete list of the binary quadratic forms for the soluble 3-dimensional $\mathbb{Z}_p$-Lie algebras (for odd p) and derive formulae for Igusa's local zeta functions associated to these quadratic forms. This leads to formulae for the zeta functions of all these Lie algebras. Our computations show, in particular, that many among the soluble $\mathbb{Z}_p$-Lie algebras are isospectral (i.e. non-isomorphic but sharing the same zeta function).

As an immediate corollary to Theorem 1.1 we gain control over the real parts of the poles of zeta functions of 3-dimensional $\mathbb{Z}_p$-Lie algebras. Indeed, it is easy to see that the only candidate poles of Igusa's local zeta function of a ternary quadratic form have real part $-3/2$, -1 and $-1/2$ (see Lemma 2.2). Thus we obtain

Corollary 1.2. *If s is a pole of $\zeta_{p^i L}(s)$, then $\mathfrak{Re}(s) \in \{0, 1/2, 1, 3/2, 2\}$.*

To determine the poles of zeta functions of $\mathbb{Z}_p$-(Lie) algebras in general is a difficult and almost entirely unsolved problem. Of particular importance is the largest actually occurring real pole, as its position and order determine the asymptotics of the subalgebra growth of L. Our analysis allows us to solve this problem for the soluble $\mathbb{Z}_p$-Lie algebras of dimension 3 (see Proposition 4.1 and Corollary 4.2).

We conclude the introduction with a number of remarks.

1. Though our results are formulated for Lie algebras, our arguments only draw on the fact that a Lie algebra is *antisymmetric*; the Jacobi identity is not being used anywhere.

2. In general, no simple identity is known which relates the zeta function of a d-dimensional $\mathbb{Z}_p$-algebra L with that of $p^i L$, $i \in \mathbb{N}_0$. In [3, Theorem 2.1], du Sautoy gives such a formula for the special case $d = 3$; our Theorem 1.1 provides inter alia an alternative proof of this formula, without reference to Mann's work on probabilistic zeta functions ([16]).

3. We point out that Theorem 1.1 is a 'local result', whereas the main conclusions of the results in [20] are valid for almost all completions of a 'global object' (such as a torsion-free nilpotent group or a torsion-free ring). The main application of the approach developed in [20] is to prove that, given a torsion-free ring L (not necessarily associative or Lie), the associated local zeta function $\zeta_{\mathbb{Z}_p \otimes_{\mathbb{Z}} L}(s)$ satisfies a functional equation for almost all primes p. The occurrence of functional equations for the zeta functions of some 3-dimensional $\mathbb{Z}_p$-Lie algebras is therefore only explained by the results of [20] in case these algebras are the 'generic' completions of an algebra over $\mathbb{Z}$ (such as, for example, $\mathfrak{sl}_2(\mathbb{Z}_p)$ for odd p). This corresponds to the fact that the proof of a functional equation for Igusa's local zeta function given in [2] critically depends on good reduction modulo p.

4. The case of 3-dimensional Lie algebras is the first non-trivial one as far as subalgebra zeta functions are concerned. In dimensions 1 and 2 it is not hard to see that the concepts of subalgebra and additive sublattice coincide, so that the zeta functions coincide with the well-known zeta functions for the abelian case. The work in [20] makes essential use of generalisations of Igusa's local zeta functions to polynomial mappings and several variables. It is remarkable that Theorem 1.1, however, reduces the case of a 3-dimensional Lie algebra to the computation of the classical Igusa integral associated of a single ternary quadratic form. Things get radically more complicated in dimensions greater than 3.

5. Rather than counting all subalgebras of finite index in a ring L, one may restrict attention to subalgebras with additional algebraic properties; among the variants of $\zeta_L(s)$ that have been considered are the ideal zeta function $\zeta_L^{\triangleleft}(s)$, enumerating ideals of finite index, and the zeta function $\widehat{\zeta_L}(s)$, counting subalgebras isomorphic to L. It would be interesting to study these zeta functions of 3-dimensional Lie algebras with the methods introduced in the present paper.

Organisation and notation. We prove our main result in Section 3. In Section 2 we collect a few elementary observations about Igusa's local zeta function. The examples given in the above table are studied in detail in Section 4, where we derive the known formulae for their zeta functions using Theorem 1.1. In this section we also list the binary quadratic forms associated to 3-dimensional soluble $\mathbb{Z}_p$-algebras ($p \geq 3$) and compute their zeta functions.

Throughout this paper we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set of non-negative integers. Given a prime p, we denote by $\mathbb{Z}_p$ the ring of p-adic integers and by $\mathbb{Q}_p$ the field of p-adic numbers. We write v_p for the p-adic valuation, and $|\ |_p$ for the p-adic absolute value.

2. Preliminaries on Igusa's local zeta function

For general background on the theory of Igusa's local zeta function we refer the reader to [1, 10]. Let $f(\mathbf{x}) \in \mathbb{Z}_p[x_1, \ldots, x_n]$. The well-known connection between Igusa's local zeta function $Z_f(s)$ and the Poincaré series enumerating the numbers of solutions of $f(\mathbf{x}) \equiv 0$ modulo (p^m) mentioned in the introduction is the following. For $m \in \mathbb{N}_0$, set

$$N_m := |\{\mathbf{x} \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid f(\mathbf{x}) = 0\}|.$$

We write $t = p^{-s}$ and treat it as an independent variable. A simple computation (cf. [1, Section 2.1]) shows that the Poincaré series

$$P_f(t) := \sum_{m=0}^{\infty} N_m (p^{-n}t)^m$$

is related to Igusa's local zeta function by the formula

$$P_f(t) = \frac{1 - tZ_f(s)}{1 - t}. \tag{1}$$

Whilst the Poincaré series $P_f(t)$ counts solutions of polynomial equations in *affine* space we shall see in Section 3 below that counting subalgebras is related to counting solutions of polynomial equations in finite

projective spaces. To this end we define now, for $m \in \mathbb{N}_0$, the affine cones

$$W := \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n,$$
$$W(m) := (W + (p^m\mathbb{Z}_p)^n)/(p^m\mathbb{Z}_p)^n$$

and set

$$N_m^\star := |\{\mathbf{x} \in W(m) \mid f(\mathbf{x}) = 0\}|.$$

We will utilise a formula for

$$P_f^\star(t) := \sum_{m=0}^{\infty} N_m^\star (p^{-n}t)^m, \tag{2}$$

analogous to (1) in the special case that f is *homogeneous*. If f is homogeneous of degree d, say, we have (cf. [2, (1) on p. 1141])

$$Z_f(s) = \frac{1}{1 - p^{-n-ds}} Z_f^\star(s), \tag{3}$$

where

$$Z_f^\star(s) = \int_W |f(\mathbf{x})|_p^s \mathrm{d}\mu.$$

Lemma 2.1. *If f is homogeneous, then*

$$P_f^\star(t) = \frac{1 - p^{-n}t - tZ_f^\star(s)}{1-t}. \tag{4}$$

Proof. For $m \in \mathbb{N}_0$, set $\mu_m^\star := \mu(\{\mathbf{x} \in W \mid v_p(f(\mathbf{x})) = m\})$. We claim that

$$\mu_m^\star = \frac{N_m^\star}{p^{nm}} - \frac{N_{m+1}^\star}{p^{n(m+1)}} - \delta_{m,0}\, p^{-n}, \tag{5}$$

where $\delta_{m,0}$ denotes the Kronecker-delta. Indeed, for $m \in \mathbb{N}_0$ we may write

$$\mu_m^\star = \mu(\{\mathbf{x} \in W \mid v_p(f(\mathbf{x})) \geq m\}) - \mu(\{\mathbf{x} \in W \mid v_p(f(\mathbf{x})) \geq m+1\}).$$

We have

$$\mu(\{\mathbf{x} \in W \mid v_p(f(\mathbf{x})) \geq m\}) = \begin{cases} N_m^\star/p^{nm} & \text{if } m \geq 1, \\ \mu(W) = 1 - p^{-n} & \text{if } m = 0. \end{cases}$$

Using (5) we obtain

$$\begin{aligned} Z_f^\star(s) &= \sum_{m=0}^{\infty} \mu_m^\star p^{-ms} = \sum_{m=0}^{\infty} \left(\frac{N_m^\star}{p^{nm}} - \frac{N_{m+1}^\star}{p^{n(m+1)}} \right) p^{-ms} - p^{-n} \\ &= P_f^\star(t) - \frac{1}{t}\left(P_f^\star(t) - 1\right) - p^{-n}. \end{aligned} \tag{6}$$

The lemma follows. □

To describe the position of the poles of Igusa's local zeta function is in general a difficult and interesting problem. In the current paper we shall only work with Igusa's local zeta function associated to ternary quadratic forms. This case is well-understood:

Lemma 2.2. *Let $f(\mathbf{x}) \in \mathbb{Z}_p[x_1, x_2, x_3]$ be a ternary quadratic form. If s is a pole of $Z_f(s)$, then $\mathfrak{Re}(s) \in \{-3/2, -1, -1/2\}$.*

Proof. A ternary quadratic form defines a cone in affine 3-space over a (possibly anisotropic) conic. A resolution of singularities is achieved by blowing up the origin, yielding an exceptional divisor with numerical data $(\nu, N) = (3, 2)$. The divisors of the proper transform have numerical data $(1, 1)$ unless the conic is a double line, in which case the numerical data is $(1, 2)$. The real parts of the poles are to be found among the fractions $-\nu/N$ (cf. [1, 9] for details). □

We thank Wim Veys for pointing this out to us.

3. Proof of the main result

In this section we prove Theorem 1.1. Let L be a 3-dimensional $\mathbb{Z}_p$-Lie algebra. To compute the zeta function of L it is helpful to make the following observations: the homothety class $[\Lambda]$ of any $\mathbb{Z}_p$-lattice Λ in the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ contains a unique ($\subseteq$-) maximal sub*algebra* Λ_0 of L, and the subalgebras contained in the class $[\Lambda]$ are exactly the multiples $p^i\Lambda_0$, $i \in \mathbb{N}_0$. Thus

$$\zeta_L(s) = \frac{1}{1 - p^{-3s}} A(s), \text{ where } A(s) := \sum_{[\Lambda]} |L : \Lambda_0|^{-s}.$$

For the computation of $A(s)$ it is useful to sort the lattice classes $[\Lambda]$ by their elementary divisor type with respect to the class $[L]$, and to take advantage of the transitive action of the group $\Gamma := \mathrm{GL}_3(\mathbb{Z}_p)$ on the classes of any fixed elementary divisor type.

Write $L = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2 \oplus \mathbb{Z}_p e_3$. A sublattice $\Lambda \subseteq L$ corresponds to a right-coset ΓM, where $M \in \mathrm{GL}_3(\mathbb{Q}_p) \cap \mathrm{Mat}_3(\mathbb{Z}_p)$, the set of 3×3-matrices over $\mathbb{Z}_p$ with non-zero determinant: the lattice is generated by vectors whose coordinates with respect to the chosen basis (e_1, e_2, e_3) are encoded in the rows of M. By the elementary divisor theorem the right-coset ΓM contains a representative of the form $D\alpha^{-1}$, where $\alpha \in \Gamma$ and $D = \mathrm{diag}(D_1, D_2, D_3) = p^{r_0}\mathrm{diag}(p^{r_1+r_2}, p^{r_2}, 1)$ is a diagonal matrix with $r_i \in \mathbb{N}_0$ for $i \in \{0, 1, 2\}$. We say that the homothety class

$[\Lambda]$ is of type[1] $\mathbf{r} = (r_1, r_2) \in \mathbb{N}_0^2$ if the diagonal matrix D determined by Λ is a scalar multiple of $\mathrm{diag}(p^{r_1+r_2}, p^{r_2}, 1)$. Below we shall make a case distinction with respect to the invariant $I([\Lambda]) = \{i \in \{1,2\} \mid r_i \neq 0\}$.

The matrix α is determined only up to right-multiplication by an element of $\Gamma_{\mathbf{r}} := \mathrm{Stab}_\Gamma(\Gamma D)$, the stabiliser in Γ of the right-coset ΓD under right-multiplication. The various stabilisers will be described in detail below. A lattice class $[\Lambda]$ is thus given by the pair $\mathbf{r} \in \mathbb{N}_0^2$ and a left-coset $\alpha\Gamma_{\mathbf{r}} \in \Gamma/\Gamma_{\mathbf{r}}$.

This parametrisation allows us to give a convenient description of the index $|L : \Lambda_0|$ of the maximal subalgebra Λ_0 in the homothety class $[\Lambda]$. In order to decide whether a lattice Λ is a subalgebra it suffices to check whether the products of pairs of a given set of generators are contained in Λ. This is particularly easy to verify if the right-coset ΓM contains a diagonal matrix; in this case the condition of being a subalgebra translates into a set of divisibility conditions on quadratic polynomials in the entries of M. In general, however, the coset ΓM may not contain any diagonal element. In this case a base change – effectuated by right-multiplication with an element in the left-coset $\alpha\Gamma_{\mathbf{r}}$ – brings us into this desirable situation. (Note that this approach differs distinctly from the point of view taken e.g. in [4], where all calculations are performed with respect to a fixed basis.) More precisely, as indicated in [20, Section 3], a lattice Λ corresponding to a right-coset $\Gamma D\alpha^{-1}$ is a subalgebra of L if and only if the following congruences hold:

$$\text{(SUB)} \qquad \forall i \in \{1,2,3\} : \ D\alpha^{-1}\mathcal{R}(\alpha[i])(\alpha^{-1})^{\mathrm{t}}D \equiv 0 \mod D_i.$$

By $\alpha[i]$ we denote the i-th column of $\alpha \in \Gamma$, and $(\alpha^{-1})^{\mathrm{t}}$ is the transpose of α^{-1}. The antisymmetric 3×3-matrix of $\mathbb{Z}_p$-linear forms

$$\mathcal{R}(\mathbf{y}) := (\mathrm{L}_{ij}(\mathbf{y})) \in \mathrm{Mat}_3(\mathbb{Z}_p[\mathbf{y}]),$$

where $\mathrm{L}_{ij}(\mathbf{y}) := \lambda_{ij}^1 y_1 + \lambda_{ij}^2 y_2 + \lambda_{ij}^3 y_3$, encodes the structure constants λ_{ij}^k of the algebra L with respect to the given $\mathbb{Z}_p$-basis (e_1, e_2, e_3).

Our opening remarks now amount to observing that condition (SUB) is satisfied for all values of r_0 greater than or equal to the minimal value of r_0 with this property. The task of determining the summand $|L : \Lambda_0|^{-s}$ of $A(s)$ thus reduces to the problem of calculating this minimal value. For each subset $I \subseteq \{1,2\}$, we set

$$A_I(s) := \sum_{I([\Lambda])=I} |L : \Lambda_0|^{-s},$$

[1]Note that this terminology differs slightly from the one used in [20], where the shape of diagonal matrix D is encoded in a subset $I \subseteq [n-1]$ and a *positive* vector $(r_i) \in \mathbb{N}^{|I|}$.

so that

$$A(s) = \sum_{I \subseteq \{1,2\}} A_I(s).$$

Our aim is to compute the Dirichlet series $A_I(s)$ by investigating the condition (SUB) in each of the four cases.

Case $I = \varnothing$. Clearly in this case the condition (SUB) is trivially satisfied for all $r_0 \in \mathbb{N}_0$. As there is a unique homothety class of type $\mathbf{r} = (0,0)$, we obtain $A_\varnothing(s) = 1$.

Case $I = \{1\}$. In this case $D = p^{r_0}\mathrm{diag}(p^{r_1}, 1, 1)$, $r_1 \in \mathbb{N}$. We have

$$\Gamma_{(r_1,0)} = \begin{pmatrix} \mathbb{Z}_p^* & \mathbb{Z}_p & \mathbb{Z}_p \\ p^{r_1}\mathbb{Z}_p & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbb{Z}_p)$}} \\ p^{r_1}\mathbb{Z}_p & & \end{pmatrix}.$$

Lattice classes of type $\mathbf{r} = (r_1, 0)$ may thus be identified with the $(p^{-2} + p^{-1} + 1)p^{2r_1}$ points of the finite projective space $\mathbb{P}^2(\mathbb{Z}/p^{r_1}\mathbb{Z}) \cong \Gamma/\Gamma_{(r_1,0)}$ by taking the first column of a matrix modulo units in $\mathbb{Z}_p/(p^{r_1}\mathbb{Z}_p)$. Only the first of the three conditions in (SUB) is non-trivial:

$$D\alpha^{-1}\mathcal{R}(\alpha[1])(\alpha^{-1})^{\mathrm{t}}D \equiv 0 \mod p^{r_0+r_1}.$$

Since the matrix $\mathcal{R}$ is antisymmetric, we only need to check a single entry of the matrix on the left hand side. Cancelling a factor p^{r_0} we obtain that (SUB) holds if and only if

$$p^{r_0}\left(\alpha^{-1}\mathcal{R}(\alpha[1])(\alpha^{-1})^{\mathrm{t}}\right)_{23} \equiv 0 \mod p^{r_1}.$$

Lemma 3.1. *For* $\mathcal{R}(\mathbf{y}) = (\mathrm{L}_{ij}(\mathbf{y}))$ *and* $\alpha = (\alpha_{ij})$ *we have*

$$\det(\alpha)(\alpha^{-1}\mathcal{R}(\alpha[1])(\alpha^{-1})^{\mathrm{t}})_{23} = \mathrm{L}_{23}(\alpha[1])\alpha_{11} - \mathrm{L}_{13}(\alpha[1])\alpha_{21} + \mathrm{L}_{12}(\alpha[1])\alpha_{31}.$$

Proof. Set $\alpha^{-1} = \beta = (\beta_{ij})$. Note that $\alpha = \det(\alpha)\beta^\natural$, where $\beta^\natural$ denotes the adjoint matrix of β. It is now easy to verify that

$$\begin{aligned}(\alpha^{-1}\mathcal{R}(\alpha[1])(\alpha^{-1})^{\mathrm{t}})_{23} &= \mathrm{L}_{23}(\alpha[1])(\beta_{22}\beta_{33} - \beta_{23}\beta_{32}) - \\ &\quad \mathrm{L}_{13}(\alpha[1])(-\beta_{21}\beta_{33} + \beta_{23}\beta_{31}) + \mathrm{L}_{12}(\alpha[1])(\beta_{21}\beta_{32} - \beta_{22}\beta_{31}) \\ &= \left(\mathrm{L}_{23}(\alpha[1])\alpha_{11} - \mathrm{L}_{13}(\alpha[1])\alpha_{21} + \mathrm{L}_{12}(\alpha[1])\alpha_{31}\right) / \det(\alpha)\end{aligned}$$

as required. □

We remark that this is a very special situation: for a quadratic polynomial in the β_{ij} to be a linear polynomial in the entries of the matrix $\alpha \in \mathrm{GL}_d(\mathbb{Z}_p)$ it is necessary that $d = 3$.

As $\det(\alpha)$ is a p-adic unit, the lemma shows that in the present case it suffices to control the values taken by the p-adic valuation of the single ternary quadratic form

$$f(\mathbf{x}) := \mathrm{L}_{23}(\mathbf{x})x_1 - \mathrm{L}_{13}(\mathbf{x})x_2 + \mathrm{L}_{12}(\mathbf{x})x_3. \tag{7}$$

More precisely,

$$\text{(SUB)} \quad \Leftrightarrow \quad r_0 \geq r_1 - v_p(f(\alpha[1])).$$

Therefore

$$A_{\{1\}}(s) = \sum_{r_1=1}^{\infty} \sum_{m=0}^{\infty} N_{r_1,m} \left(p^{-s}\right)^{r_1+3(r_1-m)}, \tag{8}$$

with

$$N_{r_1,m} := |\{\mathbf{x} = (x_1 : x_2 : x_3) \in \mathbb{P}^2(\mathbb{Z}/p^{r_1}\mathbb{Z}) \mid \min\{r_1, v_p(f(\mathbf{x}))\} = m\}|.$$

Note that $N_{r_1,m} = 0$ unless $0 \leq m \leq r_1$. We shall compute $A_{\{1\}}(s)$ in terms of $Z_f(s)$ in (11) below.

Remark. It is clear that f depends on the structure constants λ_{ij}^k of L with respect to the chosen $\mathbb{Z}_p$-basis (e_1, e_2, e_3). In fact, if

$$A := \begin{pmatrix} \lambda_{23}^1 & \lambda_{31}^1 & \lambda_{12}^1 \\ \lambda_{23}^2 & \lambda_{31}^2 & \lambda_{12}^2 \\ \lambda_{23}^3 & \lambda_{31}^3 & \lambda_{12}^3 \end{pmatrix}$$

then $f(\mathbf{x}) = \mathbf{x}A\mathbf{x}^{\mathrm{t}}$. If we change the basis (e_1, e_2, e_3) to a basis (e_1', e_2', e_3'), where $e_i' = \sum_{i=1}^3 p_{ij}e_j$ for a matrix $P = (p_{ij}) \in \Gamma$, the quadratic form f is transformed into $f'(\mathbf{x}) = \mathbf{x}A'\mathbf{x}^{\mathrm{t}}$, where $A' = (\det P)P^{-1}A(P^{-1})^{\mathrm{t}}$ (cf. [17]). We call ternary quadratic forms f and f' equivalent if they are related in this way.

Case $I = \{2\}$. This case is much simpler than the previous one. Note that $D = p^{r_0}\mathrm{diag}(p^{r_2}, p^{r_2}, 1)$, $r_2 \in \mathbb{N}$. We have

$$\Gamma_{(0,r_2)} = \begin{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p) & \mathbb{Z}_p \\ & \mathbb{Z}_p \\ p^{r_2}\mathbb{Z}_p \quad p^{r_2}\mathbb{Z}_p & \mathbb{Z}_p^* \end{pmatrix}.$$

Lattice classes of type $\mathbf{r} = (0, r_2)$ may thus be identified with the $(p^{-2}+p^{-1}+1)p^{2r_2}$ points of the finite Grassmannian $\mathrm{G}(2,3)(\mathbb{Z}/p^{r_2}\mathbb{Z}) \cong \Gamma/\Gamma_{(0,r_2)}$, determined by the first two columns of a matrix modulo p^{r_2}.

One checks immediately that the subalgebra condition (SUB) is satisfied for all $r_0 \in \mathbb{N}_0$. Thus

$$A_{\{2\}}(s) = \sum_{r_2=1}^{\infty} (p^{-2} + p^{-1} + 1) \left(p^{2-2s}\right)^{r_2} = (p^{-2} + p^{-1} + 1) \frac{p^{2-2s}}{1 - p^{2-2s}}.$$

Case $I = \{1, 2\}$. We shall see that this case reduces to the case $I = \{1\}$. We have $D = p^{r_0} \mathrm{diag}(p^{r_1+r_2}, p^{r_2}, 1)$, where $r_1, r_2 \in \mathbb{N}$. Therefore

$$\Gamma_{(r_1,r_2)} = \begin{pmatrix} \mathbb{Z}_p^* & \mathbb{Z}_p & \mathbb{Z}_p \\ p^{r_1}\mathbb{Z}_p & \mathbb{Z}_p^* & \mathbb{Z}_p \\ p^{r_1+r_2}\mathbb{Z}_p & p^{r_2}\mathbb{Z}_p & \mathbb{Z}_p^* \end{pmatrix}.$$

One verifies without difficulty that, as in the case $I = \{1\}$,

$$\text{(SUB)} \quad \Leftrightarrow \quad r_0 \geq r_1 - v_p(f(\alpha[1])).$$

In other words, if the lattice class $[\Lambda]$ is of type $\mathbf{r} = (r_1, r_2) \in \mathbb{N}^2$ and given by the left-coset $\alpha\Gamma_{\mathbf{r}}$, the subalgebra condition (SUB) only depends on the left-coset $\alpha\Gamma_{(r_1,0)}$. Evidently each fibre of the natural projection

$$\Gamma/\Gamma_{\mathbf{r}} \twoheadrightarrow \Gamma/\Gamma_{(r_1,0)}$$

has cardinality $(p^{-1} + 1)p^{2r_2}$. The computation of $A_{\{1,2\}}(s)$ reduces therefore to the computation of $A_{\{1\}}(s)$. Indeed,

$$\begin{aligned} A_{\{1,2\}}(s) &= \sum_{r_2=1}^{\infty} (p^{-1} + 1) \left(p^{2-2s}\right)^{r_2} \sum_{r_1=1}^{\infty} \sum_{m=0}^{\infty} N_{r_1,m} \left(p^{-s}\right)^{r_1+3(r_1-m)} \\ &= (p^{-1} + 1) \frac{p^{2-2s}}{1 - p^{2-2s}} A_{\{1\}}(s). \end{aligned}$$

We now develop a formula for the Dirichlet series $A_{\{1\}}(s)$ in terms of Igusa's local zeta function $Z_f(s)$ associated to the ternary quadratic form f given in (7). Writing r for r_1 we express the numbers $N_{r,m}$ in our description (8) of $A_{\{1\}}(s)$ in terms of the integers $N_m^\star$, defined in Section 2. First we rephrase the $N_{r,m}$ – counting solutions of equations in finite *projective* spaces – in terms of the numbers of solutions in corresponding *affine* cones. We set

$$N_{r,m}^\star := |\{\mathbf{x} \in W(r) \mid \min\{r, v_p(f(\mathbf{x}))\} = m\}|, \tag{9}$$

observe that $N_{r,m}(1 - p^{-1})p^r = N_{r,m}^\star$ and that

$$N_{r,m}^\star = \begin{cases} \mu_m^\star p^{nr} & \text{if } m < r, \\ N_r^\star & \text{if } m = r. \end{cases} \tag{10}$$

As $n = 3$ in our specific situation, this allows us to write

$$\begin{aligned}
A_{\{1\}}(s) &= \sum_{r=1}^{\infty} \sum_{m=0}^{r} \frac{N^{\star}_{r,m}}{(1-p^{-1})p^r} (p^{-s})^{4r-3m} \\
&= \frac{1}{1-p^{-1}} \left(\sum_{r=1}^{\infty} \sum_{m=0}^{r-1} \mu^{\star}_m p^{2r} (p^{-s})^{4r-3m} + \sum_{r=1}^{\infty} p^{-r} N^{\star}_r (p^{-s})^r \right) \\
&= \frac{1}{1-p^{-1}} \left(\sum_{m=0}^{\infty} \left(\frac{N^{\star}_m}{p^{3m}} - \frac{N^{\star}_{m+1}}{p^{3(m+1)}} \right) t^{-3m} \sum_{r=m+1}^{\infty} (p^2t^4)^r \right. \\
&\qquad \left. -p^{-3} \sum_{r=1}^{\infty} (p^2t^4)^r + P^{\star}(p^2t) - 1 \right) \\
&= \frac{1}{1-p^{-1}} \left(\left(Z^{\star}(s-2) + p^{-3} \right) \frac{p^2t^4}{1-p^2t^4} - \frac{p^{-1}t^4}{1-p^2t^4} \right. \\
&\qquad \left. + P^{\star}(p^2t) - 1 \right).
\end{aligned}$$

Here we used the identities (9), (10), (5), (2) and (6) and wrote $t = p^{-s}$. Using equation (4) (in which we replace t by p^2t, i.e. s by $s-2$), we obtain

$$A_{\{1\}}(s) = \frac{1-p^{-3}}{1-p^{-1}} \left(\frac{p^2t}{1-p^2t} - \frac{Z^{\star}_f(s-2)p^2t(1-t^3)}{(1-p^{-3})(1-p^2t^4)(1-p^2t)} \right). \tag{11}$$

Notice that $Z^{\star}_f(s-2) = 0$ if L is abelian. Using the identity (3) and the observation that

$$\begin{aligned}
A_{\{1\}}(s) + A_{\{1,2\}}(s) &= A_{\{1\}}(s) \left(1 + (p^{-1}+1) \frac{p^2t^2}{1-p^2t^2} \right) \\
&= A_{\{1\}}(s) \frac{1+pt^2}{1-p^2t^2}
\end{aligned}$$

it is now immediate that

$$\zeta_L(s) = \frac{1}{1-t^3} \sum_{I \subseteq \{1,2\}} A_I(s) = \zeta_{\mathbb{Z}_p^3}(s) - \frac{Z_f(s-2)p^2t}{(1-p^2t^2)(1-p^2t)(1-p^{-1})}.$$

This proves Theorem 1.1 for $i = 0$. But passing from L to p^iL amounts to replacing f by p^if. It is clear, however, that $Z_{p^if}(s) = t^iZ_f(s)$, so that $Z_{p^if}(s-2) = (p^2t)^iZ_f(s-2)$. The result follows.

4. Explicit computations

In this section we show how Theorem 1.1 gives rise to simple computations of the zeta functions of 3-dimensional $\mathbb{Z}_p$-Lie algebras. We shall tacitly assume the notation from Section 2.

4.1. **The Heisenberg Lie algebra $\mathfrak{h}(\mathbb{Z}_p)$.** The (local) Heisenberg Lie algebra has a presentation

$$\mathfrak{h}(\mathbb{Z}_p) = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2 \oplus \mathbb{Z}_p e_3,$$

where $[e_1, e_2] = e_3$ is the only non-zero relation. We thus have

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & y_3 & \\ -y_3 & & \\ & & \end{pmatrix}.$$

The ternary quadratic form equals $f(\mathbf{x}) = x_3^2$. It is well-known that Igusa's local zeta function associated to f equals

$$Z_f(s) = \int_{\mathbb{Z}_p} |x^2|_p^s \mathrm{d}x = \frac{1 - p^{-1}}{1 - p^{-1-2s}}.$$

Using Theorem 1.1 it is now easy to confirm the formula for the local Heisenberg Lie algebra (cf. [8]):

$$\begin{aligned} \zeta_{\mathfrak{h}(\mathbb{Z}_p)}(s) &= \zeta_{\mathbb{Z}_p^3}(s) - \frac{p^2 t}{(1 - p^2t^2)(1 - p^2t)(1 - p^3t)} \\ &= \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(2s-2)\zeta_p(3s-3)^{-1}. \end{aligned}$$

4.2. **The 'simple' Lie algebra $\mathfrak{sl}_2(\mathbb{Z}_p)$.** The Lie algebra $\mathfrak{sl}_2(\mathbb{Z}_p)$ has a presentation

$$\mathfrak{sl}_2(\mathbb{Z}_p) = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2 \oplus \mathbb{Z}_p e_3,$$

where

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -2e_1, \quad [e_2, e_3] = 2e_2.$$

We obtain

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & y_3 & -2y_1 \\ -y_3 & & 2y_2 \\ 2y_1 & -2y_2 & \end{pmatrix}.$$

The relevant ternary quadratic form is thus $f(\mathbf{x}) = x_3^2 + 4x_1x_2$. Note that, for $p > 2$, f defines a smooth conic in projective 2-space which has $p + 1$ points over $\mathbb{F}_p$ and good reduction modulo p. It follows from Denef's formula for Igusa's local zeta function in this case (cf., for instance, [2, (6) on p. 1146]) that

$$Z_f(s-2) = \frac{(1 - p^{-1})(1 - p^{-1}t)}{(1 - pt^2)(1 - pt)}. \tag{12}$$

We obtain from Theorem 1.1 the known formula

$$\begin{aligned}\zeta_{\mathfrak{sl}_2(\mathbb{Z}_p)}(s) &= \zeta_{\mathbb{Z}_p^3}(s) - \frac{(1-p^{-1}t)p^2t}{(1-pt)(1-p^2t)(1-pt^2)(1-p^2t^2)} \\ &= \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-1)\zeta_p(2s-2)\zeta_p(3s-1)^{-1}\end{aligned}$$

and, more generally, the formulae for $\zeta_{p^i\mathfrak{sl}_2(\mathbb{Z}_p)}(s)$ computed in [3, Theorem 3.1].

The case $p = 2$ is nearly as simple. We shall in fact derive a formula for $Z_f(s-2)$, $f(\mathbf{x}) = x_3^2 + p^2x_1x_2$, valid for all primes p, in terms of the function $Z_{\tilde{f}}(s-2)$, where $\tilde{f}(\mathbf{x}) = x_3^2 + x_1x_2$. As 4 is a p-adic unit for odd p, the right hand side of (12) yields a formula for $Z_{\tilde{f}}(s-2)$, valid for all primes. To compute $Z_f(s)$ we define $W_3 := \{\mathbf{x} \in \mathbb{Z}_p^3 \mid x_3 \in \mathbb{Z}_p^*\}$ and write

$$Z_f(s) = \int_{W_3} |f(\mathbf{x})|_p^s \mathrm{d}\mu + \int_{\mathbb{Z}_p^3 \setminus W_3} |f(\mathbf{x})|_p^s \mathrm{d}\mu.$$

The first summand equals $\mu(W_3) = 1 - p^{-1}$. To compute the second summand, we perform a change of variable $x_3 = px_3'$, say, effectuating a change of measure $\mathrm{d}\mu = |p|_p \mathrm{d}\mu'$. Thus

$$\int_{\mathbb{Z}_p^3 \setminus W_3} |f(\mathbf{x})|_p^s \mathrm{d}\mu = \int_{\mathbb{Z}_p^3} |p^2\tilde{f}(x_1, x_2, x_3')|_p^s \, |p|_p \mathrm{d}\mu' = p^{-1-2s} Z_{\tilde{f}}(s).$$

Combining these pieces of information we obtain, for $p = 2$,

$$Z_f(s-2) = 1 - 2^{-1} + 8 \cdot 2^{-2s} Z_{\tilde{f}}(s-2).$$

Our Theorem 1.1 now confirms that, for $p = 2$,

$$\begin{aligned}\zeta_{\mathfrak{sl}_2(\mathbb{Z}_2)}(s) &= \zeta_{\mathbb{Z}_2^3}(s) - \frac{(1 - 2\cdot 2^{-s} + 6\cdot 2^{-2s})2^{2-s}}{(1-2^{1-s})(1-2^{2-s})(1-2^{1-2s})(1-2^{2-2s})} \\ &= \zeta_2(s)\zeta_2(s-1)\zeta_2(2s-2)\zeta_2(2s-1)(1 + 6\cdot 2^{-2s} - 8\cdot 2^{-3s}).\end{aligned}$$

This formula was first given in [6, 21].

4.3. **The 'simple' Lie algebra $\mathfrak{sl}_1(\Delta_p)$.** In [12], Klopsch computed the zeta function of $L = \mathfrak{sl}_1(\Delta_p)$, where Δ_p is the maximal $\mathbb{Z}_p$-order in a central $\mathbb{Q}_p$-division algebra of index 2. The Lie algebra L contains elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the relations

$$[\mathbf{i}, \mathbf{j}] = \mathbf{k}, \quad [\mathbf{i}, \mathbf{k}] = \rho\mathbf{j}, \quad [\mathbf{j}, \mathbf{k}] = -p\mathbf{i}, \tag{13}$$

where $\rho \in \{1, 2, \ldots, p-1\}$ is a non-square modulo p if p is odd and $\rho = -3$ if $p = 2$.

For $p > 2$, the triple $(e_1, e_2, e_3) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ forms a $\mathbb{Z}_p$-basis for L. We obtain in this case

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & y_3 & \rho y_2 \\ -y_3 & & -py_1 \\ -\rho y_2 & py_1 & \end{pmatrix}$$

and are thus led to study Igusa's local zeta function associated to

$$f(\mathbf{x}) = x_3^2 - \rho x_2^2 - p x_1^2.$$

The easiest way to do this may be to compute the Poincaré series $P_f^\star(t)$ (cf. (2)) and then to use the identities (4) and (3). In fact, the series $P_f^\star(t)$ has only two non-zero summands: one easily computes $N_0^\star = 1$ and $N_1^\star = p - 1$. There are, however, no solutions $\mathbf{x}$ in W of $f(\mathbf{x}) \equiv 0$ modulo (p^2). Indeed such a solution would necessarily require $x_2 \equiv x_3 \equiv 0$ modulo (p), forcing $x_1 \equiv 0$ modulo (p). Thus $N_m^\star = 0$ for $m \geq 2$. Lemma 2.1 yields

$$1 + (p-1)p^{-3}t = P_f^\star(t) = \frac{1 - p^{-3}t - tZ_f^\star(s)}{1-t}$$

and, using (3), we obtain

$$Z_f(s-2) = \frac{(1-p^{-1})(1+p^{-1}+t)}{1-pt^2}.$$

Thus

$$\begin{aligned} \zeta_{\mathfrak{sl}_1(\Delta_p)}(s) &= \zeta_{\mathbb{Z}_p^3}(s) - \frac{(1+p^{-1}+t)p^2t}{(1-pt)(1-pt^2)(1-p^2t^2)} \\ &= \zeta_p(s)\zeta_p(2s-1)\zeta_p(2s-2). \end{aligned}$$

For $p = 2$ the triple (e_1, e_2, e_3), with

$$e_1 = 2\mathbf{i}, \quad e_2 = 2\mathbf{j}, \quad e_3 = \mathbf{j} + \mathbf{k},$$

forms a $\mathbb{Z}_p$-basis for L. Using (13) we derive the following commutator relations:

$$[e_1, e_2] = -2e_2 + 4e_3, \quad [e_1, e_3] = -4e_2 + 2e_3, \quad [e_2, e_3] = -2e_1.$$

This yields

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & -2y_2 + 4y_3 & -4y_2 + 2y_3 \\ 2y_2 - 4y_3 & & -2y_1 \\ 4y_2 - 2y_3 & 2y_1 & \end{pmatrix}$$

and thus

$$f(\mathbf{x}) = -2(x_1^2 - 2(x_2^2 - x_2x_3 + x_3^2)).$$

We shall in fact derive a recursion formula for $Z_f(s)$, where

$$f(\mathbf{x}) = p(x_1^2 - pq(x_2, x_3))$$

and q is an arbitrary binary quadratic form which is anisotropic modulo p. This condition on q and p is certainly satisfied for the form $q(x_2, x_3) = x_2^2 - x_2x_3 + x_3^2$ and the prime $p = 2$. The calculation involves the zeta function $Z_{\tilde{f}}(s)$ associated to the form

$$\tilde{f}(\mathbf{x}) = px_1^2 - q(x_2, x_3)$$

and uses changes of variables similar to the one performed in Section 4.2 for $p = 2$. Setting $W_1 := \{\mathbf{x} \in \mathbb{Z}_p^3 \mid x_1 \not\equiv 0 \mod (p)\}$ and $W_{2,3} := \{\mathbf{x} \in \mathbb{Z}_p^3 \mid (x_2, x_3) \not\equiv (0,0) \mod (p)\}$ we obtain

$$\begin{aligned}
t^{-1}Z_f(s) &= \int_{W_1} |p^{-1}f(\mathbf{x})|_p^s \mathrm{d}\mu + \int_{\mathbb{Z}_p^3 \setminus W_1} |p^{-1}f(\mathbf{x})|_p^s \mathrm{d}\mu \\
&= 1 - p^{-1} + p^{-1}tZ_{\tilde{f}}(s) \\
&= 1 - p^{-1} + p^{-1}t\left(1 - p^{-2} + \int_{\mathbb{Z}_p^3 \setminus W_{2,3}} |\tilde{f}(\mathbf{x})|_p^s \mathrm{d}\mu\right) \\
&= 1 - p^{-1} + p^{-1}t\left(1 - p^{-2} + p^{-2-s}Z_{p^{-1}f}(s)\right) \\
&= 1 - p^{-1} + p^{-1}t\left(1 - p^{-2} + p^{-2}Z_f(s)\right).
\end{aligned}$$

From this we compute

$$Z_f(s-2) = \frac{(1-p^{-1})(1+(p+1)t)p^2t}{1-pt^2},$$

and thus, setting $p = 2$,

$$\begin{aligned}
\zeta_{\mathfrak{sl}_1(\Delta_2)} &= \zeta_{\mathbb{Z}_2^3}(s) - \frac{(1+3\cdot 2^{-s})2^{4-2s}}{(1-2^{2-s})(1-2^{1-2s})(1-2^{2-2s})} \\
&= \zeta_2(s)\zeta_2(2s-1)\zeta_2(2s-2)(1+6\cdot 2^{-s}+6\cdot 2^{-2s}-12\cdot 2^{-3s}),
\end{aligned}$$

confirming the results of [12, Theorem 1.1].

4.4. **Soluble Lie algebras.** In this section let $p \geq 3$. The soluble 3-dimensional $\mathbb{Z}_p$-Lie algebras have been listed by González-Sánchez and Klopsch in [14], using an analysis of conjugacy classes in $\mathrm{SL}_2(\mathbb{Z}_p)$. It suffices to consider the following maximal representatives of the respective homothety classes listed below. All others are obtained by multiplying one of the matrices of relations $\mathcal{R}(\mathbf{y})$ by a power of p; the effect of this operation on the zeta function $Z_f(s)$ is easily controlled. In the following we choose a notation similar to [14, §6]. The soluble Lie algebras to be considered are the following.

A. The abelian Lie algebra $L_0(\infty)$.
B. The Heisenberg Lie algebra $L_0(0) = \mathfrak{h}(\mathbb{Z}_p)$.
C. The non-nilpotent Lie algebra $L_1(0)$. We obtain

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & -y_2 & -y_3 \\ y_2 & & \\ y_3 & & \end{pmatrix}$$

and

$$f(\mathbf{x}) = 0.$$

D. The non-nilpotent Lie algebras $L_2(0, r, d)$ with $r \in \mathbb{N}$ and $d \in \mathbb{Z}_p$. We obtain

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & -y_2 - p^r d y_3 & -p^r y_2 - y_3 \\ y_2 + p^r d y_3 & & \\ p^r y_2 + y_3 & & \end{pmatrix}$$

and

$$f(\mathbf{x}) = p^r(x_2^2 - d x_3^2).$$

E. The non-nilpotent Lie algebras $L_3(0, r, d)$ with $r \in \mathbb{N}_0$ and $d \in \mathbb{Z}_p$. We obtain

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & -d y_3 & -y_2 - p^r y_3 \\ d y_3 & & \\ y_2 + p^r y_3 & & \end{pmatrix}$$

and

$$f(\mathbf{x}) = x_2^2 + p^r x_2 x_3 - d x_3^2.$$

F. The non-nilpotent Lie algebras $L_4(0, r)$ with $r \in \mathbb{N}_0$. We obtain

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & -p^r y_3 & -y_2 \\ p^r y_3 & & \\ y_2 & & \end{pmatrix}$$

and

$$f(\mathbf{x}) = x_2^2 - p^r x_3^2.$$

G. The non-nilpotent Lie algebras $L_5(0, r)$ with $r \in \mathbb{N}_0$. We obtain

$$\mathcal{R}(\mathbf{y}) = \begin{pmatrix} & -p^r \rho y_3 & -y_2 \\ p^r \rho y_3 & & \\ y_2 & & \end{pmatrix},$$

where $\rho \in \mathbb{Z}_p^*$ is a non-square modulo p, and

$$f(\mathbf{x}) = x_2^2 - p^r \rho x_3^2.$$

The cases A to C have already been treated. Rather than calculate the zeta function $Z_f(s)$ in each of the remaining cases, we note that, after (possibly) dividing by a power of p, completing the square and a coordinate change, this reduces to the computation of the zeta function for the polynomial $f(x_2, x_3) = x_2^2 - dx_3^2$, $d \in \mathbb{Z}_p$. We distinguish two cases. If $d = p^k\rho$, where $\rho \in \mathbb{Z}_p^*$ is a non-square modulo p, we define $Z_{\boxslash,k}(s) := Z_f(s)$. If $d = p^k u^2$, where $u \in \mathbb{Z}_p^*$, we set $Z_{\square,k}(s) := Z_f(s)$. Both cases are easily computed using

$$Z_{\boxslash,0}(s) = \frac{1-p^{-2}}{1-p^{-2}t^2}, \quad Z_{\square,0}(s) = \left(\frac{1-p^{-1}}{1-p^{-1}t}\right)^2,$$
$$Z_{\boxslash,1}(s) = Z_{\square,1}(s) = \frac{1-p^{-1}}{1-p^{-1}t}$$

and the fact that both sequences satisfy the same simple recursion equation of length two. Indeed, for $* = \square$ or $* = \boxslash$, we have, for $k \in \mathbb{N}_0$,

$$Z_{*,k+2}(s) = p^{-1}t^2 Z_{*,k}(s) + 1 - p^{-1}.$$

An elementary calculation using these observations yields

Proposition 4.1. *Let L be a soluble 3-dimensional $\mathbb{Z}_p$-Lie algebra associated to one of the families D to G. Then, for suitable $k, \iota \in \mathbb{N}_0$ and $* = \square$ or $* = \boxslash$,*

$$\zeta_L(s) = \zeta_{\mathbb{Z}_p^3}(s) - Z_{*,k}(s-2)\zeta_p(2s-2)\zeta_p(s-2)p^{(2-s)(\iota+1)}(1-p^{-1})^{-1}.$$

The abscissa of convergence α of $\zeta_L(s)$ equals 1 in all cases. If $* = \square$ and k is even then $\zeta_L(s)$ has a triple pole at $s = 1$. In all other cases $\zeta_L(s)$ has a double pole at $s = 1$.

Corollary 4.2. *Assume the setting of Proposition 4.1. For $n \in \mathbb{N}_0$ denote by σ_n the number of subalgebras of L of index at most p^n. Then there are constants $c_1, c_2 \in \mathbb{R}$, depending on L, such that, for all $n \in \mathbb{N}_0$*

$$c_1 p^n n^2 \leq \sigma_n \leq c_2 p^n n^2 \quad \text{if } * = \square \text{ and } k \text{ is even,}$$
$$c_1 p^n n \leq \sigma_n \leq c_2 p^n n \quad \text{otherwise.}$$

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