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Commuting Differential Operators and Higher-dimensional Algebraic Varieties

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Commuting differential operators and higher-dimensional algebraic varieties

Herbert Kurke, Denis Osipov*, Alexander Zheglov†

Abstract

Several algebro-geometric properties of commutative rings of partial differential operators as well as several geometric constructions are investigated.

1 Introduction

In this paper we study several algebraic-geometric properties of commutative rings of partial differential operators (PDO for short).

The study of commutative subrings of rings of germs of linear partial differential operators is related to algebraic varieties and torsion free sheaves on such varieties. At the same time the study of commutative subrings is related to the theory of exactly solvable nonlinear partial differential equations. In dimension one (or for rings in one variable) we have a rich theory connected with famous equations such as KP (Kadomtsev-Petviashvili), KdV (Korteveg-de-Vries), sin-Gordon, Toda, etc. In dimensions greater than two the theory is far to be completed (for overview see e.g. [3] and references therein).

We start with a construction of geometric data that corresponds to a commutative ring of PDO that satisfy certain conditions (see section 2). This construction was partially mentioned already in the work of Krichever [8]. The idea of Krichever lead to a construction of a free BA-module (module consisting of Baker-Akhieser functions — eigenfunctions of the ring of PDO) and was developed later by various authors (see e.g. [16], [15], [3]) to construct explicit examples of commuting matrix rings PDO (cf. the work [23], where this construction is given in rank one case in terms of a family of Krichever sheaves).

After that we compare in details this construction and the construction given in [25] in the case of operators in two variables (see section 3). For differential operators in two variables in [25] the following approach is offered. We consider a wider class of operators, namely, the operators from the completed ring $\hat{D}$ of differential operators (see [25, Sec.2.1.5]). The operators from this ring contain all usual partial differential operators, and difference operators as well. They are also linear and act on the ring of germs of analytical functions. In the work [25] all commutative subrings in $\hat{D}$ satisfying certain mild conditions are classified in terms of Parshin's modified geometric data, which includes algebraic surfaces and torsion free sheaves on such surfaces. Note that such rings contain all subrings of partial differential operators in two variables considered in section 2 after an appropriate change of variables (cf. section 3.1). The approach offered in [25] generalizes the approach of Sato in dimension one, and differs from the approach of Krichever connected with the study of the Baker-Akhieser functions. As a result of the comparison of

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two approaches we come to a problem (see problem 3.1) which is important for the problem of classification of commutative rings of PDO.

On the other hand, there is also a two-variable analogue of the KP-hierarchy which is also related to algebraic surfaces and torsion free sheaves on such surfaces as well as to a wider class of geometric data consisting of ribbons and torsion free sheaves on them (see [22], [9], [10]). In section 3.3 we compare geometric data from [9], [10] and from [25]. Namely, we show how the construction from [9], [10] can be extended to associate with a geometric data from [25] a geometric data from [9]. So, in particular, one can apply the theory of ribbons developed in [9], [10] to study rings of PDO and their isospectral deformations.

In section 3.2 we introduce a notion of Cohen-Macaulaysation of a surface. Namely, we show that for given integral two-dimensional scheme X of finite type over a field k (or over the integers) there is a "minimal" Cohen-Macaulay scheme $CM(X)$ and a finite morphism $CM(X) \to X$ (and a finite morphism form the normalization of X to $CM(X)$). This construction generalizes the known construction of normalisation of a scheme. In work [18] (see also [17]) the author introduced the analogue of the Krichever map which associate to each Parshin's geometric data (a Cohen-Macaulay surface, an ample Cartier divisor, a smooth point and a vector bundle) a pair of subspaces in the two-dimensional local field associated with the flag (surface, divisor, point). He also shows that this map is injective on such data. In works [18], [17] there was given also a combinatorial construction how to calculate cohomology groups of vector bundles in terms of these subspaces. Later this map was generalized in [9] for geometric data of ribbons. Using these results we show in section 3.3 that the image of the extended map applied to a ribbon constructed by geometric data from [25] coincides with the image of the Parshin map applied to the Cohen-Macaulaysation of this data.

At last, we recall the construction of glueing closed subschemes and give several examples that could be useful in solving the problem 3.1.

This work was originally planned as a part of [11], but, having written it, we decided to extract it in a separate paper.

Everywhere where we work over a field k we assume that k has characteristic zero.

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2 Several constructions

Let's recall some facts about rings of partial differential operators satisfying certain mild conditions (as it usually assumed in works about algebraically integrable systems).

2.1 Generalities

Let R be a commutative k-algebra, where k is a field of characteristic zero.

Then we have the filtered ring $D(R)$ of k-linear differential operators and the R-module $\mathrm{Der}(R)$ of derivations with the properties:

$$D_0(R) \subset D_1(R) \subset D_2(R) \subset \dots ; \quad D_i(R)D_j(R) \subset D_{i+j}(R); \quad \mathrm{Der}(R) \subset D_1(R).$$

The subspaces $D_i(R)$ are defined inductively as sub-R-bimodules of $\mathrm{End}_k(R)$. By definition, $D_0(R) = \mathrm{End}_R(R) = R$, and for $i \ge 0$

$$D_{i+1}(R) = \{P \in \mathrm{End}_k(R) \mid \text{ such that } \quad [P, f] \in D_i(R) \quad \text{for all} \quad f \in R\}.$$

Then we can form the graded ring

$$gr(D(R)) = \bigoplus_{i=0}^{\infty} D_i(R)/D_{i-1}(R), \quad \text{where} \quad D_{-1}(R) = 0,$$

and for $P \in D_i(R)$ the *principal symbol* $\sigma_i(P) = P \mod D_{i-1}(R)$. For $P \in D_i$, $Q \in D_j$ we have that $\sigma_i(P)\sigma_j(Q) = \sigma_{i+j}(PQ)$ and $[P,Q] \in D_{i+j-1}(R)$. Hence $gr(D(R))$ is a commutative graded R-algebra with a Poisson bracket

$$\{\sigma_i(P), \sigma_j(Q)\} = \sigma_{i+j-1}([P,Q])$$

with the usual properties.

Definition 2.1. We denote the order function from $D(R)$ to non-negative integers as

$$\mathbf{ord}(P) = \inf\{n \mid P \in D_n(R)\}.$$

2.2 Coordinates

Definition 2.2. We say that R has a system of coordinates $(x_1, \ldots, x_n) \in R^n$ if the following two conditions are satisfied.

1. The map
$$\mathrm{Der}_k(R) \to R^n \quad : \quad D \mapsto (D(x_1), \ldots, D(x_n))$$
is bijective.

2. $\displaystyle\bigcap_{D \in \mathrm{Der}_k(R)} \mathrm{Ker}(D) = k$.

In this case there are uniquely defined $\partial_1, \ldots, \partial_n \in \mathrm{Der}_k(R)$ such that

$$\partial_i(x_j) = \delta_{ij}, \qquad \mathrm{Ker}(\partial_1) \cap \ldots \cap \mathrm{Ker}(\partial_n) = k.$$

Then $\mathrm{Der}(R)$ is a free R-module with generators $\partial_1, \ldots, \partial_n$. Besides, we have $[\partial_i, \partial_j] = 0$. One checks (by induction on the grade) that

$$R[\xi_1, \ldots, \xi_n] \simeq gr(D(R)) \quad \text{by} \quad \xi_i \mapsto \partial_i \mod D_0(R) \in gr_1(D(R)).$$

Also for $P \in D_i(R)$, $Q \in D_j(R)$ we have

$$\{\sigma_i(P), \sigma_j(Q)\} = \sum_{v=1}^{n} \frac{\partial \sigma_i(P)}{\partial \xi_v} \partial_v(\sigma_j(Q)) - \sum_{v=1}^{n} \frac{\partial \sigma_j(Q)}{\partial \xi_v} \partial_v(\sigma_i(P))$$

(where we have extended ∂_v to $R[\xi_1, \ldots, \xi_n]$ by $\partial_v(\xi_l) = 0$).

A typical example of a ring with a coordinate system is the ring $k[x_1, \ldots, x_n]$ or $k[[x_1, \ldots, x_n]]$, where in the last case we have to restrict ourself to the ring of continuous differential operators and to the space of continuous derivations with respect to the usual topology on $k[[x_1, \ldots, x_n]]$ given by the maximal ideal. The ring $k[[x_1, \ldots, x_n]]$ will be important for the main part of the article.

If $(y_1, \ldots, y_n) \in R^n$ is another coordinate system, we get a new basis $(\partial'_1, \ldots, \partial'_n)$ of $\mathrm{Der}_k(R)$. Hence the change of generators is given by the matrix

$$\begin{pmatrix} \partial_1(y_1) & \ldots & \partial_n(y_1) \\ \partial_1(y_2) & \ldots & \partial_n(y_2) \\ \vdots & \ddots & \vdots \\ \partial_1(y_n) & \ldots & \partial_n(y_n) \end{pmatrix} = M,$$

as $(\partial'_1, \ldots, \partial'_n)M = (\partial_1, \ldots, \partial_n)$, $(\xi'_1, \ldots, \xi'_n)M = (\xi_1, \ldots, \xi_n)$.

2.3 Characteristic scheme

If $J \subset D$ is a right ideal, then we obtain a homogeneous ideal $\langle \sigma_i(P), P \in J \rangle$ in $gr(D)$ and a subscheme defined by this ideal in either $\mathrm{Spec}(gr(D))$ or $\mathrm{Proj}(gr(D))$. Both are called the characteristic subscheme $\mathrm{Ch}(J)$. We consider the characteristic subscheme in $\mathrm{Proj}(gr(D))$.

If we have and fix a coordinate system, then we obtain

$$\mathrm{Proj}(gr(D)) = \mathrm{Proj}(R[\xi_1, \dots, \xi_n]) = \mathrm{Spec}(R) \times_k \mathbb{P}_k^{n-1}.$$

Consider the case of the ideal $J = PD$, where P is an operator with $\mathbf{ord}(P) = m$. If $\sigma_m(P) \in k[\xi_1, \dots, \xi_n]$, then we say that *the principal symbol is constant.* In this case the characteristic scheme is essentially given by the divisor of zeros of $\sigma_m(P)$ in $\mathbb{P}_k^{n-1}$, we call it $\mathrm{Ch}_0(P)$. It is unchanged by a k-linear change of coordinates.

Lemma 2.1. *If $P_1, \dots P_n$ are operators with constant principal symbols (with respect to a coordinate system $(x_1, \dots, x_n)$) and if $\det(\partial\sigma(P_i)/\partial\xi_j) \neq 0$, then any operator Q with $[P_i, Q] = 0$, $i = 1, \dots, n$ has also a constant principal symbol.*

Proof. Let $m_i = \mathbf{ord}(P_i)$ and $m = \mathbf{ord}(Q)$. We have

$$0 = \{\sigma_{m_i}(P_i), \sigma_m(Q)\} = \sum_{v=1}^{n} \frac{\partial\sigma_{m_i}(P_i)}{\partial\xi_v} \partial_v(\sigma_m(Q)).$$

for $i = 1, \dots, n$. Since $\det(\partial\sigma_{m_i}(P_i)/\partial\xi_j) \in k[\xi_1, \dots, \xi_n]$ is not zero, we infer that $\partial_j(\sigma_m(Q)) = 0$ for $j = 1, \dots n$. Hence Q has constant principal symbol with respect to $(x_1, \dots, x_n)$. $\square$

For any subring $F \subset D$ we *define* a filtration on F which is induced by filtration of D: $F_n = F \cap D_n = \{f \in F \mid \mathbf{ord}(f) \le n\}$. We define the ring $\mathrm{gr}(F) = \bigoplus_{n=1}^{\infty} F_n/F_{n-1}$.

2.4 Geometric properties of commutative rings PDO

For the next theorem we recall some facts from algebraic geometry. For any n-dimensional irreducible projective variety X over the field k, and any Cartier divisors $E_1, \dots, E_n \in \mathrm{Div}(X)$ on X one defines the intersection index $(E_1 \cdot \ldots \cdot E_n) \in \mathbb{Z}$ on X (see, e.g., [6], [12, ch. 1.1].) Let $(E^n) = (E \cdot \ldots \cdot E)$ be the self-intersection index of a Cartier divisor $E \in \mathrm{Div}(X)$ on X, and $\mathcal{F}$ be a coherent sheaf on X. There is the asymptotic Riemann-Roch theorem (see survey in [12, ch. 1.1.D]) which says that the Euler characteristic $\chi(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mE))$ is a polynomial of degree $\le n$ in m, with

$$\chi(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mE)) = \mathrm{rk}(\mathcal{F}) \cdot \frac{(E^n)}{n!} \cdot m^n + O(m^{n-1}), \tag{1}$$

where rk is the rank of sheaf.

There is the cycle map: $\mathrm{Z} : \mathrm{Div}(X) \to \mathrm{WDiv}(X)$ from the Cartier divisors to the Weil divisors on X (see appendix 4). From the above description it follows that if $E_1, E_2 \in \mathrm{Div}(X)$ such that $\mathrm{Z}(E_1) = \mathrm{Z}(E_2)$, then the self-intersection indices $(E_1^n) = (E_2^n)$ on X. Indeed, in appendix 4 one calculates the group $\mathrm{Ker}(\mathrm{Z})$. Hence we have that $\mathcal{O}_{\tilde{X}}(\pi^* E_1) = \mathcal{O}_{\tilde{X}}(\pi^* E_2)$, where $\pi : \tilde{X} \to X$ is the normalization of X. Therefore it is enough to prove that the self-intersection index (E^n) on X is equal to the self-intersection index $(\pi^* E^n)$ on $\tilde{X}$ for any $E \in \mathrm{Div}(X)$. But it follows from formula (1) and the following exact sequence of coherent sheaves on X for any $m \in \mathbb{Z}$

$$0 \longrightarrow \mathcal{O}_X(mE) \longrightarrow \pi_* \mathcal{O}_{\tilde{X}}(m\pi^* E) \longrightarrow (\pi_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X(mE) \longrightarrow 0,$$

since $\mathrm{rk}(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)=0$ on X.

The cycle map Z restricted to the semigroup of effective Cartier divisors $\mathrm{Div}^+(X)$ is an injective map to the semigroup of effective Weil divisors $\mathrm{WDiv}^+(X)$. We *will say* that an effective Weil divisor C on X is a $\mathbb{Q}$-*Cartier* divisor on X if $lC\in \mathrm{Im}\,(\mathrm{Z}\,|_{\mathrm{Div}^+(X)})$ for some integer $l>0$.

Definition 2.3. Let C be a $\mathbb{Q}$-Cartier divisor on X. We define the self-intersection index (C^n) on X as

$$(C^n)=(G^n)/l^n, \tag{2}$$

where $G=lC$ is a Cartier divisor for some integer $l>0$.

We note that if $l>0$ is minimal such that lC is a Cartier divisor, then for any other $l'>0$ with the property $l'C$ is a Cartier divisor we have that $l\mid l'$. Therefore, using above reasonings and the property $(E_1^n)=m^n(E_2^n)$ for any $E_1=mE_2$, $E_2\in \mathrm{Div}(X)$, $m\in\mathbb{Z}$ we obtain that formula (2) does not depend on the choice of appropriate l.

Theorem 2.1. *Let $P_1,\dots,P_n\in D=k[[x_1,\dots,x_n]][\partial_1,\dots,\partial_n]$ be any commuting operators of positive order with constant principal symbols. We suppose that the characteristic divisors of $P_1,\dots,P_n$ have no common point (in $\mathbb{P}_k^{n-1}$). Let B be any commutative k-subalgebra in D which contains the operators $P_1,\dots,P_n$. We have the following properties.*

1. $\mathrm{gr}(B)\subset k[\xi_1,\dots,\xi_n]$, *and* $k[\xi_1,\dots,\xi_n]$ *is finitely generated as* $\mathrm{gr}(B)$*-module.*
2. *The rings B and $\mathrm{gr}\,B$ are finitely generated integral k-algebras of Krull dimension n.*
3. *The affine variety $U=\mathrm{Spec}\,B$ over k can be naturally completed to an n-dimensional irreducible projective variety X with boundary C which is an integral Weil divisor not contained in the singular locus of X. Moreover, C is an unirational and ample $\mathbb{Q}$-Cartier divisor.*
4. *The B-module $L=D/Dx_1+\ldots+Dx_n$, which defines a coherent sheaf on U, can be naturally extended to a torsion free coherent sheaf $\mathcal{L}$ on X. Moreover, the self-intersection index (C^n) on X is equal to $1/\mathrm{rk}(\mathcal{L})$.*

Remark 2.1. The items 1 and partially item 2 follow from [2, Ch.III, §2.9, Prop. 10]. The item 2 was proved in [8] by Krichever in connection with integrable systems. We give here an alternative proof in the spirit of pure commutative algebra.

The sheaf $\mathcal{L}$ is a Krichever sheaf in the sense of [23, introduction]. It is in some sense similar to the sheaf from family of Krichever sheaves (or Baker-Akhieser modules), confer [23].

Proof. We prove items 1 and 2. Let $m_i=\mathbf{ord}(P_i)$ and $Q\in B$ such that $m=\mathbf{ord}(Q)$. We have $[P_i,Q]=0$ for $i=1,\dots,n$. Now $(\sigma_{m_1}(P_1),\dots,\sigma_{m_n}(P_n)):\mathbb{A}^n\to\mathbb{A}^n$ is a finite morphism by Hilbert's Nullstellensatz, since the system of equations $\sigma_{m_1}(P_1)=0,\dots,\sigma_{m_n}(P_n)=0$ defines only the zero point in $\mathbb{A}^n$. Therefore $\det(\partial\sigma_{m_i}(P_i)/\partial\xi_j)\neq 0$ (via the interpretation as the map on the tangent space). Hence and by lemma 2.1, $\sigma_m(Q)$ must have constant coefficients.

Now we have

$$k[\sigma_{m_1}(P_1),\dots,\sigma_{m_n}(P_n)]\subset \mathrm{gr}(B)\subset k[\xi_1,\dots,\xi_n]. \tag{3}$$

But $k[\xi_1,\dots,\xi_n]$ is finitely generated as $k[\sigma_{m_1}(P_1),\dots,\sigma_{m_n}(P_n)]$-module. Hence $B_0=k$, and the k-algebra $\mathrm{gr}\,B$ is a finitely generated k-algebra of Krull dimension n. Besides, $k[\xi_1,\dots,\xi_n]$ is finitely generated as $\mathrm{gr}\,B$-module. From (3) it follows that $\mathrm{gr}\,B$ is a ring without zero divisors. Hence the ring B itself is without zero divisors.

It will be useful to introduce the analog of the Rees ring $\tilde{B}$ constructed by the filtration on the ring B: $\tilde{B}=\bigoplus_{n=0}^{\infty}B_ns^n$. The ring $\tilde{B}$ is a subring of the polynomial ring $B[s]$. For the fields of fractions we have $\operatorname{Quot} \tilde{B} = \operatorname{Quot} B[s]$. Besides, $\operatorname{gr} B = \tilde{B}/(s)$. Let the k-algebra $\operatorname{gr}(B)$ be generated by elements $\sigma_{m_i}(b_i)$, $i = 1, \dots, p$ as k-algebra, where $\mathbf{ord}(b_i) = m_i$. It is easy to check that the k-algebra B is generated by the elements b_i, $i = 1, \dots, p$ as k-algebra, and the k-algebra $\tilde{B}$ is generated by the elements $s, b_1 s^{m_1}, \dots, b_r s^{m_p}$ as k-algebra. Hence we can compute the Krull dimension of the ring B:

$$\dim B = \operatorname{trdeg} \operatorname{Quot} B = \operatorname{trdeg} \operatorname{Quot} \tilde{B} - 1 = \operatorname{trdeg} \operatorname{Quot}(\tilde{B}/(s)) = \operatorname{trdeg} \operatorname{Quot}(\operatorname{gr} B) = n, \quad (4)$$

since (s) is a prime ideal of height 1 in the ring $\tilde{B}$ by Krull's height theorem.

We prove now item 3. The ideal $I = \bigoplus\limits_{n=1}^{\infty} B_{n-1} s^n = (s)$ is a homogeneous ideal in the ring $\tilde{B}$, because this ideal is generated by the homogeneous element $s \in \tilde{B}$. Besides, I is a prime ideal, since $B/I = \operatorname{gr} B$ is a ring without zero divisors.

We introduce the schemes $X = \operatorname{Proj} \tilde{B}$ and $C = \operatorname{Proj} \tilde{B}/I = \operatorname{Proj} \operatorname{gr}(B)$. Since $\tilde{B}$ and $\operatorname{gr}(B)$ are integral k-algebras, X and C are integral schemes. Therefore, using (4), we have that the homogeneous prime ideal I defines an irreducible subscheme C of codimension 1 on X. Moreover, $X \setminus C = \operatorname{Spec} \tilde{B}_{(s)} = \operatorname{Spec} B$ is an affine variety. (Here $\tilde{B}_{(s)}$ is the subring of degree zero elements in the localization $\tilde{B}_s$ of the ring $\tilde{B}$ by the multiplicative system s^n, $n \in \mathbb{Z}$).

For any $n \ge 0$ we denote the homogeneous component $\tilde{B}_n = B_n s^n \subset \tilde{B}$. Since $\tilde{B}$ is a finitely generated k-algebra with $\tilde{B}_0 = k$, by [2, Ch.III, § 1.3, prop. 3] there exists an integer $d \ge 1$ such that the k-algebra $\tilde{B}^{(d)} = \bigoplus\limits_{k=0}^{\infty} \tilde{B}_{kd}$ is finitely generated by elements from $\tilde{B}_1^{(d)}$ as a k-algebra. (Here $\tilde{B}_1^{(d)} = \tilde{B}_d$, and $\dim_k \tilde{B}_1^{(d)} < \infty$ by formula (3).) Therefore the scheme $\operatorname{Proj} \tilde{B}^{(d)} \hookrightarrow \operatorname{Proj} \operatorname{Sym}_k(\tilde{B}_1^{(d)}) \simeq \mathbb{P}_k^N$ is a projective scheme over k which is an irreducible variety.

Let us show that dC is a very ample effective Cartier divisor on X. We consider the subscheme C' in X which is defined by the homogeneous ideal $I^d = (s^d)$ of the ring $\tilde{B}$. The topological space of the subscheme C' coincides with the topological space of the subscheme C (as it can be seen on an affine covering of X). We denote

$$\delta = \min \{\mathbf{ord}(f) - \mathbf{ord}(g) \mid f, g \in B,\ \mathbf{ord}(f) > \mathbf{ord}(g)\} = \gcd \{n \mid B_n/B_{n-1} \ne 0,\ n \ge 1\}.$$

Then the function $-\operatorname{ord}/\delta : (\operatorname{Quot} B)^* \to \mathbb{Z}$ is a surjective function which defines the discrete valuation on the field $\operatorname{Quot} B$. The local ring $\mathcal{O}_{X,C}$ coincides with the valuation ring of this discrete valuation:

$$\mathcal{O}_{X,C} = \tilde{B}_{(I)} = \{a s^n / b s^n \mid n \ge 0, a \in B_n,\ b \in B_n \setminus B_{n-1}\}.$$

The ideal I induces the maximal ideal in the ring $\mathcal{O}_{X,C}$, and the ideal I^d induces the d-th power of the maximal ideal. Therefore, if we will prove that the ideal I^d defines an effective Cartier divisor on X, then the cycle map on this divisor is equal to dC (see appendix 4), i.e. C is a $\mathbb{Q}$-Cartier divisor. By [4, prop. 2.4.7] we have $X = \operatorname{Proj} \tilde{B} \simeq \operatorname{Proj} \tilde{B}^{(d)}$. Under this isomorphism the subscheme C' is defined by the homogeneous ideal $I^d \cap \tilde{B}^{(d)}$ in the ring $\tilde{B}^{(d)}$. This ideal is generated by the element $s^d \in \tilde{B}_1^{(d)}$. The open affine subsets $D_+(x_i) = \operatorname{Spec} \tilde{B}_{(x_i)}^{(d)}$ with $x_i \in \tilde{B}_1^{(d)}$ define a covering of $\operatorname{Proj} \tilde{B}^{(d)}$. In every ring $\tilde{B}_{(x_i)}^{(d)}$ the ideal $(I^d \cap \tilde{B}^{(d)})_{(x_i)}$ is generated by the element s^d/x_i. Therefore the homogeneous ideal $I^d \cap \tilde{B}^{(d)}$ defines an effective Cartier divisor.

At last, the Cartier divisor dC is a very ample divisor, because C' is a hyperplane section in the embedding $X = \operatorname{Proj} \tilde{B}^{(d)} \hookrightarrow \operatorname{Proj} \operatorname{Sym}_k(\tilde{B}_1^{(d)}) \simeq \mathbb{P}_k^N$. Besides, by item 1,

$k[\xi_1, \dots \xi_n] \supset \operatorname{gr} B$, and $k[\xi_1, \dots \xi_n]$ is a finite $\operatorname{gr}(B)$ -module. Hence the divisor $C = \operatorname{Proj} \operatorname{gr}(B)$ is an unirational variety.

Since $\mathcal{O}_{X,C}$ is a regular local ring, the divisor C is not contained in the singular locus of X

We prove now item 4 . Let's define the sheaf $\mathcal{L}$. Consider the right D -module

$$L = D/(x_1 D + \dots + x_n D)$$

with filtration $L_n = (D_n + x_1 D + \dots + x_n D)/(x_1 D + \dots + x_n D)$. Then we have $L_n B_r \subseteq L_{r+n}$.

We consider another right D -module $k[\xi_1, \dots, \xi_n]$ with the action of D given (generated) as:

$$f \circ \partial_j = f \xi_j \qquad , \qquad g \circ x_i = -\frac{\partial g}{\partial \xi_i}$$

for any $1 \le i, j \le n$, $f, g \in k[\xi_1, \dots, \xi_n]$. It is easy to check that the maps

$$k[\xi_1, \dots, \xi_n] \longrightarrow L \quad : \quad \sum_{\alpha \in \mathbb{N}^n} a_\alpha \xi^\alpha \mapsto \sum_\alpha a_\alpha \partial^\alpha$$

$$L \longrightarrow k[\xi_1, \dots, \xi_n] \quad : \quad \sum_{\alpha \in \mathbb{N}^n} p_\alpha(x) \partial^\alpha \mapsto \sum_\alpha p_\alpha(0) \xi^\alpha,$$

where $a_\alpha \in k$, $p_\alpha \in k[[x_1, \dots, x_n]]$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, are isomorphisms of the corresponding D -modules (and hence B -modules). The filtration on $k[\xi_1, \dots, \xi_n]$ is the degree filtration of the polynomials. Therefore we have that for any integer $m \ge 0$

$$\dim_k L_m = \binom{m+n}{n} = \frac{(m+1) \cdot \ldots \cdot (m+n)}{n!}. \tag{5}$$

Moreover, $\operatorname{gr}(L) = \bigoplus_{n=1}^{\infty} L_n / L_{n-1} = k[\xi_1, \dots, \xi_n]$ is a finitely generated $\operatorname{gr}(B)$ -module (see item 1). Now we have by induction on the degree of filtration that if elements $\sigma_{m_1}(v_1), \dots, \sigma_{m_s}(v_q)$ (where $v_i \in L_{m_i}$, $\sigma_{m_i}(v_i) = v_i \mod L_{m_i - 1}$, $i = 1, \dots, q$) generate $\operatorname{gr}(L)$ as a $\operatorname{gr}(B)$ -module, then elements $v_1, \dots, v_q$ generate L as B -module. Hence we obtain that $\tilde{L} = \bigoplus_{m=0}^{\infty} L_m s^m$ is a finitely generated torsion free graded $\tilde{B}$ -module which is generated by elements $v_1 s^{m_1}, \dots, v_q s^{m_q}$ over the ring $\tilde{B}$. Therefore $\mathcal{L} = \operatorname{Proj} \tilde{L}$ is a torsion free coherent sheaf[1] on X (see [4, prop. 2.7.3]). Besides, the graded $\operatorname{gr} B$ -module $\operatorname{gr} L$ defines the torsion free coherent sheaf over $C = \operatorname{Proj} \operatorname{gr} B$, and the $B = \tilde{B}_{(s)}$ -module $L = \tilde{L}_{(s)}$ defines the torsion free coherent sheaf over $X \setminus C = \operatorname{Spec} B$.

We have $X = \operatorname{Proj} \tilde{B}^{(d)}$. Under this isomorphism the graded $\tilde{B}^{(d)}$ -module $\tilde{L}^{(d)} = \bigoplus_{k=0}^{\infty} \tilde{L}_{kd}$ (where $\tilde{L}_{kd} = L_{kd} s^{kd}$) gives the coherent sheaf $\mathcal{L}$ as $\operatorname{Proj} \tilde{L}^{(d)}$. We have proved that $C' = dC$ is a very ample Cartier divisor on the projective variety X . Therefore, by [7, ch. III, th. 5.2],

$$H^i(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mC')) = 0 \quad \text{for} \quad i > 0 \quad \text{and} \quad m \gg 0.$$

Also, by [7, ch. II, exerc. 5.9(b)], $H^0(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mC')) = \tilde{L}_{md}$ for $m \gg 0$. Hence and from formula (5) we obtain

$$\chi(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mC')) = \frac{(md+1) \cdot \ldots \cdot (md+n)}{n!} \quad \text{for} \quad m \gg 0.$$

From formula (1) we have that the self-intersection index $(C'^n) = d^n / \operatorname{rk}(\mathcal{L})$ on X . Hence, the self-intersection index $(C^n) = 1/\operatorname{rk}(\mathcal{L})$ on X . $\square$

[1] Here and later in the article we use the non-standard notation Proj for the quasi-coherent sheaf associated with a graded module. If M is a filtered module, then we use the notation $\tilde{M} = \bigoplus_{i=0}^{\infty} M_i s^i$ for the analog of the Rees module, as well as for filtered rings.

3 Operators in two variables

In this section we deduce several properties of the geometric data which classify the subrings in $\hat{D}$ and establish a connection of these data with geometric data consisting of ribbons and torsion free sheaves introduced in [9],[10].

We recall that the ring $k[[u,t]]$ has a natural linear topology, where the base of neighbourhoods of zero is generated by the powers of the maximal ideal of this ring.

On the two-dimensional local field $k((u))((t))$ we will consider the following discrete valuation of rank two $\nu : k((u))((t))^* \to \mathbb{Z} \oplus \mathbb{Z}$:

$$\nu(f) = (m,l) \quad \text{iff} \quad f = t^l u^m f_0, \text{ where } \quad f_0 \in k[[u]]^* + tk((u))[[t]].$$

(Here $k[[u]]^*$ means the set of invertible elements in the ring $k[[u]]$.)

We recall several definitions and results from [25].

Definition 3.1. We call $(X, C, P, \mathcal{F}, \pi, \phi)$ a geometric data of rank r if it consists of the following data:

1. X is a reduced irreducible projective algebraic surface defined over a field k;
2. C is a reduced irreducible ample $\mathbb{Q}$-Cartier divisor on X;
3. $P \in C$ is a closed k-point, which is regular on C and on X;
4. $$\pi : \widehat{\mathcal{O}}_P \longrightarrow k[[u,t]]$$
   is a ring homomorphism such that the image of the maximal ideal of the ring $\widehat{\mathcal{O}}_P$ lies in the maximal ideal (u,t) of the ring $k[[u,t]]$, and $\nu(\pi(f)) = (0,r)$, $\nu(\pi(g)) = (1,0)$, where $f \in \mathcal{O}_P$ is a local equation of the curve C in a neighbourhood of P (since P is a regular point, the ideal sheaf of C at P is generated by one element), and $g \in \mathcal{O}_P$ restricted to C is a local equation of the point P on C (Thus, g, f are generators of the maximal ideal $\mathcal{M}_P$ in $\mathcal{O}_P$).

   Once for all, we choose parameters u, t and fix them (note that $k[[u,t]]$ is a free $\widehat{\mathcal{O}}_P$-module of rank r).
5. $\mathcal{F}$ is a torsion free quasi-coherent sheaf on X.
6. $\phi : \mathcal{F}_P \hookrightarrow k[[u,t]]$ is a $\mathcal{O}_P$-module embedding such that the homomorphisms
   $$H^0(X, \mathcal{F}(nC')) \to k[[u,t]]/(u,t)^{ndr+1}$$
   obtained as compositions of natural homomorphisms
   $$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}(nC')_P \overset{f^{nd}}{\simeq} \mathcal{F}_P \overset{\phi}{\hookrightarrow} k[[u,t]] \to k[[u,t]]/(u,t)^{ndr+1},$$
   where $C' = dC$ is a very ample divisor, are isomorphisms for any $n \geq 0$.

Two geometric data $(X, C, P, \mathcal{F}, \pi_1, \phi_1)$ and $(X, C, P, \mathcal{F}, \pi_2, \phi_2)$ are identified if the images of the embeddings (obtained by means of multiplication to f^{nd} as above)

$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}_P \overset{\phi_1}{\hookrightarrow} k[[u,t]], \quad H^0(X, \mathcal{O}(nC')) \hookrightarrow \widehat{\mathcal{O}}_P \overset{\pi_1}{\hookrightarrow} k[[u,t]]$$

and

$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}_P \overset{\phi_2}{\hookrightarrow} k[[u,t]], \quad H^0(X, \mathcal{O}(nC')) \hookrightarrow \widehat{\mathcal{O}}_P \overset{\pi_2}{\hookrightarrow} k[[u,t]]$$

coincide for any $n \geq 0$. The set of all quintets of rank r is denoted by $\mathcal{Q}_r$.

Remark 3.1. We would like to emphasize that the rank r of the geometric data in general differs from the rank of the sheaf $\mathcal{F}$, see remark 3.3.

If $\mathcal{F}_P$ is a free $\mathcal{O}_P$-module of rank r, then ϕ induces an isomorphism $\widehat{\mathcal{F}}_P \simeq k[[u,t]]$ of $\widehat{\mathcal{O}}_P$-modules. This condition is satisfied if $\mathcal{F}$ is a coherent sheaf of rank r, see corollary 3.1 below.

Definition 3.2. We define a category $\mathcal{Q}$ of geometric data as follows:

1. The set of objects is defined by
$$Ob(\mathcal{Q}) = \bigcup_{r \in \mathbb{N}} \mathcal{Q}_r.$$

2. A morphism
$$(\beta, \psi) : (X_1, C_1, P_1, \mathcal{F}_1, \pi_1, \phi_1) \to (X_2, C_2, P_2, \mathcal{F}_2, \pi_2, \phi_2)$$
of two objects consists of a morphism $\beta : X_1 \to X_2$ of surfaces and a homomorphism $\psi : \mathcal{F}_2 \to \beta_* \mathcal{F}_1$ of sheaves on X_2 such that:

   (a) $\beta|_{C_1} : C_1 \to C_2$ is a morphism of curves;

   (b)
   $$\beta(P_1) = P_2.$$

   (c) There exists a continuous ring isomorphism $h : k[[u,t]] \to k[[u,t]]$ such that
   $$h(u) = u \mod (u^2) + (t), \quad h(t) = t \mod (ut) + (t^2),$$
   and the following commutative diagram holds:
   $$\begin{array}{ccc} k[[u,t]] & \xrightarrow{h} & k[[u,t]] \\ \uparrow \pi_2 & & \uparrow \pi_1 \\ \widehat{\mathcal{O}}_{X_2,P_2} & \xrightarrow{\beta^{\sharp}_{P_1}} & \widehat{\mathcal{O}}_{X_1,P_1} \end{array}$$

   (d) Let's denote by $\beta_*(\phi_1)$ a composition of morphisms of $\mathcal{O}_{P_2}$-modules
   $$\beta_*(\phi_1) : \beta_* \mathcal{F}_{1P_2} \to \mathcal{F}_{1P_1} \hookrightarrow k[[u,t]].$$
   There is a $k[[u,t]]$-module isomorphism $\xi : k[[u,t]] \simeq h_*(k[[u,t]])$ such that the following commutative diagram of morphisms of $\mathcal{O}_{P_2}$-modules holds:
   $$\begin{array}{ccc} \mathcal{F}_{2P_2} & \xrightarrow{\psi} & \beta_* \mathcal{F}_{1P_2} \\ \downarrow \phi_2 & & \downarrow \beta_*(\phi_1) \\ k[[u,t]] & \xrightarrow{\xi} & h_*(k[[u,t]]) = k[[u,t]] \end{array}$$

Theorem 3.1. *[25, th.3.4] There is a one to one correspondence between the set of classes of equivalent* 1*-quasi elliptic strongly admissible rings (see definitions 2.18, 3.4, 2.11 in [25]) and the set of isomorphism classes of geometric data* $\mathcal{M}$ *(see definitions 3.1, 3.2).*

Definition 3.3. (cf. [25, Def.3.14]) Given a geometric data $(X, C, P, \mathcal{F}, \pi, \phi)$ we define a pair of subspaces

$$W, A \subset k[[u]]((t))$$

as follows:

Let f^d be a local generator of the ideal $\mathcal{O}_X(-C')_P$, where $C' = dC$ is a very ample Cartier divisor (cf. definition 3.1, item 6). Then $\nu(\pi(f^d)) = (0, r^d)$ in the ring $k[[u,t]]$ and therefore $\pi(f^d)^{-1} \in k[[u]]((t))$. So, we have natural embeddings for any $n > 0$

$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}(nC')_P \simeq f^{-nd}(\mathcal{F}_P) \hookrightarrow k[[u]]((t)),$$

where the last embedding is the embedding $f^{-nd}\mathcal{F}_P \overset{\phi}{\hookrightarrow} f^{-nd}k[[u,t]] \hookrightarrow k[[u]]((t))$ (cf. definition 3.1, item 6). Hence we have the embedding

$$\chi_1 \ : \ H^0(X \backslash C, \mathcal{F}) \simeq \varinjlim_{n>0} H^0(X, \mathcal{F}(nC')) \hookrightarrow k[[u]]((t)).$$

We define $W \stackrel{\text{def}}{=} \chi_1(H^0(X\backslash C, \mathcal{F}))$. Analogously the embedding $H^0(X\backslash C, \mathcal{O}) \hookrightarrow k[[u]]((t))$ is defined (and we'll denote it also by χ_1). We define $A \stackrel{\text{def}}{=} \chi_1(H^0(X\backslash C, \mathcal{O}))$.

As it follows from the definition, $A \subset k[[u']]((t')) = k[[u]]((t^r))$, where $t' = \pi(f)$, $u' = \pi(g)$ (cf. definition 3.1, item 4). Thus, on A there is a filtration A_n induced by the filtration $t'^{-n}k[[u']][[t']]$ on the space $k[[u']]((t'))$:

$$A_n = A \cap t'^{-n}k[[u']][[t']] = A \cap t^{-nr}k[[u]][[t]]$$

We have $X \simeq \mathrm{Proj}(\tilde{A})$, where $\tilde{A} = \bigoplus_{n=0}^{\infty} A_n s^n$ (cf. [25, lemma 3.3, th.3.3]). The similar filtration is defined on the space $W \subset k[[u]]((t))$:

$$W_n = W \cap t^{-nr}k[[u]][[t]]$$

And the sheaf $\mathcal{F} \simeq \mathrm{Proj}(\tilde{W})$, where $\tilde{W} = \bigoplus_{n=0}^{\infty} W_n s^n$. Note that we have $W_{nd} \simeq H^0(X, \mathcal{F}(nC'))$ by definition 3.1, item 6 and by construction of the map χ_1.

3.1 Geometric properties of geometric data

Recall that rings of commuting partial differential operators with constant higher symbols give examples of 1-quasi elliptic strongly admissible rings after appropriate change of variables (see section 3.1 and lemma 2.6 in [25]). Below we give several properties of surfaces and sheaves from geometric data in definition 3.1.

Theorem 3.2. *Let X, C be a surface and a divisor from geometric data in definition 3.1. Then X is Cohen-Macaulay outside a finite set of points disjunct from C.*

Proof. If we have a geometric data from definition 3.1, we can define a ring $A \subset k[[u']]((t'))$ (see definition 3.3 above or [25, def.3.14], [25, th.3.3]), a filtration A_i defined by the discrete valuation $\nu_{t'}$ on the field $k((u'))((t'))$:

$$A_i = \{a \in A \mid \nu_{t'}(a) \ge -i\}, \quad i \ge 0$$

which satisfy the following property: $A_{di} \simeq H^0(X, \mathcal{O}_X(idC))$ for all $i \ge 0$, where d is a minimal natural number such that dC is a very ample Cartier divisor. So, $X \simeq \mathrm{Proj}\, \tilde{A}^{(d)} \simeq \mathrm{Proj}\, \tilde{A}$, where $\tilde{A}=\bigoplus_{i=0}^{\infty} A_i s^i$, and C is defined by the homogeneous ideal $I=\tilde{A}(-1)=(s)$ in the ring $\tilde{A}$. We note that the ring $\tilde{A}$ is finitely generated over k, since the ring $\tilde{A}^{(d)}=\bigoplus_{i=0}^{\infty} A_{di}s^{di}$ is finitely generated over k as a graded ring which is equivalent to the homogeneous coordinate ring of the projective surface X, and the modules $\tilde{A}^{(d,l)}=\bigoplus_{i=0}^{\infty} A_{di+l}s^{di+l}$, $0<l<d$ are naturally isomorphic to the ideals in $\tilde{A}^{(d)}$, which are finitely generated. The curve C is covered by all affine subsets $\operatorname{Spec}\tilde{A}_{(x_i)}$, where $x_i\in A_d$, $\nu_{t'}(x_i)=-d$. (The ring $\tilde{A}_{(x_i)}$ is the subring of homogeneous elements in the localization ring of $\tilde{A}$ with respect to the multiplicatively closed system $\{x_i^m\}_{m\in\mathbb{Z}}$.)

First let's show that each point on the curve C is Cohen-Macaulay on X. It is enough to show that the ideal (s^d/x_i) in the ring $\tilde{A}_{(x_i)}$ is $I_{(x_i)}$-primary for all x_i with the above properties. (The element s^d/x_i is non-zero divisor in the ring $\tilde{A}_{(x_i)}$. Therefore to prove the Cohen-Macaulay property it is enough to find a non-invertible non-zero divisor in the ring $\tilde{A}_{(x_i)}/(s^d/x_i)$. But all zero-divisors in the last ring coincide with $I_{x_i}/(s^d/x_i)$ if (s^d/x_i) is a $I_{(x_i)}$-primary ideal, and all these zero-divisors are nilpotent. Then by [1, prop. 4.7] and by Krull's theorem $\operatorname{ht} I_{(x_i)}=1$. Thus, $\dim \tilde{A}_{(x_i)}/I_{(x_i)}=1$ and there exist non-invertible non-zero divisors). Assume that elements as^{dk}/x_i^k and bs^{dl}/x_i^l are from $\tilde{A}_{(x_i)}$, but not from the ideal (s^d/x_i), and

$$\frac{as^{dk}}{x_i^k}\cdot\frac{bs^{dl}}{x_i^l}=\frac{cs^{(k+l-1)d}}{x_i^{k+l-1}}\cdot\frac{s^d}{x_i}\in\left(\frac{s^d}{x_i}\right)\subset I_{(x_i)}.$$

We must show that $as^{dk}/x_i^k, bs^{dl}/x_i^l\in I_{(x_i)}$. Since $I_{(x_i)}$ is a prime ideal, without loss of generality we can assume that the element $g=as^{dk}/x_i^k\in I_{(x_i)}$. Note that any element $y\in I_{(x_i)}$ satisfies the property $\nu_{t'}(y)>0$. Then we have $\nu_{t'}(a)=-kd+j$, where $0<j<d$, because $g\in I_{(x_i)}$ and $g\notin (s^d/x_i)$ (if $j\ge d$, then $\nu_{t'}(a)\le (k-1)d$ and therefore $a\in A_{(k-1)d}\subset A_{kd}$, thus $as^{dk}=(as^{d(k-1)})s^d$ and $as^{dk}/x_i^k\in(s^d/x_i)$, a contradiction). Then we have

$$g^d=\frac{a^d s^{kd^2-jd}}{x_i^{kd-j}}\cdot\frac{s^{dj}}{x_i^j}\in\left(\frac{s^d}{x_i}\right),$$

and $a^d s^{kd^2-jd}/x_i^{kd-j}\notin I_{(x_i)}$, because $\nu_{t'}(a^d/x_i^{kd-j})=0$.

If $g_1=bs^{dl}/x_i^l\notin I_{(x_i)}$, then we obtain

$$g_1^d\frac{a^d s^{kd^2-jd}}{x_i^{kd-j}}\notin I_{(x_i)}.$$

But on the other hand,

$$\frac{a^d s^{kd^2-jd}}{x_i^{kd-j}}g_1^d=g^d\frac{x_i^l}{s^{dj}}g_1^d=\frac{c^d s^{(k+l-1)d^2}}{x_i^{(k+l-1)d}}\cdot\frac{s^{d^2-dj}}{x_i^{d-j}}\in I_{(x_i)},$$

a contradiction. Thus, $g_1\in I_{(x_i)}$, and therefore (s^d/x_i) is a $I_{(x_i)}$-primary ideal.

Now let V denote an open subscheme in X, such that $P\in V$ (P is a smooth point from definition 3.1) and V is normal (hence, Cohen-Macaulay). Then $X\backslash V$ is a closed subscheme with each irreducible component of dimension not greater than one. Let E be an irreducible component of dimension one. Let e denote the prime ideal of E in the ring A (the generic point of E belongs to the affine set $X\backslash C=\operatorname{Spec} A$). Take an element $a\in e$. Making an appropriate localization by a multiplicatively closed subset $S\subset A$, using [1, prop. 4.9.], we come to a ring A_S, where the primary decomposition of $(a)_S$ does not contain associated embedded ideals. So, all points on $\operatorname{Spec} A_S \cap E$ are Cohen-Macaulay. Therefore, there can be only finite number of not Cohen-Macaulay points on X. Since all points on C are Cohen-Macaulay, we can find an open $U \supset C$ such that U is a Cohen-Macaulay scheme. $\square$

Corollary 3.1. *Let $\mathcal{F}$ be a sheaf from geometric data of rank r in definition 3.1. If $\mathcal{F}$ is a coherent sheaf, then it is Cohen-Macaulay along the curve C. In particular, $\mathcal{F}_P$ is a free $\mathcal{O}_P$-module.*

Proof. Since the sheaf $\mathcal{F}$ is torsion free, then $\dim \mathcal{F}_Q = \dim(\mathcal{O}_Q/\mathrm{Ann}\,(\mathcal{F}_Q)) = 2$ for any point $Q \in C$. So, we have to show that $\operatorname{depth} \mathcal{F}_Q = 2$.

In the proof of theorem 3.2 we have shown that for any $Q \in C$ there is a regular sequence in $\mathcal{O}_{X,Q}$ coming from the sequence $s^d/x_i, bs^{dl}/x_i^l$ from $\tilde{A}_{(x_i)}$ for some i, where $bs^{dl}/x_i^l \notin I_{(x_i)}$ or, equivalently, $\nu_{t'}(b) = -dl$. Let's show that this sequence is regular also for $\mathcal{F}_Q$.

As we have already remind in definition 3.3, $\mathcal{F} \simeq \operatorname{Proj}(\tilde{W})$. So, it is enough to prove that the element bs^{dl}/x_i^l is not a zero divisor in the module $\tilde{W}_{(x_i)}/(s^d/x_i)\tilde{W}_{(x_i)}$. Let $g = as^{dk}/x_i^k \in \tilde{W}_{(x_i)}$ be an element such that $g \notin (s^d/x_i)\tilde{W}_{(x_i)}$. Note that this is equivalent to the condition $\nu_t(a) = -dkr + j$, where $0 \le j < dr$ (see analogous arguments in the proof of theorem). But then $\nu_t(ab) = -d(k+l)r + j$, whence by the same reason

$$\frac{as^{dk}}{x_i^k} \cdot \frac{bs^{dl}}{x_i^l} \notin \left(\frac{s^d}{x_i}\right) \tilde{W}_{(x_i)}.$$

Thus, bs^{dl}/x_i^l is not a zero divisor and $\operatorname{depth} \mathcal{F}_Q = 2$ for any $Q \in C$.

The last assertion follows from [14, ch. 6, § 16, exer. 4], because P is a regular point. $\square$

In view of proposition 2.1 it is important to compare the sheaf $\mathcal{L}$ there (which is an analogue of the Baker-Akhieser module) with the sheaf $\mathcal{F}$ appearing in the definition 3.1. The following proposition gives a criterion answering the question when the sheaf $\mathcal{F}$ from geometric data of rank r is a coherent sheaf of rank r on X.

Proposition 3.1. *Let $(X, C, P, \mathcal{F}, \pi, \phi)$ be a geometric data of rank r from definition 3.1. The sheaf $\mathcal{F}$ is coherent of rank r on X if and only if the self-intersection index $(C^2) = r$ on X.*

Proof. As we have already remind in the proof of theorem 3.2 (cf. [25] and definition 3.3), there are subspaces A, W such that $X \simeq \operatorname{Proj}(\tilde{A})$, $\mathcal{F} \simeq \operatorname{Proj}(\tilde{W})$. So, if the sheaf $\mathcal{F}$ is coherent of rank r, then we have by [7, ch.II, ex. 5.9] $H^0(X, \mathcal{F}(nC')) \simeq W_{nd} \simeq k[u,t]/(u,t)^{ndr+1}$ for $n \gg 0$. So, as in the proof of theorem 2.1, item 4 we obtain

$$\chi(X, \mathcal{F}(nC')) = \frac{(ndr+1)\cdot(ndr+2)}{2} \quad \text{for} \quad n \gg 0$$

and from formula (1) it follows $C^2 = r$.

Conversely, let the self-intersection index $(C^2) = r$ on X. Let $C' = dC$ be a very ample Cartier divisor. Then for any coherent sheaf $\mathcal{F}'$ the coefficient at the degree n^2 of the polynomial $\chi(X, \mathcal{F}'(nC'))$ is equal to $d^2(\operatorname{rk}\mathcal{F}')r/2$ (see formula (1)). Consider the sheaf $\mathcal{F}' = \operatorname{Proj}(\tilde{W}')$, where $\tilde{W}'$ is a graded $\tilde{A}$-submodule in $\tilde{W}$ generated by elements from W_n for sufficiently big n. Note that $\operatorname{rk}\mathcal{F}' \ge r$. Indeed, there are elements $w_1, \dots, w_r$ in W_n with $\nu_t(w_1) = -1, \dots, \nu_t(w_r) = -r$ (because for $n = md$, $m \gg 0$, by definitions 3.1 and 3.3 we have $W_{md} \simeq H^0(X, \mathcal{F}(mC')) \simeq k[[u,t]]/(u,t)^{mdr+1}$ and $W_{md} = f^{-md}H^0(X, \mathcal{F}(mC'))$, where the space is considered as a subspace in $k[[u,t]]$ through the embedding from definition 3.1, item 6) and therefore they are linearly independent over $\tilde{A}$. So, there is an embedding $\tilde{A}^{\oplus r} \hookrightarrow \tilde{W}'$ and since Proj is an exact functor (see [4, prop. 2.5.4]), we obtain an embedding $\mathcal{O}_X^{\oplus r} \hookrightarrow \operatorname{Proj}(\tilde{W}')$, hence $\operatorname{rk}\mathcal{F}' \ge r$. The same arguments show that the sheaf $\mathcal{F}'|_C = \operatorname{Proj}(\operatorname{gr}\tilde{W}')$ on C has rank greater or equal to r.

On the other hand, for big n we have

$$\chi(X, \mathcal{F}'(nC')) = \dim_k W'_{nd} \le \dim_k k[u,t]/(u,t)^{ndr+1},$$

since $W'_{nd} \subset W_{nd} \simeq k[u,t]/(u,t)^{ndr+1}$ by definition 3.3. So, the coefficient at the degree n^2 of the polynomial $\chi(X, \mathcal{F}'(nC'))$ is less or equal than $d^2r^2/2$. Hence, $\mathrm{rk}\,\mathcal{F}' = r$ and $\mathrm{rk}\,\mathcal{F}'|_C = r$. Then we also have $\chi(C, \mathcal{F}'|_C(nC')) = r^2d^2n + c(\mathcal{F}')$, where $c(\mathcal{F}') \in \mathbb{Z}$.

Now consider two such coherent sheaves $\mathcal{F}'_1 \subset \mathcal{F}'_2 \subset \mathcal{F}$. Then on C we have the exact sequences

$$0 \to A(nC') \to \mathcal{F}'_1|_C(nC') \to \mathcal{F}'_2|_C(nC') \to B(nC') \to 0$$

for all n, where A and B are coherent sheaves with finite support. Hence, we have

$$H^1(C, \mathcal{F}'_1|_C(nC')) = H^1(C, (\mathcal{F}'_1/A)|_C(nC')),$$

and for all $n \gg 0$ such that $H^1(C, (\mathcal{F}'_1/A)|_C(nC')) = 0$ we have $H^1(C, \mathcal{F}'_2|_C(nC')) = 0$. Let's fix such n_0. Note that this number depends only on $\mathcal{F}'_1$, not on $\mathcal{F}'_2$.

So, for all $n \ge n_0$ and for all coherent sheaves $\mathcal{F}'_2 \supset \mathcal{F}'_1$ we have $H^1(C, \mathcal{F}'_2|_C(nC')) = 0$. Take some $q > n_0$ and consider the sheaf $\mathcal{F}'_2 = \mathrm{Proj}(\tilde{W}')$, where $\tilde{W}'$ is a graded $\tilde{A}$-submodule in $\tilde{W}$ generated by elements from W_{qd}. Then we have

$$\chi(C, \mathcal{F}'_2|_C(qC')) = \dim H^0(C, \mathcal{F}'_2|_C(qC')) = qd^2r^2 + c(\mathcal{F}'_2)$$

for some $c(\mathcal{F}'_2) \in \mathbb{Z}$.

Note that for all n we have $\mathrm{Proj}(\tilde{W}'(nd)) \simeq \mathrm{Proj}(\tilde{W}'^{(d)}(n))$ by [4, prop. 2.4.7] (recall that $\tilde{W}'(nd)$ is equivalent to $\oplus_{i=0}^{\infty} W'_{i+nd}s^i$), and $\mathrm{Proj}(\tilde{W}'^{(d)}(n)) \simeq \mathrm{Proj}(\tilde{W}'^{(d)})(n) \simeq \mathcal{F}'_2(nC')$ by [7, ch. II, prop. 5.12]. So,

$$H^0(X, \mathcal{F}'_2(nC')) = H^0(X, \mathrm{Proj}(\tilde{W}'(nd))).$$

Lemma 3.1. *(cf. [25, lemma 3.5]) We have*

$$H^0(X, \mathrm{Proj}(\tilde{W}'(nd))) = W'_{nd}.$$

Proof. By definition, we have $W'_{nd} = \tilde{W}'(nd)_0 \subset H^0(X, \mathrm{Proj}(\tilde{W}'(nd)))$.

Let $a \in H^0(X, \mathrm{Proj}(\tilde{W}'(nd)))$, $a \notin W'_{nd}$. Then $a = (a_1, \dots, a_k)$, where $a_i \in (\tilde{W}'(nd))_{(x_i)}$, and $x_i \in \tilde{A}_d$ are generators of the space $\tilde{A}_d$ such that $x_1 = s^d$, $x_i = x'_i s^d$ (where $x'_i \in A_d$) and $a_i = a_j$ in $\tilde{A}_{x_i x_j}$.

We have $a_i = \tilde{a}_i / x_i^{k_i}$ (where $\tilde{a}_i = a'_i s^{k_i d}$, $a'_i \in W'_{k_i d + nd}$), $a_1 = \tilde{a}_1 / s^{k_1}$ and $k_1 > 0$ since $a \notin W'_{nd}$. Indeed, if $\tilde{a}_1 \in \tilde{W}'(nd)_0 = W'_{nd}$, then $a = \tilde{a}_1$ since $\tilde{W}'$ is a torsion free $\tilde{A}$-module, a contradiction. So, we have

$$a'_1 \in W'_{k_1+nd} \backslash W'_{k_1+nd-1}.$$

Then for $x'_i \in A_d \backslash A_{d-1}$ (such a generator x_i exists because all elements from $A_{d-1} \subset A_d$ lie in the ideal that defines the divisor C) we have ${x'_i}^{k_i} \in A_{dk_i} \backslash A_{dk_i-1}$ and therefore $a'_1 {x'_i}^{k_i} \in {W'}_{k_1+dk_i+nd} \backslash {W'}_{k_1+dk_i+nd-1}$. On the other hand, we have the equality $\tilde{a}_1 x_i^{k_i} = \tilde{a}_i s^{k_1}$, hence $a'_1 {x'_i}^{k_i} = a'_i$, but

$$a'_i \in {W'}_{dk_i+nd} \subset W'_{k_1+dk_i+nd-1},$$

a contradiction. So, $a \in W'_{nd}$. $\square$

Now we have that

$$H^0(C, \mathcal{F}'_2|_C(qC')) \supset H^0(X, \mathcal{F}'_2(qC'))/H^0(X, \mathcal{F}'_2((q-1)C')).$$

By lemma and by definition of the sheaf $\mathcal{F}_2'$ we have

$$H^0(X, \mathcal{F}_2'(qC'))/H^0(X, \mathcal{F}_2'((q-1)C')) = W'_{qd}/W'_{(q-1)d} = W_{qd}/W_{(q-1)d}.$$

So, we obtain $c(\mathcal{F}_2') \ge \dim(W_{qd}/W_{(q-1)d}) - qr^2d^2$.

On the other hand, for big n we have

$$H^0(C, \mathcal{F}_2'|_C(nC')) = H^0(X, \mathcal{F}_2'(nC'))/H^0(X, \mathcal{F}_2'((n-1)C')) = W'_{nd}/W'_{(n-1)d} \subset W_{nd}/W_{(n-1)d}$$

and $\dim_k(W_{nd}/W_{(n-1)d}) - nr^2d^2 = \mathrm{const} = l$ for all $n \ge 0$. Therefore

$$c(\mathcal{F}_2') = \dim H^0(C, \mathcal{F}_2'|_C(nC')) - nr^2d^2 \le l.$$

Hence $c(\mathcal{F}_2') = l$, $W'_{nd}/W'_{(n-1)d} = W_{nd}/W_{(n-1)d}$ for all $n \ge 0$, and consequently $\tilde{W}' = \tilde{W}$, i.e. $\mathcal{F} = \mathcal{F}_2'$ is a coherent sheaf of rank r on X . □

Remark 3.2. The sheaf $\mathcal{F}$ from geometric data of rank r may be not coherent, as the following example shows. Let $W = \langle u^it^{-j} \quad | i,j \ge 0, i-j \le 0\rangle$ and $A = k[ut^{-2}, t^{-2}]$ be two subspaces in $k[[u]]((t))$. Then it is easy to see that W is not a finitely generated A-module. So, the geometric data constructed by these subspaces (see [25, th.3.3]) will contain a quasicoherent, but not coherent sheaf $\mathcal{F}$.

The coherence of the sheaf $\mathcal{L}$ constructed by a ring of partial differential operators in theorem 2.1 followed from special conditions on the ring (see item 1 of this theorem). These conditions may be not true for a general 1-quasi elliptic strongly admissible ring (see theorem 3.1) even if such a ring is a ring of partial differential operators, as the example above shows: indeed, the ring A above corresponds to the ring $B = k[\partial_2^2, \partial_1\partial_2]$, see [25, th.3.2]. Nevertheless, the proposition above guarantees that for a surface and a divisor satisfying certain geometric conditions the sheaf $\mathcal{F}$ must be coherent.

Now a natural question arises: how are connected the variety X , the divisor C and the sheaf $\mathcal{L}$ from theorem 2.1 (if $\dim X = 2$), constructed by a ring of partial differential operators from this theorem, and corresponding objects of the geometric data constructed by the same ring in theorem 3.1? The proposition below gives the answer.

Proposition 3.2. *Let B be a commutative subring of partial differential operators in two variables which satisfies the conditions from theorem 2.1 and from theorem 3.1 (cf. the beginning of this subsection). Then the triple $(X, C, \mathcal{L})$ from theorem 2.1 is isomorphic to the triple $(X, C, \mathcal{F})$ (a part of geometric data) from theorem 3.1.*

Proof. Recall that the surface and divisor from theorem 3.1 are constructed by the graded ring $\tilde{A}$ which is defined by the ring $A \subset k[[u]]((t))$ (cf. definition 3.3 and see proof of theorem 3.3 in [25]). The ring A and the subspace W are defined by the ring SBS^{-1} and the space $W_0S^{-1} = DS^{-1} \mod (x_1, \dots, x_n)$ and by change of variables $\psi_1 \colon A = \psi_1(SBS^{-1})$, $W = \psi_1(W_0S^{-1})$ (see section 3.1 and remark 3.2 from [25]), where S is an invertible zeroth order pseudo-differential operator defined in [25, lemma 2.11]. So, there is a natural isomorphism $\alpha : B \to A$, $b \mapsto \psi_1(SbS^{-1})$ which induces an isomorphism $\tilde{B} \simeq \tilde{A}$, whence we have the isomorphism of surfaces and divisors.

Moreover, note that the map $\varphi : L \to W$, $l \mapsto \psi_1(lS^{-1} \mod x_1D + \ldots + x_nD)$ gives an isomorphism between the B-module L and A-module W , because SBS^{-1} is a subring of pseudo-differential operators with constant coefficients, see the proof of theorem 3.2 in [25], and

$$\begin{aligned}\varphi(lb) = \psi_1(lbS^{-1} \mod x_1D + \ldots + x_nD) = \psi_1(lS^{-1}(SbS^{-1}) \mod x_1D + \ldots + x_nD) = \\ = \psi_1(lS^{-1} \mod x_1D + \ldots + x_nD)\psi_1(SbS^{-1}) = \varphi(l)\alpha(b).\end{aligned}$$

So, φ induces an isomorphism of sheaves $\mathcal{L}$ and $\mathcal{F}$, and this isomorphism is compatible with the isomorphism of surfaces. □

Remark 3.3. The rank of the sheaf $\mathcal{F}$ from geometric data in definition 3.1 may be greater than the rank of the data even if the sheaf $\mathcal{F}$ is coherent (as it easily follows from the arguments of proposition 3.1, the rank of $\mathcal{F}$ can not be less than the rank of the data).

For example, consider the ring of PDO $B = k[\partial_2, \partial_1\partial_2 + \partial_1^2]$. It satisfies the conditions of theorem 2.1. This is also a strongly admissible ring in the sense of [25, def. 2.11] and $N_B = 1$, thus the rank of the corresponding geometric data is one (see [25, th.3.3]). On the other hand, for big m we have $\dim_k B_m \sim m^2/4$ and $\dim_k L_m \sim m^2/2$ (in the notation of theorem 2.1). Therefore, $\mathrm{rk}\,\mathcal{F} = 2$ (see the proof of theorem 2.1).

So, in general we have $\mathrm{rk}\,\mathcal{F} \ge r$, where r is the rank of the data.

As it follows from propositions above, theorem 2.1 and theorem 3.1, to find new explicit examples of partial differential operators in two variables, it is important to solve the following algebro-geometric problem (especially in view of finding new quantum algebraically completely integrable systems).

Problem 3.1. It would be nice to classify (and/or to find a way how to construct) projective irreducible surfaces X that have an irreducible ample effective $\mathbb{Q}$-Cartier divisor C not contained in the singular locus of X, with the self-intersection index $(C^2) = 1$ on X, and such that there are coherent sheaves satisfying the conditions of item 6 of definition 3.1. Since any unirational curve is a rational curve (by Lüroth's theorem), the curve C has to be a rational curve (see item 3 of theorem 2.1)

3.2 Cohen-Macaulaysation

For a Noetherian domain A define

$$A' = \bigcap_{\mathrm{ht}\,\wp=1} A_\wp$$

to be the intersection of all localizations with respect to prime ideals of height 1. We will say that $\mathrm{depth}(A) > 1$ if the condition (S_2) from [13, ch. 7, § 17, (17.I)] holds, i.e. it holds $\mathrm{depth}(A_\wp) \ge \inf(2, \mathrm{ht}(\wp))$ for all $\wp \in \mathrm{Spec}(A)$. Then one proves

Lemma 3.2. *Assume* $\dim(A) > 1$. *Then* $A' = A$ *if and only if* $\mathrm{depth}(A) > 1$.

Proof. If $A' = A$, then for any non-zero non-invertible $f \in A$ we have (since A is a domain)

$$fA = \bigcap_{\mathrm{ht}\,\wp=1} fA_\wp = \bigcap_{\mathrm{ht}\,\wp=1} (fA_\wp \bigcap A),$$

and the ideals $fA_\wp$ either coincide with $A_\wp$ or are $\wp$-primary in the rings $A_\wp$, since the rings $A_\wp$ are Noetherian local rings of dimension one. So, the ideals $fA_\wp \bigcap A$ in A either coincide with A or are $\wp$-primary in A. Thus, there is a primary decomposition for fA without embedded components (cf. [1, th. 4.10]) and therefore there are non-invertible non-zero divisors in $A_\wp/fA_\wp$ for any $\wp \supset f$ of height one, because $\dim A > 1$ (cf. [1, prop. 4.7]). Hence $\mathrm{depth}(A) > 1$.

Now assume $\mathrm{depth}(A) > 1$. If $x \in A'$ then the set of all elements $s \in A$ such that $sx \in A$ is an ideal, not contained in any prime ideal of height 1. Since $\mathrm{depth}(A) > 1$, there is a regular sequence (a, b) in this ideal. Since $a(xb) - b(xa) = 0$, it follows that $xa \in aA$, thus $x \in A$. $\square$

Assume A has the following property:

(∗) Every prime ideal of height 1 in the normalization of A intersects A in a prime ideal of height 1.

This property is satisfied for example for domains of finite type over a field or over the integers by [1, prop. 5.6, corol. 5.8, th. 5.10, th. 5.11] and by [21, vol. I, ch. V, th. 9].

Then for every Noetherian domain B between A and its normalization with $\operatorname{depth}(B) > 1$ we have that A' is contained in B (since $B_\wp \supset A_{\wp \cap A}$ for any prime ideal $\wp$ of B of height one by (*) and $B = B'$ by lemma 3.2).

Lemma 3.3. *Assume* $\dim A > 1$ *,* A *satisfies* $(*)$ *and has the property that its normalization is a finite* A*-module. Then we have*

(i) A' *is a finite* A*-module;*

(ii) $\operatorname{depth}(A') > 1$ *and* A' *is contained in any subdomain of the normalization which contains* A *and has depth greater than one;*

(iii) for a non-zero $f \in A$ *we have* $A[1/f]' = A'[1/f]$ *.*

Proof. As we have seen above, A' is contained in any subdomain of the normalization which contains A and has depth greater than 1. Since the normalization of A is a finite A-module, A' is also a finite A-module. To prove that $\operatorname{depth}(A') > 1$ we can argue as in the proof of lemma 3.2. For any non-zero non-invertible $f \in A'$ we have

$$fA' = \bigcap_{\operatorname{ht} \wp = 1} fA_\wp = \bigcap_{\operatorname{ht} \wp = 1} (fA_\wp \bigcap A'),$$

and the ideals $fA_\wp$ either coincide with $A_\wp$ or are $\wp$-primary in the rings $A_\wp$, since the rings $A_\wp$ are Noetherian local rings of dimension one. So, the ideals $fA_\wp \bigcap A'$ in A' either coincide with A' or are $\wp'$-primary in A', where $\wp'$ is the prime ideal $\wp A_\wp \cap A'$. Note that $\operatorname{ht}(\wp') = 1$. Indeed, $\wp' \cap A = \wp$ and $\operatorname{ht}(\wp) = 1$. If $\operatorname{ht} \wp' > 1$ then there is a prime ideal $\wp_1 \subset \wp'$ of height one such that $\wp_1 \cap A = \wp$ (since A' is integral over A, $\wp_1 \cap A \neq 0$). But then $\wp_1 = \wp'$ by [1, corol. 5.9]. Thus, there is a primary decomposition for fA' without embedded components and therefore there are non-invertible non-zero divisors in $A'_\wp / fA'_\wp$ for any $\wp \supset f$ of height one, because $\dim A > 1$ (cf. [1, prop. 4.7]). Hence $\operatorname{depth}(A') > 1$.

To prove (iii), first note that A' is the intersection of

1) all localisations with respect to all prime ideals of height 1 that don't contain f;

2) a finite number of localisations with respect to prime ideals of height 1 that contain f.

Finite intersections and localisations commute, and the localisation of rings in item 2) with respect to f is the quotient field $\operatorname{Quot} A$. So, we can omit all these components of the intersection. The rings in item 1) coincide with the localisations of $A[1/f]$ with respect to the same ideals. So, the equality $A'[1/f] = A[1/f]'$ follows. □

Remark 3.4. For 2-dimensional domains the property $\operatorname{depth}(A) > 1$ is equivalent with the property that A is a Cohen-Macaulay ring, see [13, ch. 7, § 17, (17.I)].

Now we have the following geometric interpretation of the facts above. Let X be a two-dimensional integral scheme which is of finite type over a field or over the integers. Let $P(X)$ be the subspace of all points of height 1 with the restricted structure sheaf. Then the direct image of the structure sheaf of $P(X)$ under the embedding of $P(X)$ into X is a coherent sheaf of algebras on X (since $P(X)(U) = A'$ for any affine $U = \operatorname{Spec} A$). Let $CM(X)$ be the relative spectrum over X of this algebra. Then, using lemma 3.3 one obtains:

(i) $CM(X)$ is an integral schema, finite and birational over X, and is a Cohen-Macaulay scheme.

(ii) Any integral X-scheme Y which is finite and birational over X and which is Cohen-Macaulay, admits a unique factorization over $CM(X)$.

We will also *call* the scheme $CM(X)$ as the *Cohen-Macaulaysation* of the scheme X.

3.3 Geometric ribbons

In this section we give a construction of a ribbon and a torsion free sheaf on it (see [9]) determined by the geometric data defined in 3.1. This construction generalizes the constructions of ribbons coming from geometric data given in [10].

We recall the general definition of ribbon from [9].

Let S be a Noetherian base scheme.

Definition 3.4 ([9])**.** A ribbon $(C,\mathcal{A})$ over S is given by the following data.

1. A flat family of reduced algebraic curves $\tau : C \to S$.
2. A sheaf $\mathcal{A}$ of commutative $\tau^{-1}\mathcal{O}_S$-algebras on C.
3. A descending sheaf filtration $(\mathcal{A}_i)_{i\in\mathbb{Z}}$ of $\mathcal{A}$ by $\tau^{-1}\mathcal{O}_S$-submodules which satisfies the following axioms:
   (a) $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$, $1\in\mathcal{A}_0$ (thus $\mathcal{A}_0$ is a subring, and for any $i\in\mathbb{Z}$ the sheaf $\mathcal{A}_i$ is a $\mathcal{A}_0$-submodule);
   (b) $\mathcal{A}_0/\mathcal{A}_1$ is the structure sheaf $\mathcal{O}_C$ of C;
   (c) for each i the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}$ (which is a $\mathcal{A}_0/\mathcal{A}_1$-module by (3a)) is a coherent sheaf on C, flat over S, and for any $s\in S$ the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}\mid_{C_s}$ has no coherent subsheaf with finite support, and is isomorphic to $\mathcal{O}_{C_S}$ on a dense open set;
   (d) $\mathcal{A} = \varinjlim_{i\in\mathbb{Z}}\mathcal{A}_i$, and $\mathcal{A}_i = \varprojlim_{j>0}\mathcal{A}_i/\mathcal{A}_{i+j}$ for each i.

Let $(X,C,P,\mathcal{F},\pi,\phi)$ be a geometric data from definition 3.1 and $\mathcal{F}$ is a coherent sheaf of rank r on X (r is not necessary equal to the rank of data).

Then we can define a ribbon $\overset{\circ}{X}_\infty = (C,\mathcal{A})$ and a torsion free sheaf $\mathcal{F}$ on it as follows. Recall that subspaces W, A in $k[[u]]((t))$ are defined by this data (see definition 3.3) and $X\simeq \mathrm{Proj}(\tilde{A})$, $\mathcal{F}\simeq \mathrm{Proj}(\tilde{W})$.

Let's define a family of torsion free coherent sheaves $\mathcal{A}_i = \mathrm{Proj}(\tilde{A}(i))$ on X, where $\tilde{A}(i)_k = \tilde{A}_{k+i}$. Then we have $\mathcal{A}_i\subset\mathcal{A}_{i+1}$. Let $X_\infty = (C,\hat{\mathcal{A}}_0)$ be a formal scheme, the formal completion of X along C. Since the functor $\mathcal{A}_i\mapsto\hat{\mathcal{A}}_i$ is an exact functor (see e.g. [7, Corol.9.8]), we have $\hat{\mathcal{A}}_i\subset\hat{\mathcal{A}}_{i+1}$ for all i. Obviously, for all i the sheaves $\mathcal{A}_{i+1}/\mathcal{A}_i\simeq\hat{\mathcal{A}}_{i+1}/\hat{\mathcal{A}}_i$ are torsion free on C and $\mathcal{A}_0/\mathcal{A}_{-1}\simeq\mathcal{O}_C$. The multiplication in the ring A induces a multiplication map $\mathcal{A}_i\mathcal{A}_j\subset\mathcal{A}_{i+j}$ and hence a multiplication map $\hat{\mathcal{A}}_i\hat{\mathcal{A}}_j\subset\hat{\mathcal{A}}_{i+j}$. So, the sheaf $\mathcal{A}=\varinjlim\hat{\mathcal{A}}_i$ defines a structure of a ribbon $(C,\mathcal{A})$ according to the definition 3.4. Analogously, let's define the sheaf $\mathcal{N}=\varinjlim\hat{\mathcal{F}}_i$, where $\mathcal{F}_i = \mathrm{Proj}(\tilde{W}(i))$. Obviously, $\mathcal{F}_i/\mathcal{F}_{i-1}$ are torsion free coherent sheaves for all i.

Let's show that $\mathcal{N}$ is a torsion free sheaf of rank r on the ribbon $(C,\mathcal{A})$ in the sense of [9, def. 11] and that the point P is smooth for the sheaf $\mathcal{N}$ in the sense of [9, def. 12]. Since P is a smooth point on C, we have to check that the map

$$(\widehat{\mathcal{F}_i/\mathcal{F}_{i-1}})_P\otimes_{\hat{\mathcal{O}}_{C,P}}(\widehat{\mathcal{A}_j/\mathcal{A}_{j-1}})_P\longrightarrow(\widehat{\mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}})_P$$

induced by the multiplication map $\mathcal{F}_i\cdot\mathcal{A}_j\subset\mathcal{F}_{i+j}$ is an isomorphism. This will follow from the fact that $(\mathcal{F}_i/\mathcal{F}_{i-1})_P\simeq\mathcal{O}_{C,P}^r$, $(\mathcal{A}_j/\mathcal{A}_{j-1})_P\simeq\mathcal{O}_{C,P}$. From the last facts it follows also that $\mathcal{N}$ is a torsion free sheaf of rank r (cf. corollary 3.1).

To prove the last fact let's note that it is enough to show it (without loss of generality) only for the sheaves $\mathcal{F}_i$ and only for $i\le 0$. Indeed, if $i>0$, then for $n>0$ sufficiently big such that $i-nd<0$ we have $\mathcal{F}_i\otimes_{\mathcal{O}_X}\mathcal{A}_{-nd}\simeq\mathcal{F}_{i-nd}$, since $\mathcal{A}_{-nd}$ is an invertible sheaf corresponding to the very ample Cartier divisor $-ndC$ (cf. [25, lemma 3.3]).

By corollary 3.1 we have $\mathcal{F}_{0,P} \simeq \mathcal{O}^r_{X,P}$ and therefore $(\mathcal{F}_0/\mathcal{F}_{-1})_P \simeq \mathcal{O}^r_{C,P}$. Now by induction on $-i$ we have for each $i<0$ the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}_{i-1,P} & \longrightarrow & \mathcal{F}_{i,P} & \longrightarrow & (\mathcal{F}_i/\mathcal{F}_{i-1})_P & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{A}^r_{-1,P} & \longrightarrow & \mathcal{O}^r_{X,P} & \longrightarrow & \mathcal{O}^r_{C,P} & \longrightarrow 0,
\end{array}$$

where $\mathcal{A}_{-1,P} \simeq \mathcal{O}_{X,P}$, because $\mathcal{A}_{-1}$ is the ideal sheaf of the cycle C and this sheaf is invertible in a neighbourhood of P, the second and the third vertical arrows are isomorphisms defined by induction, and the first arrow is the map induced by the second map on the kernels. Hence, it must be an isomorphism of $\mathcal{O}_{X,P}$-modules.

So, we obtain the following proposition:

Proposition 3.3. *Given a geometric data $q=(X,C,P,\mathcal{F},\pi,\phi)$ from definition 3.1 with a coherent sheaf of rank r, a geometric data consisting of a ribbon $(C,\mathcal{A})$ over a field k, a torsion free sheaf $\mathcal{N}$ of rank r, a smooth k-point P of the sheaf $\mathcal{N}$, formal local parameters u',t' and a trivialization $e_P: \hat{\mathcal{N}}_{0,P} \to \hat{\mathcal{A}}^r_{0,P} \simeq k[[u',t']]^r$ is defined (cf. [9, def.14]). So, we have a map*

$$\Phi : q \mapsto \tilde{q}.$$

The construction of the data $\tilde{q}$ generalizes the construction of a geometric ribbon given in [9, ex. 1]. The data $\tilde{q}$ satisfies conditions of [9, th.1].

Note that if we start with a data $q'=(CM(X),C,P,\mathcal{F},\pi,\phi)$, where $CM(X)$ is a Cohen-Macaulaysation of X (see section 3.2), then the data $\tilde{q'}=\tilde{q}$ (as it follows from the construction of the data, theorem 3.2 and from results of section 3.2). Remarkably, the following is true:

Proposition 3.4. *If $q,q' \in \mathcal{Q}_r$ and surfaces X,X' are Cohen-Macaulay, and the data $(C,\mathcal{A},P,u,t)$, $(C',\mathcal{A}',P',u',t')$ constructed by the map Φ are isomorphic (cf. [9, def.14]), then X is isomorphic to X'.*

Proof. The idea of the proof is to apply arguments from [17, th.5,6] to the data $(X,dC,\tilde{P},\mathcal{O}_X)$, $(X',d'C',\tilde{P}',\mathcal{O}_{X'})$, where dC, $d'C'$ are ample Cartier divisors and $\tilde{P}$, $\tilde{P}'$ are ample Cartier divisors on dC, $d'C'$ induced by Cartier divisors P, P' on C, C' and local parameters u, u' (cf. [9, lemma 5]). Since the ribbon data are isomorphic, their images under the generalized Krichever map coincide (see [9, th.1]). In this situation the algebras $A_{(0)}(\mathcal{O}_X)$, $A_{(0)}(\mathcal{O}_{X'})$ coincide (see the proof of [17, th.6]), therefore the surfaces X,X' defined by these algebras will be isomorphic. □

3.4 Glueing construction

As in the one-dimensional case one gets interesting solutions of the KP-equation by algebraic curves which are obtained from glueing points (to get cuspidal or nodal curves, with non-trivial Jacobians and compactified Jacobians, see [14, 19]), we hope that a similar construction for surfaces, starting from $\mathbb{P}^2$ or rational or other surfaces yields examples in our case.

We need a construction where we glue curves on a surface, or glue points of a curve on a surface. The problem is: given a projective surface $\tilde{X}$, a one-dimensional closed subscheme Y and a surjective finite morphism $p: Y \to C$ to a curve C, find a surface X with a curve $C \subset X$ and a morphism $n: \tilde{X} \to X$ such that $n(Y)=C$ (and $n|_Y$ is a given morphism p) and $n: \tilde{X}\backslash Y \simeq X\backslash C$ is an isomorphism.

In the work [5] is given a construction of glueing closed subschemes on a given scheme. It can be done for many schemes with mild conditions, and the construction is a natural generalization of the construction for curves given in [20]:

Theorem 3.3. *([5, th.5.4]) Let X' be a scheme, Y' be a closed subscheme in X' and $g: Y' \to Y$ be a finite morphism. Consider the ringed space $X = X' \sqcup_{Y'} Y$ (the amalgam) and the cocartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \Big\downarrow & & \Big\downarrow u \\ X' & \xrightarrow{f} & X \end{array}$$

Suppose that the schemes X' and Y satisfy the following property:

(AF) *All finite sets of points are contained in an open affine set.*

Then:
a) X is a scheme satisfying (AF);
b) the square above is cartesian;
c) the morphism f is finite and u is a closed embedding;
d) f induces an isomorphism of $X' - Y'$ on $X - Y$.

Often the schemes after the glueing procedure are proper but not projective (see [5, § 6] and also [5, prop. 5.6]). Here we rewrite the construction from [5] for surfaces with some extra conditions in another, but equivalent way.

We will need some more assumptions. The idea is to construct X first as a topological space by taking the quotient space with respect to the equivalence relation $\triangle_X \cup (id \times p)^{-1}(\Gamma_p)$ in $X \times X$ (here $\Gamma_p \subset Y \times C$ is a graph of p). Then we have a topological quotient map $\tilde{X} \to X$, but it is not obvious how to make X to a scheme.

For this we make the assumption that p extends to a morphism $\tilde{p}: \tilde{X}_0 \to C$, where $\tilde{X}_0 \subset \tilde{X}$ is a (Zariski) open neighbourhood of Y.

Without loss of generality we may assume that $\tilde{X}_0 = \tilde{X}$: by replacing $\tilde{X}$ by the closure of the graph of $\tilde{p}$ in $\tilde{X} \times C$ we can modify $\tilde{X}$ to $\tilde{\tilde{X}} \to \tilde{X}$ such that $\tilde{p}\sigma$ extends to a morphism of $\tilde{\tilde{X}}$ to C, this modification is outside of $\tilde{X}_0$ and after the glueing construction one can reverse this modification.

Then, making this topological construction as above, we get a factorization of the underlying continue map of $\tilde{p}$ as

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{p}} & C \\ \downarrow n & \nearrow & \\ X & & \end{array}$$

and one should define $\mathcal{O}_X = p^{-1}\mathcal{O}_C + n_* I_Y$ ($I_Y \subset \mathcal{O}_{\tilde{X}}$ ideal sheaf of Y).

Description with affine covering: for each $c \in C$ there exists an affine neighbourhood $V \subset C$ and an affine neighbourhood $\tilde{U}$ of $p^{-1}(c)$ in $\tilde{X}$ such that $\tilde{U} \cap Y = \tilde{p}^{-1}(V) \cap Y$ (first: take any affine neighbourhood U of $p^{-1}(c)$, then $Y \backslash U$ is closed and $F = p(Y \backslash U)$ is closed in C and disjoint to c. Then we take $V \subset C \backslash F$ an affine neighbourhood of c, and $U = \tilde{U} \cap \tilde{p}^{-1}(V)$). Then the diagram of maps

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & V \\ \cup & \nearrow & \\ \tilde{U} \cap Y & & \end{array}$$

corresponds to homomorphisms of coordinate rings

$$\begin{array}{ccc} \tilde{A} & \longleftarrow & R \\ \downarrow & \swarrow & \\ \tilde{A}/I & & \end{array}$$

and we define $A = R + I \subset \tilde{A}$. We have to prove that A is a finitely generated R-algebra (and $\mathrm{Spec}(A) \subset X$ as a topological space).

Lemma 3.4. *Let $R \subset \tilde{A}$ be finitely generated domains over a field k of Krull dimension 1 and 2, I an ideal in $\tilde{A}$ such that $R \subset \tilde{A}/I$ is finite. Then $A = R + I \subset \tilde{A}$ is finitely generated and $\tilde{A}$ is finite over A.*

Proof. We can find $f_1, f_2 \in \tilde{A}$ such that
1) $\tilde{A}$ is finite over $E = k[f_1, f_2]$;
2) $f_1 \in I$ (see [2, ch.V, §3, th.1]).

Since $\tilde{A}/I$ is finite over R, there exist $a_1, \ldots, a_k \in A$ such that $f_2^k + a_1 f_2^{k-1} + \ldots + a_k = b \in I$; hence if $A_0 = k[f_1, a_1, \ldots, a_k, b]$ then $A_0 \subset R + I$ and $\tilde{A}$ is finite over A_0 ($A_0 \subset A_0[f_2] \subset \tilde{A}$), therefore $A = R + I$ is finitely generated over k. □

So, $\tilde{A}$ is a (partial) normalization and I is the conductor from A to $\tilde{A}$. One checks easily that
1) $\tilde{X} \xrightarrow{n} X$ is universally closed;
2) X is separated:

$$\begin{array}{ccccc} \tilde{X} \times \tilde{X} & \stackrel{(1)}{\longrightarrow} & \tilde{X} \times X & \stackrel{(2)}{\longrightarrow} & X \times X \\ \cup & & \cup & & \cup \\ \tilde{\triangle} & \longrightarrow & \Gamma_n & \longrightarrow & \triangle \end{array}$$

Since (1) and (2) are closed maps and Γ_n, $\triangle$ are locally closed, then both maps are closed embeddings.

So, the map n is proper, and X is a separated scheme of finite type over k. Since n is also finite surjective and $\tilde{X}$ is proper over k, it follows that the structure map $X \to \mathrm{Spec}\, k$ is universally closed, hence X is a proper over k scheme.

Having this construction and general theorem 3.3 we can give examples:

Example 3.1. Consider $\tilde{X} = \mathbb{P}^2$ with a morphism $p : \tilde{X} - (0:0:1) \to C = \mathbb{P}^1$, $p(x:y:z) = (x:y)$ and two lines $Y = 2\mathbb{P}^1$, where $\mathbb{P}^1 = (x:y:0)$ in $\mathbb{P}^2$. Clearly, Y is an ample Cartier divisor on $\tilde{X}$.

In this case a Cartier divisor $\tilde{C}$ on the glued surface X is defined which is given by the same equations in the induced local covering (so, $Y = n^*\tilde{C}$). Since Y is an ample divisor and n is a finite surjective morphism of proper schemes, the divisor $\tilde{C}$ is also ample (see [7, ex.5.7.,ch.3]). Therefore, X is projective, and the cycle map from section 4 gives $Z(\tilde{C}) = 2C$. So, C is an ample $\mathbb{Q}$-Cartier divisor on X.

Example 3.2. Let X' be a projective surface over k, let Y' be a finite number of closed points on X'. Let $Y = \mathrm{Spec}\, k$ be a point and $g : Y' \to Y$ a finite morphism. Then the scheme X constructed as in theorem 3.3 is proper over k (see e.g. [5, 6.1]) and is projective by the same arguments as in the previous example.

4 Appendix: cycle map

For reader's convenience, we'll give below some facts about the cycle map for singular projective surfaces (cf. [6, §2.1]).

Let X be a projective irreducible n-dimensional variety over a field k. Let $\mathrm{Div}(X)$ be a group of Cartier divisors (equal to $H^0(X,\ k(X)^*/\mathcal{O}_X^*)$). Thus, a divisor D is given by data (U_α, f_α), where $f_\alpha \in k(X)^*$, $\{U_\alpha\}$ is an open covering of X, and $f_\alpha / f_\beta \in \mathcal{O}_X^*(U_{\alpha\beta})$.

If $C \subset X$ is an irreducible closed subvariety of codimension 1, define the order

$$\mathrm{ord}_C(D) = l_{\mathcal{O}_{X,C}}\left(\frac{1}{f_\alpha}\mathcal{O}_{X,C}/\mathcal{O}_{X,C} \cap \frac{1}{f_\alpha}\mathcal{O}_{X,C}\right) - l_{\mathcal{O}_{X,C}}\left(\mathcal{O}_{X,C}/\mathcal{O}_{X,C} \cap \frac{1}{f_\alpha}\mathcal{O}_{X,C}\right),$$

where $l_{\mathcal{O}_{X,C}}$ denotes the length of a $\mathcal{O}_{X,C}$-module, and α is chosen so that C meet U_α (the choice is irrelevant for the definition of $\mathrm{ord}_C(D)$).

Set the cycle corresponding to a divisor $D \in \mathrm{Div}(X)$ as

$$\mathrm{Z}(D) = \sum_{\mathrm{codim}\, C=1} \mathrm{ord}_C(D)\, C.$$

So, we have a homomorphism $\mathrm{Z} : \mathrm{Div}(X) \to Z^1(X)$ to the group of cycles of codimension one on X (which is the free abelian group generated by all irreducible subvarieties of codimension 1 on X). The group $Z^1(X)$ is also called the group of Weil divisors on X and is denoted also by $\mathrm{WDiv}(X)$. Let's compute the kernel of the homomorphism Z.

If R is a one-dimensional local domain and $g \in \mathrm{Quot}(R)$, then we have to compute

$$L_g = l_R(R/R \cap gR) - l_R(gR/R \cap gR),$$

which can be also expressed as follows. If we choose an element $a \in R$ such that $ag = b \in R$, then

$$\begin{aligned} L_g &= l_R(aR/aR \cap bR) - l_R(bR/aR \cap bR) = \\ &= l_R(R/aR \cap bR) - l_R(R/aR) - (l_R(R/aR \cap bR) - l_R(R/bR)) = l_R(R/bR) - l_R(R/aR). \end{aligned}$$

One checks that if $R \subset \tilde{R} \subset \mathrm{Quot}(R)$, and if $\tilde{R}$ is a finitely generated R-module, then

$$L_g = l_R(R/bR) - l_R(R/aR) = l_R(\tilde{R}/b\tilde{R}) - l_R(\tilde{R}/a\tilde{R}).$$

If $\tilde{R}$ is the integral closure of R, then one sees that $L_g = 0$ if and only if $b/a = g \in \tilde{R}^*$. Thus

$$\mathrm{Ker}(\mathrm{Div}(X) \xrightarrow{\mathrm{Z}} Z^1(X)) = H^0(X, \pi_* \mathcal{O}^*_{\tilde{X}}/\mathcal{O}^*_X) \simeq \mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(\tilde{X})),$$

where $\tilde{X} \xrightarrow{\pi} X$ is the normalization of X. (We used an exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{O}^*_X \longrightarrow \pi_* \mathcal{O}^*_{\tilde{X}} \longrightarrow \pi_* \mathcal{O}^*_{\tilde{X}}/\mathcal{O}^*_X \longrightarrow 0,$$

and the fact from [13, Th. 38]: if A is an integrally closed Noetherian domain, then $A = \bigcap\limits_{\mathrm{ht}\, \nu=1} A_\nu$, where the intersection is taken over all prime ideals of height one.)

One can define the semigroup of effective Cartier divisors $\mathrm{Div}^+(X) \subset \mathrm{Div}(X)$ (given by data (U_α, f_α), where $f_\alpha \in \mathcal{O}_X(U_\alpha) \cap k(X)^*$), and the semigroup of effective Weil divisors $\mathrm{WDiv}^+(\mathrm{X}) \subset \mathrm{WDiv}(\mathrm{X})$ (given by the formal finite sums of cycles of codimension one with positive integer coefficients). It is easy to see that $\mathrm{Z}(\mathrm{Div}^+(X)) \subset \mathrm{WDiv}^+(\mathrm{X})$. Moreover, from the above description of the map Z it follows that $\mathrm{Z}\,|_{\mathrm{Div}^+(X)}$ is an injective map.

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