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A CATEGORICAL MODEL FOR THE VIRTUAL BRAID GROUP

LOUIS H. KAUFFMAN AND SOFIA LAMBROPOULOU

1. Introduction

This paper gives a new interpretation of the virtual braid group in terms of a strict monoidal category SC that is freely generated by one object $*$ and three morphisms $\mu : * \otimes * \longrightarrow * \otimes *$, $\mu' : * \otimes * \longrightarrow * \otimes *$, and $v : * \otimes * \longrightarrow * \otimes *$. This basic structure, subjected to appropriate relations can be understood via morphisms μ_{ij} defined in terms of the generating morpisms, where the symbol μ_{ij} can be interpreted as an abstract string or connection between strands i and j in a diagram that otherwise is an identity on n strands. That is, μ_{ij} is a diagrammatically a decorated identity braid where the decoration consists in a connection between the i strand and the j strand. The μ_{ij} satisfy the algebraic Yang-Baxter equation in the sense that for $i < j < k$, $\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}$. The other generators of this category are elements v_i that can be depicted as virtual crossings between strings i and $i+1$. The v_i generate the symmetric group S_n. An n-strand diagram that is a product of these generators is a morphism from $[n]$ to $[n]$ where the symbol $[n]$ is an ordered row of n points that constitute the top or the bottom of a diagram involving n strands. In terms of the definition of the monoidal category $[n] = * \otimes * \cdots * \otimes *$ for a tensor product of n $*$'s.

The virtual braid group on n strands is isomorphic to the group of morphisms in SC from $[n]$ to $[n]$. The point of this categorical formulation of the virtual braid groups is that we see how these groups form a natural extension of the symmetric groups by formal elements that satisfy the algebraic Yang-Baxter equation. The category we desribe is a natural structure for an algebraist interested in exploring formal properties of the algebraic Yang-Baxter equation, and it is directly related to more topological points of view about virtual links and virtual braids.

In fact, this very category, under differrent motivation, was constructed in [15] where the intent was to construct a category that would be naturally associated with a Hopf algebra on the one hand, and would receive topological tangles, knots and links under a functor from the tangle category to the Hopf algebra category. The present category, giving the structure of the virtual braid group, is a subcategory of that category associated

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with a general Hopf algebra. We explain this relationship in detail in Section 6 of the present paper.

Without the concept of virtuality, the direct relationship of the algebraic Yang-Baxter equation with the braid groups would not be apparent. We see, that, from an algebraic point of view, the virtual braid group is an entirely natural construction. It is the universal algebraic structure related to viewing solutions to the algebraic Yang-Baxter equation inside tensor products of algebras and endowing these tensor products with the natural permutation action of the symmetric group. See Remark 10 of [2] and references therein for another earlier observation of the relationship of the algebraic Yang-Baxter equation with the pure virtual braid group.

We develop this model for the virtual braid group by first recalling its usual definition motivated by virtual knot theory. We then proceed to reformulate the virtual braid group in terms of the above mentioned generators. By the time we reach Theorem 1, we have reformulated the virtual braid group in terms of the new generators. We then use this approach to give a presentation of the pure virtual braid group in Theorem 2.

More precisely, in Section 2 we give a presentation for the virtual braid group in terms of our stringy model. We start by describing the usual presentation of the virtual braid group in terms of classical braid generators and virtual generators that act as permutations between pairs of adjacent strands in the braid. Elementary connection strings (see Figure 6) are defined as elementary pure braids – products of braid generators and virtual generators. We then generalize the notion of connecting string and show that it has the formal diagrammatic property of being stretched and contracted as shown in Figure 8. With these constructions we then rewrite presentations for the virtual braid group and, in Section 3, show how the connection with strings generate the pure virtual braid group with a set of relations that correspond to the algebraic Yang-Baxter equation. See Theorem 2.

In Section 4 we construct the String Category alluded to in the first paragraph of this introduction. In Section 5 we detail the relationship with the algebraic Yang-Baxter equation, show how to use solutions of the algebraic Yang-Baxter equation to obtain representations of the pure virtual braid group and virtual braid group. In our point of view the entire virtual braid group can be seen as a natural extension of the pure virtual braid group by a category of permutation operators. The pure virtual braid groups themselves are seen to be a natural monoidal category associated with solutions of the algebraic Yang-Baxter equation. This gives an essentially categorical point of view for understanding the nature of the virtual braid group. In starting our discussion of the virtual braid group from virtual knot theory we began with the motivation that virtual crossings are artifacts of a planar representation of knots and links that are embedded in thickened surfaces. This is a correct point of view, but it does not speak directly to the algebraic structure of the virtual braid group, where the virtual part of the group is the symmetric group generated by the virtual crossings. In the braiding context the virtual crossings are permutation operators and it is conceptually important to have a point of view in which their role is natural in a categorical and algebraic sense. This is what we have done in reformulating the virtual braid group in terms of the category of string connectors and associated permutation operators.

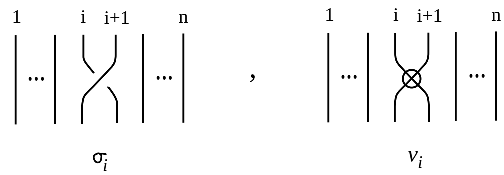

FIGURE 1. The generators of VB_n

In Section 6 we discuss a generalization of the virtual braiding formalism to categories that can be used for obtaining invariants of knots and links via Hopf algebras. The invariants we obtain are invariants of *rotational virtual knots and links* where the term rotational means that we do not allow the use of the first virtual Reidemeister move. See Figure 17. As a monoidal category of virtual tangles, the rules for rotational virtuals are shown in Figure 20. This is a most convenient category for working with virtual knots and links as it generalizes the virtual braid group, and every quantum link invariant for classical knots and links extends to an invariant for rotational virtual knots and links. In this section we show how a generalization of the string connectors defined previously in the paper enable the construction of quantum link invariants associated with a Hopf algebra in a generalization of the Kauffman-Radford formulation of the Hennings invariant.

2. A Stringy Presentation for the Virtual Braid Group

2.1. **The virtual braid group.** Let's begin with a presentation for the virtual braid group. The set of isotopy classes of virtual braids on n strands forms a group, the *virtual braid group* denoted VB_n, that was introduced in [16]. The group operation is the usual braid multiplication (form bb' by attaching the bottom strand ends of b to the top strand ends of b'). VB_n is generated by the usual braid generators $\sigma_1, \dots, \sigma_{n-1}$ and by the virtual generators $v_1, \dots, v_{n-1}$, where each virtual crossing v_i has the form of the braid generator σ_i with the crossing replaced by a virtual crossing. See Figure 1 for illustrations. Recall that in virtual crossings we do not distinguish between under and over crossing. Thus, VB_n is an extension of the classical braid group B_n by the symmetric group S_n, whereby v_i corresponds to the elementary transposition $(i, i+1)$.

Among themselves the braid generators satisfy the usual *braiding relations*:

$$\begin{array}{llll} \text{(B1)} & \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1}, & \\ \text{(B2)} & \sigma_i\sigma_j & = & \sigma_j\sigma_i, & \text{for } j \neq i \pm 1. \end{array}$$

Among themselves, the virtual generators are a presentation for the symmetric group S_n, so they satisfy the following *virtual relations:*

$$\begin{array}{lllll} \text{(S1)} & v_iv_{i+1}v_i & = & v_{i+1}v_iv_{i+1}, & \\ \text{(S2)} & v_iv_j & = & v_jv_i, & \text{for } j \neq i \pm 1, \\ \text{(S3)} & {v_i}^2 & = & 1. & \end{array}$$

The *mixed relations* between virtual generators and braiding generators are as follows:

$$\begin{array}{lllll} \text{(M1)} & v_i\sigma_{i+1}v_i & = & v_{i+1}\sigma_iv_{i+1}, & \\ \text{(M2)} & \sigma_iv_j & = & v_j\sigma_i, & \text{for } j \neq i \pm 1. \end{array}$$

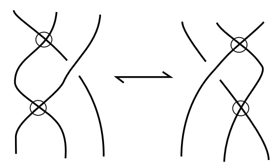

FIGURE 2. The local detour

To summarize, the virtual braid group VB_n has the following presentation [16].

$$VB_n = \left\langle \begin{array}{c} \sigma_1, \ldots, \sigma_{n-1}, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} (B1), (B2), \\ (S1), (S2), (S3), \\ (M1), (M2) \end{array} \right\rangle \tag{1}$$

It is worth noting at this point that the virtual braid group VB_n does not embed in the classical braid group B_n, since the virtual braid group contains torsion elements (the v_i have order two) and it is well–known that B_n does not. But the classical braid group embeds in the virtual braid group just as classical knots embed in virtual knots. This fact may be most easily deduced from [24], and can also be seen from [26] and [6]. For reference to previous work on virtual braids the reader should consult [3, 5, 9, 10, 11, 16, 17, 18, 13, 14, 23, 24, 26, 29, 32, 33, 34, 19] and references therein. For work on welded braids, see [6, 14, 19].

Further, for Markov–type theorems for virtual braids, giving sets of moves on virtual braids that generate the same equivalence classes as the oriented virtual link types of their closures, see [14] and [20]. Such theorems are important for understanding the structure and classification of virtual knots and links.

The second mixed relation in the presentation of the virtual braid group will be called the *local detour relation* and it is illustrated in Figure 2. Note that the following relations are also local detour moves for virtual braids and they are easy consequences of the above.

$$\begin{array}{rcl} v_i v_{i+1} {\sigma_i}^{\pm 1} & = & {\sigma_{i+1}}^{\pm 1} v_i v_{i+1}, \\ {\sigma_i}^{\pm 1} v_{i+1} v_i & = & v_{i+1} v_i {\sigma_{i+1}}^{\pm 1}. \end{array} \tag{2}$$

This set of relations taken together define the basic local isotopies for virtual braids. Note that each relation is a braided version of a local virtual link isotopy. The local detour move is written equivalently:

$$\sigma_{i+1} = v_i v_{i+1} \sigma_i v_{i+1} v_i. \tag{3}$$

Notice that Eq. 3 is the braid detour move of the ith strand around the crossing between the $(i+1)$-st and the $(i+2)$-nd strand (see first two illustrations in Figure 3) and it provides an inductive way of expressing all braiding generators in terms of the first braiding generator σ_1 and the virtual generators $v_1, \ldots, v_{n-1}$ (see first and last illustrations in Figure 3), that is:

$$\sigma_j = (v_{j-1} \ldots v_2 v_1)\,(v_j \ldots v_3 v_2)\,\sigma_1\,(v_2 v_3 \ldots v_j)\,(v_1 v_2 \ldots v_{j-1}). \tag{4}$$

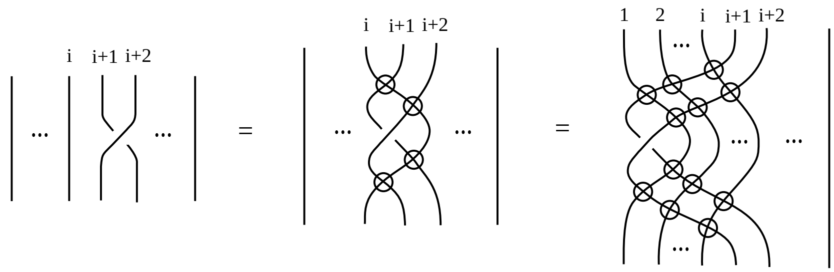

FIGURE 3. Detouring the crossing σ_{i+1}

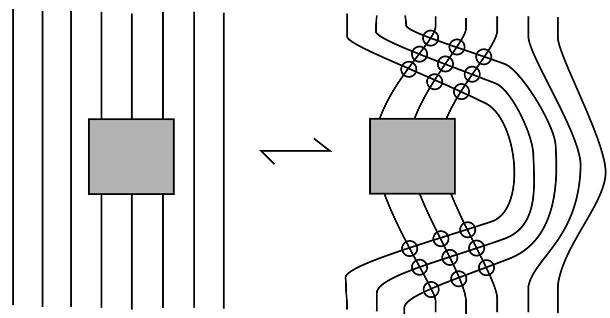

FIGURE 4. Detouring a box

In [19] we derive the following reduced presentation for VB_n:

$$VB_n = \left\langle \begin{array}{c} \sigma_1, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} (S1), (S2), (S3) \\ \sigma_1 v_j = v_j \sigma_1, \quad \text{for } j > 2 \\ v_1\sigma_1 v_1\, v_2\sigma_1 v_2\, v_1\sigma_1 v_1 = v_2\sigma_1 v_2\, v_1\sigma_1 v_1\, v_2\sigma_1 v_2 \\ \sigma_1\, v_2v_3v_1v_2\sigma_1v_2v_1v_3v_2 = v_2v_3v_1v_2\sigma_1v_2v_1v_3v_2\,\sigma_1 \end{array} \right\rangle \tag{5}$$

The local detour move gives rise to a generalized *detour move*, by which any box in the braid can be detoured to any position in the braid, see Figure 4.

Finally, it is worth recalling that in virtual knot theory there are "forbidden moves" involving two real crossings and one virtual. More precisely, there are two types of forbidden moves: One with an over arc, denoted F_1 and another with an under arc, denoted F_2. See [16] for explanations and interpretations. Variants of the forbidden moves are illustrated in Figure 5. So, relations of the types:

$$\sigma_i v_{i+1} \sigma_i^{-1} = \sigma_{i+1}^{-1} v_i \sigma_{i+1} \quad (F1) \qquad \text{and} \qquad \sigma_i^{-1} v_{i+1} \sigma_i = \sigma_{i+1} v_i \sigma_{i+1}^{-1} \quad (F2) \tag{6}$$

are not valid in virtual knot theory.

2.2. We now wish to describe a new set of generators and relations for the virtual braid group that makes it particularly easy to describe the pure virtual braid group, VP_n. In order to accomplish this aim, we introduce the following elements of VP_n, for $i = 1, \ldots, n-1$.

$$\mu_{i,i+1} := \sigma_i v_i \tag{7}$$

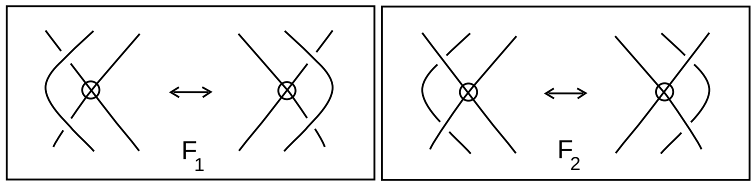

FIGURE 5. The forbidden moves

We indicate $\mu_{i,i+1}$ by a connecting string between the i-th and $(i+1)$-st strands with a dark vertex on the i-th strand, a dark vertex on the $(i+1)$-st strand, and an arrow from left to right. View Figure 6. The inverses $\mu_{i,i+1}^{-1} = v_i\sigma_i^{-1}$ have same directional arrows but are indicated by using white vertices. Note that, by detouring it to the leftmost position of the braid, we can write $\mu_{i,i+1}$ in terms of μ_{12} conjugated by a virtual word:

$$\mu_{i,i+1} = (v_{i-1}\ldots v_2v_1)(v_i\ldots v_3v_2)\mu_{12}(v_2v_3\ldots v_i)(v_1v_2\ldots v_{i-1}). \tag{8}$$

We also introduce the elements

$$\mu_{i+1,i} := v_i\sigma_i = v_i\mu_{i,i+1}v_i \tag{9}$$

We indicate $\mu_{i+1,i}$ by a connecting string between the i-th and $(i+1)$-st strands, with a dark vertex on the i-th strand, a dark vertex on the $(i+1)$-st strand, and an arrow from right to left (reversing the direction from $\mu_{i,i+1}$), view Figure 6. An illustration of Eq. 9 (see top of Figure 7) explains the reversing of the direction of the arrow in the graphical interpretation of $\mu_{i+1,i}$. The inverses $\mu_{i+1,i}^{-1} = \sigma_i^{-1}v_i$ have same directional arrows but are indicated by using white vertices. Note that an analogous equation to Eq. 8 holds:

$$\mu_{i+1,i} = (v_{i-1}\ldots v_2v_1)(v_i\ldots v_3v_2)\mu_{21}(v_2v_3\ldots v_i)(v_1v_2\ldots v_{i-1}) \tag{10}$$

Definition 1. The pure virtual braids $\mu_{i,i+1}, \mu_{i+1,i}$ and their inverses shall be called *elementary connecting strings*.

From Eqs. 7 and 9 follow directly the relations:

$$v_i\mu_{i+1,i} = \mu_{i,i+1}v_i \quad \text{and} \quad \mu_{i+1,i}^{-1}v_i = v_i\mu_{i,i+1}^{-1}, \tag{11}$$

also illustrated in Figure 7.

Further, we generalize the notion of a connecting string and define, for $i < j$, the element μ_{ij} (a connecting string from strand i to strand j) by the formula

$$\mu_{ij} := v_{j-1}v_{j-2}\cdots v_{i+1}\,\mu_{i,i+1}\,v_{i+1}\cdots v_{j-2}v_{j-1}. \tag{12}$$

In a diagram μ_{ij} is denoted by a connecting string from strand i to strand j, with dark vertices on these two strands and an arrow pointing from left to right, view Figure 8.

We also generalize, for $i < j$, the elements $\mu_{i+1,i}$ to the elements:

$$\mu_{ji} := t_{ij}\,\mu_{ij}\,t_{ij} \tag{13}$$

where $t_{ij} = v_iv_{i+1}\ldots v_j\ldots v_{i+1}v_i$ is the element of S_n (generated by the v_i's) that interchanges strands i and j, leaving all other strands fixed. We denote μ_{ji} by a connecting

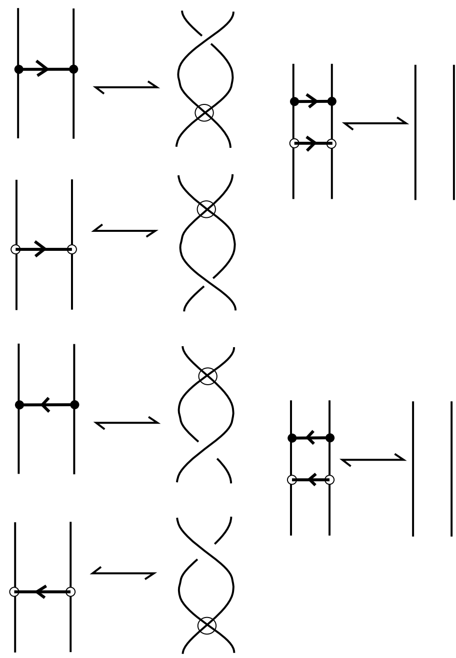

FIGURE 6. The elementary connecting strings $\mu_{i,i+1}$, $\mu_{i+1,i}$ and their inverses

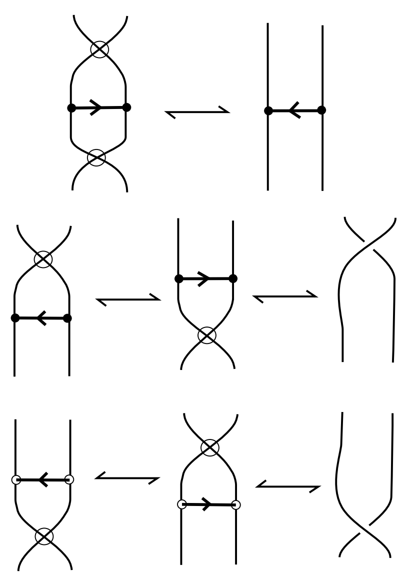

FIGURE 7. Relations between the elementary connecting strings

string from strand i to strand j, with dark vertices, and an arrow pointing from right to left. Figure 9 illustrates the example $\mu_{31} = v_2 v_1 v_2 \mu_{13} v_2 v_1 v_2$. It is easily verified that

$$\mu_{ji} = v_{j-1} v_{j-2} \cdots v_{i+1}\, \mu_{i+1,i}\, v_{i+1} \cdots v_{j-2} v_{j-1} \tag{14}$$

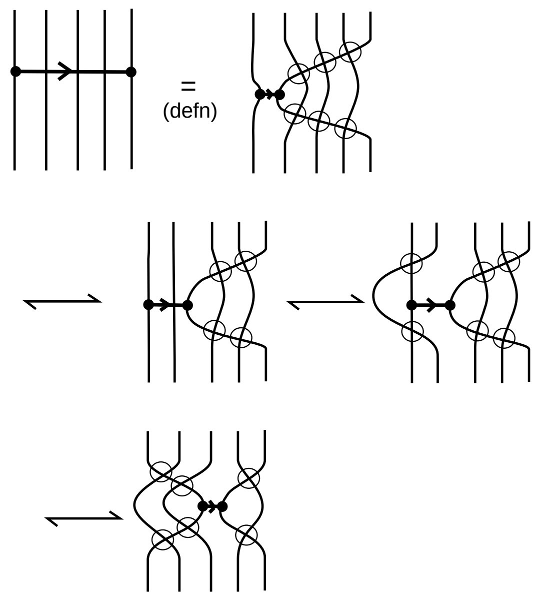

FIGURE 8. Connecting strings

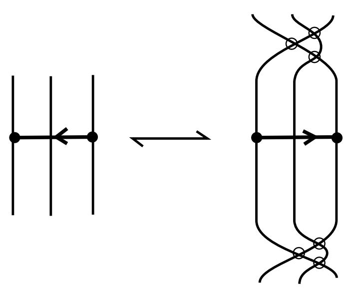

FIGURE 9. The exchange of labels between μ_{ij} and μ_{ji}

The inverses of the elements μ_{ij} and μ_{ji} have same directional arrows respectively, but white dotted vertices.

Definition 2. The elements μ_{ij}, μ_{ji} and their inverses shall be called *connecting strings.*

With the above conventions we can speak of connecting strings μ_{rs} for any r, s. It is important to have the elements μ_{ji} when $j > i$, but in the algebra they are all defined in terms of the μ_{ij}. The importance of having the elements μ_{ji} will become clear when we restrict to the pure virtual braid group.

Remark 1. In the definition of μ_{ij} if we consider $\mu_{i,i+1}$ as a virtual box inside the virtual braid we can use the (generalized) detour moves to bring it to any position, as Figure 8 illustrates. This means that the contraction of μ_{ij} to $\mu_{i,i+1}$ may be pulled anywhere between the i-th and the j-th strands. By the same reasoning the contraction of μ_{ji} to $\mu_{i+1,i}$ may be also pulled anywhere between the i-th and the j-th strands.

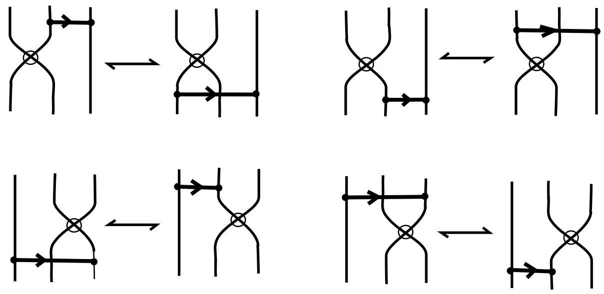

FIGURE 10. Slide moves

2.3. We shall next give some relations satisfied by the connecting strings. Before that we need the following remark.

Remark 2. The symmetric group S_n clearly acts on VB_n by conjugation. By their definition (Eqs. 7, 9, 24, 14), this action on connecting strings translates into permuting their indices, that is, a permutation $\tau \in S_n$ acting on μ_{rs} will change it to $\mu_{\tau(r),\tau(s)}$. This means that S_n acts by conjugation also on the subgroup of VB_n generated by the μ_{ij}'s. Moreover, by Eqs. 8, 9, all connecting strings may be obtained by the action of S_n on μ_{12}. Note that for $\sigma \in S_n$ we regard σ both as a product of the elements v_i and as a permutation of the set $\{1, 2, 3, \ldots, n\}$.

Further, any relation in VB_n transforms into a valid relation after acting on it an element of S_n. In particular, a commuting relation between connecting strings will be transformed to a new commuting relation between connecting strings.

Lemma 1. *The following relations hold in VB_n for all i.*

(1) $v_i\mu_{i,i+1} = \mu_{i+1,i}v_i \quad , \quad v_i\mu_{i+1,i} = \mu_{i,i+1}v_i$
(2) $v_{i+1}\mu_{i,i+1} = \mu_{i,i+2}v_{i+1} \quad , \quad v_{i+1}\mu_{i+1,i} = \mu_{i+2,i}v_{i+1}$
(3) $v_{i-1}\mu_{i,i+1} = \mu_{i-1,i+1}v_{i-1} \quad , \quad v_{i-1}\mu_{i+1,i} = \mu_{i+1,i-1}v_{i-1}$
(4) $v_j\mu_{i,i+1} = \mu_{i,i+1}v_j \quad , \quad v_j\mu_{i+1,i} = \mu_{i+1,i}v_j, \quad j \neq i-1,\, i,\, i+1.$

The above local relations generalize to similar ones involving different indices. Relations 1 are generalized by Eq. 13, reflecting the mutual reversing of μ_{ij} and μ_{ji}, recall Figures 7 and 9. Relations 2 and 3 are the local slide moves*, as illustrated in Figure 10, and they generalize to the* slide moves *coming from the defining equations: $\mu_{i+1,k} = v_i\mu_{ik}v_i$ for any $k < i$ or $k > i+1$. Relations 4 and their generalizations: $v_j\mu_{ik} = \mu_{ik}v_j$ for any $k \neq i$ and $j \neq i-1,\, i,\, k-1,\, k$, are all commuting relations. All these relations result from the action of any $\tau \in S_n$ on μ_{12}:*

$$\boldsymbol{\tau^{-1}\,\mu_{12}\,\tau = \mu_{\tau(1),\tau(2)}} \quad . \tag{15}$$

Proof. All relations 1,2 and 3 follow directly from the definitions of the elements μ_{ij} and μ_{ji}. For example, $v_{i+1}\mu_{i,i+1} = \mu_{i,i+2}v_{i+1}$ is equivalent to the defining relation $\mu_{i,i+2} = v_{i+1}\mu_{i,i+1}v_{i+1}$. Figure 11 illustrates the proof of a local slide move. Relations 4 follow immediately from the commuting relations (S2) and (M2) of VB_n. The generalizations of all types of moves follow from the local ones after using detour moves. Finally, the

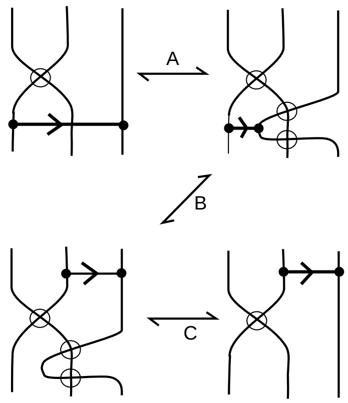

FIGURE 11. Proving a local slide move

derivation of all relations from the action of S_n on μ_{12} is explained in Remark 2 and, more precisely, by the Eqs. 8, 24, 10, 14. □

Lemma 2. *The following commuting relations among connecting strings hold in VB_n.*

(1) $\boldsymbol{\mu_{12}\mu_{34} = \mu_{34}\mu_{12}}$
(2) $\mu_{14}\mu_{23} = \mu_{23}\mu_{14}$ (*action by* (324))
(3) $\mu_{13}\mu_{24} = \mu_{24}\mu_{13}$ (*action by* (23))

The above local relations generalize to commuting relations of the form:

$$\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i,j\} \cap \{k,l\} = \emptyset. \tag{16}$$

All the above commuting relations result from relation 1 by actions of permutations (indicated for relations 2, 3 to the right of each relation). Moreover, for any choice of four strands there are exactly 24 *such commuting relations that preserve the four strands.*

Proof. Relation 1 clearly rest on the virtual braid commuting relations (B2) and (M2). We shall show how relation 2 reduces to relation 1. In the proof we underline in each step the generators of VB_n on which virtual braid relations are applied.

$$\begin{aligned}
\mu_{i,i+3}\mu_{i+1,i+2} &= v_{i+2}v_{i+1}\mu_{i,i+1}\underline{v_{i+1}v_{i+2}\mu_{i+1,i+2}} \\
&\overset{detour}{=} v_{i+2}v_{i+1}\underline{\mu_{i,i+1}\mu_{i+2,i+3}}v_{i+1}v_{i+2} \\
&\overset{(1)}{=} \underline{v_{i+2}v_{i+1}\mu_{i+2,i+3}}\mu_{i,i+1}v_{i+1}v_{i+2} \\
&\overset{detour}{=} \mu_{i+1,i+2}v_{i+2}v_{i+1}\mu_{i,i+1}v_{i+1}v_{i+2} \\
&= \mu_{i+1,i+2}\mu_{i,i+3}.
\end{aligned}$$

Figure 12 illustrates how relation 3 also reduces to relation 1. Notice now that relations 2 and 3 can be derived from relation 1 by conjugation by the permutations (324) and (23) respectively. Let us see how this works specifically for relation 2: the indices of relation 1 against the indices of relation 2 induce the permutation $(324) = v_2v_3$. This means that conjugating relation 1 by the word v_2v_3 will yield relation 2.

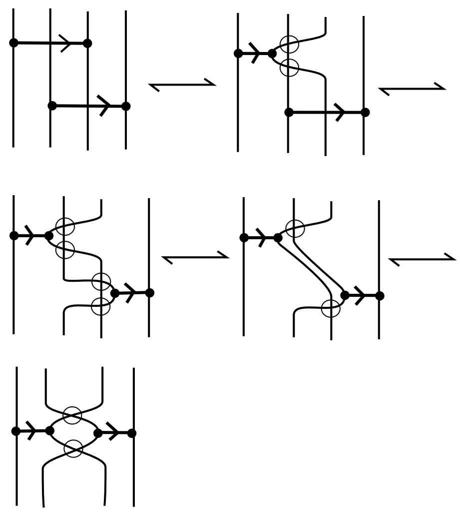

FIGURE 12. A local commuting relation

Notice also that there are 24 commuting relations in total involving the strands $\{1, 2, 3, 4\}$ and indices in any order. Likewise for any choice of four strands. The derivation of all relations from the action of S_n on relation 1 is clear from Remark 2. □

Lemma 3. *The following* stringy braid relations *hold in VB_n.*

(1) $\boldsymbol{\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}}$
(2) $\mu_{21}\mu_{23}\mu_{13} = \mu_{13}\mu_{23}\mu_{21}$ (*action by* (12))
(3) $\mu_{13}\mu_{12}\mu_{32} = \mu_{32}\mu_{12}\mu_{13}$ (*action by* (23))
(4) $\mu_{32}\mu_{31}\mu_{21} = \mu_{21}\mu_{31}\mu_{32}$ (*action by* (13))
(5) $\mu_{23}\mu_{21}\mu_{31} = \mu_{31}\mu_{21}\mu_{23}$ (*action by* (123))
(6) $\mu_{31}\mu_{32}\mu_{12} = \mu_{12}\mu_{32}\mu_{31}$ (*action by* (132))

The above relations generalize to three-term relations of the form:

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \textit{for any distinct } i, j, k. \tag{17}$$

All six relations stated above result from the action on relation 1 by permutations of S_n, which only permute the indices $\{1, 2, 3\}$. These permutations are indicated to the right of each relation. Moreover, for any choice of three strands there are exactly six relations analogous to the above, which all result from relation 1 by actions of appropriate permutations that preserve the three strands each time.

Proof. Figure 13 illustrates relation 1. Relation 1 rests on the braid relations (B1) of VB_n. Indeed, let us prove one relation of this type. See also Figure 14 for a pictorial proof.

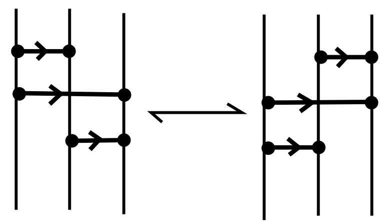

FIGURE 13. The stringy braid relation

$$
\begin{aligned}
\mu_{i+1,i+2}\mu_{i,i+2}\mu_{i,i+1} &= (\sigma_{i+1}\underline{v_{i+1})(v_{i+1}}\sigma_i\underline{v_i v_{i+1})(\sigma_i}v_i) \\
&\overset{(S3,M1)}{=} \underline{\sigma_{i+1}\sigma_i\sigma_{i+1}v_i v_{i+1}v_i} \\
&\overset{(B1,S1)}{=} \sigma_i\sigma_{i+1}\underline{\sigma_i v_{i+1}v_i}v_{i+1} \\
&\overset{(M1,S3)}{=} \sigma_i\sigma_{i+1}\underline{v_{i+1}v_i v_{i+1}}v_{i+1}\sigma_{i+1}v_{i+1} \\
&\overset{(S1)}{=} \sigma_i\underline{\sigma_{i+1}v_i v_{i+1}}v_i v_{i+1}\sigma_{i+1}v_{i+1} \\
&\overset{(M1)}{=} (\sigma_i v_i)(v_{i+1}\sigma_i v_i v_{i+1})(\sigma_{i+1}v_{i+1}) \\
&= \mu_{i,i+1}\mu_{i,i+2}\mu_{i+1,i+2}.
\end{aligned}
$$

The other five stated relations follow from relation 1. Indeed, substituting the μ_{ji}'s from Eqs. 9 and 13, and drawing the two sides of a relation we notice that there is always a region where, by the slide relations, all three connecting strings become consecutive without any of them having to be reversed, thus enabling application of the first relation. This diagrammatic argument confirms the fact that all six relations are derived from the first one by the action of appropriate elements of S_n. Let us see how this works specifically for relation 5: the indices of relation 1 against the indices of relation 5 induce the permutation $(123) = v_2v_1$. This means that conjugating relation 1 by the word v_2v_1 will yield relation 5. Finally, the derivation of all stringy braid relations from the action of S_n on relation 1 is clear from Remark 2. □

Another remark is now due.

Remark 3. The forbidden moves are naturally forbidden also in the stringy category. For example, the forbidden relations $F1, F2$ of Eq. 6 translate into the following corresponding *forbidden stringy relations* $SF1, SF2$:

(18) $\mu_{i,i+2}\mu_{i+1,i+2} = \mu_{i+1,i+2}\mu_{i,i+2} \quad (SF1)$ and $\mu_{i,i+2}\mu_{i,i+1} = \mu_{i,i+1}\mu_{i,i+2} \quad (SF2)$

which, together with all similar relations arising from conjugating the above by permutations, are not valid in the stringy category. See Figure 15 for illustrations.

2.4. The stringy presentation. We will now define an abstract stringy presentation for VB_n that starts from the concept of connecting string and recaptures the virtual braid group. By Eq. 7 we have

(19) $$\sigma_i = \mu_{i,i+1}v_i$$

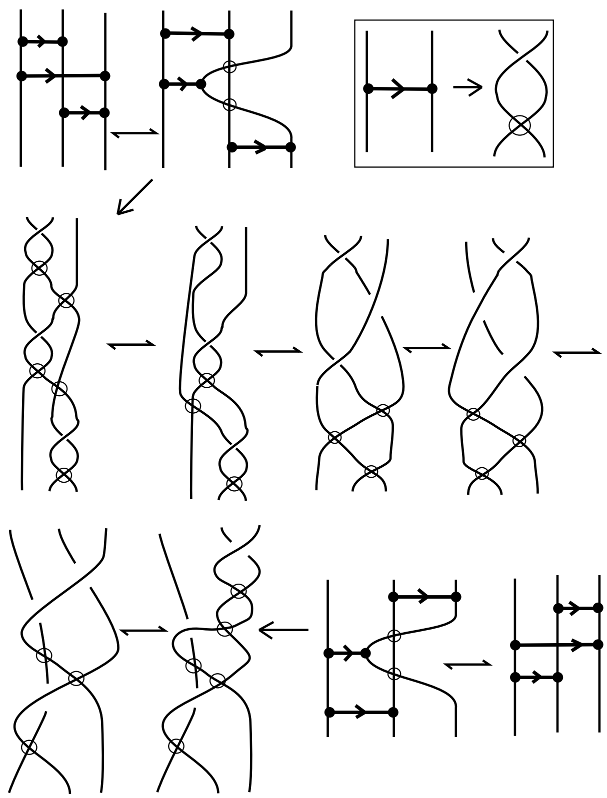

FIGURE 14. Proof of the stringy braid relation

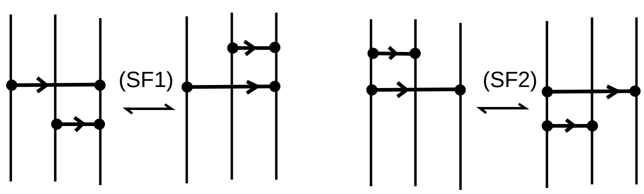

FIGURE 15. Stringy forbidden moves

so, the connecting strings μ_{ij} can be taken as an alternate set of generators of the virtual braid group, along with the virtual generators v_i. The relations in this new presentation consist in the results we proved above in Lemmas 1, 2, 3 describing the interaction of connecting strings with virtual crossings, the commutation properties of connecting strings, the stringy braiding relations, and the usual relations $(S1), (S2), (S3)$ in the symmetric group S_n. For the work below, recall that we have defined the element $t_{ij} = v_i v_{i+1} \dots v_j \dots v_{i+1} v_i$ that corresponds to the transposition (ij) in S_n.

In any presentation of a group G containing the elements $\{v_1, \dots, v_{n-1}\}$ and the relations $(S1), (S2), (S3)$ among them, we have an action of the symmetric group S_n on the group G defined by conjugation by an element τ in S_n, expressed in terms of the v_i:

$$g^\tau = \tau g \tau^{-1}$$

for g in G. In particular, we can consider $t_{ij}\, g\, t_{ij}$ as the action by the transposition t_{ij} on an element g of G. We will use this action to define a stringy model of the virtual braid group.

Definition 3. Let VS_n denote the following stringy group presentation.

$$VS_n = \left\langle \begin{array}{l} \mu_{ij}, \quad 1 \leq i \neq j \leq n, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} \tau \mu_{ij} \tau^{-1} = \mu_{\tau(i),\tau(j)}, \quad \tau \in S_n \\ \mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12} \\ \mu_{12}\mu_{34} = \mu_{34}\mu_{12} \\ (S1), (S2), (S3) \end{array} \right\rangle \tag{20}$$

We can now state the following theorem.

Theorem 1. *The stringy group VS_n is isomorphic to the virtual braid group VB_n.*

Proof. First we define a homomorphism $F : VB_n \longrightarrow VS_n$ by $F(v_i) = v_i$ and $F(\sigma_i) = \mu_{i,i+1} v_i$, and extend the map to be a homomorphism on words in the generators of the virtual braid group. In order to show that this map is well-defined, we must show that it preserves the relations in the virtual braid group. Since $F(v_i) = v_i$, the relations among the v_i with themselves are preserved identically. The commuting relations in the braid group are $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 2$. Thus we must show that

$$\mu_{i,i+1} v_i \mu_{j,j+1} v_j = \mu_{j,j+1} v_j \mu_{i,i+1} v_i.$$

But this follows immediately from relations 4 of Lemma 1 and from Lemma 2. The mixed commuting relations $(M2)$ follow also directly from relations $(S2)$ and relations 4 of Lemma 1. This completes the verification that the commuting relations in the virtual braid group are compatible with F.

The detour moves $(M2)$ in the virtual braid group go under F to the slide relations of Lemma 1. We illustrate this in Figure 16.

It remains to prove that the braiding relations $(B1)$ carry over to VS_n under F. Indeed:

$$\begin{aligned} F(\sigma_i \sigma_{i+1} \sigma_i) &= \mu_{i,i+1} v_i \mu_{i+1,i+2} v_{i+1} \mu_{i,i+1} v_i \\ &\overset{Lemma\ 1}{=} \mu_{i,i+1} \mu_{i,i+2} \mu_{i+1,i+2} v_i v_{i+1} v_i, \end{aligned}$$

while

$$\begin{aligned} F(\sigma_{i+1} \sigma_i \sigma_{i+1}) &= \mu_{i+1,i+2} v_{i+1} \mu_{i,i+1} v_i \mu_{i+1,i+2} v_{i+1} \\ &\overset{Lemma\ 1}{=} \mu_{i+1,i+2} \mu_{i,i+2} \mu_{i,i+1} v_{i+1} v_i v_{i+1}. \end{aligned}$$

and the two expressions are equal from Lemma 3 and relations $(S1)$. This completes the proof that the mapping F is a well-defined homomorphism of groups.

We now define an inverse mapping $G : VS_n \longrightarrow VB_n$ by $G(v_i) = v_i$ and $G(\mu_{i,i+1}) = \sigma_i v_i$. At this stage we have two pieces of work to accomplish: We must extend G to all of VB_n and we must show that G is well-defined and that it preserves the relations in the group presentation. This will be done in the next paragraphs.

First of all, note that we have the VS_n relations:

$$\tau^{-1}\, \mu_{ij}\, \tau = \mu_{\tau(i),\tau(j)}$$

for all τ in S_n. In particular, this means that if $\tau(1) = i$ and $\tau(2) = j$, then

$$\mu_{ij} = \tau^{-1}\, \mu_{12}\, \tau.$$

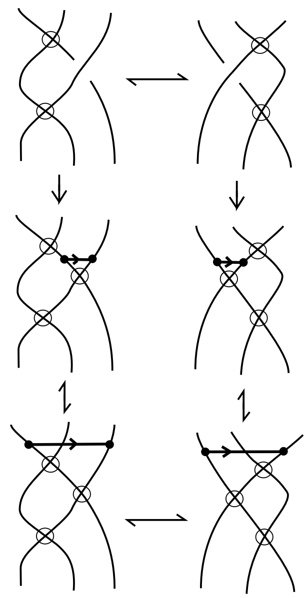

FIGURE 16. The detour moves correspond to the slide moves in the stringy category

Thus we can define

$$G(\mu_{ij}) = \tau^{-1}G(\mu_{12})\tau = \tau^{-1}\sigma_1 v_1 \tau.$$

It is easy to see that this is well-defined by noting that if λ is another permutation such that $\lambda(1) = i$ and $\lambda(2) = j$, then $\lambda = \tau\gamma$ where γ is a permutation that fixes 1 and 2. But such a permutation commutes with $\sigma_1 v_1$ as is easy to see in the virtual braid group. Hence λ can replace τ in the formula for $G(\mu_{ij})$ with no change. We leave it as an exercise for the reader to check that our definition of $G(\mu_{i,i+1})$ in the previous paragraph agrees with the present definition. This completes the definition of the map G. We now need to see that it respects the other relations in VB_n.

We must show that

$$G(\mu_{12}\mu_{34}) = G(\mu_{34}\mu_{12}).$$

Just note that

$$G(\mu_{12}\mu_{34}) = \sigma_1 v_1 \sigma_3 v_3 = \sigma_3 v_3 \sigma_1 v_1 = G(\mu_{34}\mu_{12}),$$

by the commuting relations in the virtual braid group.

Finally, we must prove

$$G(\mu_{12}\mu_{13}\mu_{23}) = G(\mu_{23}\mu_{13}\mu_{12}).$$

Note that $\mu_{13} = v_2\mu_{12}v_2$, so we must prove that in the virtual braid group,

$$\sigma_1 v_1 v_2 \sigma_1 v_1 v_2 \sigma_2 v_2 = \sigma_2 v_2 v_2 \sigma_1 v_1 v_2 \sigma_2 v_2.$$

Figure 14 illustrates how this identity follows via braiding and detour moves.

We have verified that the mapping G is well-defined and, by definition, the compositions $F \circ G$ and $G \circ F$ are the identity on VS_n and VB_n. Therefore VS_n and VB_n are isomorphic groups. This completes the proof of the Theorem. □

Finally, we also give below a reduced presentation for VB_n, which derives immediately from (5).

Proposition 1. *The following is a reduced stringy presentation for VB_n:*

$$(21)\quad VB_n = \left\langle \begin{array}{c} \mu_{12}, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} \mu_{12}v_j = v_j\mu_{12}, \quad \textit{for } j > 2 \\ \mu_{12}\, v_2\mu_{12}v_2\, v_1v_2\mu_{12}v_2v_1 = v_1v_2\mu_{12}v_2v_1\, v_2\mu_{12}v_2\, \mu_{12} \\ \mu_{12}\, v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2 = v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2\, \mu_{12} \\ (S1), (S2), (S3) \end{array} \right\rangle$$

Note that the second relation is the stringy braid relation 1 of Lemma 3 and the third relation is the commuting relation 1 of Lemma 2.

3. The Pure Virtual Braid Group

3.1. A presentation for the pure virtual braid group. From presentation Eq. 1 of VB_n we have a surjective homomorphism

$$\pi : VB_n \longrightarrow S_n$$

defined by

$$\pi(\sigma_i) = \pi(v_i) = v_i.$$

For a virtual braid b, we refer to $\pi(b)$ as the *permutation associated with the virtual braid* b, and we define the *pure virtual braid group* VP_n to be the kernel of the homomorphism π. Hence, VP_n is a normal subgroup of VB_n of index $n!$. So, $VP_n \cdot S_n = VB_n$. Moreover, $VP_n \cap S_n = \{id\}$. Hence, $VB_n = VP_n \rtimes S_n$. Equivalently, we have the exact sequence

$$1 \longrightarrow VP_n \longrightarrow VB_n \longrightarrow S_n \longrightarrow 1.$$

A presentation for VP_n can be now derived immediately from the stringy presentation of VB_n as an application of the Reidemeister-Schreier process [7, 25, 30]. To see this, we first need the following.

Lemma 4. *The subgroup VP_n of VB_n is generated by the elements μ_{ij} for all $i \neq j$.*

Proof. Indeed, by Eqs. 7 and 9, $\sigma_i = \mu_{i,i+1}v_i = v_i\mu_{i+1,i}$. So, any element $b \in VB_n$ can be written as a product in the μ_{ij}'s and the v_k's. Furthermore, by the slide relations of Lemma 1, all μ_{ij}'s can pass to the top of the braid, leaving at the bottom a word τ in the v_k's, such that $\tau = \pi(b)$. Thus, if $b \in VP_n$ then τ must be the identity permutation. This completes the proof of the Lemma. □

We can now give a stringy presentation of VP_n.

Theorem 2. *The following is a presentation for the pure virtual braid group.*

$$(22)\qquad VP_n = \left\langle\; \mu_{rs}, \quad r \neq s \;\middle|\; \begin{array}{l} \mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \textit{for all distinct } i, j, k \\ \mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i, j\} \cap \{k, l\} = \emptyset \end{array} \right\rangle$$

Proof. Having reformulated the presentation of the virtual braid group, the proof is now a direct application of the Reidemeister-Schreier technique [7, 25, 30]. The relations in VP_n arise as conjugations of the relations in VB_n by coset representatives of VP_n in VB_n, which are the elements of S_n. The relations $(S1), (S2), (S3)$ describe S_n and are used for choosing the coset representatives. We now describe the process from the point of view of covering spaces. We have $VP_n \subset VB_n$ as a normal subgroup with the subgroup S_n acting on it by conjugation. VP_n is the fundamental group of the covering space E of a cell complex B with fundamental group VB_n, where E has group of deck transformations S_n. Since the elements of the symmetric group lift to paths in the covering space, the relations $\tau\mu_{ij}\tau^{-1} = \mu_{\tau(i),\tau(j)}$ serve to describe the action of the symmetric group on the loops in the covering space (these loops are the lifts of the elements μ_{ij}). We choose basic relations in VP_n to be the lifts at a specific basepoint of the braiding relation $\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}$ and the commuting relation $\mu_{12}\mu_{34} = \mu_{34}\mu_{12}$. All other relations are obtained from these by the action of S_n, and all relations constitute the two orbits of the basic relations under this action. For example the relations

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}$$

constitute the orbit under the action of S_n on the single basic braiding relation

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}.$$

The same pattern applies to the commuting relations. This gives the statement of the Theorem and completes the proof. □

3.2. Semi-Direct Product Structure. The virtual braid group and the pure virtual braid group can be described in terms of semi-direct products of groups, just as is begun in the paper by Bardakov [1] and continued in [8]. In this section we remark that these decompositions are based on the following algebra: The Yang-Baxter relation has the generic form

$$\mu_{i,i+1}\mu_{i,i+2}\mu_{i+1,i+2} = \mu_{i+1,i+2}\mu_{i,i+2}\mu_{i,+1}$$

which is abstractly in the form

$$ABC = CBA$$

and can be rewritten in the form $B^{-1}ABC = B^{-1}CBA$ or

$$A^B = C^BAC^{-1}.$$

This allows one to rewrite some of the Yang-Baxter relations in terms of the conjugation action of the group on itself, and this is the key to the structural work pioneered by Bardakov.

4. A String Category for the Virtual Braid Group

In this section we summarize our results by pointing out that the string connectors and the virtual crossings can be regarded as generators of a category whose algebraic structure yields the virtual braid group and the pure virtual braid group.

For this purpose we define a strict monoidal category with generating morphisms μ_{ij} where this symbol is intepreted as an abstract string or connection between strands i and j in a diagram that otherwise is an identity braid on n strands just as defined in the previous sections. The other generators of this category are morphisms v_i that are interpreted as virtual crossings between strings i and $i+1$. The generators v_i have all the relations for transpositions generating the symmetric group. Compositions of these elements generate the morphisms of the category. The relations among these morphisms are exactly the relations described for the v_k and the μ_{ij} in the previous sections.

Consider the strict monoidal category freely generated by one object $*$ and three morphisms $\mu : * \otimes * \longrightarrow * \otimes *$, $\mu' : * \otimes * \longrightarrow * \otimes *$, and $v : * \otimes * \longrightarrow * \otimes *$. Let $\mu_{12} = \mu \otimes id_*$, $\mu_{21} = \mu' \otimes id_*$, $v_1 = v \otimes id_*, v_2 = id_* \otimes v$ and let

$$v_i = id_* \otimes \cdots \otimes id_* \otimes v \otimes id_* \otimes \cdots \otimes id_*$$

where v occurs in the i-th place in this tensor product. More generally, it is understood that μ_{12} can stand for $\mu \otimes id_* \otimes \cdots \otimes id_*$ and that μ_{21} can stand for $\mu' \otimes id_* \otimes \cdots \otimes id_*$ for an arbitrary number of tensor factors.

Quotient this category by the following relations.

(1) $\mu\mu' = id_{*\otimes *} = \mu'\mu$,
(2) $vv = id_*$,
(3) $\mu_{12}v_j = v_j\mu_{12}$, for $j > 2$,
(4) $\mu_{12}\, v_2\mu_{12}v_2\, v_1v_2\mu_{12}v_2v_1 = v_1v_2\mu_{12}v_2v_1\, v_2\mu_{12}v_2\, \mu_{12}$,
(5) $\mu_{12}\, v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2 = v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2\, \mu_{12}$,
(6) $v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1}$,
(7) $v_iv_j = v_jv_i$, for $j \neq i \pm 1$.

This quotient is called the *String Category* and denoted SC. The category SC is still strict monoidal.

To recapture the connecting string morphisms, we follow the formalism of the previous sections. Define

$$\mu_{i,i+1} = id_* \otimes \cdots \otimes id_* \otimes \mu \otimes id_* \otimes \cdots \otimes id_*$$

where μ occurs in the i and $i+1$ places in the tensor product and define

$$\mu_{i+1,i} = id_* \otimes \cdots \otimes id_* \otimes \mu' \otimes id_* \otimes \cdots \otimes id_*$$

where μ' occurs in the i and $i+1$ places in the tensor product. Define, for $i < j$, the element μ_{ij} by the formula

$$\mu_{ij} = v_{j-1}v_{j-2}\cdots v_{i+1}\, \mu_{i,i+1}\, v_{i+1}\cdots v_{j-2}v_{j-1}. \tag{23}$$

and define

$$\mu_{ji} = v_{j-1}v_{j-2}\cdots v_{i+1}\, \mu_{i+1,i}\, v_{i+1}\cdots v_{j-2}v_{j-1}. \tag{24}$$

Remark 4. Note that, in this notation, relation 4 becomes the algebraic Yang-Baxter equation

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12},$$

and relation 5 becomes the commuting relation

$$\mu_{12}\mu_{34} = \mu_{34}\mu_{12}.$$

Then one has, as conseqeunces, the general algebraic Yang-Baxter equation and commuting relations, as we have described them in earlier sections of the paper.

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \text{for all distinct } i, j, k$$

and

$$\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i, j\} \cap \{k, l\} = \emptyset.$$

The morphisms v_i effect the action of the symmetric group and the category models the pure virtual braid group in the following precise sense:

By Proposition 1, for any positive integer n, the group of endomorphisms of the object $*^{\otimes n}$ is isomorphic to VB_n. In particular any monoidal functor

$$F : SC \longrightarrow Mod_k$$

gives rise to a representation of VB_n :

$$f \in End_{SC}(*^{\otimes n}) \simeq VB_n \longmapsto F(f) \in End_k(A^{\otimes n})$$

where $A = F(*)$.

Remark. For each natural number n, the symbols

$$[n] = * \otimes * \otimes \cdots \otimes *$$

with n $*$'s are the objects in thecategory. One can regard $[n]$ as an ordered row of n points that constitute the top or the bottom of a diagram involving n strands. Diagrammatically, μ_{ij} consists in n parallel strands with a string connector between the i-th and j-th strands directed from i to j. Similarly, v_i corresponds to a diagram of n strands where there is a virtual crossing between the i-th and $(i + 1)$-st strands. An n-strand diagram that is a product of these generators is regarded as a morphism from $[n]$ to $[n]$ for n any natural number. Note that we interpret μ_{ij} and v_i diagrammatically according to the conventions previously established in this paper.

The virtual braid group on n strands is isomorphic to the group of morphisms in the String Category from $[n]$ to $[n]$. The point of this categorical formulation of the virtual braid groups is that we see how these groups form a natural extension of the symmetric groups by formal elements that satisfy the algebraic Yang-Baxter equation. The category we desribe is a natural structure for an algebraist interested in exploring formal properties of the algebraic Yang-Baxter equation.

For the reader who would like to take the String Category as a starting point for the theory of virtual braids, here is a description of how to read our figures for that purpose. Figure 1 illustrates the permutation generators v_i for the String Category. The braiding elements σ_i will be defined in terms of the string generators. Elementary connecting strings are given in Figure 6. Note that, it is implicit in Figure 6 how to define the braiding elements σ_i by composing string generators with permutations (virtual crossings). See also Figure 7, which illustrates basic relationships among string generators, permutations and braiding operators. Figure 8 illustrates the general connecting strings and their relations with the permutation operators. In particular, Figure 8 shows how any string connection can be written in terms of a basic string generator and a product of permutations. Figure 9 illustrates how μ_{ij} and μ_{ji} are related diagrammatically. Figures 10, 11 and 12 show the basic slide relations between string connections and permutations. Figure 13 illustrates the algebraic Yang-Baxter relation as it occurs for the string connectors.

5. Representations of the Virtual and Pure Virtual Braid Groups

5.1. Let A be an algebra over a ground ring k. Let $\rho \in A \otimes A$ be an element of the tensor product of A with itself. Then ρ has the form given by the following equation

$$\rho = \sum_{i=1}^{N} e_i \otimes e^i \tag{25}$$

where e_i and e^j are elements of the algebra A. We will write this sum symbolically as

$$\rho = \sum e \otimes e' \tag{26}$$

where it is understood that this is short-hand for the above specific summation.

We then define, for $i < j$, $\rho_{ij} \in A^{\otimes n}$ by the equation

$$\rho_{ij} = \sum 1_A \otimes \cdots \otimes 1_A \otimes e \otimes 1_A \otimes \cdots \otimes 1_A \otimes e' \otimes 1_A \otimes \cdots \otimes 1_A \tag{27}$$

where the e occurs in the i-th tensor factor and the e' occurs in the j-th tensor factor.

If $i > j$ we define ρ_{ij} by reversing the roles of e and e' as shown in the next equation

$$\rho_{ij} = \sum 1_A \otimes \cdots \otimes 1_A \otimes e' \otimes 1_A \otimes \cdots \otimes 1_A \otimes e \otimes 1_A \otimes \cdots \otimes 1_A \tag{28}$$

where e' occurs in the j-th tensor factor, and e occurs in the i-th tensor factor.

We say that ρ is a *solution to the algebraic Yang-Baxter equation* if it satisfies the equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12} \tag{29}$$

in $A^{\otimes n}$. It is immediately obvious that if ρ satisfies the algebraic Yang-Baxter equation, then, for any pairwise distinct i, j, k we have

$$\rho_{ij}\rho_{ik}\rho_{jk} = \rho_{jk}\rho_{ik}\rho_{ij}, \tag{30}$$

and that the equations obtained from this particular equation by permuting the indices i, j, k remain true. All such equations derive from permutations of any given instance of the algebraic Yang-Baxter equation.

The following proposition is an immediate consequence of our presentation for the pure virtual braid group.

Proposition 2. *Let VP_n denote the pure virtual braid group with generators μ_{ij} and relations as given in Theorem 2 of Section 3. Let A be an algebra with an invertible algebraic solution to the Yang-Baxter equation denoted by $\rho \in A \otimes A$ as described above. Define $rep : VP_n \longrightarrow A^{\otimes n}$ by the equation*

$$rep(\mu_{ij}) = \rho_{ij}.$$

Then rep is a representation of the the virtual braid group to the tensor algebra $A^{\otimes n}$.

Proof. Note that it follows at once from the definitions of the ρ_{ij} that $\rho_{ij}\rho_{kl} = \rho_{kl}\rho_{ij}$ whenever the sets $\{i, j\}$ and $\{k, l\}$ are disjoint. Thus, we have shown that the ρ_{ij} satisfy all the relations in the pure virtual braid group. This completes the proof of the Proposition. □

Next, we show how to obtain representations of the full virtual braid group. To this purpose, consider the algebra $Aut(A^{\otimes n})$ of linear automorphisms of $A^{\otimes n}$ as a module over A. Assume that we are given an invertible solution to the algebraic Yang-Baxter equation, $\rho \in A \otimes A$, and define $\tilde{\rho}_{ij} : A^{\otimes n} \longrightarrow A^{\otimes n}$ by the equation $\tilde{\rho}_{ij}(\alpha) = \rho_{ij}\alpha$ where $\alpha \in A^{\otimes n}$. Since ρ is invertible, $\tilde{\rho}_{ij} \in Aut(A^{\otimes n})$. Let $P_{ij} : A^{\otimes n} \longrightarrow A^{\otimes n}$ be the mapping that interchanges the i-th and j-th tensor factors. We let P_i denote $P_{i,i+1}$. Then $P_{ij} \in Aut(A^{\otimes n})$. We now define $Rep : VB_n \longrightarrow Aut(A^{\otimes n})$ by the equations

$$Rep(\mu_{ij}) = \tilde{\rho}_{ij}$$

and

$$Rep(v_i) = P_i.$$

Here we use our presentation (20) for the virtual braid group.

Proposition 3. *With the above conventions the mapping $Rep : VB_n \longrightarrow Aut(A^{\otimes n})$ is a representation of the virtual braid group to a subgroup of $Aut(A^{\otimes n})$.*

Proof. It is clear that the elements P_i obey all the relations in the symmetric group S_n. Thus it remains to show that letting $\lambda = Rep(\tau)$ where τ is an element of S_n, the relations

$$\lambda\rho_{ij}\lambda^{-1} = \tilde{\rho}_{\lambda(i),\lambda(j)}, \quad \tau \in S_n$$

are satisfied in $Aut(A^{\otimes n})$. Since ρ_{ij} is defined via the placement of the e and e' factors in the summation for ρ on the i-th and j-th strands, these relations are immediate. This completes the proof of the proposition. □

Remark 5. The method we have described for constructing a representation of the virtual braid group from an algebraic solution to the Yang-Baxter equation generalizes the well-known construction of a representation of the classical Artin braid group from a solution to the Yang-Baxter equation in braided form. In the usual method for constructing the classical representation, one composes the algebraic solution with a permutation, obtaining a solution to the braiding equation $(B1)$. This is the same as our relation

$$\sigma_i = \mu_{i,i+1}v_i$$

between the braiding element σ_i and the stringy generator $\mu_{i,i+1}$ for the pure virtual braid group. Without the concept of virtuality, the direct relationship of the algebraic Yang-Baxter equation with the braid groups would not be apparent. We see, that, from an algebraic point of view, the virtual braid group is an entirely natural construction. It is the universal algebraic structure related to viewing solutions to the algebraic Yang-Baxter equation inside tensor products of algebras and endowing these tensor products with the natural permutation action of the symmetric group.

Solutions to the algebraic version of the Yang-Baxter equation are usually thought of as deformations of the identity mapping on a two-fold tensor product $A \otimes A$. We think of a braiding operator as a deformation of a transposition, and so one goes between the algebraic and braided versions of such operators by composition with a transposition.

The Artin braid group B_n is motivated by a combination of topological considerations and the desire for a group structure that is very close to the structure of the symmetric group S_n. We have seen that the virtual braid group VB_n is motivated at first by a natural extension of the Artin braid group in the context of virtual knot theory, but now we see a different motivation for the virtual braid group. Given that one studies the algebraic Yang-Baxter equation in the context of tensor powers of an algebra A, it is thoroughly natural to study the compositions of algebraic braiding operators placed in two out of the n tensor lines (the stringy generators) and to let the permutation group of the tensor lines act on this algebra. As we have seen in (20), this is precisely the virtual braid group. Viewed in this way, the virtual braid group has nothing to do with the plane and nothing to do with virtual crossings. It is a natural group associated with the structure of algebraic braiding.

5.2. A Representation Category for the Virtual Braid Group. We now give a categorical interpretation of virtual knot theory and the virtual braid group in terms of these representation modules. For A as above, let $End(A^{\otimes n})$ denote the linear endomorphisms of $A^{\otimes n}$ as a module over A. View $End(A^{\otimes n})$ as a category with generating morphisms:

(1) $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n \in A^{\otimes n}$ acting on $A^{\otimes n}$ by left multiplication,
(2) the elements of the symmetric group S_n, generated by transpositions of adjacent tensor factors.

This category has one object. In making the representation of VB_n we have used the stringy generators μ_{ij} and mapped them to sums of morphisms of the first type above. The virtual braid group VB_n described via (20), can be viewed as a category with one object and generators μ_{ij} and v_k. Of course any associative algebra can be seen as a single object category with morphisms the elements of the algebra. But here we have a pictorial representation of the morphisms as stringy braid diagrams. These diagrams can be generalized to include the algebraic category $End(A^{\otimes n})$ by letting algebra elements decorate the lines and taking the transpositions of the form $P_{i,i+1}$ as represented by v_i via a diagram of lines i and $i+1$ virtually crossing over one another. In this view the virtual crossing is interpreted as a generator of the symmetric group. The virtual crossings have not disappeared. They have become part of the embedded symmetry of the structure of the virtual braid group. This is in sharp contrast to the role of the virtual crossings in the original form of the virtual braid group. There the virtual crossings appear as artifacts of the presentation of virtual knots in the plane where those knots acquire extra crossings that are not really part of the essential structure of the virtual knot. Nevertheless, these same crossings appear crucially in the virtual braid group, and turn into the generators of the symmetric group embedded in the virtual braid group. With the use of the full set of μ_{ij} in (20) the detour moves and other remnants of the virtual crossings as artifacts have completely disappeared into the permutation action. We will continue the categorical discussion for the virtual braid group after first discussing certain aspects of knot theory and the tangle categories.

The representations of VB_n that we have here derived can be interpreted as follows.

Theorem 3. *Let $\rho \in A \otimes A$ be a solution of the algebraic Yang-Baxter equation, where A is an algebra over a commutative ring k. Define a monoidal functor*

$$F_A : SC \longrightarrow Mod_k$$

by setting $F_A(*) = A$, $F_A(\mu) = \tilde{\rho}$, and $F_A(v_i) = P$, where the endomorphisms $\tilde{\rho}$ and P of $A \otimes_k A$ are given by

$$\tilde{\rho}(x \otimes y) = \rho(x \otimes y)$$

and

$$P(x \otimes y) = y \otimes x$$

for all $x, y \in A$.

Proof. The proof follows from the previous discussion. □

Remark 6. *In the case A is a bialgebra (so that the category Mod_A of modules over A is monoidal), it would be interesting to address the following question: When does the above functor $F_A : SC \longrightarrow Mod_k$ lift to a monoidal functor $\tilde{F}_A : SC \longrightarrow Mod_A$ (that is such that $U \circ \tilde{F}_A = F_A$ as monoidal functors, where $U : Mod_A \longrightarrow Mod_k$ is the forgetful functor)?*

5.3. Virtual Hecke Algebra. From the point of view of the theory of braids the Hecke algebra $H_n(q)$ is a quotient of the group ring $Z[q, q^{-1}][B_n]$ of the Artin braid group by the ideal generated by $\sigma_i^2 - z\sigma_i - 1$. This corresponds to the identity $\sigma_i - \sigma_i^{-1} = z1$, which is sometimes regarded diagrammatically as a skein identity for calculating knot polynomials. By the same token, we define the *virtual Hecke algebra* $VH_n(q)$ to be the quotient of the group ring $Z[q, q^{-1}][VB_n]$ by the ideal generated by $\sigma_i^2 - z\sigma_i - 1$ for $i = 1, 2, \ldots n-1$. There are difficulties in extending structure theorems for the Hecke algebra to corresponding structure theorems for the virtual Hecke algebra, but some matters of representations do generalize directly. In particular, if R is a solution to the Yang-Baxter equation with $R : W \otimes W \longrightarrow W$, where W is a module over $Z[q, q^{-1}]$, then one has a corresponding representation $Rep : WH_n(q) \longrightarrow Aut(W^{\otimes n})$. This representation is specified as follows.

$$Rep(\sigma_i) = \sum 1 \otimes \cdots \otimes 1 \otimes R \otimes 1 \otimes \cdots \otimes 1 \tag{31}$$

where R operates in the i-th and $i + 1$-st tensor factors, and

$$Rep(v_i) = \sum 1 \otimes \cdots \otimes 1 \otimes P \otimes 1 \otimes \cdots \otimes 1 \tag{32}$$

where P acts to permute the i-th and $(i + 1)$-st tensor factors. It is easy to see that this gives a representation of the virtual Hecke algebra.

One can hope that the presence of such representations would shed light on the existence of a generalization of the Ocneanu trace [12] on the Hecke algebra to a corresponding trace and link invariant using the virtual Hecke algebra. At this point there is an issue about the nature of the generalization. One can aim for a trace on the virtual Hecke algebra that is compatible with the Markov Theorem for virtual knots and links as formulated in [14, 20]. This is the trace that is most difficult to achieve. A simpler trace is possible by working in rotational virtual knot theory [16]. See the next section for a discussion of unoriented quantum invariants for rotational virtuals. We will report on the relation of this approach with the Markov Theorem in a separate paper.

Another line of investigation is suggested by translating the basic Hecke algebra relation into the language of stringy connections. We have $\sigma = \mu v$ for the abstract relation between a braiding generator, a connector and a virtual element. Thus, the Hecke relation $\sigma^2 = z\sigma + 1$ becomes

$$\mu v \mu = z\mu + v,$$

and it is possible to work in the presentation (20) of the virtual braid group to find a structure theory for the virtual Hecke algebra.

6. Rotational Virtual Links, Quantum Algebras, Hopf algebras and the Tangle Category

This section will show how the ideas and methods of this paper fit together with representations of quantum algebras (to be defined below) and Hopf algebras and invariants of virtual links. We begin with a quick review of the theory of virtual links (in relation to virtual braids), and we construct the virtual tangle category. This category is a natural generalization of the virtual braid group. A functor from the virtual tangle category to an algebraic category will form a generalization of the representations of virtual braid groups that we have discussed in the previous section. This functor is related to (rotational) invariants of virtual knots and links. It is not hard to see that the construction given in this section defines a category (for arbitrary Hopf algebras) that generalizes the String Category given earlier in this paper. The category that we define here contains virtual crossings, special elements that satisfy the algebraic Yang-Baxter equation and also cup and cap operators. The subcategory without the cup and cap operators and without any (symbolic) algebra elements except those involved with the algebraic Yang-Baxter operators is isomorphic to the String Category.

We begin with Figure 17. This figure illustrates the moves on virtual knot and link diagrams that serve to define the theory of virtual knots and links. Two knot or link diagrams with virtual and classical crossings are said to be virtually isotopic if one can be obtained from the other by a finite sequence of these moves. In the figure the moves are divided into type A, B and C moves. Moves of type A are the classical Reidemeister moves. Note that these are essentially the same as corresponding moves in the Artin braid group except for the boxed move involving a loop in the diagram. The move involving this loop is usually called the *first* Reidemeister move. When we forbid the first Reidemeister move, the equivalence relation is called *regular isotopy.* The moves of type B are purely virtual and (except for the move involving a virtual loop) correspond to the properties of virtual crossings in the virtual braid group. We call the equivalence relation that forbids both the virtual loop move and the classcial loop move *virtual regular isotopy.* Finally, we have moves of type C. These are the basic detour moves, and they correspond to the mixed moves in the virtual braid group.

In this section we will work with virtual knots and links up to virtual regular isotopy. In addition to the usual kinds of virtual phenomena, we will see some extra features in looking at this equivalence relation. We shall say that a virtual knot or link is *rotationally knotted* or *rotationally linked* if it is not equivalent to an unknot or an unlink under virtual

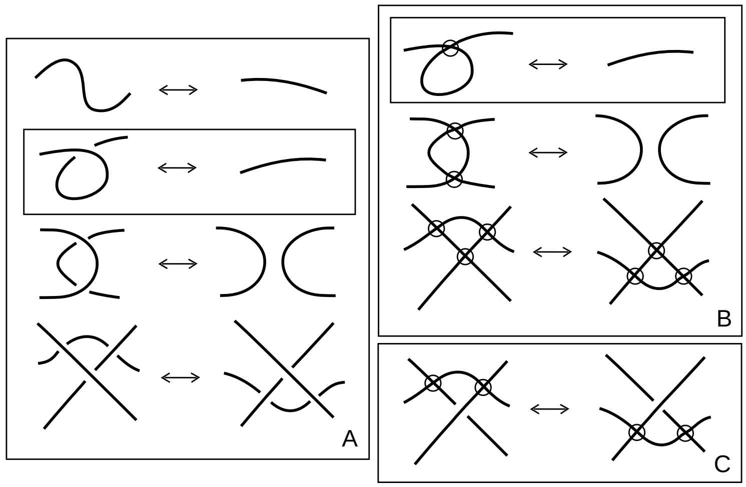

FIGURE 17. Virtual moves

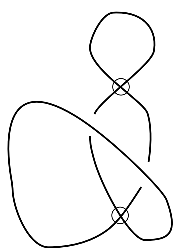

FIGURE 18. A rotational virtual knot

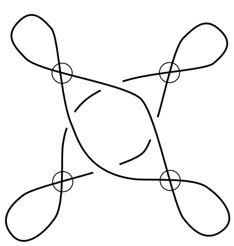

FIGURE 19. A rotational virtual link

regular isotopy. View Figure 18 and Figure 19. In the first figure we illustrate a rotational virtual knot, and in the second we show a rotational virtual link. Both the knot and the link are kept from being trivial by the presence of flat loops as discussed above. There is much more to say about rotational virtuals, and we refer the reader to [16] for some steps in this direction.

The advantage in studying virtual knots up to virtual regular isotopy is that all so-called quantum link invariants generalize to invariants of virtual regular isotopy. This means that virtual regular isotopy is a natural equivalence relation for studying topology associated with solutions to the Yang-Baxter equation. Here we create a context by defining the

Virtual Tangle Category, VTC, as indicated in Figure 20. The tangle category is generated by the morphisms shown in the box at the top of this figure. These generators are: a single identity line, right-handed and left-handed crossings, a cap and a cup, a virtual crossing. The objects in the tangle category consist in the set of $[n]$ where n is equal to 0 or to a natural number $1, 2, 3, \cdots$. For a morphism $[n] \longrightarrow [m]$, the numbers n and m denote, respectively, the number of free arcs at the bottom and at the top of the diagram that represents the morphism. The morphisms are like braids except that they can (due to the presence of the cups and caps) have different numbers of free ends at the top and the bottom of a diagram. The sense in which the elementary morphisms (line, cup, cap, crossings) generate the tangle category is shown in Figure 21. This figure shows a virtual trefoil as a morphism from [0] to [0] in the category. The tensor product of morphisms is the horizontal juxtaposition of their diagrams. Each of the seven horizontal segments of the figure represents one of the elementary morphisms tensored with the identity line. The segments are ordered so that the number of lower free ends on each segment is equal to the number of upper free ends on the segment below it. Consequently there is a well-defined composition of all of the segments and this composition is a morphism $[0] \longrightarrow [0]$ that represents the knot. The basic equivalences of morphisms are shown in Figure 20. Note that II, III, V are formally equivalent to the rules for the virtual braids. The zero-th move is a cancellation of consecutive maxima and minima, and the move IV is a swing move in both virtual and classical relations of crossings to maxima and minima. It should be clear that the tangle category is a generalization of the virtual braid group with a natural inclusion of the virtual braids as special tangles in the category. Standard braid closure and the plat closure of braids have natural definitions as tangle operations. It is not hard to see that any virtual knot or link can be represented in the tangle category as a morphism from [0] to [0], and that *two virtual links are virtually regular isotopic if and only if their tangle representatives are equivalent in the tangle category.* Note that none of the rules for equivalence in the tangle category involve either a classical loop or a virtual loop. This means that the virtual tangle category is a natural home for the theory of rotational virtual knots and links.

Now we shift to a category associated with an algebra that is directly related to our representations of the virtual braid group. We take the following definition [21, 15]: A *quantum algebra* A is an algebra over a commutative ground ring k with an invertible mapping $s : A \longrightarrow A$ that is an *antipode*, that is $s(ab) = s(b)s(a)$ for all a and b in A, and there is an element $\rho \in A \otimes A$ satisfying the algebraic Yang-Baxter equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

We further assume that ρ is invertible and that

$$\rho^{-1} = (1_A \otimes s) \circ \rho = (s \otimes 1_A) \circ \rho.$$

The multiplication in the algebra is usually denoted by $m : A \otimes A \longrightarrow A$ and is assumed to be associative. It is also assumed that the algebra has a multiplicative unit element. The defining properties of a quantum algebra are part of the properties of a Hopf algebra, but a Hopf algebra has a comultiplication $\Delta : A \otimes A \longrightarrow A$ that is a homomorphism of algebras, plus a list of further relations, including a fundamental relationship between the multiplication, the comultiplication and the antipode. In the interests of simplicity,

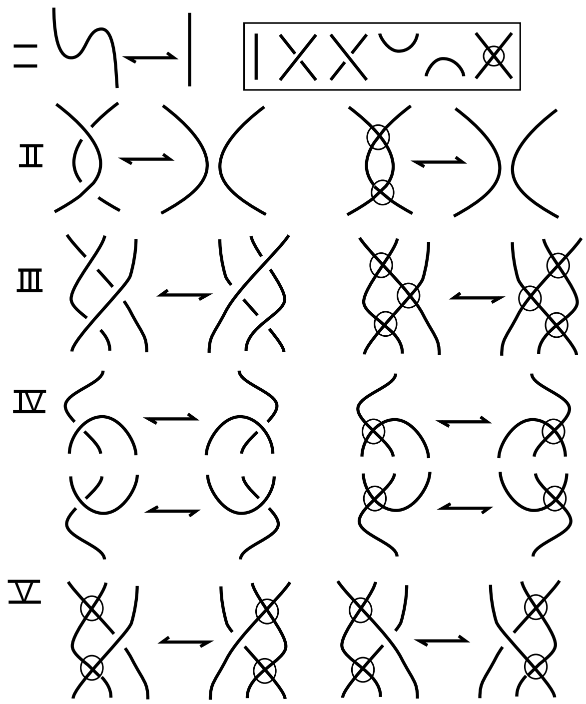

FIGURE 20. Regular isotopy with respect to the vertical direction

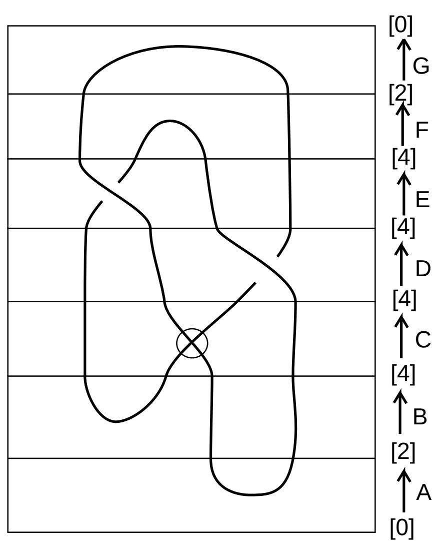

FIGURE 21. Virtual trefoil as a morphism in the tangle category

we shall restrict ourselves to quantum algebras here, but most of the remarks that follow apply to Hopf algebras, and particularly quasi-triangular Hopf algebras. Information on Hopf algebras is included at the end of this section. See [15] for more about these connections.

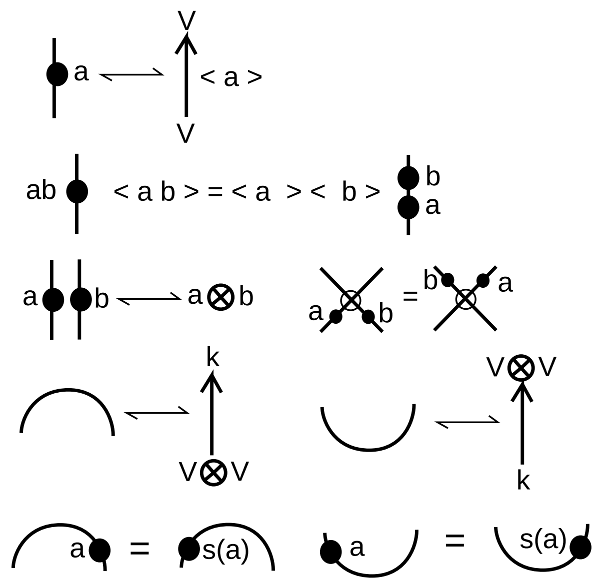

FIGURE 22. Morphisms in $Cat(A)$

We construct a category $Cat(A)$ associated with a quantum algebra A. This category is a very close relative to the virtual tangle category. $Cat(A)$ differs from the tangle category in that it has only virtual crossings, and there are labeled vertical lines that carry elements of the algebra A. See Figure 22. Given $a, b \in A$ we compose the morphisms corresponding to a and b by taking a line labeled ab to be their composition. In other words, if $\langle x \rangle$ denotes the morphism in $Cat(A)$ associated with $x \in A$, then

$$\langle a \rangle \circ \langle b \rangle = \langle ab \rangle.$$

As for the additive structure in the algebra, we extend the category to an additive category by formally adding the generating morphisms. In the Figure 22 we illustrate the composition of such morphisms and we illustrate a number of other defining features of the category $Cat(A)$.

We illustrate how the tensor product of elements $a \otimes b$ is represented by parallel vertical lines with a labeling the left line and b labeling the right line. We indicate that the virtual crossing acts as a permutation in relation to the tensor product of algebra morphisms. That is, we illustrate that

$$\langle a \rangle \otimes \langle b \rangle \circ P = P \circ \langle b \rangle \otimes \langle a \rangle.$$

Here P denotes the virtual crossing of two segments. A crossing of two undecorated segments is regarded as a morphism $P : V \otimes V \longrightarrow V \otimes V$ (see remark below). Since the lines interchange, we expect P to behave as the permutation of the two tensor factors.

We show the notation V for the object $[1]$ in this category and we use $V \otimes V = [2]$, $V \otimes V \otimes V = [3]$ and so on for all the natural number objects in the category. We write $[0] = k$, identifying the ground ring with the “empty object” $[0]$. It is then axiomatic that $k \otimes V = V \otimes k = V$. Morphisms are indicated both diagrammatically and in terms of arrows and objects in this figure. Finally, the figure indicates the arrow and object forms of the cup and the cap, and crucial axioms relating the antipode with the cup and the cap.

A cap (see Figure 22) is regarded as a morphism from $V \otimes V$ to k, while a cup is regarded as a morphism form k to $V \otimes V$. As in the case of the crossing the relevance of these morphisms to the category is entirely encoded in their properties. The basic property of the cup and the cap is that *if one "slides" a decoration across the maximum or minimum in a counterclockwise turn, then the antipode s of the algebra is applied to the decoration.* In categorical terms this property says

$$Cap \circ (\langle 1 \rangle \otimes a) = Cap \circ (\langle sa \rangle \otimes 1)$$

and

$$(\langle a \rangle \otimes 1) \circ Cup = (1 \otimes \langle sa \rangle) \circ Cup.$$

Here 1 denotes the identity morphism for [0]. These properties and other naturality properties of the cups and the caps are illustrated in Figure 22 and Figure 23. These naturality properties of the flat diagrams include regular homotopy of immersions, as illustrated in these figures.

In Figure 23 we see how this property of the cups and the caps leads to a diagrammatic interpretation of the antipode. In the figure we see that the antipode $s(a)$ is represented by composing with a cap and a cup on either side of the morphism for a. In term of the composition of morphisms diagram becomes

$$\langle sa \rangle = (Cap \otimes 1) \circ (1 \otimes \langle a \rangle \otimes 1) \circ (1 \otimes Cup).$$

Similarly,

$$\langle s^{-1}a \rangle = (1 \otimes Cap) \circ (1 \otimes \langle a \rangle \otimes 1) \circ (Cup \otimes 1).$$

This, in turn, leads to the interpretation of the flat curl as an element G in A such that $s^2(a) = GaG^{-1}$ for all a in A. G is a flat curl diagram interpreted as a morphism in the category. We see that formally it is natural to interpret G as an element of A. In a so-called *ribbon Hopf algebra* there is such an element already in the algebra. In the general case it is natural to extend the algebra to contain such an element.

We are now in a position to describe a functor F from the virtual tangle category VTC to $Cat(A)$. (The virtual tangle category is defined for virtual link diagrams without decorations. It has the same objects as $Cat(A)$.) F simply decorates each positive crossing of the tangle (with respect to the vertical - see Figure 25) with the Yang-Baxter element (given by the quantum algebra A) $\rho = \Sigma e \otimes e^{'}$ and each negative crossing (with respect to the vertical) with $\rho^{-1} = \Sigma s(e) \otimes e^{'}$. The form of the decoration is indicated in Figure 25. Since we have labelled the negative crossing with the inverse Yang-Baxter element, it follows at once that the two crossings are mapped to inverse elements in the category of the algebra. Furthermore, this association is a direct generalization of our mapping of the virtual braid group to the stringy connection presentation.

One further comment is in order about the antipode. In Figure 26 we show that our axiomatic assumption about the antipode (the sliding rule around maxima and minima) actually demands that the inverse of ρ is $(s \otimes 1_A) \circ \rho = (1_A \otimes s) \circ \rho$. This follows by examining the form of the inverse of the positive crossing in the tangle category by turning that crossing to produce an identity between the positive crossing and the negative

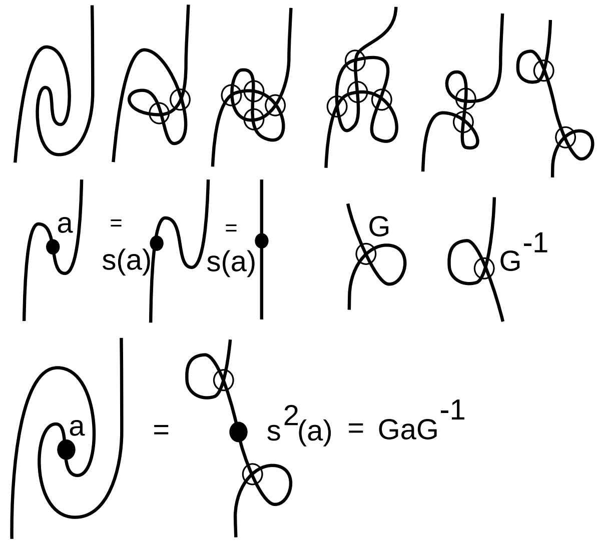

FIGURE 23. Diagrammatics of the antipode

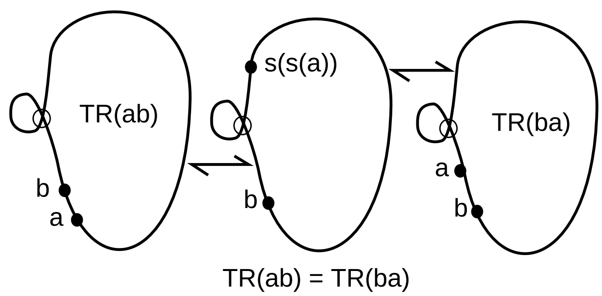

FIGURE 24. Formal trace

crossing twisted with additional maxima and minima. This relationship shows that the structure of quantum algebra is nearly forced if we wish to construct a functor of this kind.

The key point about this category is that, because quantum algebra elements can be moved around the diagram, we can concentrate all the algebra in one place. Because the flat curls are identified with either G or G^{-1}, we can use regular homotopy of immersions to bring each component of a virtual link diagram to the form of a circle with a single concentrated decoration (involving a sum over many products) and a reduced pattern of flat curls that can be encoded as a power of the special element G. Once the underlying curve of a link component is converted to a loop with total turn zero as in Figure 24 then we can think of such a loop with algebra labeling the loop as a representative for a formal trace of that algebra and call it $TR(X)$ as in the figure. In the figure we illustrate that for such a labeling $TR(ab) = TR(ba)$, thus one can take a product of algebra elements on a zero-rotation loop up to cyclic order of the product. In situations where we choose

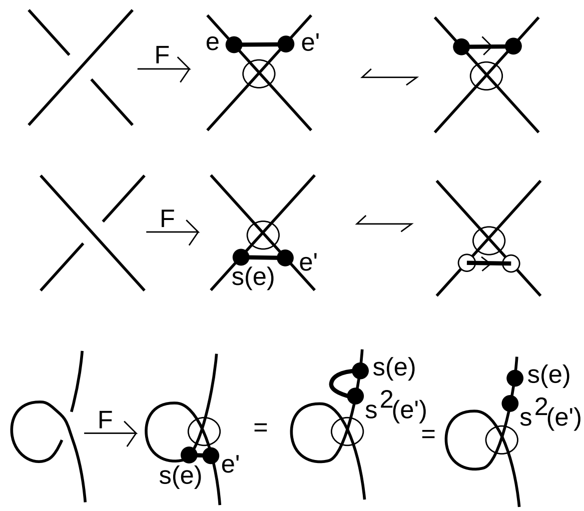

FIGURE 25. The functor $F : T \longrightarrow Cat(A)$

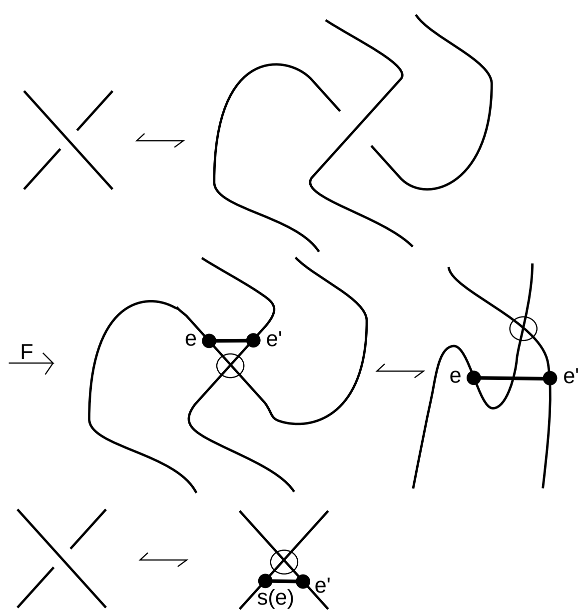

FIGURE 26. Inverse and antipode

a representation of the algebra or in the case of finite dimensional Hopf algebras where one can use right integrals [15], there are ways to make actual evaluations of such traces. Here we use them formally to indicate the result of concentrating the algebra on the loop.

Note that the category $C(A)$ of a quantum algebra A can be generalized to an abstract category with labels, virtual crossings, and with stringy connections that satisfy the algebraic Yang-Baxter equation. Each such stringy connection has a left label e or $s(e)$ and a right label e'. We retain the formalism of the antipode as a formal replacement

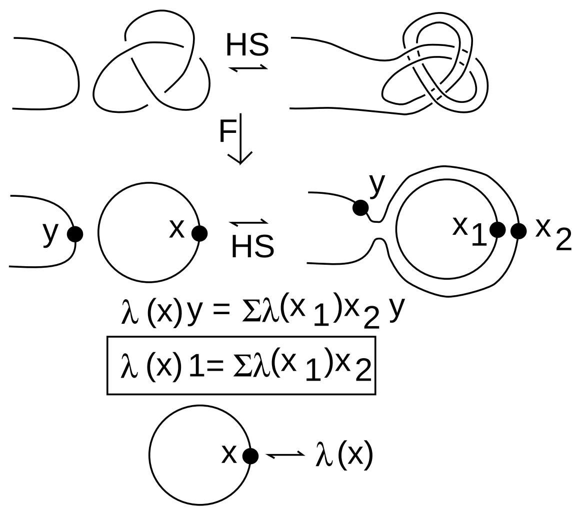

FIGURE 27. The Kirby move

for adjoining a label with a cup and a cap. The resulting *abstract algebra category* will be denoted by $\overline{C(A)}$. Since we take this category with no further relations, the functor $F : VTang \longrightarrow \overline{C(A)}$ is an equivalence of categories. This functor is the direct analog of our reformulation of the virtual braid group in terms of stringy connectors.

In Figure 27 we illustrate how one can use this concentration of algebra on the loop in the context of a Hopf algebra that has a right integral. The right integral is a function $\lambda : A \longrightarrow k$ satisfying

$$\lambda(x)1_A = \Sigma\lambda(x_1)x_2$$

where the coproduct in the Hopf algebra has formula $\Delta(x) = \Sigma x_1 \otimes x_2$. Here we point out how the use of the coproduct corresponds to doubling the lines in the diagram, and that if one were to associate the function λ with a circle with rotation number one, then the resulting link evaluation will be invariant under the so-called Kirby move. The Kirby move replaces two link components with new ones by doubling one component and connecting one of the components of the double with the other component. It turns out that classical framed links L have an associated compact oriented three manifold $M(L)$ and that two links related by Kirby moves have homeomorphic three-manifolds. Thus the evaluation of links using the right integral yields invariants of three-manifolds. We only sketch this point of view here, and refer the reader to [15].

In Figure 28 we illustrate the entire functorial process for the virtual trefoil of Figure 21. The virtual trefoil is denoted by K, and we find that $F(K)$ reduces to a zero-rotation circle with the inscription $e's(f)s^2(e)s^3(f')G^2$. We can, therefore, write the equation

$$F(K) = TR[e's(f)s^2(e)s^3(f')G^2].$$

Another way to think about this trace expression is to regard it as a Gauss code for the knot that has extra structure. The chords in the Gauss diagram are the stringy connections of the beginning of this paper, generalized to the algebra category $Cat(A)$. The powers of the antipode and the power of G keep track of rotational features in

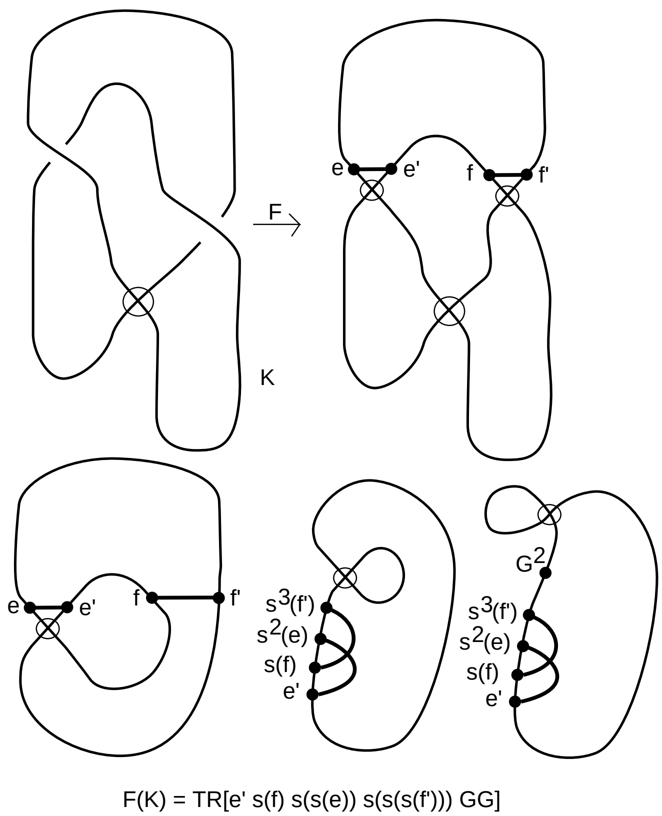

FIGURE 28. The functor $F : T \longrightarrow Cat(A)$ applied to a virtual trefoil

the diagram as it lives in the tangle category up to regular isotopy. We now see that the mapping of the virtual braid group to the braid group generated by permutations and stringy connections has been generalized to the functor F taking the virtual tangle category to the abstract category of a quantum algebra. We regard this generalization as an appropriate context for thinking about virtual knots, links and braids.

6.1. Hopf Algebra. This section is added for reference about Hopf algebras. Quasitriangular Hopf algebras are an important special case of the quantum algebras discussed in this section.

Recall that a Hopf algebra A [31] is a bialgebra over a commutative ring k that has an associative multiplication and a coassociative comultiplication and is equipped with a counit, a unit and an antipode. The ring k is usually taken to be a field. A is an algebra with multiplication $m : A \otimes A \longrightarrow A$. The associative law for m is expressed by the equation $m(m \otimes 1_A) = m(1_A \otimes m)$ where 1_A denotes the identity map on A.

The coproduct $\Delta : A \longrightarrow A \otimes A$ is an algebra homomorphism and is coassociative in the sense that $(\Delta \otimes 1_A)\Delta = (1_A \otimes \Delta)\Delta$.

The unit is a mapping from k to A taking 1 in k to 1 in A, and thereby defining an action of k on A. It will be convenient to just identify the units in k and in A, and to ignore the name of the map that gives the unit.

The counit is an algebra mapping from A to k denoted by $\epsilon : A \longrightarrow k$. The following formulas for the counit dualize the structure inherent in the unit: $(\epsilon \otimes 1_A)\Delta = 1_A = (1_A \otimes \epsilon)\Delta$.

It is convenient to write formally

$$\Delta(x) = \sum x_1 \otimes x_2 \in A \otimes A$$

to indicate the decomposition of the coproduct of x into a sum of first and second factors in the two-fold tensor product of A with itself. We shall often drop the summation sign and write

$$\Delta(x) = x_1 \otimes x_2.$$

The antipode is a mapping $s : A \longrightarrow A$ satisfying the equations $m(1_A \otimes s)\Delta(x) = \epsilon(x)1$, and $m(s \otimes 1_A)\Delta(x) = \epsilon(x)1$ where 1 on the right hand side of these equations denotes the unit of k as identified with the unit of A. It is a consequence of this definition that $s(xy) = s(y)s(x)$ for all x and y in A.

A quasitriangular Hopf algebra A [4] is a Hopf algebra with an element $\rho \in A \otimes A$ satisfying the following equations:

1) $\rho\Delta = \Delta'\rho$ where Δ' is the composition of Δ with the map on $A \otimes A$ that switches the two factors.

2)

$$\rho_{13}\rho_{12} = (1_A \otimes \Delta)\rho,$$
$$\rho_{13}\rho_{23} = (\Delta \otimes 1_A)\rho.$$

Remark 7. The symbol ρ_{ij} denotes the placement of the first and second tensor factors of ρ in the i and j places in a triple tensor product. For example, if $\rho = \sum e \otimes e'$ then

$$\rho_{13} = \sum e \otimes 1_A \otimes e'.$$

These conditions imply that ρ has an inverse, and that

$$\rho^{-1} = (1_A \otimes s^{-1})\rho = (s \otimes 1_A)\rho.$$

It follows easily from the axioms of the quasitriangular Hopf algebra that ρ satisfies the Yang-Baxter equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

A less obvious fact about quasitriangular Hopf algebras is that there exists an element u such that u is invertible and $s^2(x) = uxu^{-1}$ for all x in A. In fact, we may take $u = \sum s(e')e$ where $\rho = \sum e \otimes e'$. As we shall see, this result, originally due to Drinfeld [4], follows from the diagrammatic categorical context of this paper.

An element G in a Hopf algebra is said to be *grouplike* if $\Delta(G) = G \otimes G$ and $\epsilon(G) = 1$ (from which it follows that G is invertible and $s(G) = G^{-1}$). A quasitriangular Hopf algebra is said to be a *ribbon Hopf algebra* [27], [15] if there exists a grouplike element G such that (with u as in the previous paragraph) $v = G^{-1}u$ is in the center of A and $s(u) = G^{-1}uG^{-1}$. We call G a special grouplike element of A.

Since $v = G^{-1}u$ is central, $vx = xv$ for all x in A. Therefore $G^{-1}ux = xG^{-1}u$. We know that $s^2(x) = uxu^{-1}$. Thus $s^2(x) = GxG^{-1}$ for all x in A. Similarly, $s(v) = s(G^{-1}u) = s(u)s(G^{-1}) = G^{-1}uG^{-1}G = G^{-1}u = v$. Thus the square of the antipode is represented as conjugation by the special grouplike element in a ribbon Hopf algebra, and the central element $v = G^{-1}u$ is invariant under the antipode.

References

[1] V. G. Bardakov, The virtual and universal braids, *Fund. Math.*, **184** (2004), pp. 1-18.

[2] V. G. Bardakov and P. Bellingeri, Combinatorial properties of virtual braids, *Topology and Its Applications*, **156** (2009), pp. 1071 - 1082.

[3] J.S.Carter, S. Kamada and M. Saito, Stable equivalence of knots on surfaces and virtual knot cobordisms, in "Knots 2000 Korea, Vol. 1 (Yongpyong)", *J. Knot Theory & Ramifications* **11** (2002), No. 3, 311–320.

[4] V.G. Drinfeld, Quantum groups. *Proceedings of the International Congress of Mathematicians, Berkeley, California, USA*(1987), 798- 820.

[5] H. Dye and L. H. Kauffman, Minimal surface representations of virtual knots and links, arXiv:math.AT/0401035 v1 Jan. 2004.

[6] R. Fenn, R. Rimanyi, C. Rourke, The braid permutation group, *Topology* **36** (1997), 123–135.

[7] R. Fox, A quick trip through knot theory, in "Topology of Three-Manifolds" ed. by M.K.Fort, Prentice-Hall Pub. (1962).

[8] E. Godelle and L. Paris, $K(\pi, 1)$ and word problems for infinite type Artin-Tits groups, and applications to virtual braid groups, arXiv:1007.1365 v1, 8 Jul 2010.

[9] Mikhail Goussarov, Michael Polyak and Oleg Viro, Finite type invariants of classical and virtual knots, *Topology* **39** (2000), 1045–1068.

[10] D. Hrencecin, "On Filamentations and Virtual Knot Invariants" Ph.D Thesis, Unviversity of Illinois at Chicago (2001).

[11] D. Hrencecin and L. H. Kauffman, "On Filamentations and Virtual Knots", *Topology and Its Applications*, **134** (2003), 23–52.

[12] Jones, V. F. R. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math.* (2) 126 (1987), no. 2, 335–388.

[13] T. Kadokami, Detecting non-triviality of virtual links. J. Knot Theory Ramifications 12 (2003), no. 6, 781–803.

[14] S. Kamada, Braid presentation of virtual knots and welded knots, (preprint, March 2000).

[15] L.H. Kauffman and D.E. Radford, Invariants of 3- manifolds derived from finite dimensional Hopf algebras. *Journal of Knot Theory and Its Ramifications*, Vol. 4, No. 1 (1995), 131-162.

[16] Louis H. Kauffman, Virtual Knot Theory , *European J. Comb.* (1999) Vol. 20, 663-690.

[17] Louis H. Kauffman, A Survey of Virtual Knot Theory, *Proceedings of Knots in Hellas '98*, World Sci. Pub. 2000, 143–202.

[18] Louis H. Kauffman, Detecting Virtual Knots, *Atti. Sem. Mat. Fis. Univ. Modena Supplemento al Vol. IL* (2001), 241–282.

[19] L. H. Kauffman, S. Lambropoulou, Virtual Braids, *Fundamenta Mathematicae* 184 (2004), 159–186.

[20] L.H. Kauffman, S. Lambropoulou, Virtual braids and the L-move, *J. Knot Theory Ramifications* 15 (2006), no. 6, 773–811.

[21] L. H. Kauffman, *Knots and Physics*, World Scientific Publishers (1991), Second Edition 1994 (723 pages), Third Edition 2001.
[22] R. Kirby, A calculus for framed links in S^3, *Invent. Math.*, **45**, (1978), pp. 35-56.
[23] T. Kishino and S. Satoh, A note on non-classical virtual knots, (to appear in *J. Knot Theory & Ramifications*).
[24] Greg Kuperberg, What is a virtual link? arXiv:math.GT / 0208039 v1 5 Aug 2002,*Algebraic and Geometric Topology*, 2003, **3**, 587-591.
[25] W. Magnus, A. Karass, D. Solitar, Combinatorial Group Theory, Dover Pub. (1966,1976).
[26] V. Manturov, "O raspoznavanii virtual'nykh kos" (On the Recognition of Virtual Braids), Zapiski Nauchnykh Seminarov POMI (Petersburg Branch of Russ. Acad. Sci. Inst. Seminar Notes), 299. *Geometry and Topology* **8** (2003), 267–286.
[27] N. Yu. Reshetikhin and V.G. Turaev, Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.* Vol. 127 (1990) 1-26.
[28] N.Y. Reshetikhin and V. Turaev, Invariants of Three-Manifolds via link polynomials and quantum groups, *Invent. Math.*,Vol.103,1991, pp. 547-597.
[29] S. Satoh, Virtual knot presentation of ribbon torus-knots, *J. Knot Theory & Ramifications* **9** (2000), No.4, 531–542.
[30] Classical Topology and Combinatorial Group Theory, Springer-Verlag (1980,1993).
[31] M.E. Sweedler, *Hopf Algebras.* Mathematics Lecture Notes Series, Benjamin, New York, 1969.
[32] V. Turaev, Virtual strings and their cobordisms, arXiv.math.GT/0311185.
[33] V. V. Vershinin, On the homology of virtual braids and the Burau representation, arXiv:math.GT/9904089 v1 18 Apr 1999.
[34] V. V. Vershinin, On the singular braid monoid, arXiv:Math.GR/0309339 v1 20 Sep 2003.

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