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Spatial-Skin Effect for a Spectral Problem with "Slightly Heavy" Concentrated Masses in a Thick Cascade Junction

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Spatial-Skin Effect for a Spectral Problem with “Slightly Heavy” Concentrated Masses in a Thick Cascade Junction

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Abstract

The asymptotic behavior (as $\varepsilon \to 0$) of eigenvalues and eigenfunctions of a boundary-value problem for the Laplace operator in a thick cascade junction with concentrated masses is studied. This cascade junction consists of the junction’s body and a great number $5N = \mathcal{O}(\varepsilon^{-1})$ of $\varepsilon-$alternating thin rods belonging to two classes. One class consists of rods of finite length and the second one consists of rods of small length of order $\mathcal{O}(\varepsilon)$. The density of the junction is of order $\mathcal{O}(\varepsilon^{-\alpha})$ on the rods from the second class and $\mathcal{O}(1)$ outside of them. There exist five qualitatively different cases in the asymptotic behavior of eigenvibrations as $\varepsilon \to 0$, namely the case of “light” concentrated masses ($\alpha \in (0,1)$), “middle” concentrated masses ($\alpha = 1$), “slightly heavy” concentrated masses ($\alpha \in (1,2)$), “intermediate heavy” concentrated masses ($\alpha = 2$), and “very heavy” concentrated masses ($\alpha > 2$).

In the paper we study the influence of the concentrated masses on the asymptotic behavior of the eigen-magnitudes if $\alpha \in (1,2)$.

Mathematical Subject Classification: 35B27, 35B25, 47A75, 74K30

Contents

1 Introduction

In present paper we continue our investigation of a spectral problem with concentrated masses in a new kind of thick junctions, namely *thick cascade junctions*, which we have begun in [1, 2].

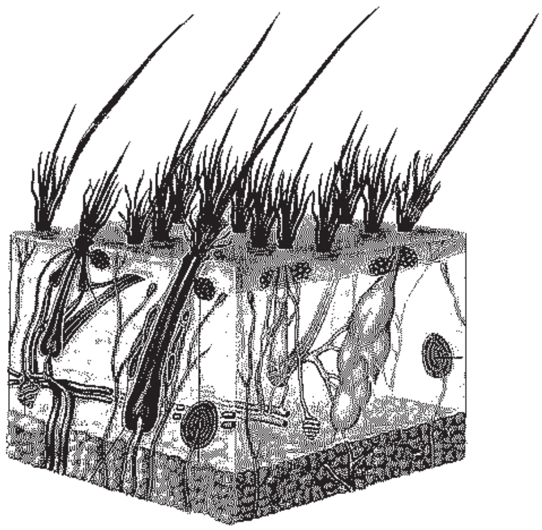

Figure 1: Skin with hairs of two classes.

Thick cascade junctions are prototypes of widely used nanotechnological, microtechnical, modern engineering constructions (microstrip radiator, ferrite-filled rod radiator), as well as many physical and biological systems with very distinct characteristic scales, for instance construction of a bowel with different levels of absorption on various parts of the bowel trunks, structure of an animal's fell consisting of hair and undercoat (down hair) with different thermal conductivities (see Figure 1).

Vibration systems with a concentration of masses on a small set of diameter $\mathcal{O}(\varepsilon)$ have been studied for a long time. It was experimentally established that such concentration leads to the big reduction of the main frequency and to the big localization of vibrations. The new impulse in this research was given by E. Sánchez-Palencia in the paper [3], in which the effect of local vibrations was mathematically described. After this paper, many articles appeared. The reader can find widely presented bibliography on spectral problems with concentrated masses and problems in thick junctions in [1, 2].

In the present paper the case of "slightly heavy" concentrated masses is considered, for which we discover a new spatial–skin effect for eigenvibrations of thick cascade junctions.

1.1 Statement of the problem

Let $a,\ b_1,\ b_2,\ h_1,\ h_2$ be positive numbers such that

$$0 < b_1 < b_2 < \frac{1}{2}, \quad 0 < b_1 - \frac{h_1}{2}, \quad b_1 + \frac{h_1}{2} < b_2 - \frac{h_1}{2}, \quad b_2 + \frac{h_1}{2} < \frac{1}{2} - \frac{h_2}{2}.$$

These inequalities mean that the intervals

$$\left(b_1 - \frac{h_1}{2}\,,\, b_1 + \frac{h_1}{2}\right), \quad \left(b_2 - \frac{h_1}{2}\,,\, b_2 + \frac{h_1}{2}\right), \quad \left(\frac{1-h_2}{2}\,,\, \frac{1+h_2}{2}\right),$$
$$\left(1 - b_2 - \frac{h_1}{2}\,,\, 1 - b_2 + \frac{h_1}{2}\right), \quad \left(1 - b_1 - \frac{h_1}{2}\,,\, 1 - b_1 + \frac{h_1}{2}\right)$$

are not intersected and they belong to $(0,1)$. Let us divide the segment $[0,a]$ into N equal segments $[\varepsilon j, \varepsilon(j+1)],\ j = 0,\ldots,N-1$. Here N is a big positive integer, hence the value $\varepsilon = a/N$ is a small discrete parameter.

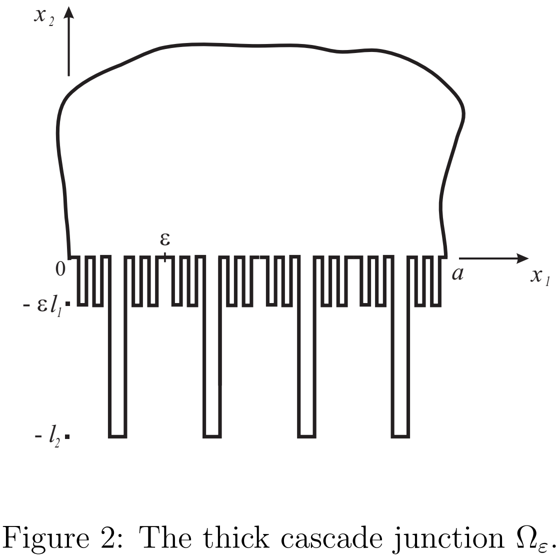

Figure 2: The thick cascade junction Ω_ε.

A model *thick cascade junction* Ω_ε (see Fig. 2) consists of the junction's body

$$\Omega_0 = \{x \in \mathbb{R}^2 :\ 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1)\,\},$$

where $\gamma \in C^1([0,a])$, $\min_{[0,a]}\gamma =: \gamma_0 > 0$, and a large number of thin rods

$$G_j^{(1)}(d_k,\varepsilon) = \left\{ x\in\mathbb{R}^2 : \ |x_1 - \varepsilon\,(j+d_k)| < \frac{\varepsilon h_1}{2}, \quad x_2 \in (-\varepsilon\, l_1, 0] \right\}, \quad k=1,\ldots,4,$$

$$G_j^{(2)}(\varepsilon) = \left\{ x\in\mathbb{R}^2 : \ |x_1 - \varepsilon\,(j+\frac{1}{2})| < \frac{\varepsilon h_2}{2}, \quad x_2 \in (-l_2, 0] \right\}, \quad j=0,1,\ldots,N-1,$$

where $d_1=b_1$, $d_2=b_2$, $d_3=1-b_2$, $d_4=1-b_1$, that is $\Omega_\varepsilon = \Omega_0\cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where

$$G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1}\Big(\bigcup_{k=1}^{4} G_j^{(1)}(d_k,\varepsilon)\Big), \qquad G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(\varepsilon).$$

Thus the number of the thin rods is equal to $5N$; the thin rods are divided into two classes $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ subject to their length and thickness. The length and thickness of the rods from the first class are equal to εl_1 and εh_1 respectively, and these magnitudes are equal to l_2 and εh_2 for the rods from the second class. In addition, the thin rods from each classes are ε-periodically alternated along the segment $I_0 = \{x : x_1\in[0,a], \quad x_2 = 0\}$.

Only vibrations of Ω_ε depending on time by the factor $\exp(-i\sqrt{\lambda}\ t)$ will be considered. Hence we have to investigate the corresponding spectral problem

$$\left\{\begin{array}{rcll} -\Delta_x\ u(\varepsilon,x) &=& \lambda(\varepsilon)\ \rho_\varepsilon(x)u(\varepsilon,x), & x\in\Omega_\varepsilon;\\ -\partial_\nu u(\varepsilon,x) &=& 0, & x\in\Upsilon_\varepsilon^{(1)}\cup\Upsilon_\varepsilon^{(2)}\cup\Gamma_\varepsilon;\\ u(\varepsilon,x) &=& 0, & x\in\Gamma_1;\\ [u]_{|x_2=0} &=& [\partial_{x_2}u]_{|x_2=0} = 0, & x_1\in Q_\varepsilon. \end{array}\right. \tag{1.1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; the brackets denote the jump of the enclosed quantities; $\Upsilon_\varepsilon^{(i)}$ is the union of the lateral sides and the lower bases of the rods from the $i-$th class, $i=1,2$; $\Gamma_1 = \{x : \ x_2 = \gamma(x_1), \ x_1\in[0,a]\}$; $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \big(\Upsilon_\varepsilon^{(1)}\cup\Upsilon_\varepsilon^{(2)}\cup\Gamma_1\big)$; the density

$$\rho_\varepsilon(x) = \left\{\begin{array}{ll} 1, & x\in\Omega_0\cup G_\varepsilon^{(2)},\\ \varepsilon^{-\alpha}, & x\in G_\varepsilon^{(1)}; \end{array}\right.$$

the parameter $\alpha\in\mathbb{R}$ (if $\alpha>0$, then concentrated masses are presented on the thin rods from the first class $G_\varepsilon^{(1)}$); $Q_\varepsilon = Q_\varepsilon^{(1)}\cup Q_\varepsilon^{(2)}$, $Q_\varepsilon^{(i)} = G_\varepsilon^{(i)}\cap\{x : x_2=0\}$, $i=1,2$.

Obviously, that for each fixed value of ε there is a sequence of eigenvalues

$$0<\lambda_1(\varepsilon)<\lambda_2(\varepsilon)\le\ldots\le\lambda_n(\varepsilon)\le\cdots\to+\infty \quad (\text{as} \quad n\to\infty) \tag{1.2}$$

of problem (1.1). The corresponding eigenfunctions $\{u_n(\varepsilon,\cdot) : \ n\in\mathbb{N}\}$, which belong to $\mathcal{H}_\varepsilon$, can be orthonormalized as follows

$$(u_n,u_m)_{L^2(\Omega_0\cup G_\varepsilon^{(2)})} + \varepsilon^{-\alpha}(u_n,u_m)_{L^2(G_\varepsilon^{(1)})} = \delta_{n,m}, \quad \{n,\,m\}\in\mathbb{N}. \tag{1.3}$$

Here and below $\delta_{n,m}$ is the Kronecker delta, $\mathcal{H}_\varepsilon$ is the Sobolev space $\{u\in H^1(\Omega_\varepsilon) : u|_{\Gamma_1} = 0$ *in sense of the trace*$\}$ with the scalar product

$$(u,v)_{\mathcal{H}_\varepsilon} := \int_{\Omega_\varepsilon} \nabla u\cdot\nabla v\,dx \quad \forall\ u,v\in\mathcal{H}_\varepsilon.$$

Our aim is to study the asymptotic behavior of the eigenvalues $\{\lambda_n(\varepsilon) : n \in \mathbb{N}\}$ and the eigenfunctions $\{u_n(\varepsilon, \cdot) : n \in \mathbb{N}\}$ as $\varepsilon \to 0$, i.e., when the number of the attached thin rods from each class infinitely increases and their thickness decreases to zero.

It should be noted that the limit process is accompanied by the concentrated masses on the rods from the first class. In fact, we have two kinds of perturbations for problem (1.1): the domain perturbation and the density perturbation. We are going to study the influence of both these factors on the asymptotic behavior of the eigenvalues and eigenfunctions as well.

1.2 The outline of results

We establish five qualitatively different cases in the asymptotic behavior of eigenvalues and eigenfunctions of problem (1.1) as $\varepsilon \to 0$, namely the case of "light" concentrated masses ($\alpha \in (0,1)$), "middle" concentrated masses ($\alpha = 1$), "slightly heavy" concentrated masses ($\alpha \in (1,2)$), "intermediate heavy" concentrated masses ($\alpha = 2$), and "very heavy" concentrated masses ($\alpha > 2$). In the present paper we consider the cases $\alpha \in (1,2)$.

In [1, 2] we studied the cases of "light" and "middle" concentrated masses. In these cases the perturbation of domain plays the leading role in the asymptotic behavior.

If $\alpha \in (0,1)$, then the spectrum of the homogenized problem coincides with the spectrum of the problem in domain without concentrated masses (see for instance the papers by T.A. Mel'nyk and S.A. Nazarov [4, 5, 6], where it was discovered a remarkable peculiarity in the geometric structure of the spectrum (the presence of *lacunas*)). The concentrated masses bring the influence only from the second term of the asymptotic expansion, in particular the asymptotic ansatz for an eigenvalue is as follows

$$\lambda(\varepsilon) \approx \lambda_0 + \varepsilon^{1-\alpha}\lambda_{1-\alpha} + \varepsilon\lambda_1 + \varepsilon^{2-\alpha}\lambda_{2-\alpha} + \dots \tag{1.4}$$

The concentrated masses are revealed in the corresponding homogenized spectral problem in the case $\alpha = 1$. This influence appears through the term $4h_1 l_1 \lambda_0\, v_0^+(x_1, 0)$ with the spectral parameter λ_0 in the jump of the derivatives in the joint zone, i.e.

$$\partial_{x_2} v_0^+(x_1, 0) - h_2 \partial_{x_2} v_0^-(x_1, 0) = -4h_1 l_1 \lambda_0\, v_0^+(x_1, 0), \quad x_1 \in (0, a). \tag{1.5}$$

This term shows also the influence of the geometrical structure of the thin rectangles from the first class on the asymptotics. In this case the asymptotic ansatz for an eigenvalue has the form

$$\lambda(\varepsilon) \approx \lambda_0 + \varepsilon\lambda_1 + \dots \tag{1.6}$$

It turned out that we cannot directly substitute $\alpha = 1$ in (1.4) to obtain ansatz (1.6).

In [1, 2] we proved the Hausdorff, low- and high-frequency convergences of the spectrum of problem (1.1) to the spectrum of the corresponding homogenized problem as $\varepsilon \to 0$, in both cases $\alpha \in (0,1)$ and $\alpha = 1$. Also we deduced the corresponding asymptotic estimates both for the eigenfunctions and eigenvalues. In addition, as in paper [7], we discovered *pseudovibrations* for problem (1.1), having quickly oscillating character, and in which different rods of the junction vibrate individually, i.e. each rod has its own frequency.

Here we consider the case $\alpha \in (1,2)$. In this case the concentrated masses begin to play the leading role in the asymptotic behavior of the eigenvalues and the eigenfunctions. The principal differences between this and previous cases are the following: all eigenvalues $\{\lambda_n(\varepsilon)\}$ converge to zero with the rate $\varepsilon^{\alpha-1}$ as ε tends to zero and the asymptotic ansatz for an eigenvalue is the following:

$$\lambda(\varepsilon) \approx \varepsilon^{\alpha-1}\lambda_0 + \varepsilon^{2\alpha-2}\lambda_{\alpha-1} + \dots; \tag{1.7}$$

the amplitude of the corresponding eigenvibrations is of order $\mathcal{O}(\varepsilon^{\frac{\alpha-1}{2}})$ (see Lemma 2.1) and these vibrations have a new type of the skin effect which we call *spatial-skin effect*. It means that the vibrations of thin rods from the second class repeat the shape of the vibrations of the joint zone (see Figure 3).

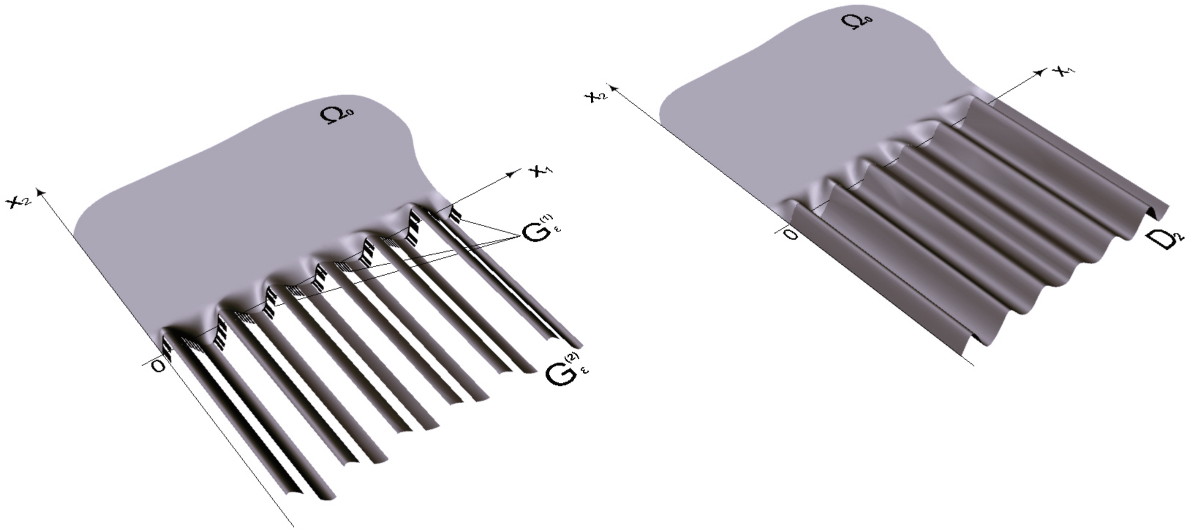

Figure 3: The initial and homogenized eigenvibrations with spatial-skin effect.

As far as we know, for the first time the skin effect for systems with many concentrated masses was discovered in [8] (we discuss this paper in more detail in Conclusions).

The structure of the present paper is the following. In Section 2 we construct the leading terms of the asymptotics both for eigenfunctions and eigenvalues. It should be noted that the asymptotics for an eigenfunction consists of three parts, namely the outer asymptotic expansion in the junction's body, the outer asymptotic expansions in each thin rectangles from the second class and the inner asymptotic expansion in the joint zone of the body and thin rectangles of both classes. Then in Subsection 2.3, using the method of matching of asymptotic expansions (see [9]), we derive the corresponding homogenized spectral problem and correctors for the eigenfunctions. In Subsection 2.4 we find residuals of the approximations of eigen-elements and estimate them. Sections 3 is devoted to the justification of the asymptotics. In Conclusions we discuss several generalizations and applications of results obtained in this paper.

2 Construction of the asymptotics

2.1 Asymptotic Estimates for the Eigenvalues and Eigenfunctions

Using the approach of Lemma 1 from the paper [10], we prove the following lemma.

Lemma 2.1. *For any fixed $n \in \mathbb{N}$ there exist constants C_0, C_1 and ε_0 such that for all value of ε from the interval $(0, \varepsilon_0)$ the following estimates hold*

$$0 < \lambda_n(\varepsilon) \le C_0\, \varepsilon^{\alpha-1}, \tag{2.1}$$

$$\|u_n\|_{\mathcal{H}_\varepsilon} \le C_1\, \varepsilon^{\frac{\alpha-1}{2}}. \tag{2.2}$$

In addition, there is a positive constant c_0 (depending neither on ε nor n) such that for all $n \in \mathbb{N}$ and ε

$$0 < c_0\, \varepsilon^{\alpha-1} \le \lambda_n(\varepsilon). \tag{2.3}$$

Proof. Let $\mathcal{L}_n(\widetilde{\phi}_1, \dots, \widetilde{\phi}_n)$ be the n-dimensional subspace of $\mathcal{H}_\varepsilon$, which is spanned on n linearly independent functions

$$\widetilde{\phi}_k = \begin{cases} \phi_k(x), & x \in \Omega_0 \\ \phi_k(x_1, 0), & x \in G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}, \end{cases} \quad k = 1, \dots, n,$$

where $\phi_1, \dots, \phi_n$ are eigenfunctions of the following Steklov spectral problem

$$\begin{cases} \Delta_x\, \phi(x) = 0, & x \in \Omega_0 \\ \partial_\nu\, \phi(x) = 0, & x \in \Gamma_2, \\ \phi(x) = 0, & x \in \Gamma_1, \\ \partial_{x_2} \phi(x_1, 0) = -\mu\, \phi(x_1, 0), & x_1 \in (0, a), \end{cases}$$

which are orthonormalized in $L_2(I_0)$.

By virtue of the minimax principle for eigenvalues, we have

$$\begin{aligned} \lambda_n(\varepsilon) &= \min_{E \in \mathbf{E}_n} \max_{v \in E \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\int_{G_\varepsilon^{(2)}} v^2\, dx + \varepsilon^{-\alpha} \int_{G_\varepsilon^{(1)}} v^2\, dx} \le \varepsilon^\alpha \min_{E \in \mathbf{E}_n} \max_{v \in E \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\int_{G_\varepsilon^{(1)}} v^2\, dx} \\ &\le \varepsilon^\alpha \max_{0 \ne v \in \mathcal{L}_n} \frac{l_2 \int_{I_0} |\partial_{x_1} v(x_1, 0)|^2\, dx_1 + \mu_n \int_{I_0} v^2(x_1, 0)\, dx_1}{\varepsilon\, l_1 \int_{Q_\varepsilon^{(1)}} v^2(x_1, 0)\, dx_1} \\ &\le \varepsilon^{\alpha-1} \mu_n\, c_2 \max_{0 \ne v \in \mathcal{L}_n} \frac{\int_{I_0} v^2(x_1, 0)\, dx_1}{\int_{Q_\varepsilon^{(1)}} v^2(x_1, 0)\, dx_1}. \end{aligned} \tag{2.4}$$

Here $\mathbf{E_n}$ is a set of all subspaces of $\mathcal{H}_\varepsilon$ with dimension n, $Q_\varepsilon^{(1)} = G_\varepsilon^{(1)} \cap \{x_2 = 0\}$.

Now let us prove that for ε small enough the following inequality holds

$$\max_{0 \ne v \in \mathcal{L}_n} \frac{\int_{I_0} v^2(x_1, 0)\, dx_1}{\int_{Q_\varepsilon^{(1)}} v^2(x_1, 0)\, dx_1} \le c_3.$$

Assuming the contrary, we can find a subsequence $\{\varepsilon_m\}$ of the sequence $\{\varepsilon\}$ and a sequence $\{v_{\varepsilon_m}\} \in \mathcal{L}_n$ such that $\|v_{\varepsilon_m}\|_{L_2(I_0)} = 1$ and

$$\lim_{\varepsilon_m \to 0} \int\limits_{Q_{\varepsilon_m}^{(1)}} v_{\varepsilon_m}^2(x_1, 0)\, dx_1 = 0. \tag{2.5}$$

Since the subspace $\mathcal{L}_n$ is finite and the sequence $\{v_{\varepsilon_m}\}$ is bounded in $H^1(\Omega_0)$, we can regard that $\{v_{\varepsilon_m}\}$ converges in $H^1(\Omega_0)$ to some element $v_0 \in H^1(\Omega_0) \cap \mathcal{L}_n,\ \|v_0\|_{L_2(I_0)} = 1$. Then

$$\lim_{\varepsilon_m \to 0} \int\limits_{Q^{(1)}_{\varepsilon_m}} v^2_{\varepsilon_m}(x_1, 0)\, dx_1 = \lim_{\varepsilon_m \to 0} \int\limits_{I_0} \chi_{Q^{(1)}_{\varepsilon_m}}(x_1)\, v^2_{\varepsilon_m}(x_1, 0)\, dx_1 = 4h_1 \int\limits_{I_0} v^2_0(x_1, 0)\, dx_1 = 4h_1,$$

but this is a contradiction to (2.5). Here $\chi_{Q^{(1)}_\varepsilon}$ is the characteristic function of the set $Q^{(1)}_\varepsilon$. Thus the estimate (2.1) holds.

Then, it follows from (1.1), (1.3) and (2.1) that

$$\|u_n(\varepsilon, \cdot)\|^2_{\mathcal{H}_\varepsilon} = \lambda_n(\varepsilon) \le C_n\, \varepsilon^{\alpha-1}.$$

Using the approach from the paper [11] (see also [12]), it is easy to prove the following Friedrichs–type inequality:

$$\varepsilon^{-1} \int\limits_{G^{(1)}_\varepsilon} v^2\, dx \le C_2 \int\limits_{\Omega_\varepsilon} |\nabla v|^2\, dx \quad \forall\, v \in \mathcal{H}_\varepsilon. \tag{2.6}$$

Due to the normalization condition (1.3) we have $\lim\limits_{\varepsilon \to 0} \varepsilon^{\alpha-1} \int_{G^{(2)}_\varepsilon} u^2_1(\varepsilon, x)\, dx = 0$. Taking into account this limit and (2.6), we deduce the following:

$$\begin{aligned}
\lambda_n(\varepsilon) \ge \lambda_1(\varepsilon) &= \min_{v \in \mathcal{H}_\varepsilon \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\int_{G^{(2)}_\varepsilon} v^2\, dx + \varepsilon^{-\alpha} \int_{G^{(1)}_\varepsilon} v^2\, dx} \\
&\ge \varepsilon^{\alpha-1} \min_{v \in \mathcal{H}_\varepsilon \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\varepsilon^{\alpha-1} \int_{G^{(2)}_\varepsilon} v^2\, dx + \varepsilon^{-1} \int_{G^{(1)}_\varepsilon} v^2\, dx} \\
&\ge \frac{\varepsilon^{\alpha-1}}{2} \frac{\int_{\Omega_\varepsilon} |\nabla u^2_1(\varepsilon, x)|^2\, dx}{\varepsilon^{-1} \int_{G^{(1)}_\varepsilon} u^2_1(\varepsilon, x)^2\, dx} \ge c_0\, \varepsilon^{\alpha-1},
\end{aligned}$$

where c_0 depends neither on ε nor n. The lemma is proved. □

Remark 2.1. *Here and what it follows all constants in inequalities are independent of the parameter ε.*

2.2 Formal Asymptotics

Keeping in mind the bounds from Lemma 2.1 for "slightly heavy" concentrated masses we rescale the eigenvalues and eigenfunctions as follows:

$$\lambda(\varepsilon) = \varepsilon^{\alpha-1} \Lambda(\varepsilon), \qquad u(\varepsilon, x) = \varepsilon^{\frac{\alpha-1}{2}} v(\varepsilon, x). \tag{2.7}$$

Under this rescaling the problem (1.1) becomes

$$\left\{ \begin{array}{rcll}
-\Delta_x\, v(\varepsilon, x) &=& \varepsilon^{\alpha-1} \Lambda(\varepsilon)\, v(\varepsilon, x), & x \in \Omega_0 \cup G^{(2)}_\varepsilon; \\
-\Delta_x\, v(\varepsilon, x) &=& \varepsilon^{-1} \Lambda(\varepsilon)\, v(\varepsilon, x), & x \in G^{(1)}_\varepsilon; \\
-\partial_\nu v(\varepsilon, x) &=& 0, & x \in \Upsilon^{(1)}_\varepsilon \cup \Upsilon^{(2)}_\varepsilon \cup \Gamma_\varepsilon; \\
v(\varepsilon, x) &=& 0, & x \in \Gamma_1; \\
[v]_{|x_2=0} &=& [\partial_{x_2} v]_{|x_2=0} = 0, & x_1 \in Q_\varepsilon.
\end{array} \right. \tag{2.8}$$

Let us define an operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ that corresponds to problem (2.8) by the following equality

$$(A_\varepsilon u, v)_{\mathcal{H}_\varepsilon} = (u, v)_{\mathcal{V}_\varepsilon} \quad \forall\, u, v \in \mathcal{H}_\varepsilon, \tag{2.9}$$

where $\mathcal{V}_\varepsilon$ is the space $L^2(\Omega_\varepsilon)$ with the scalar product

$$(u, v)_{\mathcal{V}_\varepsilon} := \varepsilon^{\alpha-1} \int\limits_{\Omega_0 \cup G_\varepsilon^{(2)}} u\, v\, dx + \varepsilon^{-1} \int\limits_{G_\varepsilon^{(1)}} u\, v\, dx.$$

It is easy to see that operator A_ε is self-adjoint, positive, and compact. In addition, problem (2.8) is equivalent to the spectral problem $A_\varepsilon u = \lambda^{-1}(\varepsilon)\, u$ in $\mathcal{H}_\varepsilon$.

Therefore, for each fixed value of ε there is a sequence of eigenvalues to problem (2.8)

$$0 < \Lambda_1(\varepsilon) < \Lambda_2(\varepsilon) \le \ldots \le \Lambda_n(\varepsilon) \le \cdots \to +\infty \quad \text{as} \quad n \to \infty. \tag{2.10}$$

The corresponding eigenfunctions $\{v_n(\varepsilon, \cdot) :\ n \in \mathbb{N}\}$ can be orthonormalized in the following way:

$$(v_n, v_m)_{\mathcal{V}_\varepsilon} = \delta_{n,m}, \quad \{n,\, m\} \in \mathbb{N}. \tag{2.11}$$

Remark 2.2. *The asymptotic ansatz both for eigenvalues and eigenfunctions of problem* (2.8) *essentially depends on value of the parameter* α. *For instance, if* $\alpha \in \left(1, \frac{3}{2}\right)$, *then* $2\alpha - 2 < 1$ *and the underlined terms in* (2.12)–(2.15) *stay between terms of order* $\mathcal{O}(\varepsilon^{\alpha-1})$ *and* $\mathcal{O}(\varepsilon)$. *In the case* $\alpha = \frac{3}{2}$ *we should summarize the respective terms. In addition, the* α *is nearest to* 1, *the more terms are between* $\varepsilon^{\alpha-1}\lambda_{\alpha-1}$ *and* $\varepsilon\lambda_1$. *But the leading two terms for the asymptotic expansions are the same for* $\alpha \in (1, 2)$. *Since we are going to justify only these leading terms, the asymptotic approximations are constructed in more details for the case* $\alpha \in \left(\frac{3}{2}, 2\right)$.

Combining the algorithm of constructing asymptotics in thin domains with the methods of homogenization theory, we seek the main terms of the asymptotics for the eigenvalue $\Lambda_n(\varepsilon)$ and the eigenfunction $v_n(\varepsilon, \cdot)$ in the form (index n is omitted):

$$\Lambda(\varepsilon) \approx \lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1 + \underline{\varepsilon^{2\alpha-2}\lambda_{2\alpha-2}} + \ldots \tag{2.12}$$

$$v(\varepsilon, x) \approx v_0^+(x) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x) + \varepsilon v_1^+(x) + \underline{\varepsilon^{2\alpha-2} v_{2\alpha-2}^+(x)} + \ldots \quad \text{in domain } \Omega_0; \tag{2.13}$$

in the thin rectangle $G_j^{(2)}(\varepsilon)$ $(j = 0, \ldots, N-1)$

$$\begin{aligned} v(\varepsilon, x) \approx\ & v_0^-(x_1, x_2, \eta_1 - j) + \varepsilon^{\alpha-1} v_{\alpha-1}^-(x_1, x_2, \eta_1 - j) + \\ & + \varepsilon v_1^-(x_1, x_2, \eta_1 - j) + \underline{\varepsilon^{2\alpha-2} v_{2\alpha-2}^-(x_1, x_2, \eta_1 - j)} + \ldots, \qquad \eta_1 = \frac{x_1}{\varepsilon}; \end{aligned} \tag{2.14}$$

and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$
\begin{aligned}
v(\varepsilon,x) \approx\ & v_0^+(x_1,0) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x_1,0) + \varepsilon v_1^+(x_1,0) + \\
& + \varepsilon \Big(Z_1^{(0)}(\tfrac{x}{\varepsilon}) v_0^+(x_1,0) + \sum_{i=1}^{2} Z_1^{(i)}(\tfrac{x}{\varepsilon}) \partial_{x_i} v_0^+(x_1,0) \Big) + \underline{\varepsilon^{2\alpha-2} v_{2\alpha-2}^+(x_1,0)} + \\
& + \varepsilon^{\alpha} \Big(Z_\alpha^{(0)}(\tfrac{x}{\varepsilon}) v_0^+(x_1,0) + X_\alpha^{(0)}(\tfrac{x}{\varepsilon}) v_{\alpha-1}^+(x_1,0) + \sum_{i=1}^{2} X_\alpha^{(i)}(\tfrac{x}{\varepsilon}) \partial_{x_i} v_{\alpha-1}^+(x_1,0) \Big) + \\
& + \varepsilon^{2} \Big(\sum_{|\beta|\le 2} Z_2^{(\beta)}(\tfrac{x}{\varepsilon}) D^{\beta} v_0^+(x_1,0) + Y_2^{(0)}(\tfrac{x}{\varepsilon}) v_1^+(x_1,0) + \sum_{i=1}^{2} Y_2^{(i)}(\tfrac{x}{\varepsilon}) \partial_{x_i} v_1^+(x_1,0) \Big) + \ldots
\end{aligned}
\tag{2.15}
$$

We used the following standard notation: $\partial_{x_i} = \dfrac{\partial}{\partial x_i}$, $D^{\beta} = \dfrac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}$, where $\beta = (\beta_1, \beta_2)$, $|\beta| = \beta_1 + \beta_2$, $\beta_i \in \mathbb{Z}_+$.

Denote $\Gamma_2 := \partial\Omega_0 \setminus (\Gamma_1 \cup I_0)$. Substituting (2.12) and (2.13) in problem (2.8) and collecting terms with equal order of ε, we get:

$$
-\Delta_x\, v_0^+(x) = 0, \quad x \in \Omega_0,
$$

$$
\partial_\nu v_0^+(x)|_{x\in\Gamma_2} = 0, \qquad v_0^+(x)|_{x\in\Gamma_1} = 0.
$$

It remains to ensure the continuity of the asymptotic approximations on the interfaces between the "rectangles" and the "body". The necessity of the condition $v_0^+(x_1,0) = v_0^-(x_1,0)$ $(x \in I_0)$ is evident. Another condition appears when one constructs the junction layer. This is a standard Steklov boundary condition $\partial_{x_2} v_0^+(x_1,0) = -4h_1 l_1 \lambda_0 v_0^+(x_1,0)$ $(x \in I_0)$ and it is obtained in Subsection 2.3.

Collecting terms of order $\varepsilon^{\alpha-1}$, we have

$$
-\Delta_x\, v_{\alpha-1}^+(x) = \lambda_0 v_0^+(x), \quad x \in \Omega_0,
$$

$$
\partial_\nu v_{\alpha-1}^+(x)|_{x\in\Gamma_2} = 0, \qquad v_{\alpha-1}^+(x)|_{x\in\Gamma_1} = 0.
$$

The transmission conditions are

$$
v_{\alpha-1}^+(x_1,0) = v_{\alpha-1}^-(x_1,0), \qquad x \in I_0,
$$

and

$$
\partial_{x_2} v_{\alpha-1}^+(x_1,0) - h_2 \partial_{x_2} v_{\alpha-1}^-(x_1,0) = \mathcal{F}_{\alpha-1}(x_1), \qquad x \in I_0, \tag{2.16}
$$

where $\mathcal{F}_{\alpha-1}$ is a given function on I_0 that will be defined in Subsection 2.3.

Collecting terms of order ε, we have

$$
-\Delta_x\, v_1^+(x) = 0, \quad x \in \Omega_0,
$$

$$
\partial_\nu v_1^+(x)|_{x\in\Gamma_2} = 0, \qquad v_1^+(x)|_{x\in\Gamma_1} = 0.
$$

Next we will find the following boundary condition

$$
\partial_{x_2} v_1^+(x_1,0) = \mathcal{F}_1(x_1), \qquad x \in I_0, \tag{2.17}
$$

where $\mathcal{F}_1$ is a given function on I_0 that will be defined in Subsection 2.3.

2.2.1 Formal asymptotics on thin rectangles.

Using Taylor series for functions $\{v_\gamma^-\}$ in (2.14) and changing variable $x_1 \mapsto \eta_1$ in a neighborhood of the point $x_1 = \varepsilon(j+\frac{1}{2})$ in thin rectangle $G_j^{(2)}(\varepsilon)$, we get

$$\begin{aligned} v(\varepsilon,x) \approx W_0^{(j)}(x_2,\eta_1) + \varepsilon^{\alpha-1} W_{\alpha-1}^{(j)}(x_2,\eta_1) + \varepsilon W_1^{(j)}(x_2,\eta_1) + \\ + \varepsilon^{2\alpha-2} W_{2\alpha-2}^{(j)}(x_2,\eta_1) + \varepsilon^{\alpha} W_\alpha^{(j)}(x_2,\eta_1) + \ldots, \end{aligned} \tag{2.18}$$

where

$$W_\gamma^{(j)}(x_2,\eta_1) = v_\gamma^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big), \quad \gamma \in \{0, \alpha-1\}, \tag{2.19}$$

$$W_\gamma^{(j)}(x_2,\eta_1) = v_\gamma^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big) \frac{\partial v_{\gamma-1}^-}{\partial x_1}\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) \tag{2.20}$$

if $\gamma \in \{1, 2\alpha-2, \alpha\}$.

Substituting (2.12) and (2.18) in the problem (2.8) instead of $\Lambda_n(\varepsilon)$ and $v_n(\varepsilon,\cdot)$ respectively, collecting terms with equal powers of ε, we obtain the following boundary-value problems

$$\begin{cases} -\partial^2_{\eta_1\eta_1} W_\gamma^{(j)}(x_2,\eta_1) = 0, & \eta_1 \in \big(\frac{1-h_2}{2}, \frac{1+h_2}{2}\big), \\ \partial_{\eta_1} W_\gamma^{(j)}(x_2, \frac{1\pm h_2}{2}) = 0, \end{cases} \tag{2.21}$$

for $\gamma \in \{0, \alpha-1, 1, 2\alpha-2, \alpha\}$,

$$\begin{cases} -\partial^2_{\eta_1\eta_1} W_\gamma^{(j)}(x_2,\eta_1) = \partial^2_{x_2x_2} W_{\gamma-2}^{(j)}(x_2,\eta_1), & \eta_1 \in \big(\frac{1-h_2}{2}, \frac{1+h_2}{2}\big), \\ \partial_{\eta_1} W_\gamma^{(j)}(x_2, \frac{1\pm h_2}{2}) = 0, \end{cases} \tag{2.22}$$

for $\gamma \in \{2, 3\}$, and

$$\begin{cases} -\partial^2_{\eta_1\eta_1} W_{\alpha+1}^{(j)}(x_2,\eta_1) = \partial^2_{x_2x_2} W_{\alpha-1}^{(j)}(x_2,\eta_1) + \lambda_0 W_0^{(j)}(x_2,\eta_1), & \eta_1 \in \big(\frac{1-h_2}{2}, \frac{1+h_2}{2}\big), \\ \partial_{\eta_1} W_{\alpha+1}^{(j)}(x_2, \frac{1\pm h_2}{2}) = 0. \end{cases} \tag{2.23}$$

Here the variable x_2 is regarded as a parameter, $\partial_{\eta_1} = \dfrac{\partial}{\partial \eta_1}$.

From (2.21) we deduce that solutions $W_\gamma^{(j)}$, $\gamma \in \{0, \alpha-1, 1, 2\alpha-2, \alpha\}$, are independent of η_1. Moreover, we deduce from the solvability conditions for problems (2.22) that $W_0^{(j)}$ and $W_1^{(j)}$ are independent of x_2 in addition. Taking into account these facts and relations (2.19) and (2.20), we get that $W_0^{(j)} \equiv v_0^-\big(\varepsilon(j+\frac{1}{2})\big)$ and

$$W_1^{(j)} \equiv v_1^-\big(\varepsilon(j+\tfrac{1}{2}), \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big) \frac{\partial v_0^-}{\partial x_1}\big(\varepsilon(j+\tfrac{1}{2})\big) = \Phi_1\big(\varepsilon(j+\tfrac{1}{2})\big), \tag{2.24}$$

where the value Φ_1 will be defined in subsection 2.3.

The solvability condition for problem (2.23) give us the following equation

$$h_2\, \partial^2_{x_2x_2} v_{\alpha-1}^-(x_1,x_2) + \lambda_0 h_2\, v_0^-(x_1) = 0, \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j+\tfrac{1}{2}). \tag{2.25}$$

Since we seek for smooth functions $\{v_\gamma^-\}$ and the points $\{x_1 = \varepsilon(j+\frac{1}{2}) : j = 0, \ldots, N-1\}$ form the ε-net in the interval $(0,a)$, then differential equation (2.25) defined on N segments can be extended to the rectangle $D_2 = (0,a) \times (-l_2, 0)$.

Bearing in mind the boundary conditions of the original problem at $x_2 = -l_2$, we should add the following condition $\partial_{x_2} v_{\alpha-1}^-(x_1, -l_2) = 0$ to equation (2.25).

2.2.2 Junction-layer solutions

Let us pass to the "fast" variables $\eta = \dfrac{x}{\varepsilon}$ in (2.8). Under this transformation as $\varepsilon \to 0$ the domain Ω_0 transforms to $\{\eta : \eta_i > 0,\, i = 1, 2\}$, the thin rectangle $G_0^{(2)}(\varepsilon)$ to the semistrip

$$\Pi^- = \left(\frac{1}{2} - \frac{h_2}{2}, \frac{1}{2} + \frac{h_2}{2}\right) \times (-\infty, 0]$$

and rectangle $G_0^{(1)}(d_k, \varepsilon)$ to the fixed rectangle

$$\Pi_k = \left(d_k - \frac{h_1}{2}, d_k + \frac{h_1}{2}\right) \times (-l_1, 0].$$

Taking into account the periodic structure of Ω_ε in a neighborhood of I_0, we take the following cell of periodicity

$$\Pi = \Pi^- \cup \Pi^+ \cup \Pi_{l_1},$$

in which we will consider boundary-value problems for coefficients Z, X, Y from (2.15). Here $\Pi^+ = (0, 1) \times (0, +\infty)$, $\Pi_{l_1} := \bigcup_{k=1}^{4} \overline{\Pi}_k$ (see Fig.4).

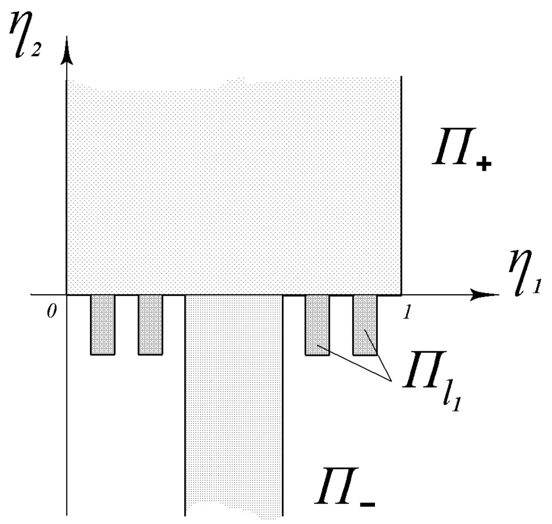

Figure 4: The cell of periodicity.

Note that for the transmission zone we have

$$
\begin{aligned}
\partial_{x_1} v(\varepsilon,x) \approx \varepsilon^0 \Big(& \partial_{x_1} v_0^+(x_1,0)\Big[1+\partial_{\eta_1} Z_1^{(1)}\Big] + \partial_{\eta_1} Z_1^{(0)} v_0^+(x_1,0) + \partial_{\eta_1} Z_1^{(2)} \partial_{x_2} v_0^+(x_1,0)\Big) + \\
& + \varepsilon^{\alpha-1}\Big(\partial_{\eta_1} Z_\alpha^{(0)} v_0^+(x_1,0) + \partial_{\eta_1} X_\alpha^{(0)} v_{\alpha-1}^+(x_1,0) + \\
& + \partial_{x_1} v_{\alpha-1}^+(x_1,0)\Big[1+\partial_{\eta_1} X_\alpha^{(1)}\Big] + \partial_{\eta_1} X_\alpha^{(2)} \partial_{x_2} v_{\alpha-1}^+(x_1,0)\Big) + \\
& + \varepsilon\Big(Z_1^{(0)}(\eta)\partial_{x_1} v_0^+(x_1,0) + \sum_{i=1}^{2} Z_1^{(i)}(\eta)\partial^2_{x_1 x_i} v_0^+(x_1,0) + \\
& + \sum_{|\beta|\le 2} \partial_{\eta_1} Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1,0) + \partial_{x_1} v_1^+(x_1,0)\Big[1+\partial_{\eta_1} Y_2^{(1)}\Big] + \\
& + \partial_{\eta_1} Y_2^{(0)} v_1^+(x_1,0) + \partial_{\eta_1} Y_2^{(2)} \partial_{x_2} v_1^+(x_1,0)\Big) + \mathcal{O}(\varepsilon^{2\alpha-2})
\end{aligned}
\tag{2.26}
$$

and

$$
\begin{aligned}
\Delta_x v(\varepsilon,x) \approx \varepsilon^{-1}\Big(& \Delta_\eta Z_1^{(0)} v_0^+(x_1,0) + \sum_{i=1}^{2} \Delta_\eta Z_1^{(i)}(\eta)\partial_{x_i} v_0^+(x_1,0)\Big) + \\
& + \varepsilon^{\alpha-2}\Big(\Delta_\eta Z_\alpha^{(0)} v_0^+(x_1,0) + \Delta_\eta X_\alpha^{(0)} v_{\alpha-1}^+(x_1,0) + \\
& + \sum_{i=1}^{2} \Delta_\eta X_\alpha^{(i)}(\eta)\partial_{x_i} v_{\alpha-1}^+(x_1,0)\Big) + \\
& + \varepsilon^0\Big(\partial^2_{x_1 x_1} v_0^+(x_1,0)\Big[1+2\partial_{\eta_1} Z_1^{(1)}\Big] + 2\partial_{\eta_1} Z_1^{(0)} \partial_{x_1} v_0^+(x_1,0) + \\
& + 2\partial_{\eta_1} Z_1^{(2)} \partial^2_{x_1 x_2} v_0^+(x_1,0) + \sum_{|\beta|\le 2} \Delta_\eta Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1,0) + \\
& + \Delta_\eta Y_2^{(0)} v_1^+(x_1,0) + \sum_{i=1}^{2} \Delta_\eta Y_2^{(i)}(\eta)\partial_{x_i} v_1^+(x_1,0)\Big) + \mathcal{O}(\varepsilon^{\alpha-1}).
\end{aligned}
\tag{2.27}
$$

Keeping in mind (2.26) and (2.27), substituting the series (2.15) and (2.12) in the problem (2.8) and collecting terms with equal powers of ε, we get problems for $Z_1^{(i)}$, $X_\alpha^{(i)}$, $Y_2^{(i)}$, $i=0,1,2$, $Z_\alpha^{(0)}$ and $Z_2^{(\beta)}$, $|\beta|\le 2$. Obviously, these solutions have to be 1-periodic in η_1. Therefore we will demand the following periodic conditions

$$
\begin{aligned}
\partial^s_{\eta_1} Z(0,\eta_2) &= \partial^s_{\eta_1} Z(1,\eta_2), \quad \eta_2>0, \quad s=0,1,\\
\partial^s_{\eta_1} Y(0,\eta_2) &= \partial^s_{\eta_1} Y(1,\eta_2), \quad \eta_2>0, \quad s=0,1,\\
\partial^s_{\eta_1} X(0,\eta_2) &= \partial^s_{\eta_1} X(1,\eta_2), \quad \eta_2>0, \quad s=0,1,
\end{aligned}
\tag{2.28}
$$

on the vertical sides of semistrip Π^+. In addition, it is easy to see that all these solutions must satisfy the Neumann conditions

$$
\begin{aligned}
\partial_{\eta_2} Z(\eta_1,0) &= 0, \quad (\eta_1,0)\in\partial\Pi, \quad \partial_{\eta_2} Z(\eta_1,-l_1)=0, \quad (\eta_1,-l_1)\in\partial\Pi,\\
\partial_{\eta_2} Y(\eta_1,0) &= 0, \quad (\eta_1,0)\in\partial\Pi, \quad \partial_{\eta_2} Y(\eta_1,-l_1)=0, \quad (\eta_1,-l_1)\in\partial\Pi,\\
\partial_{\eta_2} X(\eta_1,0) &= 0, \quad (\eta_1,0)\in\partial\Pi, \quad \partial_{\eta_2} X(\eta_1,-l_1)=0, \quad (\eta_1,-l_1)\in\partial\Pi,
\end{aligned}
\tag{2.29}
$$

on the horizontal parts of the boundary of Π.

Denote by $\partial\Pi_{\|}$ the vertical part of $\partial\Pi$ laying in $\{\eta : \ \eta_2 < 0\}$.

Thus for $Z_1^{(i)}, \ i = 0,1,2, \ Z_\alpha^{(0)}, \ X_\alpha^{(i)}, \ i = 0,1,2$, and $Z_2^{(\beta)}, \ |\beta| \le 2$, we have the following problems (to all those problems we must add the respective conditions (2.28) and (2.29)):

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_1^{(0)}(\eta) & = & \left\{ \begin{array}{ll} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0, & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_1^{(0)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.30}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_1^{(i)}(\eta) & = & 0, & \eta \in \Pi, \\ \partial_{\eta_1} Z_1^{(i)}(\eta) & = & -\delta_{1i}, & \eta \in \partial\Pi_{\|}, \quad i = 1,2; \end{array} \right. \tag{2.31}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_\alpha^{(0)}(\eta) & = & \left\{ \begin{array}{ll} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_{\alpha-1}, & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_\alpha^{(0)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.32}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_2^{(0,0)}(\eta) & = & \left\{ \begin{array}{ll} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_1 + \lambda_0 Z_1^{(0)}(\eta), & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_2^{(0,0)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.33}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_2^{(1,0)}(\eta) & = & \left\{ \begin{array}{ll} 2\partial_{\eta_1} Z_1^{(0)}(\eta), & \eta \in \Pi^+ \cup \Pi^-, \\ 2\partial_{\eta_1} Z_1^{(0)}(\eta) + \lambda_0 Z_1^{(1)}(\eta), & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_2^{(1,0)}(\eta) & = & -Z_1^{(0)}(\eta), \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.34}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_2^{(0,1)}(\eta) & = & \left\{ \begin{array}{ll} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0 Z_1^{(2)}(\eta), & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_2^{(0,1)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.35}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_2^{(0,2)}(\eta) & = & 0, & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(0,2)}(\eta) & = & 0, & \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.36}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_2^{(1,1)}(\eta) & = & 2\partial_{\eta_1} Z_1^{(2)}(\eta), & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(1,1)}(\eta) & = & -Z_1^{(2)}(\eta), & \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.37}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_2^{(2,0)}(\eta) & = & 1 + 2\partial_{\eta_1} Z_1^{(1)}(\eta), & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(2,0)}(\eta) & = & -Z_1^{(1)}(\eta), & \eta \in \partial\Pi_{\|}. \end{array} \right. \tag{2.38}$$

Also it should be noted that $X_\alpha^{(k)} \equiv Y_2^{(k)} \equiv Z_1^{(k)}, \ k = 0,1,2$.

The existence and the main asymptotic relations for the functions $\{Z_1^{(i)}\}$, $\{Z_2^{(\beta)}\}$ can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity [13, 14, 15, 16]. The proofs are substantially simplified if the polynomial property of the corresponding quasilinear forms is employed [17]. However, if a domain, where we consider a boundary-value problem, has some symmetry, then we can define more exactly the asymptotic relations and detect other properties of junction-layer solutions (see Lemma 4.1 and Corollary 4.1 from [18]). Using this approach, one can prove the following lemma.

Lemma 2.2. *There exist solutions* $Z_1^{(i)} \in H^1_{loc,\eta_2}(\Pi)$, $i = 0, 1, 2$, *of the problems* (2.30), (2.31), $Z_\alpha^{(0)} \in H^1_{loc,\eta_2}(\Pi)$ *of the problem* (2.32) *and* $Z_2^{(\beta)} \in H^1_{loc,\eta_2}(\Pi)$, $|\beta| \le 2$ *of the problems* (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), *which have the following differentiable asymptotics*

$$Z_1^{(0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2 + C_1^{(0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.39}$$

$$Z_\alpha^{(0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_{\alpha-1}}{h_2}\,\eta_2 + C_\alpha^{(0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.40}$$

$$Z_1^{(1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \Big(-\eta_1 + \dfrac{1}{2}\Big) + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.41}$$

$$Z_1^{(2)}(\eta) = \begin{cases} \eta_2 + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2} + C_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.42}$$

$$Z_2^{(0,0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_1 + \mu_{(0,0)}}{h_2}\,\eta_2 + C_2^{(0,0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.43}$$

$$Z_2^{(1,0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2\Big(-\eta_1 + \dfrac{1}{2}\Big) + C_2^{(1,0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases}$$

$$Z_2^{(0,1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)d\eta}{h_2}\,\eta_2 + C_2^{(0,1)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases}$$

$$Z_2^{(0,2)}(\eta) = \begin{cases} \eta_2 + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2} + C_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases}$$

$$Z_2^{(1,1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2}\Big(-\eta_1 + \dfrac{1}{2}\Big) + C_2^{(1,1)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.44}$$

$$Z_2^{(2,0)}(\eta) = \begin{cases} -\dfrac{1}{2}\eta_2^2 + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\mu_{(2,0)}}{h_2}\eta_2 + C_2^{(2,0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.45}$$

where

$$\mu_{(0,0)} = \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(0)}(\eta) d\eta,$$

$$\mu_{(2,0)} = 2 \int\limits_{\Pi^+} \partial_{\eta_1} Z_1^{(1)}(\eta)\ d\eta + \int\limits_{\Pi_{l_1} \cup \Pi^-} (1 + \partial_{\eta_1} Z_1^{(1)}(\eta))\ d\eta. \tag{2.46}$$

Moreover functions $Z_1^{(1)}$, $Z_2^{(1,0)}$, $Z_2^{(1,1)}$ are odd in η_1 with respect to $\frac{1}{2}$; functions $Z_1^{(0)}$, $Z_\alpha^{(0)}$, $Z_1^{(2)}$, $Z_2^{(0,0)}$, $Z_2^{(0,1)}$, $Z_2^{(0,2)}$ and $Z_2^{(2,0)}$ are even in η_1 with respect to $\frac{1}{2}$.

For the proof we refer to our previous paper [2].

2.3 Homogenized problem and correctors

We have formally constructed the leading terms of the asymptotic expansions (2.13), (2.14), (2.15) in three different parts of the junction Ω_ε. Now we apply the method of matching of asymptotic expansions to complete the constructions. Following this method (see, for instance [9]), the asymptotics of the external expansions (2.13) and (2.14) as $x_2 \to \pm 0$ have to coincide with the corresponding asymptotics of the internal expansion (2.15) as $\eta_2 \to \pm\infty$ respectively.

Writing down the Taylor series for v_0^+, $v_{\alpha-1}^+$ and v_1^+ with respect to x_2 in a neighborhood of the point $(x_1, 0)$, where $x_1 \in (0, a)$, and passing to the variables $\eta_2 = \varepsilon^{-1} x_2$, we derive

$$\begin{aligned} v(\varepsilon, x) &= v_0^+(x_1, 0) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x_1, 0) + \varepsilon\Big(\eta_2 \partial_{x_2} v_0^+(x_1, 0) + v_1^+(x_1, 0)\Big) + \\ &+ \varepsilon^{2\alpha-2} v_{2\alpha-2}^+(x_1, 0) + \varepsilon^\alpha \eta_2\, \partial_{x_2} v_{\alpha-1}^+(x_1, 0) + \\ &+ \varepsilon^2 \Big(\frac{1}{2} \eta_2^2 \partial_{x_2 x_2}^2 v_0^+(x_1, 0) + \eta_2 \partial_{x_2} v_1^+(x_1, 0)\Big) + \vartheta_{up}^{ex}(\varepsilon, \eta_2), \end{aligned} \tag{2.47}$$

where $\vartheta_{up}^{ex}(\varepsilon, \eta_2) = \mathcal{O}(\max(\varepsilon^3 \eta_2^3, \varepsilon^{2\alpha-1}\eta_2))$ as $x_2 \equiv \varepsilon\eta_2 \to +0$. Bearing in mind the asymptotics of the functions $Z_1^{(k)}$, $X_\alpha^{(k)}$, $Y_2^{(k)}$ $(k = 0, 1, 2)$, $Z_\alpha^{(0)}$, $Z_2^{(\beta)}$ $(|\beta| < 2)$, as $\eta_2 \to +\infty$ (see (2.39)–(2.45)), we write down the asymptotics

$$\begin{aligned} v(\varepsilon, x) &= v_0^+(x_1, 0) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x_1, 0) + \varepsilon\Big(v_1^+(x_1, 0) + \eta_2 \partial_{x_2} v_0^+(x_1, 0)\Big) + \varepsilon^{2\alpha-2} v_{2\alpha-2}^+(x_1, 0) + \\ &+ \varepsilon^\alpha \eta_2 \partial_{x_2} v_{\alpha-1}^+(x_1, 0) + \varepsilon^2 \left(\eta_2 \partial_{x_2 x_2}^2 v_0^+(x_1, 0) - \frac{1}{2} \eta_2^2 \partial_{x_1 x_1}^2 v_0^+(x_1, 0) + \eta_2 \partial_{x_2} v_1^+(x_1, 0)\right) + \\ &+ \vartheta_{up}^{in}(\varepsilon, \eta_2), \end{aligned} \tag{2.48}$$

where $\vartheta_{up}^{in}(\varepsilon, \eta_2) = \mathcal{O}(\max(\varepsilon^3 \eta_2^3, \varepsilon^{2\alpha-1}\eta_2))$ as $\eta_2 \to +\infty$. Thus, the leading terms in (2.47) and (2.48) coincide till order $\mathcal{O}(\varepsilon^\alpha)$.

To match (2.14) and (2.15) we write down (2.14) as $x_2 \to -0$ in fast variables:

$$\begin{aligned} v(\varepsilon, x) &= v_0^-(x_1) + \varepsilon^{\alpha-1} v_{\alpha-1}^-(x_1, 0) + \\ &+ \varepsilon\Big(\underbrace{\Phi_1(x_1)} + T(\eta_1) \partial_{x_1} v_0^-(x_1, 0)\Big) + \varepsilon^{2\alpha-2} v_{2\alpha-2}^-(x_1, 0) + \\ &+ \varepsilon^\alpha \Big(\eta_2 \partial_{x_2} v_{\alpha-1}^-(x_1, 0) + T(\eta_1) \partial_{x_1} v_{\alpha-1}^-(x_1, 0)\Big) + \vartheta_{down}^{ex}(\varepsilon, \eta_2), \end{aligned} \tag{2.49}$$

where $\vartheta^{ex}_{down}(\varepsilon,\eta_2) = \mathcal{O}(\max(\varepsilon^3\eta_2^3, \varepsilon^{2\alpha-1}\eta_2))$ as $x_2 \equiv \varepsilon\eta_2 \to -0$ and keeping in mind the asymptotics of the functions $Z_1^{(k)}$, $X_\alpha^{(k)}$, $Y_2^{(k)}$ $(k=0,1,2)$, $Z_\alpha^{(0)}$, $Z_2^{(\beta)}$ $(|\beta|<2)$, as $\eta_2 \to -\infty$ we rewrite (2.15) as $\eta_2 \to -\infty$ in the form

$$
\begin{aligned}
v(\varepsilon,x) = {} & v_0^+(x_1,0) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x_1,0) + \varepsilon\, \underbrace{v_1^+(x_1,0)} + \\
& + \varepsilon\Bigg(T(\eta_1)\partial_{x_1} v_0^+(x_1,0) + \overbrace{\frac{\eta_2}{h_2}\,\partial_{x_2} v_0^+(x_1,0)} + \underbrace{C_1^{(2)}\,\partial_{x_2} v_0^+(x_1,0)} + \\
& + \overbrace{\frac{4h_1 l_1 \lambda_0}{h_2}\eta_2\, v_0^+(x_1,0)} + \underbrace{C_1^{(0)}\, v_0^+(x_1,0)}\ \Bigg) + \varepsilon^{2\alpha-2} v_{2\alpha-2}^+(x_1,0) + \\
& + \varepsilon^\alpha\Bigg(T(\eta_1)\partial_{x_1} v_{\alpha-1}^+(x_1,0) + \Big(\frac{\eta_2}{h_2} + C_1^{(2)}\Big)\partial_{x_2} v_{\alpha-1}^+(x_1,0) + \\
& + \Big(\frac{4h_1 l_1 \lambda_{\alpha-1}}{h_2}\eta_2 + C_\alpha^{(0)}\Big) v_0^+(x_1,0)\Big) + \Big(\frac{4h_1 l_1 \lambda_0}{h_2}\eta_2 + C_1^{(0)}\Big) v_{\alpha-1}^+(x_1,0)\Big)\Bigg) + \\
& + \varepsilon^2\Bigg(\Big(\frac{4h_1 l_1 \lambda_1 + \mu_{(0,0)}}{h_2}\eta_2 + C_2^{(0,0)}\Big) v_0^+(x_1,0) + \\
& + \Big(\frac{4h_1 l_1 \lambda_0}{h_2}\, T(\eta_1)\eta_2 + C_2^{(1,0)}\Big)\partial_{x_1} v_0^+(x_1,0) + \Big(\frac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}\, d\eta}{h_2}\eta_2 + C_2^{0,1)}\Big)\partial_{x_2} v_0^+(x_1,0) \\
& + \Big(\frac{\mu_{(2,0)}}{h_2}\eta_2 + C_2^{(2,0)}\Big)\partial^2_{x_1x_1} v_0^+(x_1,0) + \Big(\frac{\eta_2}{h_2} T(\eta_1) + C_2^{(1,1)}\Big)\partial^2_{x_1x_2} v_0^+(x_1,0) + \\
& + \Big(\frac{\eta_2}{h_2} + C_1^{(2)}\Big)\partial^2_{x_2x_2} v_0^+(x_1,0) + \frac{4h_1 l_1 \lambda_0}{h_2}\eta_2\, v_1^+(x_1,0) + C_1^{(0)}\, v_1^+(x_1,0) + \\
& + T(\eta_1)\partial_{x_1} v_1^+(x_1,0) + \frac{\eta_2}{h_2}\,\partial_{x_2} v_1^+(x_1,0) + C_1^{(2)}\,\partial_{x_2} v_1^+(x_1,0)\ \Bigg) + \vartheta^{in}_{down}(\varepsilon,\eta_2),
\end{aligned}
\tag{2.50}
$$

where $\vartheta^{in}_{down}(\varepsilon,\eta_2) = \mathcal{O}(\max(\varepsilon^3\eta_2^3, \varepsilon^{2\alpha-1}\eta_2))$ as $\eta_2 \to -\infty$, $T(\eta_1) = -\eta_1 + \frac{1}{2} + [\eta_1]$ and $[\eta_1]$ is the entire part of the number η_1, the constants $\mu_{(0,0)}$ and $\mu_{(2,0)}$ are defined in (2.46).

Equating the corresponding coefficients in (2.49) and (2.50) at ε^0 and $\varepsilon^{\alpha-1}$, we get $v_0^+(x_1,0) = v_0^-(x_1)$ and $v_{\alpha-1}^+(x_1,0) = v_{\alpha-1}^-(x_1,0)$. The same procedure at ε^1 brings us the following relations:

$$
\partial_{x_2} v_0^+(x_1,0) + 4h_1 l_1 \lambda_0\, v_0^+(x_1,0) = 0 \tag{2.51}
$$

for the over-braced terms, and

$$
\Phi_1(x_1) = v_1^+(x_1,0) + C_1^{(2)}\partial_{x_2} v_0^+(x_1,0) + C_1^{(0)} v_0^+(x_1,0) \tag{2.52}
$$

for the under-braced terms. Moreover, taking (2.24) into account, we have

$$
v_1^-(x_1, \tfrac{x_1}{\varepsilon}) = \Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\,\partial_{x_1} v_0^+(x_1,0), \qquad x \in G_\varepsilon^{(2)}. \tag{2.53}
$$

Thus, for

$$
v_0(x) = \begin{cases} v_0^+(x), & x \in \ \Omega_0, \\ v_0^-(x_1) \equiv v_0^+(x_1,0), & x \in \ D_2 = (0,a)\times(-l_2,0), \end{cases}
$$

and the number λ_0 we have the problem

$$\begin{cases} \Delta_x \, v_0^+(x) &= 0, & x \in \Omega_0 \\ \partial_\nu \, v_0^+(x) &= 0, & x \in \Gamma_2, \\ v_0^+(x) &= 0, & x \in \Gamma_1, \\ \partial_{x_2} v_0^+(x_1, 0) &= -4h_1 l_1 \lambda_0 \, v_0^+(x_1, 0), & x_1 \in (0, a), \end{cases} \tag{2.54}$$

which called *homogenized spectral problem* for problem (2.8).

Recall that the number λ_0 is called an eigenvalue of problem (2.54) if there exists a function $v_0 \in \mathcal{H}_0 := \{u \in H^1(\Omega_0) : \; u|_{\Gamma_1} = 0\}$, $v_0 \neq 0$, which is called an eigenfunction corresponding to λ_0, such that the following integral identity holds :

$$\langle v_0, \, \varphi \rangle_{\mathcal{H}_0} = \lambda_0 \, (\mathcal{T}_0 v_0, \mathcal{T}_0 \varphi)_{\mathcal{V}_0} \qquad \forall \; \varphi \in \mathcal{H}_0, \tag{2.55}$$

where $\langle v_0, \, \varphi \rangle_{\mathcal{H}_0} := \int_{\Omega_0} \nabla v_0 \cdot \nabla \varphi \, dx$ is the scalar product in $\mathcal{H}_0$; the space $\mathcal{V}_0$ is the space $L_2(I_0)$ with the following scalar product

$$(\mathcal{T}_0 v_0, \mathcal{T}_0 \varphi)_{\mathcal{V}_0} := 4h_1 l_1 \, (\mathcal{T}_0 v_0, \mathcal{T}_0 \varphi)_{L^2(I_0)};$$

and $\mathcal{T}_0 : \mathcal{H}_0 \mapsto \mathcal{V}_0$ is the trace operator.

Let $A_0 \equiv \mathcal{T}_0 \circ \mathcal{T}_0^* : \mathcal{V}_0 \longmapsto \mathcal{V}_0$, where $\mathcal{T}_0^*$ is the conjugate operator to $\mathcal{T}_0$. It is easy to verify (see for instance [19]) that A_0 is self-adjoint, positive, compact, and the spectral problem (2.54) is equivalent to the spectral problem

$$A_0 \, (\mathcal{T}_0 v_0) = \frac{1}{\lambda_0} \, \mathcal{T}_0 v_0 \quad \text{in} \;\; \mathcal{V}_0. \tag{2.56}$$

Thus, the eigenvalues of problem (2.54) form the sequence

$$0 < \lambda_0^{(1)} < \lambda_0^{(2)} \leq \ldots \leq \lambda_0^{(n)} \leq \cdots \to +\infty \quad \text{as} \quad n \to \infty \tag{2.57}$$

with the classical convention of repeated eigenvalues. The respective sequence of the corresponding eigenfunctions $\{v_0^{+,n} : \; n \in \mathbb{N}\}$ can be orthonormalized as follows

$$4h_1 l_1 \int\limits_{I_0} v_0^{+,n}(x_1, 0) \; v_0^{+,m}(x_1, 0) \, dx_1 = \delta_{n,m}, \quad \{n, \, m\} \in \mathbb{N}. \tag{2.58}$$

Next, let λ_0 be an eigenvalue of problem (2.54), v_0 is the corresponding eigenfunction normalized by (2.58).

Then we convince that the following leading terms of the asymptotic expansions (2.13), (2.14) and (2.15) are matched, if functions $\mathcal{F}_{\alpha-1}$ and $\mathcal{F}_1$ from (2.16) and (2.17) are equal respectively

$$\mathcal{F}_{\alpha-1}(x_1) = 4h_1 l_1 \lambda_0 v_{\alpha-1}^+(x_1, 0) + 4h_1 l_1 \lambda_{\alpha-1} v_0^+(x_1, 0), \quad x_1 \in I_0,$$

and

$$\begin{aligned} \mathcal{F}_1(x_1) = &- \mu_{(2,0)} \partial^2_{x_1 x_1} v_0^+(x_1, 0) - \partial^2_{x_2 x_2} v_0^+(x_1, 0) - \\ &- \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(2)}(\eta) \; d\eta \partial_{x_2} v_0^+(x_1, 0) - \Big(4h_1 l_1 \lambda_1 + \mu_{(0,0)}\Big) v_0^+(x_1, 0), \quad x_1 \in I_0 \end{aligned}$$

and the conditions (2.16) and (2.17) hold true.

Therefore, for

$$v_{\alpha-1}(x) = \begin{cases} v^+_{\alpha-1}(x), & x \in \Omega_0, \\ v^-_{\alpha-1}(x), & x \in D_2, \end{cases}$$

and $\lambda_{\alpha-1}$ we derive the following boundary-value problem:

$$\left\{\begin{aligned} -\Delta_x\, v^+_{\alpha-1}(x) &= \lambda_0 v^+_0(x), \qquad x \in \Omega_0, \\ \partial_\nu\, v^+_{\alpha-1}(x) &= 0, \quad x \in \Gamma_2; \qquad v^+_{\alpha-1}(x) = 0, \quad x \in \Gamma_1, \\ -\,\partial^2_{x_2x_2} v^-_{\alpha-1}(x_1,x_2) &= \lambda_0\, v^+_0(x_1,0), \qquad x \in D_2, \\ \partial_{x_2} v^-_{\alpha-1}(x_1,-l_2) &= 0, \qquad x_1 \in (0,a), \\ v^+_{\alpha-1}(x_1,0) &= v^-_{\alpha-1}(x_1,0), \quad x \in I_0, \\ \partial_{x_2} v^+_{\alpha-1}(x_1,0) - h_2 \partial_{x_2} v^-_{\alpha-1}(x_1,0) &= -\,4h_1 l_1 \Big(\lambda_0\, v^+_{\alpha-1}(x_1,0) + \\ &\qquad + \lambda_{\alpha-1}\, v^+_0(x_1,0)\Big), \qquad x \in I_0. \end{aligned}\right. \tag{2.59}$$

Now, solving the ordinary differential equation in D_2 with regard the first transmission condition and the corresponding boundary condition, we find that

$$v^-_{\alpha-1}(x_1,x_2) = -\lambda_0 \Big(\frac{1}{2}x_2^2 + l_2 x_2\Big) v^+_0(x_1,0) + v^+_{\alpha-1}(x_1,0).$$

and hence, the problem (2.59) can be rewritten as follows:

$$\left\{\begin{aligned} -\Delta_x\, v^+_{\alpha-1}(x) &= \lambda_0 v^+_0(x), \qquad x \in \Omega_0, \\ \partial_\nu\, v^+_{\alpha-1}(x) &= 0, \quad x \in \Gamma_2; \qquad v^+_{\alpha-1}(x) = 0, \quad x \in \Gamma_1, \\ \partial_{x_2} v^+_{\alpha-1}(x_1,0) &= -\,4h_1 l_1 \lambda_0\, v^+_{\alpha-1}(x_1,0) - \\ &\quad -\,4h_1 l_1 \lambda_{\alpha-1}\, v^+_0(x_1,0) - h_2 l_2 \lambda_0\, v^+_0(x_1,0), \qquad x \in I_0. \end{aligned}\right. \tag{2.60}$$

It is easy to see that the solution to problem (2.60) is not uniquely defined. We should choose the number $\lambda_{\alpha-1}$ to satisfy the solvability condition for the problem (2.60). Writing down the integral identity (2.55) for problem (2.54) with the test-function $v^+_{\alpha-1}$ and the respective integral identity of problem (2.60) with the test-function v^+_0, then subtracting them and bearing in mind (2.58) and (2.62), we get

$$\lambda_{\alpha-1} = -\frac{\lambda_0}{4h_1 l_1}\left(h_2 l_2 + \int\limits_{\Omega_0} (v^+_0)^2\, dx\right). \tag{2.61}$$

For the uniqueness of the solution we demand the following orthogonality condition:

$$\int\limits_{I_0} v^+_{\alpha-1}(x_1,0)\, v^+_0(x_1,0)\, dx_1 = 0. \tag{2.62}$$

Keeping in mind (2.53), for v_1^+ and λ_1 we get the following boundary-value problem

$$\begin{cases} \Delta_x\, v_1^+(x) = 0, \qquad x \in \Omega_0, \\ \partial_\nu\, v_1^+(x) = 0, \quad x \in \Gamma_2; \qquad v_1^+(x) = 0, \quad x \in \Gamma_1, \\ \partial_{x_2} v_1^+(x_1,0) = -\,4h_1 l_1 \lambda_0\, v_1^+(x_1,0) - \mu_{(2,0)} \partial^2_{x_1x_1} v_0^+(x_1,0) - \partial^2_{x_2x_2} v_0^+(x_1,0) - \\ \qquad - \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(2)}(\eta)\, d\eta\, \partial_{x_2} v_0^+(x_1,0) - \Big(4h_1 l_1 \lambda_1 + \mu_{(0,0)}\Big)\, v_0^+(x_1,0), \quad x \in I_0. \end{cases} \tag{2.63}$$

Also for this problem the solution is not uniquely defined. We act as in the previous problem. Taking into account the normalization condition (2.58) and the orthogonality condition

$$\int\limits_{I_0} v_1^+(x_1,0)\, v_0^+(x_1,0)\, dx_1 = 0,$$

we get

$$\lambda_1 = \frac{1}{4h_1 l_1}\Bigg(\mu_{(2,0)} \int\limits_{I_0} \left(\partial_{x_1} v_0^+\right)^2 dx_1 - \int\limits_{I_0} v_0^+ \partial^2_{x_2x_2} v_0^+\, dx_1 - \\ - \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(2)}(\eta)\, d\eta \int\limits_{I_0} v_0^+\, \partial_{x_2} v_0^+\, dx_1 - \mu_{(0,0)}\Bigg), \tag{2.64}$$

where $\mu_{(0,0)}$ and $\mu_{(2,0)}$ are defined in (2.46).

2.4 Asymptotic approximations

Let λ_0 be an eigenvalue of problem (2.54), v_0 is the corresponding eigenfunction, i.e., $v_0 = v_0^+$ in Ω_0, where v_0^+ is the corresponding eigenfunction to problem (2.54), and $v_0 = v_0^+(x_1,0)$ in D_2. Then we can define $\lambda_{\alpha-1}$ and λ_1 with the help of (2.61) and (2.64) respectively, and the unique solutions $v_{\alpha-1}^+$ and v_1^+ to problems (2.60) and (2.63) respectively.

Using the method of matching of asymptotic expansions for the leading terms of (2.13), (2.14) and (2.15), we construct the following approximation function

$$R_\varepsilon(x) = \begin{cases} v_0^+(x) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x) + \varepsilon v_1^+(x) + \chi_0(x_2)\, \mathcal{N}_\varepsilon^+(x_1, \frac{x}{\varepsilon}), & x \in \Omega_0; \\ v_0^+(x_1,0) + \varepsilon^{\alpha-1} v_{\alpha-1}^+(x_1,0) + \varepsilon v_1^+(x_1,0) + \mathcal{N}_{1,\varepsilon}^-(x_1, \frac{x}{\varepsilon}), & x \in G_\varepsilon^{(1)}, \\ v_0^+(x_1,0) + \varepsilon^{\alpha-1} v_{\alpha-1}^-(x) + \varepsilon\big(\Phi_1(x_1) + T(\frac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1,0)\big) + \\ \quad + \varepsilon^\alpha T(\frac{x_1}{\varepsilon})\partial_{x_1} v_{\alpha-1}^-(x) + \chi_0(x_2)\, \mathcal{N}_{2,\varepsilon}^-(x_1, \frac{x}{\varepsilon}), & x \in G_\varepsilon^{(2)}, \end{cases} \tag{2.65}$$

where χ_0 is a smooth cut-off function such that $\chi_0(x_2) = 1$ for $|x_2| \le \tau_0/2$, and $\chi_0(x_2) = 0$ for

$|x_2| \geq \tau_0$ ($\tau_0 < \min\{\gamma_0, l_2\}$ (see subsection 1.1));

$$\mathcal{N}_\varepsilon^+(x_1,\eta) = \varepsilon\bigg(Z_1^{(0)}(\eta)\, v_0^+(x_1,0) + \sum_{i=1}^{2}\big(Z_1^{(i)}(\eta) - \delta_{i,2}\eta_2\big)\partial_{x_i} v_0^+(x_1,0)\bigg) +$$

$$+ \varepsilon^\alpha\bigg(Z_\alpha^{(0)}(\eta) v_0^+(x_1,0) + X_\alpha^{(0)}(\eta) v_{\alpha-1}^+(x_1,0) + \sum_{i=1}^{2}\big(X_\alpha^{(i)}(\eta) - \delta_{i,2}\,\eta_2\big)\partial_{x_i} v_{\alpha-1}^+(x_1,0)\bigg), \quad (2.66)$$

$$\mathcal{N}_{1,\varepsilon}^-(x_1,\eta) = \varepsilon\bigg(Z_1^{(0)}(\eta) v_0^+(x_1,0) + \sum_{i=1}^{2} Z_1^{(i)}(\eta)\partial_{x_i} v_0^+(x_1,0)\bigg) +$$

$$+ \varepsilon^\alpha\bigg(Z_\alpha^{(0)}(\eta) v_0^+(x_1,0) + X_\alpha^{(0)}(\eta) v_{\alpha-1}^+(x_1,0) + \sum_{i=1}^{2} X_\alpha^{(i)}(\eta)\partial_{x_i} v_{\alpha-1}^+(x_1,0)\bigg) \quad (2.67)$$

and

$$\mathcal{N}_{2,\varepsilon}^-(x_1,\eta) = \varepsilon\bigg(\big(Z_1^{(1)}(\eta) - T(\eta_1)\big)\partial_{x_1} v_0^+(x_1,0) + \Big(Z_1^{(2)}(\eta) - \frac{\eta_2}{h_2} - C_1^{(2)}\Big)\partial_{x_2} v_0^+(x_1,0) +$$

$$+ \Big(Z_1^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2 - C_1^{(0)}\Big) v_0^+(x_1,0)\bigg) +$$

$$+ \varepsilon^\alpha\bigg(\Big(Z_\alpha^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_{\alpha-1}}{h_2}\,\eta_2\Big) v_0^+(x_1,0) + \Big(X_\alpha^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2\Big) v_{\alpha-1}^+(x_1,0) +$$

$$+ \big(X_\alpha^{(1)}(\eta) - T(\eta_1)\big)\,\partial_{x_1} v_{\alpha-1}^+(x_1,0) + \Big(X_\alpha^{(2)}(\eta) - \frac{\eta_2}{h_2}\Big)\partial_{x_2} v_{\alpha-1}^+(x_1,0)\bigg). \quad (2.68)$$

Due to (2.52) it is easy to verify that $R_\varepsilon|_{x_2=0+} = R_\varepsilon|_{x_2=0-}$ on Q_ε, i.e. $R_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_1)$. Also using (2.51) and second transmision condition in (2.59) and (2.60), one can verify that

$$\partial_{x_2} R_\varepsilon|_{x_2=0+} = \partial_{x_2} R_\varepsilon|_{x_2=0-} + \varepsilon \partial_{x_2} v_1^+(x_1,0) \quad \text{on} \quad Q_\varepsilon^{(1)} \quad (2.69)$$

$$\partial_{x_2} R_\varepsilon|_{x_2=0+} = \partial_{x_2} R_\varepsilon|_{x_2=0-} + \varepsilon \partial_{x_2} v_1^+(x_1,0) + \varepsilon^\alpha T(\tfrac{x_1}{\varepsilon})\partial^2_{x_2 x_1} v_{\alpha-1}^-(x_1,0) \quad \text{on} \quad Q_\varepsilon^{(2)}. \quad (2.70)$$

2.4.1 Discrepancies in the equation of problem (2.8)

Substituting R_ε and $\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1$ in the differential equation of problem (2.8) instead of $v(\varepsilon,\cdot)$ and $\Lambda(\varepsilon)$ respectively and calculating discrepancies with regard problems (2.30)–(2.38) and (2.54), (2.59) and (2.60), we get
in domain Ω_0 :

$$\Delta_x R_\varepsilon(x) + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big) R_\varepsilon(x) =$$

$$= \varepsilon^{2\alpha-2}\mathcal{F}_\varepsilon^+(x) + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big)\chi_0(x_2)\mathcal{N}_\varepsilon^+(x_1, \tfrac{x}{\varepsilon}) +$$

$$+ \varepsilon^{-1}\chi_0'(x_2)\big(\partial_{\eta_2}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \partial_{x_2}\big(\chi_0'(x_2)\mathcal{N}_\varepsilon^+(x_1,\tfrac{x}{\varepsilon})\big) +$$

$$+ \varepsilon^{-1}\chi_0(x_2)\big(\partial^2_{x_1\eta_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \chi_0(x_2)\partial_{x_1}\Big(\big(\partial_{x_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\Big), \quad (2.71)$$

where

$$\mathcal{F}_\varepsilon^+ = \lambda_0 v_{\alpha-1}^+ + \lambda_{\alpha-1} v_0^+ + \varepsilon(\lambda_{\alpha-1} v_1^+ + \lambda_1 v_{\alpha-1}^+) + \varepsilon^{2-\alpha}(\lambda_1 v_0^+ + \lambda_0 v_1^+) + \varepsilon^{\alpha-1}\lambda_{\alpha-1} v_{\alpha-1}^+ + \varepsilon^{3-\alpha}\lambda_1 v_1^+;$$

in $G_\varepsilon^{(1)}$:

$$\begin{aligned}\Delta_x R_\varepsilon(x) + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big)R_\varepsilon(x) &= \varepsilon^{\alpha-1}\partial^2_{x_1x_1} v_{\alpha-1}^+(x_1,0) + \varepsilon\partial^2_{x_1x_1} v_1^+(x_1,0) + \\ &+ \mathcal{F}_{1,\varepsilon}^-(x_1) + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big)\mathcal{N}_{1,\varepsilon}^-(x_1, \tfrac{x}{\varepsilon}) + \\ &+ \varepsilon^{-1}\big(\partial^2_{x_1\eta_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \partial_{x_1}\big(\big(\partial_{x_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\big), \end{aligned} \tag{2.72}$$

where

$$\begin{aligned}\mathcal{F}_{1,\varepsilon}^-(x_1) &= \lambda_1 v_0^+(x_1,0) + \lambda_0 v_1^+(x_1,0) + \varepsilon^{2\alpha-3}\lambda_{\alpha-1} v_{\alpha-1}^+(x_1,0) + \\ &+ \varepsilon\lambda_1 v_1^+(x_1,0) + \varepsilon^{\alpha-1}(\lambda_{\alpha-1} v_1^+(x_1,0) + \lambda_1 v_{\alpha-1}^+(x_1,0));\end{aligned}$$

and in $G_\varepsilon^{(2)}$:

$$\begin{aligned}\Delta_x R_\varepsilon(x) + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big)R_\varepsilon(x) &= \varepsilon^{\alpha} T(\tfrac{x_1}{\varepsilon})\partial^3_{x_1x_2x_2} v_{\alpha-1}^+(x) + \\ + \varepsilon\partial_{x_1}\big(\partial_{x_1}\Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v_0^+(x_1,0)\big) &+ \varepsilon^\alpha\partial_{x_1}\big(T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v_{\alpha-1}^+(x)\big) + \\ + \varepsilon^{\alpha-1}\big(\varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big)R_\varepsilon(x) &+ \varepsilon^{\alpha-1}\lambda_0\big(R_\varepsilon(x) - v_0^+(x_1,0)\big) + \\ + \varepsilon^{-1}\chi_0'(x_2)\big(\partial_{\eta_2}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} &+ \partial_{x_2}\big(\chi_0'(x_2)\mathcal{N}_{2,\varepsilon}^-(x_1, \tfrac{x}{\varepsilon})\big) + \\ + \varepsilon^{-1}\chi_0(x_2)\big(\partial^2_{x_1\eta_1}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} &+ \chi_0(x_2)\partial_{x_1}\big(\big(\partial_{x_1}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\big). \end{aligned} \tag{2.73}$$

2.4.2 Discrepancies on the boundary

It easy to checked that $R_\varepsilon = 0$ on Γ_1 and $\partial_\nu R_\varepsilon = 0$ on the whole boundary $\partial\Omega_\varepsilon \setminus \Gamma_1$, except its vertical parts, on which

$$\partial_{x_1} R_\varepsilon(x) = \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \tag{2.74}$$

on the vertical parts of $\partial\Omega_0$,

$$\partial_{x_1} R_\varepsilon(x) = \varepsilon\partial_{x_1} v_1^+(x_1,0) + \big(\partial_{x_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \tag{2.75}$$

on the vertical parts of $\partial G_\varepsilon^{(1)}$, and

$$\begin{aligned}\partial_{x_1} R_\varepsilon(x) =& \varepsilon\big(\partial_{x_1}\Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v_0^+(x_1,0)\big) + \varepsilon^\alpha\big(T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v_{\alpha-1}^+(x)\big) \\ &+ \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\end{aligned} \tag{2.76}$$

on the vertical parts of $\partial G_\varepsilon^{(2)}$.

2.4.3 Discrepancies in the integral identity

Multiplying (2.71)-(2.73) with arbitrary function $\psi \in \mathcal{H}_\varepsilon$, integrating by parts and taking (2.69), (2.70), (2.74)-(2.76) into account, we deduce

$$-\int_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi \, dx + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big) \int_{\Omega_0 \cup G_\varepsilon^{(2)}} R_\varepsilon \, \psi \, dx + \\ + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big) \int_{G_\varepsilon^{(1)}} R_\varepsilon \psi \, dx = \ell_\varepsilon(\psi), \quad (2.77)$$

where the linear functional ℓ_ε is defined as follows

$$\ell_\varepsilon(\psi) := \varepsilon \int_{Q_\varepsilon} \partial_{x_2} v_1^+(x_1,0)\, \psi(x_1,0)\, dx + \varepsilon^\alpha \lambda_0 \int_{G_\varepsilon^{(2)}} T(\tfrac{x_1}{\varepsilon})(x_2 + l_2)\partial_{x_1} v_0^+(x_1,0)\, \psi \, dx + \\ + \varepsilon^{2\alpha-2} \int_{\Omega_0} \mathcal{F}_\varepsilon^+ \, \psi \, dx + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big) \int_{\Omega_0} \chi_0(x_2)\mathcal{N}_\varepsilon^+(x_1, \tfrac{x}{\varepsilon})\, \psi \, dx + \\ + \int_{G_\varepsilon^{(1)}} \mathcal{F}_{1,\varepsilon}^-(x_1)\, \psi \, dx + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \varepsilon\lambda_1\big) \int_{G_\varepsilon^{(1)}} \mathcal{N}_{1,\varepsilon}^-(x_1, \tfrac{x}{\varepsilon})\, \psi \, dx + \\ + \varepsilon^{\alpha-1} \int_{G_\varepsilon^{(1)}} \partial^2_{x_1 x_1} v_{\alpha-1}^+(x_1,0)\, \psi \, dx - \varepsilon \int_{G_\varepsilon^{(1)}} \partial_{x_1} v_1^+(x_1,0)\, \partial_{x_1}\psi \, dx \\ - \varepsilon \int_{G_\varepsilon^{(2)}} \big(\partial_{x_1}\Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v_0^+(x_1,0)\big)\, \partial_{x_1}\psi \, dx - \varepsilon^\alpha \int_{G_\varepsilon^{(2)}} T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v_{\alpha-1}^+(x)\, \partial_{x_1}\psi \, dx + \\ + \varepsilon^{2\alpha-2}\big(\lambda_{\alpha-1} + \varepsilon^{2-\alpha}\lambda_1\big) \int_{G_\varepsilon^{(2)}} R_\varepsilon \, \psi \, dx + \varepsilon^{\alpha-1}\lambda_0 \int_{G_\varepsilon^{(2)}} \big(R_\varepsilon - v_0^+(x_1,0)\big)\, \psi \, dx + \\ + \varepsilon^{-1} \int_{\Omega_0 \cup G_\varepsilon^{(2)}} \chi_0'(x_2)\big(\partial_{\eta_2}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\, \psi \, dx - \int_{\Omega_0 \cup G_\varepsilon^{(2)}} \chi_0'(x_2)\mathcal{N}_\varepsilon(x_1, \frac{x}{\varepsilon})\, \partial_{x_2}\psi \, dx + \\ + \varepsilon^{-1} \int_{\Omega_0 \cup G_\varepsilon^{(2)}} \chi_0(x_2)\big(\partial^2_{x_1\eta_1}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\, \psi \, dx - \int_{\Omega_0 \cup G_\varepsilon^{(2)}} \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\, \partial_{x_1}\psi \, dx + \\ + \varepsilon^{-1} \int_{G_\varepsilon^{(1)}} \big(\partial^2_{x_1\eta_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\, \psi \, dx - \int_{G_\varepsilon^{(1)}} \big(\partial_{x_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\, \partial_{x_1}\psi \, dx. \quad (2.78)$$

Here $\mathcal{N}_\varepsilon$ coincides with $\mathcal{N}_\varepsilon^+$ on Ω_0 and with $\mathcal{N}_{2,\varepsilon}^-$ on $G_\varepsilon^{(2)}$.

Let us estimate $|\ell_\varepsilon(\psi)|$. It is easy to see that the integrals in the first and second lines of (2.78) are of order $\mathcal{O}(\varepsilon)$. Obviously also, that

$$\varepsilon^{\alpha-1}\Big| \int_{G_\varepsilon^{(1)}} \partial_{x_1} v_{\alpha-1}^+(x_1,0)\, \partial_{x_1}\psi \, dx \Big| \le \varepsilon^{\alpha-1} \|\partial_{x_1} v_{\alpha-1}^+(x_1,0)\|_{L^2(G_\varepsilon^{(1)})} \|\psi\|_{L^2(G_\varepsilon^{(1)})} \le \varepsilon^{\alpha-\frac{1}{2}} C_1 \|\psi\|_{\mathcal{H}_\varepsilon}.$$

Similarly, $\varepsilon\Big|\int_{G_\varepsilon^{(1)}} \partial_{x_1} v_1^+(x_1,0)\,\partial_{x_1}\psi\,dx\Big| \le \varepsilon^{\frac{3}{2}} C_2 \|\psi\|_{\mathcal{H}_\varepsilon}$. Due to (2.6)

$$\Big|\int_{G_\varepsilon^{(1)}} \mathcal{F}_{1,\varepsilon}^-(x_1)\,\psi\,dx\Big| \le \varepsilon C_3 \|\psi\|_{\mathcal{H}_\varepsilon}.$$

Remark 2.3. *Recall that according to Remark 2.2 the parameter α is from the interval $(\frac{3}{2},2)$. The same estimate holds if $\alpha=\frac{3}{2}$. If $\alpha\in(1,\frac{3}{2})$, then $\Big|\int_{G_\varepsilon^{(1)}} \mathcal{F}_{1,\varepsilon}^-(x_1)\,\psi\,dx\Big| \le \varepsilon^{2\alpha-2} C_3 \|\psi\|_{\mathcal{H}_\varepsilon}$.*

The main term in the second integral from the third line of (2.78) can be estimated with the help of (2.6) by the following way

$$\lambda_0\Big|\int_{G_\varepsilon^{(1)}} Z_1^{(0)}(\tfrac{x}{\varepsilon})\,v_0^+(x_1,0)\,\psi\,dx\Big| \le \varepsilon^{\frac{1}{2}} C_1 \|\psi\|_{\mathcal{H}_\varepsilon}\sqrt{\int_{G_\varepsilon^{(1)}} \Big|Z_1^{(0)}(\tfrac{x}{\varepsilon})\Big|^2 dx} \le$$

$$\le \varepsilon C_2 \|\psi\|_{\mathcal{H}_\varepsilon}\sqrt{\int_{\Pi_{l_1}} \Big|Z_1^{(0)}(\eta)\Big|^2 d\eta} \le \varepsilon C_3 \|\psi\|_{\mathcal{H}_\varepsilon}. \qquad (2.79)$$

According to Remark 2.3 one can verify that all integrals from the fifth and sixth lines of (2.78) are of order $\mathcal{O}(\varepsilon)$ as well.

Due to the asymptotic relations in (2.39)-(2.45) as $\eta_2\to\pm\infty$, the terms in the seventh line of (2.78) are exponentially small.

Thanks to Lemma 3.1 ([6]) the integrals in the eighth line of (2.78) are of order $\mathcal{O}(\varepsilon^{1-\delta})$, where δ is arbitrary positive number.

The first integral in the last line of (2.78) can be estimated similarly as in (2.79). As a result it is of order $\mathcal{O}(\varepsilon^1)$. Obviously, that the last integral in (2.78) is of order $\mathcal{O}(\varepsilon^{\frac{3}{2}})$.

Thus, we have

$$|\ell_\varepsilon(\psi)| \le c(\delta)\,\varepsilon^{1-\delta}\|\psi\|_{\mathcal{H}_\varepsilon}. \qquad (2.80)$$

With the help of operator $A_\varepsilon:\mathcal{H}_\varepsilon\mapsto\mathcal{H}_\varepsilon$ defined in (2.9) we deduce from (2.77) and (2.80) the following inequality

$$\|\,R_\varepsilon - (\lambda_0+\varepsilon^{\alpha-1}\lambda_{\alpha-1}+\varepsilon\lambda_1)A_\varepsilon R_\varepsilon\,\|_{\mathcal{H}_\varepsilon} \le c(\delta)\,\varepsilon^{1-\delta}, \qquad (2.81)$$

where δ is arbitrary positive number.

3 Justification of the asymptotics

To justify the asymptotic approximations constructed above, we use the scheme proposed in [19] for investigation of the asymptotic behavior of the eigenvalues and eigenvectors of an family of abstract operators $\{A_\epsilon : H_\epsilon\mapsto H_\epsilon\}_{\varepsilon>0}$ in the limit passage as $\epsilon\to 0$. This scheme generalizes procedure of justification of the asymptotic behavior of eigenvalues and eigenvectors of boundary value problems in perturbed domains that was proposed in [20].

In our case this is the family of operators $\{A_\varepsilon:\mathcal{H}_\varepsilon\mapsto\mathcal{H}_\varepsilon\}_{\varepsilon>0}$ defined in (2.9). Recall that the operator A_ε corresponds to problem (2.8).

For thick junctions there exist no extension operators that would be bounded uniformly in ε in the Sobolev space H^1 (see [6]). But as was shown in [6], for eigenfunctions of spectral problems in thick junctions it was possible to construct special extensions that are bounded on each eigenfunction. A such extension operator was constructed for eigenfunctions of problem (1.1) in the case when the parameter $\alpha \in (0,1]$ in our papers [1, 2]. Repeating word for word the proof of Theorem 4.1 (see [1, 2]), we get the following result.

Theorem 3.1. *There exists an extension operator* $\mathbf{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto H^1(\Omega, \Gamma_1)$ *such that for any eigenfunction* $v_n(\varepsilon, \cdot)$ *normalized by (2.11) there exist positive constants* C_n *and* ε_n *such that for all values of the parameter* ε *from the interval* $(0, \varepsilon_n)$ *the following estimates hold:*

$$\| \mathbf{P}_\varepsilon v_n(\varepsilon, \cdot) \|_{H^1(\Omega,\Gamma_1)} \le C_n \| v_n(\varepsilon, \cdot) \|_{\mathcal{H}_\varepsilon} \le C_n', \tag{3.1}$$

where Ω *is the interior of the union* $\overline{\Omega}_0 \cup \overline{D}_2$.

3.1 Condition $\mathbf{D_1} - \mathbf{D_5}$

For the convenience of readers we write here the conditions of the scheme from paper [19], which are modified under problems (2.8) and (2.54).

Let $N(\frac{1}{\mu}, \mathrm{A}_0)$ denote the proper subspace corresponding to the eigenvalue $\frac{1}{\mu}$ of operator A_0 defined in (2.56) and let $\{(v_n(\varepsilon, \cdot), \Lambda_n(\varepsilon)) : \ \varepsilon > 0\}$ denote the sequence whose components are the eigenfunction v_n ($\|v_n\|_{\mathcal{V}_\varepsilon} = 1$) and the corresponding characteristic number of operator A_ε.

Condition $\mathbf{D_1}$. There exists a linear operator $\mathrm{S}_\varepsilon : \mathcal{H}_0 \mapsto \mathcal{H}_\varepsilon$ such that

$$\|\mathrm{S}_\varepsilon u\|_{\mathcal{H}_\varepsilon} \le c_1 \|u\|_{\mathcal{H}_0}, \quad \forall u \in \mathcal{H}_0,$$

where the constant c_1 is independent of ε and u.

Condition $\mathbf{D_2}$. There exists a linear operator $\mathrm{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_0$ such that

$$\forall\, n \in \mathbb{N} \ \exists\, c_2 > 0 \ \ \exists\, \varepsilon_0 > 0 \ \ \forall \varepsilon \in (0, \varepsilon_0) : \quad \|\mathrm{P}_\varepsilon v_n(\varepsilon, \cdot)\|_{\mathcal{H}_0} \le c_2 \, \|v_n(\varepsilon, \cdot)\|_{\mathcal{H}_\varepsilon}.$$

Condition $\mathbf{D_3}$. For an arbitrary sequence $\{(v_n(\varepsilon, \cdot), \Lambda_n(\varepsilon)) : \ \varepsilon > 0\}$ and any subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ such that $\mathrm{P}_{\varepsilon'} v_n(\varepsilon', \cdot) \to v_n^0$ weakly in $\mathcal{H}_0$, one has

$$\lim_{\varepsilon' \to 0} \left(v_n(\varepsilon', \cdot), \mathrm{S}_{\varepsilon'} \varphi\right)_{\mathcal{H}_{\varepsilon'}} = \left(v_n^0, \varphi\right)_{\mathcal{H}_0} \quad \forall \varphi \in \mathcal{H}_0.$$

Condition $\mathbf{D_4}$. If for certain functions $w^\varepsilon, v^\varepsilon \in \mathcal{H}_\varepsilon$ one has $\mathrm{P}_\varepsilon w^\varepsilon \to w^0$ and $\mathrm{P}_\varepsilon v^\varepsilon \to v^0$ weakly in $\mathcal{H}_0$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} \left(w^\varepsilon, v^\varepsilon\right)_{\mathcal{V}_\varepsilon} = \left(\mathcal{T}_0 w^0, \mathcal{T}_0 v^0\right)_{\mathcal{V}_0}.$$

If $v \in \mathcal{H}_0$, then $\mathrm{P}_\varepsilon(\mathrm{S}_\varepsilon v) \to v$ weakly in $\mathcal{H}_0$ as $\varepsilon \to 0$.

Condition $\mathbf{D_5}$. There exists a number $\delta_0 > 0$ such that for any $\frac{1}{\mu} \in \sigma(A_0)$ there exists a linear operator $\mathcal{R}_\varepsilon : N(\frac{1}{\mu}, A_0) \mapsto \mathcal{H}_\varepsilon$ such that for every eigenfunction $v \in N(\frac{1}{\mu}, A_0)$, normalized by $\|\mathcal{T}_0 v\|_{\mathcal{V}_0} = 1$, we have

$$\mathcal{R}_\varepsilon v = \mathrm{S}_\varepsilon v + \mathcal{O}(\varepsilon) \ \text{ in } \mathcal{V}_\varepsilon \quad \text{and} \quad \|\mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon} = c_v + \mathcal{O}(\varepsilon);$$

in addition, there exist constants c_3, ε_0, $\mu_{\alpha-1}$, μ_1 such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\|\mathcal{R}_\varepsilon v - (\mu + \varepsilon^{\alpha-1}\mu_{\alpha-1} + \varepsilon \mu_1)\, A_\varepsilon(\mathcal{R}_\varepsilon v)\|_{\mathcal{H}_\varepsilon} \le c_3 \varepsilon^{\delta_0}.$$

To clarify these conditions, we use the following diagram

$$
\begin{array}{ccc}
\mathcal{H}_\varepsilon & \xrightarrow{J_\varepsilon} & \mathcal{V}_\varepsilon \\
{\scriptstyle \mathrm{P}_\varepsilon}\Big\downarrow \ \Big\uparrow{\scriptstyle \mathrm{S}_\varepsilon} & & \\
\mathcal{H}_0 & \xrightarrow{\mathcal{T}_0} & \mathcal{V}_0
\end{array}
$$

where operator $J_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{V}_\varepsilon$ is the identical imbedding operator, operator $\mathcal{T}_0 : \mathcal{H}_0 \mapsto \mathcal{V}_0$ is the trace operator (see (2.55)). Conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ are some connection conditions between spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ that defined in subsection 1.1 and 2.3 respectively. If conditions $\mathbf{D_3}$ and $\mathbf{D_4}$ are satisfied, then it means that spectral problem (2.54) is the homogenized problem for problem (2.8). Condition $\mathbf{D_5}$ means that it is possible to construct asymptotic approximations near points of the spectrum of operator A_0.

Now let us verify conditions $\mathbf{D_1} - \mathbf{D_5}$ for our problems (2.8) and (2.54). The operator $S_\varepsilon : \mathcal{H}_0 \mapsto \mathcal{H}_\varepsilon$ assigns to each function $v \in \mathcal{H}_0$ its bounded extension Ev to $H^1(\Omega, \Gamma_1)$ and then restricts Ev to Ω_ε, i.e., $S_\varepsilon = (Ev)|_{\Omega_\varepsilon}$. Clearly, S_ε is uniformly bounded with respect to ε. Thus condition $\mathbf{D_1}$ is satisfied.

The operator $\mathrm{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_0$ from condition $\mathbf{D_2}$ is associated with the restriction of the extension operator $\mathbf{P}_\varepsilon$ from Theorem 3.1 to domain Ω_0, i.e. $\mathrm{P}_\varepsilon v_n = (\mathbf{P}_\varepsilon v_n)|_{\Omega_0}$.

Let us verify condition $\mathbf{D_3}$. Consider the sequence $\{v_n(\varepsilon, \cdot)\}_{\varepsilon>0}$ for any fixed index $n \in \mathbb{N}$. Due to Theorem 3.1 there exists some subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$ (again denoted by $\{\varepsilon\}$) such that $\mathbf{P}_\varepsilon v_n(\varepsilon, \cdot) \to v_0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$. Since

$$
\int\limits_{D_2} \chi_{h_2}(\tfrac{x_1}{\varepsilon})\, \partial_{x_2} \mathbf{P}_\varepsilon(v_n(\varepsilon, x))\, \phi(x)\, dx = - \int\limits_{D_2} \chi_{h_2}(\tfrac{x_1}{\varepsilon})\, \mathbf{P}_\varepsilon(v_n(\varepsilon, x))\, \partial_{x_2}\phi\, dx \quad \forall\ \phi \in C_0^\infty(D_2),
$$

we get

$$
\chi_{h_2}(\tfrac{x_1}{\varepsilon}) \partial_{x_2} \mathbf{P}_\varepsilon(v_n(\varepsilon, x)) \to h_2\, \partial_{x_2} v_n^0(x) \quad \text{weakly in} \ \ L^2(D_2) \quad \text{as} \ \ \varepsilon \to 0. \tag{3.2}
$$

Here $\chi_{h_2}(\eta_1)$ $(\eta_1 \in \mathbb{R})$ is 1-periodic function that equals 1 on the interval $\left(\frac{1-h_2}{2},\ \frac{1+h_2}{2}\right)$ and vanishing on the rest of the segment $[0, 1]$.

If we consider the corresponding integral identity for problem (2.8) with the following test function

$$
\psi(x) = \left\{ \begin{array}{rl} 0, & x \in \Omega_0 \cup G_\varepsilon^{(1)}, \\ \varepsilon T\left(\frac{x_1}{\varepsilon}\right)\phi(x), & x \in G_\varepsilon^{(2)}, \end{array} \right. \qquad \phi \in C_0^\infty(D_2),
$$

where T is defined in (2.50), we get

$$
\int\limits_{D_2} \chi_{h_2}(x_1/\varepsilon) \partial_{x_1} \mathbf{P}_\varepsilon(v_n(\varepsilon, x))\, \phi\, dx = \mathcal{O}(\varepsilon) \quad \text{as} \ \ \varepsilon \to 0. \tag{3.3}
$$

Due to the second inequality in (3.1), it is easy to verify that

$$
\int\limits_{G_\varepsilon^{(1)}} \nabla v_n(\varepsilon, x) \cdot \nabla \varphi(x)\, dx \to 0 \quad \text{as} \ \ \varepsilon \to 0 \quad \forall\ \varphi \in H^1(\Omega, \Gamma_1). \tag{3.4}
$$

The corresponding integral identity for problem (2.8) with a test function $\phi \in C_0^\infty(D_2)$ reads as follows

$$\int\limits_{G_\varepsilon^{(2)}} \nabla v_n(\varepsilon, x) \cdot \nabla \phi(x)\, dx = \varepsilon^{\alpha-1} \Lambda_n(\varepsilon) \int\limits_{G_\varepsilon^{(2)}} v_n(\varepsilon, x)\, \phi(x)\, dx \tag{3.5}$$

for ε small enough. Taking into account limits (3.2), (3.3) and the boundedness of $\Lambda_n(\varepsilon)$ with respect to ε (see Lemma 2.1), we deduce from (3.5 that

$$h_2 \int\limits_{D_2} \partial_{x_2} v_n^0\, \partial_{x_2} \phi\, dx = 0 \quad \forall\, \phi \in C_0^\infty(D_2),$$

i.e., $\partial_{x_2} v_n^0$ is some function of x_1 a.e. in D_2. On the the other hand $\partial_{x_2} v_n^0|_{x_2=-l_2} = 0$, because $\partial_{x_2} v_n(\varepsilon, \cdot)|_{x_2=-l_2} = 0$. Therefore, $v_n^0(x) = v_n^0(x_1, 0)$ for a.e. $x \in D_2$.

Thus, we ascertain that

$$\lim_{\varepsilon \to 0} \big(v_n(\varepsilon, \cdot), S_\varepsilon \varphi\big)_{\mathcal{H}_\varepsilon} = \lim_{\varepsilon \to 0} \Big(\int\limits_{\Omega_0} \nabla v_n(\varepsilon, x) \cdot \nabla \varphi\, dx + \int\limits_{G_\varepsilon^{(1)}} \nabla v_n(\varepsilon, x) \cdot \nabla (E\varphi)|_{G_\varepsilon^{(1)}}\, dx +$$

$$+ \int\limits_{D_2} \chi_{h_2}(\tfrac{x_1}{\varepsilon}) \nabla\big(\partial_{x_2} \mathbf{P}_\varepsilon(v_n(\varepsilon, x))\big) \cdot \nabla(E\varphi)|_{D_2}\, dx \Big) = \int\limits_{\Omega_0} \nabla v_n^0(x) \cdot \nabla \varphi\, dx = \big(v, \varphi\big)_{\mathcal{H}_0} \quad \forall\, \varphi \in \mathcal{H}_0,$$

i.e., condition $\mathbf{D_3}$ is satisfied.

Let for certain functions $u^\varepsilon, v^\varepsilon \in \mathcal{H}_\varepsilon$ one has $\mathbf{P}_\varepsilon u^\varepsilon \to u^0$ and $\mathbf{P}_\varepsilon v^\varepsilon \to v^0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$. Then

$$\lim_{\varepsilon \to 0} \big(u^\varepsilon, v^\varepsilon\big)_{\mathcal{V}_\varepsilon} = \lim_{\varepsilon \to 0} \varepsilon^{-1} \int\limits_{G_\varepsilon^{(1)}} u^\varepsilon v^\varepsilon\, dx. \tag{3.6}$$

With the help of the inequality

$$\varepsilon^{-1} \int\limits_{G_\varepsilon^{(1)}} \big(\varphi(x) - \varphi(x_1, 0)\big)^2\, dx \le \varepsilon\, l_1 \int\limits_{G_\varepsilon^{(1)}} \big(\partial_{x_2} \varphi(x)\big)^2\, dx \quad \forall\, \varphi \in H^1(G_\varepsilon^{(1)}),$$

we deduce that $\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{G_\varepsilon^{(1)}} u^\varepsilon v^\varepsilon\, dx = 4 h_1 l_1 \int_{I_0} u^0(x_1, 0)\, v^0(x_1, 0)\, dx_1$. This means that the first part of condition $\mathbf{D_4}$ holds. The second of condition $\mathbf{D_4}$ in our case is obvious.

Condition $\mathbf{D_5}$, in fact, has been verified in subsection 2.4, namely the action of the operator $\mathcal{R}_\varepsilon$ in $\mathbf{D_5}$ is the construction of the approximating function R_ε on the basis of an eigenfunction of the homogenized problem (2.54). Furthermore, the approximating function satisfies inequality (2.67) that is analog of the corresponding inequality in condition $\mathbf{D_5}$.

3.2 The main results

Thus, all conditions $\mathbf{D_1}$–$\mathbf{D_5}$ of the scheme from [19] are satisfied. Applying this scheme and taking into account (2.7), we get the following theorems.

Theorem 3.2. *For any $n \in \mathbb{N}$*

$$\varepsilon^{1-\alpha}\lambda_n(\varepsilon) \to \lambda_0^{(n)} \quad as \ \ \varepsilon \to 0,$$

where $\{\lambda_n(\varepsilon) : n \in \mathbb{N}\}$ is the ordered sequence (1.2) *of eigenvalues of problem* (1.1), $\{\lambda_0^{(n)}\}$ *is the ordered sequence* (2.57) *of eigenvalues of the homogenized problem* (2.54).

There exists a subsequence of the sequence $\{\varepsilon\}$ (again denoted by $\{\varepsilon\}$) such that

$$\forall\, n \in \mathbb{N} \quad \varepsilon^{-\frac{\alpha-1}{2}}\mathbf{P}_\varepsilon u_n(\varepsilon,\cdot) \to v_0^n \quad weakly\ in\ H^1(\Omega,\Gamma_1) \qquad as \ \ \varepsilon \to 0,$$

where

$$v_0^n(x) = \begin{cases} v_0^{+,n}(x), & x \in \ \Omega_0, \\ v_0^{+,n}(x_1,0), & x \in \ D_2 = (0,a)\times(-l_2,0), \end{cases}$$

$\{u_n(\varepsilon,\cdot) : n \in \mathbb{N}\}$ *is the sequence of eigenfunctions that are orthonormalized with relations* (1.3), $\{v_0^{+,n} : \ n \in \mathbb{N}\}$ *are eigenfunctions of the homogenized problem* (2.54) *that satisfy the following orthonormalized conditions*

$$\left(v_0^{+,n}, v_0^{+,k}\right)_{\mathcal{V}_0} = 4h_1 l_1 \int\limits_{I_0} v_0^{+,n}(x_1,0)\, v_0^{+,k}(x_1,0)\, dx_1 = \delta_{n,k}, \qquad n,k \in \mathbb{N}.$$

Let $\lambda_0^{(n+1)} = \ldots = \lambda_0^{(n+r)}$ be an $r-$multiple eigenvalue of the homogenized problem (2.54) and the corresponding eigenfunctions $v_0^{(+,n+1)}, \ldots, v_0^{(+,n+r)}$ are orthonormalized in $\mathcal{V}_0$. Using formulas (2.61) and (2.64), we successively construct next terms $\varepsilon^{\alpha-1}\lambda_{\alpha-1}^{(n+i)}$ and $\varepsilon\,\lambda_1^{(n+i)}$, of the asymptotic expansion (2.12) and define the unique solutions $v_{\alpha-1}^{+,n+i}$ and $v_1^{+,n+i}$ to problems (2.60) and (2.63) respectively, $i = 1,\ldots,r$. Denote by

$$\Lambda_i^{(n)}(\varepsilon) := \lambda_0^{(n+i)} + \varepsilon^{\alpha-1}\lambda_{\alpha-1}^{(n+i)} + \varepsilon\lambda_1^{(n+i)}$$

the partial sum of (2.12).

We formulate the following theorem under assumption that all $\Lambda_1^{(n)}(\varepsilon), \ldots, \Lambda_r^{(n)}(\varepsilon)$ are different and

$$\Lambda_1^{(n)}(\varepsilon) < \Lambda_2^{(n)}(\varepsilon) < \ldots < \Lambda_r^{(n)}(\varepsilon) \tag{3.7}$$

for ε small enough.

For more general case when $\{\Lambda_i^{(n)}(\varepsilon) : \ i = 1,\ldots,r\}$ are splited into k groups the formulation is similar as in Theorems 5.4 and 5.6 from our paper [2].

Theorem 3.3. *Let inequalities (3.7) are satisfied. Then for any $\delta > 0$, $i \in \{1,\ldots,r\}$ and ε small enough, we have*

$$\left|\varepsilon^{1-\alpha}\lambda_{n+i}(\varepsilon) - \left(\lambda_0^{(n+i)} + \varepsilon^{\alpha-1}\lambda_{\alpha-1}^{(n+i)}\right)\right| \ \le \ C_1(n,\delta)\,\varepsilon^{1-\delta},$$

and

$$\left\|\varepsilon^{-\frac{\alpha-1}{2}}u_{n+i}(\varepsilon,\cdot) - \frac{R_\varepsilon^{(n+i)}}{\|R_\varepsilon^{(n+i)}\|_{\mathcal{H}_\varepsilon}}\right\|_{H^1(\Omega_\varepsilon)} \le C_2(n,\delta)\ \varepsilon^{1-\delta},$$

where $R_\varepsilon^{(n+i)}$ is the approximating function constructed by formula (2.65) *with the help of solutions $v_0^{+,n+i}$, $v_{\alpha-1}^{+,n+i}$ and $v_1^{+,n+i}$.*

Remark 3.1. *If $\alpha \in (1,\frac{3}{2})$, then the estimates in Theorem 3.3 are of order $\mathcal{O}(\varepsilon^{2\alpha-2})$ (see Remark 2.3).*

4 Conclusions and Remarks

In the papers [21, 8] the authors considered vibrations of a domain containing many small regions of high density of order $\mathcal{O}(\varepsilon^{-\alpha})$, periodically situated along the boundary. The main assumption is that $\varepsilon << \eta(\varepsilon)$, $\eta(\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$, where ε is the diameter of a concentrated mass and η is the distance between them. In addition the homogeneous Dirichlet conditions are set on some part of the boundaries of the concentrated masses. Three qualitatively different cases in the asymptotic behavior of eigen-magnitudes were found: $\alpha \in (0,2)$, $\alpha = 2$ and $\alpha > 2$. Only convergence theorems were proved for eigenvalues and eigenfunctions as $\varepsilon \to 0$. In [8], under the assumption that $\alpha > 2$ the authors discovered eigenvibrations located near the boundary in a such way that the corresponding eigenfunctions, on a microscopic scale, present a *skin effect* in the concentrated masses.

In Remark 6 from the paper [21] one can get to know that the results of the paper as well as the technique can not be applied directly to the problems with concentrated masses situated near the boundary, if the distance between them is of the same order as the diameter of the masses.

Using the approach of the present paper it is easy to study the mentioned problems in the case $\varepsilon = \theta\eta$, where $\theta \in (0,1)$ is some fixed number. Moreover we can consider spectral problems with more general settings, for instance, in domains with adjoint masses or with masses situated near the boundary (see Figure 5, concentrated masses are in black color). For

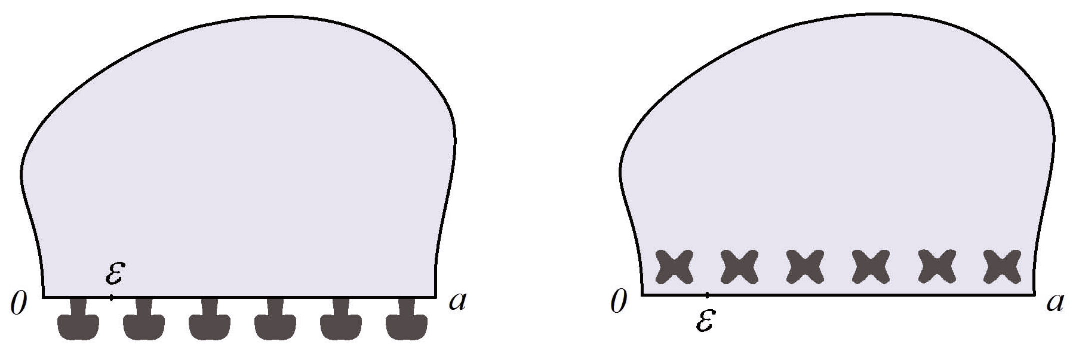

Figure 5: Domains with concentrated masses near the boundary.

these problems there will exist five qualitatively different cases (mentioned in the Introduction) and the skin-effect occurs for heavy masses starting from $\alpha > 1$. For such kind of problems we can construct the asymptotic approximations for eigenelements and prove asymptotic estimates similar those from Theorem 3.3. Also the asymptotic investigation proposed in the present paper can be applied to spectral problems with concentrated masses both in $3D$-thick cascade junctions and $3D$-domains like in Figure 5.

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