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The McKay-Conjecture for exceptional groups and odd primes

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Abstract

Let $\mathbf{G}$ be a simply-connected simple algebraic group over an algebraically closed field of characteristic p with a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ and $G := \mathbf{G}^F$, such that the root system is of exceptional type or G is a Suzuki-group or Steinberg's triality group. We show that all irreducible characters of $\mathrm{C}_G(\mathbf{S})$, the centraliser of $\mathbf{S}$ in G, extend to their inertia group in $\mathrm{N}_G(\mathbf{S})$, where $\mathbf{S}$ is any F-stable Sylow torus of $(\mathbf{G}, F)$. Together with the work in [17] this implies that the McKay-conjecture is true for G and odd primes ℓ different from the defining characteristic. Moreover it shows important properties of the associated simple groups, which are relevant for the proof that the associated simple groups are good in the sense of Isaacs, Malle and Navarro, as defined in [15].

1 Introduction

The McKay-conjecture claims that for any finite group H and any prime ℓ, the equation $|\mathrm{Irr}_{\ell'}(H)| = |\mathrm{Irr}_{\ell'}(\mathrm{N}_H(P))|$ holds for a Sylow ℓ-subgroup P of H, the normaliser $\mathrm{N}_H(P)$ of P in H and $\mathrm{Irr}_{\ell'}(H) := \{\chi \in \mathrm{Irr}(H) \mid \ell \nmid \chi(1)\}$. This paper proves this equality for exceptional groups H and odd primes ℓ, different from the defining characteristic of H. Furthermore it aims at making a step towards proving this longstanding conjecture by showing a certain extensibility property (Theorem A). With [17], this implies for any exceptional group of Lie type H the existence of a bijection $\mathrm{Irr}_{\ell'}(H) \to \mathrm{Irr}_{\ell'}(N)$ for a specified group $\mathrm{N}_H(P) \leq N \lneq H$. This is a part of a set of conditions on simple group, which according to [15], imply the McKay-conjecture for all finite groups. Theorem B gives some more technical details related to this condition.

In [15] the authors formulated a condition on simple groups, which verifies the McKay-conjecture. More precisely they show that a group H fulfils the McKay-conjecture for a prime ℓ, if all simple groups involved in H are good for ℓ.

Roughly speaking a simple group S is good for ℓ if for the maximal perfect central ℓ'-extension G of S the following conditions hold:

- There exists a group $N \lneq G$ with $N \geq \mathrm{N}_G(P)$ for some Sylow ℓ-subgroup P of G with $\mathrm{Aut}(G) = \langle \mathrm{N}_{\mathrm{Aut}(G)}(N), \mathrm{Inn}(G) \rangle$, where $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$ denote the groups of inner automorphism and all automorphisms of G respectively.

- There exists an $\mathrm{N}_{\mathrm{Aut}(G)}(P)$-equivariant bijection $' : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N)$.

- For each character $\chi \in \mathrm{Irr}_{\ell'}(G)$ one finds a group $\mathcal{M} \rhd G$ such that $\mathcal{M}$ induces all automorphisms stabilising χ on G and $\mathrm{C}_{\mathcal{M}}(G)$ is abelian. The characters χ and χ' have extensions $\widetilde{\chi}$ and $\widetilde{\chi}'$ to $\langle G, \mathrm{C}_{\mathcal{M}}(G) \rangle$ and $\langle N, \mathrm{C}_{\mathcal{M}}(G) \rangle$ respectively, such that $\mathrm{Irr}\left(\mathrm{C}_{\mathcal{M}}(G) \mid \widetilde{\chi} \right) = \mathrm{Irr}\left(\mathrm{C}_{\mathcal{M}}(G) \mid \widetilde{\chi}' \right)$, both characters are invariant in $\mathcal{M}$ and $\mathrm{N}_{\mathcal{M}}(N)$ respectively, and their associated elements in the Schur multipliers $\mathrm{M}\left(\mathcal{M}/\langle \mathrm{C}_{\mathcal{M}}(G), G \rangle \right)$ and $\mathrm{M}\left(\mathrm{N}_{\mathcal{M}}(N)/\langle \mathrm{C}_{\mathcal{M}}(G), N \rangle \right)$ are equal under the canonical isomorphism.

In [18] it was proven that all simple groups that are not of Lie type are good for every prime ℓ. If S is a simple group of Lie type, G can be assumed to be the fixed point subgroup of a simply-connected simple algebraic group under a Frobenius map. It was shown in [17] that, while ℓ differs from the defining characteristic, the normalisers of specific Sylow tori in G can be chosen as N to fulfil the first condition. Furthermore the desired bijection from the second condition was defined under the assumption that all irreducible characters χ of L extend to their inertia group in N, where L is the centraliser of the before mentioned Sylow torus $\mathbf{S}$. We verify this assumption for certain groups G.

The Sylow tori were first introduced in [5] and play a key role in the d-Harish-Chandra theory.

Theorem A *Let* $\mathbf{G}$ *be a simply-connected simple algebraic group defined over* $\overline{\mathbb{F}}_q$ *and* $F : \mathbf{G} \to \mathbf{G}$ *a Frobenius map, such that the root system of* $\mathbf{G}$ *is exceptional or such that* $\mathbf{G}^F$ *is a Suzuki-group or Steinberg's triality group. Furthermore let* $\mathbf{S}$ *be a Sylow torus of* $(\mathbf{G}, F)$. *Then all irreducible characters of* $L := \mathrm{C}_{\mathbf{G}^F}(\mathbf{S})$ *extend to their inertia group in* $N := \mathrm{N}_{\mathbf{G}^F}(\mathbf{S})$.

For the proof we determine the structure of the groups L and N, which enables us to prove the following.

Theorem B *Assume the setting of theorem A. Let* $\kappa \in \mathrm{Irr}(N)$, $\chi \in \mathrm{Irr}(L \mid \kappa)$ *and* $\phi \in \mathrm{Aut}(G)$ *stabilising* L, N *and* $\mathrm{Irr}(L \mid \kappa)$ *such that there exists an extension* $\widetilde{\chi}$ *of* χ *to* $\langle \phi, L \rangle$ *with* $\mathrm{I}_N(\chi) = \mathrm{I}_N(\widetilde{\chi})$. *Then* $\kappa^\phi = \kappa$.

With this result one should be able to describe how automorphisms act on $\mathrm{Irr}_{\ell'}(N)$. This is relevant for proving that the bijection is equivariant. The third condition is still being worked on.

Moreover theorem A enables us to verify the ordinary McKay-conjecture for exceptional groups of Lie type and odd primes. For $\ell = 3$ and $\mathbf{G}^F \in \left\{ \mathsf{G}_2(q), \, {}^2\mathsf{F}_4(2^{2f+1}) \right\}$ we refer to [1] and [16]. In all other cases the verification is an easy consequence of [17] and theorem A.

Theorem C *Let* $\mathbf{G}$ *and* $F : \mathbf{G} \to \mathbf{G}$ *be as in theorem A and* ℓ *an odd prime with* $\ell \nmid q$. *Then* $\left| \mathrm{Irr}_{\ell'} \left(\mathbf{G}^F \mid \nu \right) \right| = \left| \mathrm{Irr}_{\ell'} \left(\mathrm{N}_{\mathbf{G}^F}(P) \mid \nu \right) \right|$ *for every character* $\nu \in \mathrm{Irr} \left(\mathrm{Z}(\mathbf{G}^F) \right)$.

This paper is structured in the following way: We begin by recalling the basic definitions concerning Sylow tori and generic groups and introduce the general setting. Afterwards we define Sylow twists, a useful tool in the construction of centralisers and normalisers of Sylow tori.

We precede with proving theorem A in special cases: under the assumption that $\mathbf{S}$ is a Sylow 1-torus of $(\mathbf{G}, F)$ and F is a standard Frobenius endomorphism (section 5), in the situation where $\mathrm{C}_{\mathbf{G}^F}(\mathbf{S})$ is abelian (section 6) and in the remaining cases (section 7).

In sections 8 and 9 we prove theorem B. The proof of theorem C is given in the final section.

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2 Notation and settings

In this section we introduce the general framework for our further calculations, at the same time we establish some settings which we will later assume.

Setting 2.1 *Let* p *be a prime number and* $\mathbf{G}$ *be a simply-connected simple algebraic group with irreducible root system* R *over the algebraic closure* $\overline{\mathbb{F}}_p$ *of the field* $\mathbb{F}_p$. *We choose a fundamental system* R_F *of* R. *According to [23, 24] the group* $\mathbf{G}$ *can be seen as a finitely presented group with the generators* $x_\alpha(t)$ *(*$t \in \overline{\mathbb{F}}_p$, $\alpha \in R$*). We use the notation of [24, 8], in particular the definition of the elements* $h_\alpha(t')$, $n_\alpha(t')$ *and* $x_\alpha(t)$ *(*$t \in \overline{\mathbb{F}}_p$, $t' \in \overline{\mathbb{F}}_p^*$, $\alpha \in R$, *where* $\overline{\mathbb{F}}_p^*$ *denotes the multiplicative group of* $\overline{\mathbb{F}}_p$ *.)*

In this situation $\mathbf{T} := \left\langle h_\alpha(t) \mid \alpha \in R, t \in \overline{\mathbb{F}}_p^* \right\rangle$ *is a maximal torus of* $\mathbf{G}$ *with normaliser* $\mathbf{N} := \mathrm{N}_{\mathbf{G}}(\mathbf{T}) = \left\langle n_\alpha(t) \mid \alpha \in R, t \in \overline{\mathbb{F}}_p^* \right\rangle$. *The factor group* $\mathbf{N}/\mathbf{T}$ *is the Weyl group* W *of* $\mathbf{G}$, *hence isomorphic to the reflection group of* R. *We denote the corresponding epimorphism by*

$$\rho : \mathbf{N} \to W \ \ with \ n_\alpha(t) \mapsto w_\alpha \ for \ \alpha \in R, \ t \in \overline{\mathbb{F}}_p^*,$$

where w_α *is the reflection along the root* $\alpha \in R$.

The extended Weyl group $V := \langle n_\alpha(-1) \mid \alpha \in R \rangle$ *introduced in [25] plays a significant role in our further considerations. Due to the relation*

$$n_\alpha(t) = h_\alpha(t) n_\alpha(-1)$$

this group fulfils $\mathbf{N} = \langle V, \mathbf{T} \rangle$. *The Steinberg relations imply that* $H := \langle h_\alpha(-1) \mid \alpha \in R \rangle$ *satisfies* $H = V \cap \mathbf{T}$ *and has order* $(2, q-1)^{|R_F|}$. *(We denote the greatest common divisor of* $a, b \in \mathbb{Z}$ *(*$a, b > 0$*) by* (a, b).*)*

3

A finite reductive group is the subgroup of fixed points of a Frobenius endomorphism. We mainly use the following ones.

Setting 2.2 *We assume setting 2.1.*

(a) *For every power q of p a Frobenius endomorphism $F_0 : \mathbf{G} \to \mathbf{G}$ is defined by*

$$x_\alpha(t) \mapsto x_\alpha(t^q) \quad (\alpha \in R, \ t \in \overline{\mathbb{F}}_p) \ .$$

This map is often called a standard Frobenius map. If the Dynkin diagram associated to R_F has a length preserving symmetry $\sigma : R_F \to R_F$, the associated automorphism $\Gamma_0 : \mathbf{G} \to \mathbf{G}$ acts via

$$x_\alpha(t) \mapsto x_{\sigma(\alpha)}(t) \quad (\pm\alpha \in R_F, \ t \in \overline{\mathbb{F}}_p) \ .$$

The composition $\Gamma_0 \circ F_0$ is also a Frobenius map on $\mathbf{G}$ in the sense of [9, 1.17].

Let $\Gamma \in \langle \Gamma_0 \rangle$ and $F = F_0 \circ \Gamma$. The maximal torus $\mathbf{T}$ of $\mathbf{G}$ defined in 2.1 is then F-stable.

(b) *The triple $(\mathbf{G}, \mathbf{T}, F)$ defines a quintupel $\mathbb{G} := (X, R, Y, R^\vee, W\phi)$ called the generic group, where $X := \mathrm{Hom}(\mathbf{T}, \overline{\mathbb{F}}_p^*)$, $Y := \mathrm{Hom}(\overline{\mathbb{F}}_p^*, \mathbf{T})$, $R^\vee$ are the coroots of $\mathbf{G}$, ϕ is the automorphism of Y, respectively $\overline{Y} := Y \otimes \mathbb{C}$, induced by F, and $W\phi$ its coset in the automorphisms of Y. In addition there exists a perfect pairing $\langle -, - \rangle : X \times Y \to \mathbb{Z}$ with $\langle \alpha, \alpha^\vee \rangle = 2$ (see [6, section 2]).*

While the root datum $(X, R, Y, R^\vee)$ determines the reductive algebraic group over $\overline{\mathbb{F}}_p$ up to isomorphism, the group $\mathbf{G}^F$ is similarly determined by the corresponding generic group $\mathbb{G}$ and the prime power q. Thus we may also denote $\mathbf{G}^F$ by $\mathbb{G}(q)$. The polynomial order $|\mathbb{G}|$ of $\mathbb{G}$ is a monic polynomial of the form

$$|\mathbb{G}|\,(x) = x^N \prod_{d \in \mathbb{Z}_{>0}} \Phi_d^{a(d)}(x) \in \mathbb{Z}[x]$$

for some $N \in \mathbb{Z}_{\geq 0}$, the d-th cyclotomic polynomial Φ_d and $a(d) \in \mathbb{Z}_{\geq 0}$ and fulfils $|\mathbb{G}(q')| = |\mathbb{G}|\,(q')$ for every prime power q'.

Besides these reductive groups we also prove statements about Suzuki- and Ree-groups, for which we have the following, very similar setting.

Setting 2.3 *We assume setting 2.1 with $\mathbf{G} \in \left\{ \mathsf{B}_2(\overline{\mathbb{F}}_2), \mathsf{F}_4(\overline{\mathbb{F}}_2), \mathsf{G}_2(\overline{\mathbb{F}}_3) \right\}$ and $q := p^{2a+1}$ $(a \in \mathbb{Z}, 0 \leq a)$ and F_0 the standard Frobenius endomorphism of $\mathbf{G}$ associated to p^a. Then there exists an automorphism $\Gamma_0 : \mathbf{G} \to \mathbf{G}$ induced from the nontrivial symmetry σ of the Coxeter diagram of R_F acting via*

$$x_\alpha(t) \mapsto \begin{cases} x_{\sigma(\alpha)}(t^p) & \text{if } \alpha \text{ is short} \\ x_{\sigma(\alpha)}(t) & \text{otherwise} \end{cases} \quad (\pm\alpha \in R_F, \ t \in \overline{\mathbb{F}}_p) \ .$$

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The map $F := F_0 \circ \Gamma_0$ is also a Frobenius endomorphism and the group $\mathbf{G}^F$ is a Suzuki- or Ree-group. According to [5] the triple $(\mathbf{G}, \mathbf{T}, F)$ defines a quintupel $\mathbb{G} := (X, R, Y, R^\vee, W\phi)$ as well, called the (tp)-generic group, where X is the character lattice of $\mathbf{T}$, Y the coroot lattice, $\sqrt{q}\phi$ is the automorphism of $\mathbb{Z}[\sqrt{p}] \otimes_\mathbb{Z} Y$ induced by F and $W\phi$ a coset in the automorphisms of $\mathbb{Z}[\sqrt{p}] \otimes_\mathbb{Z} Y$.

In this case the polynomial order $|\mathbb{G}|$ is monic with $|\mathbb{G}|(x) = x^N \prod_\Psi \Psi^{a_\Psi}(x) \in \mathbb{Z}[\sqrt{p}][x]$ for some $N \in \mathbb{Z}_{\geq 0}$, the (tp)-cyclotomic polynomials Ψ, which have been defined in [5, 3.13], and $a_\Psi \in \mathbb{Z}_{\geq 0}$. It has similar properties.

We further mention some constructions of generic groups which will be used in the following sections.

Remark 2.4 (Tori and Levi subgroups of generic groups) *In [5] some substructures of generic groups are defined. They correspond to $\mathbf{G}^F$-conjugacy classes of F-stable subgroups of $\mathbf{G}^F$.*

Generic groups of the form $(^X/_{Y'^\perp}, Y', w\phi]_{Y'}) = (^X/_{Y'^\perp}, \emptyset, Y', \emptyset, w\phi]_{Y'})$ with $w \in W$ and a $w\phi$-stable sublattice Y' of Y are called tori of $\mathbb{G}$. (The sublattice $Y'^\perp$ of X is defined by the bilinear form on $X \times Y$ in the natural way.) The generic order of a generic torus $\mathbb{T} = (X, Y, \phi)$ coincides with the characteristic polynomial of ϕ on $Y \otimes \mathbb{C}$, i.e.,

$$|\mathbb{T}|(x) = \det_{Y \otimes \mathbb{C}} (x\phi - 1).$$

Analogously, the Levi subgroups of $\mathbb{G}$ are generic groups of the form $(X, R', Y, R'^\vee, W_{R'}w\phi)$ for $w\phi \in W\phi$ and a $w\phi$-stable parabolic subroot system R' of R. (The group $W_{R'}$ is the subgroup of W generated by the reflections along the roots of R'.) Furthermore one can associate to a generic torus $\mathbb{S} = (X', Y', w\phi]_{Y'})$ ($w \in W$) of $\mathbb{G}$ its centraliser in $\mathbb{G}$. This is the Levi subgroup $C_\mathbb{G}(\mathbb{S}) = (X, R', Y, R'^\vee, W_{R'}w\phi)$ with $R' := R \cap Y'^\perp$.

Generic groups of F-stable tori and F-stable Levi subgroups of $\mathbf{G}$ are generic tori and generic Levi subgroups of $\mathbb{G}$ respectively. Also taking the centraliser of a torus and computing the generic group commute with each other.

In the setting 2.2 the Sylow d-tori of $\mathbb{G}$ are defined to be the tori $\mathbb{S} := (X', Y', w\phi)$ of $\mathbb{G}$ with $|\mathbb{S}| = \Phi_d^{a(d)}$. In the situation described in 2.3 we call a torus $\mathbb{S}$ with $|\mathbb{S}| = \Psi^{a_\Psi}$ a Ψ-torus, where Ψ is a (tp)-cyclotomic polynomial. An F-stable torus of $\mathbf{G}$ whose generic group is a Sylow torus, is also called a Sylow torus. The existence and conjugacy in $\mathbf{G}^F$ of all Sylow d-tori of $(\mathbf{G}, F)$ was proven in [5, 3.4].

In our further considerations we will mainly concentrate an groups associated to Sylow tori.

Definition 2.5 *Let $\mathbf{S}$ be a Sylow d-torus or a Sylow Ψ-torus of $(\mathbf{G}, \mathbf{T}, F)$ respectively. We call $C_{\mathbf{G}^F}(\mathbf{S})$ Sylow d-Levi subgroup or Sylow Ψ-Levi subgroup respectively and $N_{\mathbf{G}^F}(\mathbf{S})$ the corresponding Sylow d-normaliser or the corresponding Sylow Ψ-normaliser respectively. By abuse of notation we call a group Sylow torus, if there exists $d \in \mathbb{Z}$ or a (tp)-cyclotomic polynomial Ψ, such that $\mathbf{S}$ is Sylow d-torus or Sylow Ψ-torus respectively.*

We call a Sylow torus **S** *a regular Sylow torus of* (**G**, *F*), *if its centraliser in* **G** *is a torus. One can show that a Sylow d-Levi subgroup is a torus iff d is a regular number of* $W\phi$ *in the sense of Springer, [22]. By abuse of notation we call d a regular number of* (**G**, *F*) *in this case.*

In several proofs we will not deal with Sylow *d*-Levi subgroups and Sylow *d*-normalisers of (**G**, *F*), instead we will deal with isomorphic groups which we get by the following remark.

2.6 *Let L be a Sylow d-Levi subgroup of* (**G**, *F*) *and L′ a Sylow d-Levi subgroup of* (**G**, *gF*) *for some* $g \in$ **G**. *Because of Lang's theorem [23, 4.4.17] there exists an element* g' *with* $g'F(g')^{-1} = g$. *Conjugating with* g' *maps the Sylow tori of* (**G**, *F*) *to the Sylow tori of* (**G**, *gF*). *As all Sylow d-tori of* (**G**, *F*) *are conjugate to each other in* **G**, *there exists an inner automorphism of* **G** *mapping L onto L′, i.e., there exists an element* $x \in$ **G** *with* $L^x = L'$. *This element also induces an isomorphism of the corresponding Sylow d-normalisers. Obviously the analogous statement about Sylow Ψ-tori is true.*

3 Sylow twists and some constructions

In the proof of theorem A we need a good way to construct Sylow *d*-Levi subgroups and Sylow *d*-normalisers. This will be done with the help of Sylow *d*-twists.

Definition 3.1 (Sylow *d*-twist and Sylow Ψ-twist) *(a) We assume setting 2.2 and let d be a positive integer. An automorphism of the form* $v\Gamma$ $(v \in$ **N**) *is called Sylow d-twist of* (**G**, *F*), *if*

$$\Phi_d^{a(d)} \big| \det_{\overline{Y}}(x\rho(v)\phi - 1),$$

i.e., $\Phi_d^{a(d)}$ *is a divisor of the characteristic polynomial of* $\rho(v)\phi$ *on* $\overline{Y}$.

(b) We assume to have the situation described in 2.3 and Ψ to be a (tp)-cyclotomic polynomial. An automorphism of the form $v\Gamma_0$ $(v \in$ **N**) *is called Sylow Ψ-twist of* (**G**, *F*) *if*

$$\Psi^{a_\Psi} \big| \det_{\overline{Y}}(x\rho(v)\phi - 1).$$

Altogether we call such automorphisms Sylow twist of (**G**, *F*).

In the case where *d* is a regular number of (**G**, *F*), Sylow twists can be constructed with the help of morphisms between the associated braid group, the group *V* and the Weyl group *W*:

Remark 3.2 *Assume setting 2.2. Let d be a regular number of* (**G**, *F*) *and* ϕ_W *the automorphism of W induced by F.*

(a) Then $v\Gamma$ *is a Sylow twist of* (**G**, *F*) *iff* $\rho(v)\phi_W$ *is a regular element of* $W\phi_W$.

(b) Let $\mathbf{B}$ be the braid group associated to W with generators $s_1, \ldots, s_{|R_F|}$, $\phi_{\mathbf{B}}$ the automorphism of $\mathbf{B}$ associated to ϕ_W, $\mathbf{w}_0 \in \mathbf{B}$ the element corresponding to the longest element w_0 in W, $\mathbf{w}\phi_{\mathbf{B}}$ a good d-th $\phi_{\mathbf{B}}$-root of $\mathbf{w}_0{}^2$ and $\tau : \mathbf{B} \to V$ the epimorphism with

$$s_i \mapsto n_{\alpha_i}(-1) \text{ for all } 1 \leq i \leq |R_F|.$$

Then $\tau(\mathbf{w})\Gamma$ is a Sylow twist of $(\mathbf{G}, F)$. We call the element $\tau(\mathbf{w}_0) = \widetilde{\mathbf{w}_0}$ the canonical representative of w_0 in $\mathbf{G}$.

Proof. This follows from the fact that $\rho(\tau(\mathbf{w}))$ is a regular element of W by [6, 3.14], hence every $n \in \rho^{-1}(w)$ is a Sylow twist according to the definition of regular elements in [22, 4.2]. Analogously [6, 6.6] and [22, 6.4] prove the equivalent result if Γ is nontrivial. $\square$

A Sylow d-twist $v\Gamma$ determines a generic Sylow d-torus of $\mathbb{G}$ and the corresponding algebraic torus of $(\mathbf{G}, vF)$, where vF acts via $x \mapsto F(x^v)$.

Lemma 3.3 (Construction of Sylow d-tori) *Assume setting 2.2. Let d be a regular number of $(\mathbf{G}, F)$ and $v\Gamma$ a Sylow d-twist. Further let $Y' := Y \cap \ker_{\overline{Y}}(\Phi_d(\rho(v)\phi))$, i.e., $Y' \otimes \mathbb{C}$ is the product of the eigenspaces of $\rho(v)\phi$ to primitive d-th roots of unity, and $X' := {}^{X}/{}_{Y'^{\perp}}$. Then the generic torus $\mathbb{S} := (X', Y', \rho(v)\phi)$ is a Sylow d-torus of $\mathbb{G}$ and*

$$\mathbf{S} := \left\{ t \in \mathbf{T} \ \middle| \ \lambda(t) = 1 \text{ for all } \lambda \in Y'^{\perp} \right\}$$

a Sylow d-torus of $(\mathbf{G}, vF)$.

Proof. According to the definition of Sylow d-twist and the order of a generic torus mentioned in 2.4 the triple $\mathbb{S}$ is a Sylow d-torus of $\mathbb{G}$. By definition $(\mathbf{S}, F|_{\mathbf{S}})$ corresponds to the generic group $\mathbb{S}$. $\square$

The associated Sylow d-Levi subgroup of $(\mathbf{G}, vF)$ is the following.

Lemma 3.4 (Associated Sylow d-Levi subgroup) *Let $\mathbf{X}_\alpha$ be the root subgroup corresponding to $\alpha \in R$. Then:*

(a) $C_{\mathbb{G}}(\mathbb{S}) = (X, R', Y, R'^{\vee}, W'\rho(v)\phi)$ *with* $R' := R \cap Y'^{\perp}$ *and* $W' := W_{R'}$,

(b) $C_{\mathbf{G}}(\mathbf{S}) = \langle \mathbf{T}, \mathbf{X}_\alpha \,|\, \alpha \in R' \rangle.$

Proof. The formula in (a) is the definition of $C_{\mathbb{G}}(\mathbb{S})$.

According to [10, Proposition 1.14] the group $\mathbf{L} := C_{\mathbf{G}}(\mathbf{S})$ is reductive. The maximal vF-stable torus $\mathbf{T}$ lies in $\mathbf{L} := \langle \mathbf{T}, \mathbf{X}_\alpha \,|\, \alpha \in R' \rangle$ and $(X, Y, \rho(v)\phi)$ is the complete root datum of $(\mathbf{T}, vF)$. Furthermore the root subgroups of $\mathbf{L}$ correspond to R'. $\square$

Now it remains to compute the associated Sylow normaliser. For a generic Levi subgroup $\mathbb{L}$ or Sylow normaliser $\mathbb{L} = (X, R', Y, R'^{\vee}, W_{R'}w\phi)$ of $\mathbb{G}$

$$W_{\mathbb{G}}(\mathbb{L}) = \left\{ w' \in W \mid w'(R') = R' \ (w'W_{R'})^{w\phi} = w'W_{R'} \right\}/W_{R'}$$

is called the *relative Weyl group*, see [5, 1B and 2.1.(3)]. This group coincides with $\mathrm{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}$. Furthermore we use the group $\mathbf{N} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ from 2.1.

Lemma 3.5 (Sylow normaliser) *Assume setting 2.2. Let $\mathbf{L} := \mathrm{C}_{\mathbf{G}}(\mathbf{S})$. Then*

(a) $\mathrm{N}_{\mathbf{G}^{vF}}(\mathbf{S}) = \mathbf{N}^{vF}$, if $R' = \emptyset$ and

(b) $\mathrm{N}_{\mathbf{G}^{vF}}(\mathbf{S}) = \langle U, \mathbf{L}^{vF} \rangle$ for every group $U \le \mathrm{N}_{\mathbf{G}^{vF}}(\mathbf{T})$ fulfilling

$$\langle \rho(U), W' \rangle / W' = W_{\mathbb{G}}(\mathbb{L}) \ and \ U \cap \mathbf{L} \le \mathbf{T}.$$

Proof. According to the definition of $\mathbf{L}$ and the properties of Sylow tori we have $N := \mathrm{N}_{\mathbf{G}^{vF}}(\mathbf{S}) = \mathrm{N}_{\mathbf{G}^{vF}}(\mathbf{L})$.

If $R' = \emptyset$ this shows $\mathbf{N}^{vF} \le \mathbf{N}$. According to [9, 3.3.6] the groups $\mathbf{N}^{vF}/\mathbf{T}^{vF}$ and $W_{\mathbb{G}}(\mathbb{L})$ coincide. Together with $W_{\mathbb{G}}(\mathbb{L}) = \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}$ from [6, 2.1.(3)] this proves (a).

Let U be as in (b) and $u \in U$. Because of $\mathbf{L} = \langle \mathbf{T}, \mathbf{X}_\alpha \mid \alpha \in R' \rangle$ it suffices to prove $\mathbf{T}^u = \mathbf{T}$ and $\mathbf{X}_\alpha^u \le \mathbf{L}$ for all $\alpha \in R'$. According to the properties of $\rho(U)$ the Steinberg relations imply $\mathbf{X}_\alpha^u \le \mathbf{L}$ for all $\alpha \in R'$, see for example [9, 2.5.15]. This shows $\langle U, L \rangle \le \mathrm{N}_{\mathbf{G}^{vF}}(\mathbf{S})$ with $L := \mathbf{L}^{vF}$. Because of $W_{\mathbb{G}}(\mathbb{L}) = \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}$ the equation $U \cap \mathbf{L} \le \mathbf{T}$ implies

$$\langle U, L \rangle / L \cong U/(U \cap L) \cong \rho(U) \cong \langle \rho(U), W' \rangle / W' = W_{\mathbb{G}}(\mathbb{L}) = N/L,$$

where we use the equation $\rho(U) \cap W' = \rho(U) \cap (\mathrm{N}_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{L}) \le \rho(\mathbf{T}) = 1$. $\square$

If $\mathbf{N}^{vF} = \langle \mathrm{C}_V(vF), \mathbf{T}^{vF} \rangle$, we call $v\Gamma$ *good Sylow twist*. This property is equivalent to $\rho(\mathrm{C}_V(v\Gamma)) = \mathrm{C}_W(\rho(v)\phi)$. The constructions introduced above can easily be transferred to setting 2.3. We omit the details.

4 Good and very good Sylow twists

In this and the two following sections we show theorem A in the case where $\mathbf{S}$ is a regular Sylow torus of $(\mathbf{G}, F)$.

Definition 4.1 (Maximal Extensibility) *Let $L \lhd N$ be finite groups and $\chi \in \mathrm{Irr}(L)$. An extension of χ to its inertia group $\mathrm{I}_N(\chi)$ in N is called maximal extension of χ in N and in this situation we call χ maximal extendible. If every character $\chi \in \mathrm{Irr}(L)$ is maximal extendible, we say that maximal extensibility holds with respect to $L \lhd N$.*

For example maximal extensibility holds with respect to $L \lhd N$ whenever N/L is cyclic by [14, 11.22]. Extending the standard notation we denote the group $\langle x \in U \mid \chi^x = \chi \rangle$ by $\mathrm{I}_U(\chi)$ for every subgroup $U \le N$. Very often we need the following well-known fact, which shortens our proofs that the characters are extendible..

Lemma 4.2 *Let $L \triangleleft N$, $U \leq N$ with $\langle L, U \rangle = N$ and L abelian. If maximal extensibility holds with respect to $U \cap L \triangleleft U$, it also holds with respect to $L \triangleleft N$.*

Proof. Let $\lambda \in \mathrm{Irr}(L)$ and $\widetilde{\delta}$ a maximal extension of $\delta := \lambda|_{U \cap L}$ to $\mathrm{I}_U(\delta)$. Then

$$\widetilde{\lambda}(vl) := \widetilde{\delta}(v)\lambda(l) \text{ for every } u \in \mathrm{I}_U(\lambda), l \in L$$

defines a maximal extension of λ.$\square$

In the situation of theorem A this lemma has the following consequence.

Corollary 4.3 *Assume setting 2.2 or 2.3. Let $v\Gamma$ be a good Sylow twist and let maximal extensibility hold with respect to $\mathrm{C}_H(v\Gamma) \triangleleft \mathrm{C}_V(v\Gamma)$. Then maximal extensibility holds with respect to $\mathbf{T}^{vF} \triangleleft \mathbf{N}^{vF}$, as well.*

Proof. By the assumptions on $v\Gamma$, the groups $\mathbf{T}^{vF}$, $\mathrm{C}_V(v\Gamma)$ and $\mathbf{N}^{vF}$ satisfy the assumptions on L, U and N in lemma 4.2, so the claim is a direct consequence of that result. $\square$

Sylow twists, for which the above assumption about maximal extensibility holds, will be called *very good*. In the next two sections we prove the following theorem, which implies theorem A, whenever $\mathbf{S}$ is a regular Sylow torus of $(\mathbf{G}, F)$.

Theorem D   *(a) Assume the situation of 2.2 with an exceptional root system R or such that $\mathbf{G}^F$ is Steinberg's triality group. Let d be a regular number for $(\mathbf{G}, F)$. Then there exists a very good Sylow d-twist of $(\mathbf{G}, F)$.*

  (b) Assume the situation of 2.3 and Ψ a (tp)-cyclotomic polynomial dividing the generic order associated to $(\mathbf{G}, F)$. Then there exists a very good Sylow Ψ-twist of $(\mathbf{G}, F)$.

There are two situations where one can easily verify the existence of very good Sylow twists.

Lemma 4.4 *Assume $2 | q$. Then theorem D holds.*

Proof. In the situation of part (a) we choose w to be a regular element of order d in W. For Ψ in (b) there exists a Sylow Ψ-torus $(X', Y', w\phi)$.

As the characteristic of the underlying field is 2 the groups V and W coincide. see 2.1. By lemma 3.2 we know that $n \in \rho(w)^{-1} \cap V$ is a Sylow d-twist. Furthermore $V = W$ implies that $v\Gamma$ is a very good Sylow twist, as $\mathrm{C}_H(n) = 1$. An analogous statement also holds in the situation of Suzuki- and Ree-groups.$\square$

The next situation concerns the cases where the relative Weyl group is cyclic. In this situation the extensibility property is guaranteed.

Lemma 4.5 *Assume setting 2.1.*

(a) *Let L be a Sylow Levi subgroup and N the associated Sylow normaliser such that N/L is cyclic. Then maximal extensibility holds with respect to $L \triangleleft N$.*

(b) *Every good Sylow twist $v\Gamma$ with cyclic $\mathrm{C}_W(\rho(v)\phi)$ is very good.*

Proof. This is a consequence of maximal extensibility with cyclic factor groups, see [14, 11.22]. $\square$

The following lemma is a sufficient criteria to assure, that a Sylow twist is very good.

Lemma 4.6 *Assume setting 2.2. Let d be a regular number of $(\mathbf{G}, F)$, j an odd divisor of d and $v\Gamma$ a Sylow d-twist of $(\mathbf{G}, F)$. Then $v\Gamma$ is very good, if $(v\Gamma)^j$ is a very good Sylow $\frac{d}{j}$-twist of $(\mathbf{G}, \Gamma^j \circ F_0)$.*

Proof. We begin by verifying that $v\Gamma$ is a good Sylow d-twist, i.e., $\rho(U) = \mathrm{C}_W(\rho(v)\Gamma)$ with $U := \mathrm{C}_V(v\Gamma)$.

The inclusion $\rho(U) \leq \mathrm{C}_W(\rho(v)\phi)$ is clear. Let $w \in \mathrm{C}_W(\rho(v)\phi)$. Because of $\mathrm{C}_W(\rho(v)\phi) \leq \mathrm{C}_W((\rho(v)\phi)^j)$ there exists an element $n \in U_j := \mathrm{C}_V((v\Gamma)^j) \leq U$ with $\rho(n) = w$.

As $v\Gamma$ induces on $K_j := \mathrm{C}_H((v\Gamma)^j)$ an automorphism of order j with fixed point subgroup $K := \mathrm{C}_H(v\Gamma)$, we can use the equation $K_j = K \times [K_j, v\Gamma]$ from [13, 14.5 (c)] about coprime actions. Let $k_2 \in K$ and $k_1 \in K_j$ such that $n^{-1}n^{v\Gamma} = k_2[k_1, v\Gamma]$.

This implies $(nk_1)^{v\Gamma} = nk_2[k_1, v\Gamma]k_1^{v\Gamma} = nk_1k_2$ and $nk_1 = (nk_1)^{(v\Gamma)^j} = nk_1(k_2)^j$. Because of $2 \nmid j$ this shows $k_2 = 1_K$, $nk_1 \in U$ and $w \in \rho(U)$. Hence $v\phi$ is a good Sylow d-twist of $(\mathbf{G}, F)$.

Now we have to show maximal extensibility with respect to $K \triangleleft U$. Let $\chi \in \mathrm{Irr}(K)$ and $\widetilde{\chi}$ an extension of χ on K_j with $\widetilde{\chi}|_{[K_j, v\phi]} = 1$.
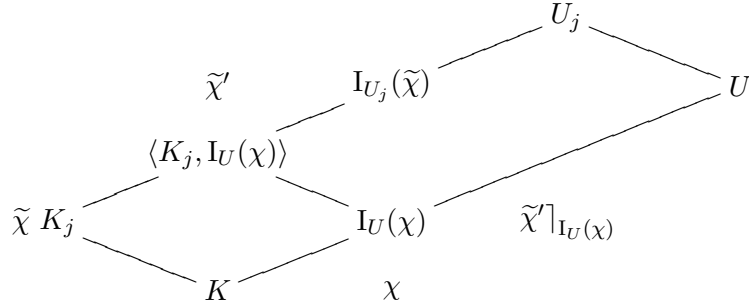
We know $U \leq \mathrm{N}_V(K_j)$ and $U \leq \mathrm{N}_{U_j}([K_j, v\phi])$ from the definitions of the groups U and K_j. This implies $\mathrm{I}_{U_j}(\widetilde{\chi}) \geq \mathrm{I}_U(\chi)$. According to the preliminaries there exists an extension $\widetilde{\chi}'$ of $\widetilde{\chi}$ to $\mathrm{I}_{U_j}(\widetilde{\chi})$, whose restriction to $\mathrm{I}_U(\chi)$ is a maximal extension of χ. $\square$

5 A special regular case

In this section we give a proof for the case $d = 1$ and $F = F_0$. Because of lemma 4.4 we further assume the underlying characteristic p to be odd. In this situation we establish a strong connection between closed subroot systems of $R^\vee$ and the characters $\mathrm{Irr}(H)$ and use this link to prove the following proposition.

Proposition 5.1 *Assume setting 2.2 with $F = F_0$ and $p \neq 2$. Then $1_{\mathbf{G}}$ is a very good Sylow 1-twist of $(\mathbf{G}, F)$.*

According to corollary 4.3 it suffices for the proof to show that maximal extensibility holds with respect to $H \lhd V$. There exists the following connection between subroot system and $\mathrm{Irr}\,(H)$.

Lemma 5.2 *Let $\chi \in \mathrm{Irr}\,(H)$.*

(a) $\mathcal{R}(\chi) := \{\alpha \in R^{\vee} \mid h_{\alpha^{\vee}}(-1) \in \ker(\chi)\}$ is a closed subroot system of $R^{\vee}$.

(b) $\ker(\chi) = \langle h_{\alpha}(-1) \mid \alpha^{\vee} \in \mathcal{R}(\chi) \rangle$.

Proof. Part (a) follows from the equation

$$(1) \qquad\qquad h_{\alpha^{\vee}}(-1) h_{\beta^{\vee}}(-1) = h_{(\alpha+\beta)^{\vee}}(-1),$$

which holds according to [12, 1.12.1].

For the equation in (b) we choose a set of elements which generate $\ker(\chi)$ and verify that they also lie in the group of the right hand side. Because of the inclusion $\ker(\chi) \subseteq \langle h_{\alpha}(-1) \mid \alpha^{\vee} \in \mathcal{R}(\chi) \rangle$ this suffices for the proof.

The group H is the direct product of the groups $\langle h_{\alpha}(-1) \rangle$ $(\alpha \in R_F)$. The group $\ker(\chi)$ is generated by the elements

- $h_{\beta}(-1)$ $(\beta \in R_F)$ fulfilling $\chi(h_{\beta}(-1)) = 1$ and

- $h_{\beta}(-1) h_{\beta'}(-1)$ $(\beta, \beta' \in R_F)$ with $\chi(h_{\beta}(-1)) = \chi(h_{\beta'}(-1)) = -1$.

Elements of the first type obviously lie in $\langle h_{\alpha}(-1) \mid \alpha^{\vee} \in \mathcal{R}(\chi) \rangle$.

Let $\beta, \beta' \in R_F \setminus \mathcal{R}(\chi)^{\vee}$ such that there exists a cycle-free path in the Dynkin diagram of $R_F{}^{\vee}$ with starting point $\beta^{\vee}$ and endpoint $\beta'^{\vee}$ crossing only vertices corresponding to roots $\delta \in R_F{}^{\vee} \cap \mathcal{R}(\chi)$. Let $\mathcal{M}$ be the subset of all these roots δ. It is well-known that $\gamma := \beta^{\vee} + \sum_{\alpha \in \mathcal{M}} \alpha + \beta'^{\vee}$ is a root in $R^{\vee}$. According to (1) the element $h_{\gamma^{\vee}}(-1)$ lies in $\ker(\chi)$. This fact implies $\gamma \in \mathcal{R}(\chi)$ and

$$h_{\beta}(-1) h_{\beta'}(-1) \in \left\langle h_{\alpha}(-1) \mid \alpha^{\vee} \in \mathcal{R}(\chi) \right\rangle.$$

All elements of the second type are products of such elements: For each pair of roots $\beta, \beta' \in R_F \setminus \mathcal{R}(\chi)^{\vee}$ there exists a cycle-free path in the Dynkin diagram with the coroot β as starting and β' as end point. The path intersects the set $R_F \setminus \mathcal{R}(\chi)^{\vee}$ in the roots $\{\gamma_1, \ldots, \gamma_l\}$, say with this order. The pairs $(\beta^{\vee}, \gamma_1)$, ..., $(\gamma_l, \beta'^{\vee})$ satisfy our above assumption, such that the corresponding elements in H like $h_{\beta}(-1) h_{\gamma_1}(-1)$ lie in $\langle h_{\alpha}(-1) \mid \alpha^{\vee} \in \mathcal{R}(\chi) \rangle$. The equation

$$h_{\beta}(-1) h_{\beta'}(-1) = (h_{\beta}(-1) h_{\gamma_1}(-1)) \cdots (h_{\gamma_l}(-1) h_{\beta'}(-1))$$

shows $h_{\beta}(-1) h_{\beta'}(-1) \in \langle h_{\alpha}(-1) \mid \alpha^{\vee} \in \mathcal{R}(\chi) \rangle$ and the equality stated in (b). $\square$

Moreover relation (1) implies

$$(2) \qquad \ker(\chi) = \left\langle h_\alpha(-1) \,\middle|\, \alpha \in \mathcal{R}(\chi)_F^\vee \right\rangle$$

for any choice of $\mathcal{R}(\chi)_F$ simple roots of $\mathcal{R}(\chi)$.

¿From this fact we can already deduce all possible subroot systems $\mathcal{R}(\chi)$ ($\chi \in \mathrm{Irr}\,(H)$) up to W-conjugacy, if R is a root system of type A_l. This result also produces some useful information in the remaining cases.

Lemma 5.3 *Let R be a root system of type A_l, $R_F = \{\alpha_1, \ldots, \alpha_l\}$, and $1 \neq \chi \in \mathrm{Irr}\,(H)$. Then there exists $1 \leq j \leq l$ and $n \in V$ such that*

$$\chi^n(h_{\alpha_i}(-1)) = \begin{cases} 1 & i \neq j, \\ -1 & i = j \end{cases} \quad \text{for any } 1 \leq i \leq l.$$

Proof. We know from [3] that every maximal closed subroot system R' of $R^\vee = R$ has a system of simple roots with $l - 1$ elements. Furthermore every closed subroot systems R' of R with $|R'_F| = l - 1$ must be a maximal one.

The closed subroot system $\mathcal{R}(\chi)$ is a maximal closed subroot system: As $\ker(\chi)$ is an elementary abelian group with 2^{l-1} elements and every set which generates $\ker(\chi)$ consists of at least $l - 1$ elements, equation (2) implies that the fundamental system of $\mathcal{R}(\chi)$ has at least $l - 1$ elements.

As $\mathcal{R}(\chi)$ is a maximal closed subroot system of R there exists an element $w \in W$ and $j \in \{1, \ldots, l\}$ such that the set $\{\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l\}$ is a system of simple roots for $\mathcal{R}(\chi)^w$. This is a consequence of the theorem of Borel-de Siebenthal, see [3].

Because of $\rho(V) = W$ we can choose an element $n \in V$ with $\rho(n) = w^{-1}$. According to the Steinberg relations j and n fulfil the equation stated above. $\square$

Remark 5.4 *The proof shows that $\mathcal{R}(\chi)$ is a maximal closed subroot system and any one can occur.*

Although the lemma deals only for root systems of type A_l it is also helpful with root systems of type D_4 or E_i ($6 \leq i \leq 8$).

Lemma 5.5 *Let R be a root system of type B_2, D_4, E_i ($6 \leq i \leq 8$), F_4 or G_2 and $\chi \in \mathrm{Irr}\,(H)$. Then $\mathrm{Stab}_W(\mathcal{R}(\chi)) = \langle w_\alpha \,|\, \alpha \in \mathcal{R}(\chi) \rangle$.*

Proof. This statement is mainly verified by computer calculations with the computer algebra system Magma [4]. First assume that R is not a simply-laced root system, i.e., is of type B_2, F_4 or G_2. For these root systems one calculates the group $\mathrm{Stab}_W(\mathcal{R}(\chi))/\langle w_\alpha \,|\, \alpha \in \mathcal{R}(\chi) \rangle$ and verifies the statement for all characters $\chi \in \mathrm{Irr}\,(H)$.

In the remaining cases the root system R is of type D_i ($i = 4$) or E_i ($6 \leq i \leq 8$) with root system R_F. There exists a subroot system R' of type A_{i-1} with a system of simple

roots $R'_F \subset R_F$. Without loss of generality we can assume that $\chi]_{\langle h_\alpha(-1)|\alpha \in R'_F\rangle}$ is trivial or of the form described in lemma 5.3, i.e., there exists a root $\beta \in R'_F$ with

$$\chi(h_\alpha(-1)) = \begin{cases} 1 & \alpha \in R'_F \setminus \{\beta\}, \\ -1 & \alpha = \beta. \end{cases}$$

There are only a few characters of $\mathrm{Irr}\,(H)$ fulfilling this condition, but every irreducible character of H is conjugate to such a character in V. For these characters we verify the statement above. $\square$

This statement is useful for constructing maximal extensions of χ in V:

Lemma 5.6 *Let R be a root system of type D_4, E_i $(6 \leq i \leq 8)$, F_4 or G_2 and H and V be as in 2.1. Then maximal extensibility holds with respect to $H \lhd V$.*

Proof. Let $\chi \in \mathrm{Irr}\,(H)$ be nontrivial. We first show $\rho(\mathrm{I}_V(\chi)) = \mathrm{Stab}_W(\mathcal{R}(\chi))$.

According to the Steinberg relations we have $h_\alpha(-1)^{n^{-1}} = h_{\rho(n)(\alpha)}(-1)$ for all $n \in V$. This implies
$$\rho(n)(\mathcal{R}(\chi)) = \mathcal{R}(\chi) \text{ for all } n \in \mathrm{I}_V(\chi),$$
hence $\rho(\mathrm{I}_V(\chi)) \leq \mathrm{Stab}_W(\mathcal{R}(\chi))$.

Each element $n \in V$ with $\rho(n) \in \mathrm{Stab}_W(\mathcal{R}(\chi))$ stabilises the set $\{h_\alpha(-1) \mid \alpha \in \mathcal{R}(\chi)^\vee\}$ by lemma 5.2 (b). This implies $\ker(\chi)^n = \ker(\chi)$. As a character of order 2 the character χ is uniquely determined by its kernel, hence $n \in \mathrm{I}_V(\chi)$. According to the previous lemma we have $\mathrm{I}_V(\chi) = \langle H, n_\alpha(-1) \mid \alpha \in \mathcal{R}(\chi)^\vee\rangle$. Calculations with the Steinberg relations show

$$\left\langle n_\alpha(-1) \,\middle|\, \alpha \in \mathcal{R}(\chi)^\vee \right\rangle \cap H = \left\langle h_\alpha(-1) \,\middle|\, \alpha \in \mathcal{R}(\chi)^\vee \right\rangle \leq \ker(\chi).$$

Hence there exists an extension $\widetilde{\chi} \in \mathrm{Irr}\,(\mathrm{I}_V(\chi))$ of χ with

$$\widetilde{\chi}(n_\alpha(-1)) = 1 \text{ for all } \alpha \in \mathcal{R}(\chi)^\vee.$$

Therefore maximal extensibility holds with respect to $H \lhd V$, if R is a root system of the given type. $\square$

Taken together, the above section proves proposition 5.1.

6 The remaining regular cases

We have already proven theorem D in the special case of proposition 5.1. For the remaining cases we use a mixture of computer calculations and reductions as well as the lemmas in section 4. We start with some results about Sylow 2-twists.

Lemma 6.1 *Assume setting 2.2 with $\mathbf{G}^F$ as in theorem A and $2 \nmid q$. Furthermore let $\mathbf{B}$, $\mathbf{w}_0$ and $\widetilde{\mathbf{w}}_0$ be defined like in remark 3.2. Then the following statements hold:*

(a) *The element* $\widetilde{\mathbf{w}}_0$ *is a very good Sylow* 2-*twist of* $(\mathbf{G}, F)$, *if* $\phi = \mathrm{id}_{\overline{Y}}$.

(b) *The element* $\widetilde{\mathbf{w}}_0 \Gamma$ *is a very good Sylow* 2-*twist of* $(\mathbf{G}, F)$ *and* Γ *is a very good Sylow* 1-*twist of* $(\mathbf{G}, F)$, *if* $o(\phi) = 2$ *and* R *is a root system of type* E_6.

Proof. According to remark 3.2 the element $\widetilde{\mathbf{w}}_0$ is a Sylow 2-twist with $\mathrm{C}_V(\widetilde{\mathbf{w}}_0) = V$, if R is not of type E_6. The longest element w_0 of W lies in the centre of W. According to [11, 1.5.1] this implies $\mathbf{w}_0 \in \mathrm{Z}(\mathbf{B})$, which moreover shows $\mathrm{C}_V(\widetilde{\mathbf{w}}_0) = V$. Because of $\mathrm{C}_V(\widetilde{\mathbf{w}}_0) = V$ and $\mathrm{C}_H(\widetilde{\mathbf{w}}_0) = H$ the element $\widetilde{\mathbf{w}}_0$ is a good Sylow 2-twist and a very good one according to proposition 5.1.

In the case, where R is a root system of type E_6, computer calculations show that $U := \mathrm{C}_V(\widetilde{\mathbf{w}}_0)$ is isomorphic to the extended Weyl group V' of the simply-connected algebraic group with a root system of type F_4 over $\overline{\mathbb{F}}_q$. The group H', the normal toral subgroup of V', is the image of $K := \mathrm{C}_H(\widetilde{\mathbf{w}}_0)$. As $\mathrm{C}_W(w_0)$ is a Coxeter group of type F_4, $\widetilde{\mathbf{w}}_0$ is a good Sylow 2-twist. Because of lemma 5.1 maximal extensibility holds with respect to $H' \lhd V'$ and $K \lhd U$. This shows (a).

For the proof of (b) we assume R to be a root system of type E_6. According to [11, 1.5.1] we have $\mathrm{C}_V(\widetilde{\mathbf{w}}_0 \Gamma) = V$. Here we can again use the arguments of (a) in order to show that $\widetilde{\mathbf{w}}_0 \Gamma$ is a very good Sylow 2-twist. The equation $\mathrm{C}_V(\widetilde{\mathbf{w}}_0 \Gamma) = V$ implies $\mathrm{C}_V(\widetilde{\mathbf{w}}_0) = \mathrm{C}_V(\Gamma)$, which shows that Γ is a very good Sylow 2-twist. $\square$

Before we concentrate on the remaining cases we should make a remark on two ideas that shorten the computer calculations. The extended Weyl group may have quite a huge order and is implemented in Magma [4] in such a way that group theory algorithms cannot be used anymore.

6.2 (Tools for computations) *As H is an elementary-abelian* 2-*group, the groups* $\mathrm{I}_N(\chi)$ *and* $\mathrm{N}_N(\ker(\chi))$ *coincide for every character* $\chi \in \mathrm{Irr}\,(H)$. *The character* χ *can be extended to* $\mathrm{I}_N(\chi)$ *if* $[\mathrm{I}_N(\chi), \mathrm{I}_N(\chi)] \cap H \leq \ker(\chi)$, *as* χ *is a linear character.*

We verify theorem A in the case where $\mathbf{S}$ is a regular Sylow torus of $(\mathbf{G}, F)$. Although the arguments of the proofs sometimes coincide we deal with the cases where F is a standard Frobenius endomorphism, a Frobenius endomorphism induced from a nontrivial graph automorphism, and where $\mathbf{G}^F$ is a Suzuki- or Ree-group in separate propositions.

Proposition 6.3 *We assume setting 2.2 where $F = F_0$ and $\mathbf{G}^F$ is one of the groups mentioned in theorem A. Let d be a regular number of $(\mathbf{G}, F)$. Then there exists a very good Sylow twist of $(\mathbf{G}, F)$.*

Proof. We have already proven the statement for $d \in \{1, 2\}$ in proposition 5.1 and lemma 6.1

Assume first that $\mathrm{C}_W(w)$ is cyclic for a regular element $w \in W$ of order d. Let w' be a generating element of $\mathrm{C}_W(w)$ and $j \in \mathbb{Z}$ such that $w = (w')^j$. Let s_i ($1 \leq i \leq l := |R_F|$) be the reflections along the simple roots of R in W and $\mathrm{r} : W \to \mathbf{B}$ be the map with $\mathrm{r}(s_{i_1} \cdots s_{i_k}) = \mathrm{s}_{i_1} \cdots \mathrm{s}_{i_k}$ ($i_1, \ldots, i_k \in \{1, \ldots, l\}$) whenever $s_{i_1} \cdot \ldots \cdot s_{i_l}$ is a reduced

expression in W. Then the element $v = \tau(\mathrm{r}(w')^j)$ is a very good Sylow d-twist of $(\mathbf{G}, F)$, as $\tau(\mathrm{r}(w'))$ lies in $\mathrm{C}_V(v)$ by construction. This deals with the case where $\mathrm{C}_W(w)$ is cyclic.

Type	d	$\mathrm{C}_W(\rho(v))$ is cyclic	verified with	reduced to
E_6	3			$d = 1$ with $v = 1_V$
	4		$v = \tau(\mathrm{r}(w')^2)$	
	6			$d = 2$ with $v = \widetilde{\mathbf{w}_0}$
	$8, 9, 12$	$\times$		
E_7	$3, 5$			$d = 1$ with $v = 1_V$
	6			$d = 2$ with $v = \widetilde{\mathbf{w}_0}$
	$7, 9, 14, 18$	$\times$		
E_8	$3, 5$			$d = 1$ with $v = 1_V$
	$6, 10$			$d = 2$ with $v = \widetilde{\mathbf{w}_0}$
	4		$v = \tau(\mathrm{r}(w)^6)$	
	8		$v = \tau(\mathrm{r}(w)^3)$	
	12			$d = 4$ with $v = \tau(\mathrm{r}(w)^6)$
	$15, 20, 24, 30$	$\times$		
F_4	3			$d = 1$ with $v = 1_V$
	4		$v = \tau(\mathrm{r}(c^3))$	
	6			$d = 2$ with $v = \widetilde{\mathbf{w}_0}$
	$8, 12$	$\times$		
G_2	3			$d = 1$ with $v = 1_V$
	6			$d = 2$ with $v = \widetilde{\mathbf{w}_0}$

Table 1: Proof scheme for the regular numbers $d > 2$ when $F = F_0$

For some d we verified the existence of very good Sylow d-twists of $(\mathbf{G}, F)$ by computer calculations. Hereby we restrict ourselves to regular numbers $d > 2$ of $(\mathbf{G}, F)$, which are 2-powers and whose associated relative Weyl group is non-cyclic. We obtain Sylow d-twists from good roots of $\mathbf{w}_0{}^2$ according to remark 3.2. One can find a list of such elements in [6, Appendix 1]. We use the notation of the elements introduced there. Table 1 reflects which element v can be chosen for the calculations.

In the remaining cases we use theorem 4.6. We can find a good root $\mathbf{w}$ of $\mathbf{w}_0{}^2$ in the table of [6, Appendix 1], such that $\tau(\mathbf{w}^j)$ is a very good Sylow d-twist of $(\mathbf{G}, F)$ and j the $2'$-part of d, the biggest odd integer dividing d. According to remark 3.2 the element $\tau(\mathbf{w})$ is a Sylow d-twist and proposition 4.6 shows that it is a very good Sylow d-twist of $(\mathbf{G}, F)$. $\square$

In the next proposition we assume that F is a Frobenius endomorphism induced by a nontrivial graph automorphism.

Proposition 6.4 *Assume the situation of theorem D (a) with $\Gamma \neq \mathrm{id}_{\mathbf{G}}$ and $2 \nmid q$. Then there exists a very good Sylow d-twist of $(\mathbf{G}, F)$.*

Proof. The main ingredients of the proof are the same as in the previous proposition. Only the arguments in the situation where the relative Weyl group is cyclic have to be revisited.

For $d \in \{1, 2\}$ the statement was proven in lemma 6.1 (b), when R is a root system of type E_6. For some regular numbers we may again use theorem 4.6.

If d is a regular number such that the relative Weyl group of a Sylow d-Levi subgroup of $(\mathbf{G}, F)$ is cyclic, we have to prove that good Sylow d-twists exist. These are then very good according to lemma 4.5.

If R is a root system of type E_6, the regular numbers with this property are $\{8, 12, 18\}$. Let $\mathbf{w}\phi_{\mathbf{B}}$ be a good d-th $\phi_{\mathbf{B}}$-root of $\mathbf{w}_0{}^2$ from [6, Appendix 1], where $\phi_{\mathbf{B}}$ is the automorphism of $\mathbf{B}$ induced on the braid group $\mathbf{B}$ of R by ϕ. According to [6, Appendix 1] this element can be assumed to be $\phi_{\mathbf{B}}$-invariant and to have order d. The element $v := \tau(\mathbf{w})$ fulfils $v \in \mathrm{C}_V(v\Gamma)$, $\langle v \rangle \leq \mathrm{C}_V(v\Gamma)$ and $\langle \rho(v) \rangle \leq \rho(\mathrm{C}_V(v\Gamma))$. Because of $\mathrm{C}_W(\rho(v\Gamma)) \cong \mathrm{C}_d$ and $\mathrm{C}_d \cong \langle \rho(v) \rangle \leq \rho(\mathrm{C}_V(v\Gamma)) \leq \mathrm{C}_W(v\Gamma)) \cong \mathrm{C}_d$ the element $v\Gamma$ is a good Sylow d-twist of $(\mathbf{G}, F)$, where C_i denotes the cyclic group of order i.

If R is a root system of type D_4 and $o(\Gamma) = 3$, we again use $v\Gamma$ with $v := \tau(\mathbf{w})$ as a Sylow d-twist, where $\mathbf{w}$ is a good d-th ϕ-root of $\mathbf{w}_0{}^2$ from the list in [6, Appendix 1]. We can calculate $\mathrm{C}_W(\rho(v)\Gamma) \cong \mathrm{C}_4$. As $(v\Gamma)^3 \in \mathrm{C}_V(v\Gamma)$, we get

$$\mathrm{C}_4 \cong \left\langle (v\Gamma)^3 \right\rangle \leq \rho(\mathrm{C}_V(v\Gamma)).$$

This proves that $v\Gamma$ is a good Sylow d-twist and shows that there exists a very good Sylow d-twist of $(\mathbf{G}, F)$, if the relative Weyl group is cyclic.

Type	d	$\mathrm{C}_W(\rho(v)\Gamma)$ is cyclic	verified for	reduced to
${}^2\mathsf{E}_6$	3			$d = 1$ with $v\Gamma = \Gamma$
	4		$v\Gamma = \tau(\mathrm{r}(c^3))\Gamma$	
	6			$d = 2$ with $v\Gamma = \widetilde{\mathbf{w}_0}\Gamma$
	$8, 12, 18$	$\times$		
${}^3\mathsf{D}_4$	1		$v\Gamma = \Gamma$	
	2		$v\Gamma = \widetilde{\mathbf{w}_0}\Gamma$	
	3			$d = 1$ with $v = 1_V$
	6			$d = 2$ with $v = \widetilde{\mathbf{w}_0}$
	12	$\times$		

Table 2: Proof scheme when d is regular and $\Gamma \neq 1$

In the remaining cases we verify that a Sylow twist constructed with a good root of $\mathbf{w}_0{}^2$ is very good or we can deduce the existence of a very good Sylow twist with the help of proposition 4.6. Again the procedure for the various regular numbers can be found in table 2.
$\square$

The previous two propositions show theorem D (a). The following lemma deals with part (b).

Lemma 6.5 *Assume setting 2.3 and Ψ to be a (tp)-cyclotomic polynomial such that the Sylow Ψ-torus of $(\mathbf{G}, F)$ is nontrivial. Then there exists a very good Sylow Ψ-twist of $(\mathbf{G}, F)$.*

Proof. We first show that the centraliser of each Sylow Ψ-torus is a torus. Let $(X/Y'^{\perp}, Y', w\phi)$ be a Sylow Ψ-torus with $Y' := \ker_{\overline{Y}}(\Psi(w\phi)) \cap Y$. As Ψ^{a_Ψ} is the characteristic polynomial of $w\phi$ on Y' we know that the dimension of $Y' \otimes \mathbb{C}$ and the degree of Ψ^{a_Ψ} coincide. ¿From the study of the (tp)-cyclotomic polynomials in [5] one can see that the degree of Ψ^{a_Ψ} and $\dim(\overline{Y})$ coincide and hence every Sylow torus is regular.

Furthermore ${}^2\mathsf{F}_4(q)$ and ${}^2\mathsf{B}_2(q)$ are defined over fields of even characteristic. With the methods from the proof of lemma 4.4 we find a very good Sylow Ψ-twist. ¿From [5] we know that a Sylow Ψ-torus $(X', Y', w\phi)$ of $(\mathbf{G}, F)$ exists. Then the automorphism $v\Gamma$ with the unique element $v \in V$ such that $\rho(v) = w$ is a very good Sylow Ψ-twist of $(\mathbf{G}, F)$.

In the remaining case computer calculations show that good Sylow Ψ-twists of $(\mathbf{G}, F)$ exist, which are very good as the associated relative Weyl groups are cyclic. $\square$

Altogether in the last three sections we have proven the following statement.

Proposition 6.6 *Assume setting 2.1, such that $\mathbf{G}^F$ is one of the groups mentioned in theorem A. Let $\mathbf{S}$ be a regular Sylow torus of $(\mathbf{G}, F)$, L the associated Sylow Levi subgroup and N the associated Sylow normaliser. Then maximal extensibility holds with respect to $L \lhd N$.*

Proof. By theorem D very good Sylow twists exist. By corollary 4.3 this ensures maximal extensibility. $\square$

The next section is concerned with the proof of the analogous statement when $\mathbf{S}$ is a nonregular Sylow torus of $(\mathbf{G}, F)$.

7 The nonregular cases

The aim of this section is twofold: On the one hand we verify theorem A in the remaining cases, where $\mathbf{S}$ is a nonregular Sylow torus of $(\mathbf{G}, F)$.

On the other hand we give precise informations about the structure of the Sylow Levi subgroups and Sylow normalisers. These build the foundation for the proof of theorem B in the succeeding sections.

The first three columns of table 3 represent the remaining cases: whenever $\mathbf{G}^F$ and d are as listed in the table, every Sylow d-torus of $(\mathbf{G}, F)$ is nonregular. The third column gives the isomorphism type of N/L, where L is a Sylow d-Levi subgroup and N the associated Sylow d-normaliser. The group G_8 denotes the complex reflection group introduced in [19].

As the table shows, for the proof of theorem A it suffices to restrict to the case where $d = 4$ and the underlying root system is of type E_7, since N/L is cyclic otherwise. Because

of our ambition to prove theorem B we also analyse the structure of L and N in the other cases.

Like in the proof of Proposition 5.1 we first extend a linear character of a subgroup of L and construct with this character the desired maximal extension.

We now sketch the main ideas for the following proof.

7.1 (Procedure in the nonregular case:) *We proceed in the following steps:*

- *Choose an F-stable subgroup $\mathbf{G}' \leq \mathbf{G}$ and a Sylow d-twist $v\Gamma$ with nice properties.*

- *Determine a Sylow d-Levi subgroup L of $(\mathbf{G}, vF)$ and a related one L' of $(\mathbf{G}', vF\rvert_{\mathbf{G}'})$ to obtain informations on the inner structure of L.*

- *Construct the associated Sylow d-normalisers N and N'.*

- *Deduce the maximal extensibility with respect to $L \lhd N$.*

For every nonregular number mentioned in table 3 we perform the following computations:

(a) We choose an element $w \in W$ and $v \in V$ with $\rho(v) = w$ such that

$$\dim(Y' \otimes \mathbb{C}) = a(d)\Phi_d(1)$$

for $Y' := \ker_{\overline{Y}}(\Phi_d(w\phi)) \cap Y$. Hence the automorphism $v\Gamma$ is a Sylow d-twist of $(\mathbf{G}, F)$.

(b) Let $R_1 := Y'^{\perp} \cap R$ and $R_2 := R_1^{\perp} \cap R$. Computer calculations show that $R_1 \cup R_2$ and R generate the same subspaces of $\overline{Y}$.

To these root systems we associate the algebraic groups

$$\mathbf{T}_1 := \langle h_\alpha(t) \in \mathbf{G} \mid \alpha \in R_1 \rangle \text{ and } \mathbf{G}_2 := \langle \mathbf{X}_\alpha \mid \alpha \in R_2 \rangle,$$

and define their finite counterparts $T_1 := \mathbf{T}_1^{vF}$ and $G_2 := \mathbf{G}_2^{vF}$. By the property of R_1 and R_2 mentioned above and by definition, the finite subgroup $L := \langle \mathbf{T}_1, \mathbf{G}_2 \rangle^{vF}$ forms a Sylow d-normaliser of $(\mathbf{G}, vF)$. The finite groups have the following properties:

- $[\mathbf{T}_1, \mathbf{G}_2] = 1$,
- $L_0 := \langle T_1, G_2 \rangle \lhd L$ with $L/L_0 \cong Z := T_1 \cap G_2$

The subgroup structure can be seen in the following diagram.

$$
\begin{array}{c}
L \\
\mid Z \\
L_0 = T_1 \circ_Z G_2
\end{array}
$$

T_1 $\diagup$ $\diagdown$ G_2

$$
\begin{array}{c}
Z \\
\mid \\
1
\end{array}
$$
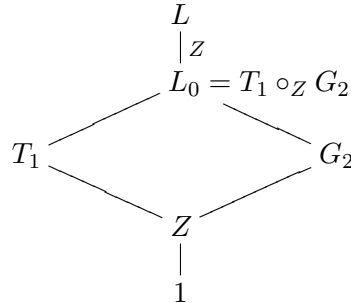
(c) Whereas the previous procedure was possible without a careful choice of w and v, the next steps depend on these choices: We can always choose v and w such that there exists a subgroup $U \leq \mathrm{C}_V(v\Gamma)$ with $U \cap L < L_0$ and $\langle U, L \rangle$ is the Sylow d-normaliser associated to L. In some cases U is a cyclic group generated by $v\Gamma$ or $(v\Gamma)^2$ respectively. Furthermore one hopes to get a quite small group U/U_0 with $U_0 := \mathrm{C}_U(G_2)$.

Performing the above procedure with the computer gives the following table, where S_3 denotes the symmetric group acting on three points. The calculations also show $U \cap \mathbf{T} \leq L_0$.

$\mathbf{G}^F$	d	N/L	Type of R_2	G_2	Z $i := \gcd(j, q-1)$	U/U_0
$\mathsf{E}_{6,sc}(q)$	5	C_5	A_1	$\mathsf{A}_1(q)$	C_i $(j := 2)$	1
$\mathsf{E}_{7,sc}(q)$	5	C_{10}	A_2	$\mathsf{A}_2(q)$	C_i $(j := 3)$	C_2
$\mathsf{E}_{7,sc}(q)$	10	C_{10}	A_2	$^2\mathsf{A}_2(q)$	C_i $(j := 3)$	C_2
$\mathsf{E}_{7,sc}(q)$	4	G_8	A_1^3	$\mathsf{A}_1(q)^3$	$\mathrm{C}_i \times \mathrm{C}_i$ $(j := 2)$	S_3
$\mathsf{E}_{7,sc}(q)$	8	C_8	A_1^3	$\mathsf{A}_1(q^2) \times \mathsf{A}_1(q)$	C_i $(j := 2)$	C_2
$\mathsf{E}_{7,sc}(q)$	12	C_{12}	A_1^3	$\mathsf{A}_1(q^3)$	1	C_3
$\mathsf{E}_{8,sc}(q)$	7	C_{14}	A_1	$\mathsf{A}_1(q)$	C_i $(j := 2)$	1
$\mathsf{E}_{8,sc}(q)$	14	C_{14}	A_1	$\mathsf{A}_1(q)$	C_i $(j := 2)$	1
$\mathsf{E}_{8,sc}(q)$	9	C_{18}	A_2	$\mathsf{A}_2(q)$	C_i $(j := 3)$	C_2
$\mathsf{E}_{8,sc}(q)$	18	C_{18}	A_2	$^2\mathsf{A}_2(q)$	C_i $(j := 3)$	C_2
$^2\mathsf{E}_{6,sc}(q)$	10	C_5	A_1	$\mathsf{A}_1(q)$	C_i $(j := 2)$	1

Table 3: The structure of Sylow Levi subgroups

During the next proof we need the following remark, which deals with characters on central products.

7.2 (Characters on central products) *A group A is the central product of $B_1 \leq A$ and $B_2 \leq A$, denoted by $A = B_1 \circ_{B_1 \cap B_2} B_2$ or $A = B_1 \circ B_2$ for short, if $[B_1, B_2] = 1$ and $A = \langle B_1, B_2 \rangle$. According to [14, 4.21] the irreducible characters of A can be deduced from $\mathrm{Irr}(B_1)$ and $\mathrm{Irr}(B_2)$: for every irreducible character $\chi \in \mathrm{Irr}(A)$ there exist unique characters $\lambda \in \mathrm{Irr}(B_1)$ and $\eta \in \mathrm{Irr}(B_2)$ with $\mathrm{Irr}(B_1 \cap B_2 \mid \lambda) = \mathrm{Irr}(B_1 \cap B_2 \mid \eta)$ and*

$$\chi(b_1 b_2) = \lambda(b_1)\eta(b_2) \text{ for all } b_1 \in B_1, \ b_2 \in B_2.$$

We denote χ by $\lambda.\eta$ in this situation. The set $\mathrm{Irr}(B_1 \cap B_2 \mid \lambda)$ is defined as $\left\{ \psi \in \mathrm{Irr}(B_1 \cap B_2) \mid (\psi]_{B_1 \cap B_2}, \lambda) \neq 0 \right\}$, where $(-, -)$ denotes the usual scalar product on class functions.

Now we finish the proof of theorem A and prove the only non-cyclic, nonregular case.

Lemma 7.3 *Assume $\mathbf{G} = \mathsf{E}_{7,sc}(\overline{\mathbb{F}}_p)$, $F : \mathbf{G} \to \mathbf{G}$ to be the standard Frobenius endomorphism and $\mathbf{S}$ to be a Sylow 4-torus of $(\mathbf{G}, F)$. Let $L := \mathrm{C}_{\mathbf{G}^F}(\mathbf{S})$ and $N := \mathrm{N}_{\mathbf{G}^F}(\mathbf{S})$. Then maximal extensibility holds with respect to $L \lhd N$.*

Proof. We use the construction of L and N above. According to the calculations $\mathbf{G}_2$ is the central product of three vF-stable groups, which are isomorphic to $\mathrm{SL}_2(\overline{\mathbb{F}}_p)$ and will be called $\mathbf{G}_{2,i}$ ($i = 1, 2, 3$). Here we denote the finite subgroups isomorphic to $\mathrm{SL}_2(q)$ by $G_{2,i} := \mathbf{G}_{2,i}^{vF}$ ($i = 1, 2, 3$). We assume $\mathbf{S}$ to be defined with a Sylow 4-twist $v \in V$ of $(\mathbf{G}, F)$. Later we will use an associated Lang map $\mathcal{L} : \mathbf{G} \to \mathbf{G}$ with $x \mapsto F(x^v)x^{-1}$.

Let $\chi \in \mathrm{Irr}(L)$ and $\chi_0 \in \mathrm{Irr}(L_0 \mid \chi)$. We first prove maximal extensibility with respect to $L_0 \lhd L$. Let $\widetilde{T}_1 := \{x \in \mathbf{T}_1 \mid \exists g \in \mathbf{G}_2 : xg \in L_{\chi_0}\}$ with $L_{\chi_0} := \mathrm{I}_L(\chi_0)$, $\widetilde{G}_{2,i} := \{x \in \mathbf{G}_{2,i} \mid \exists g \in \langle \mathbf{T}_1, \mathbf{G}_{2,j} | j \neq i\rangle : xg \in L_{\chi_0}\}$ for $i \in \{1, 2, 3\}$ and $\widetilde{L}_0 := \widetilde{T}_1 \circ \widetilde{G}_{2,1} \circ \widetilde{G}_{2,2} \circ \widetilde{G}_{2,3}$.

According to the definition $\widetilde{L}_0$ satisfies $\widetilde{L}_0 \cap L = L_{\chi_0}$ and $\mathrm{I}_{\widetilde{L}_0}(\chi_0) = \widetilde{L}_0$.

Now we extend χ_0 to $\widehat{\chi}_0 \in \mathrm{Irr}\left(\widehat{L}_0\right)$ with $\left(\widehat{\chi}\rceil_{L_{\chi_0}}\right)^L = \chi$. According to remark 7.2 we find unique characters $\lambda \in \mathrm{Irr}(T_1)$, $\eta_i \in \mathrm{Irr}(G_{2,i})$ ($i \in \{1, 2, 3\}$) with $\chi_0 = \lambda.\eta_1.\eta_2.\eta_3$. As $\widetilde{T}_1$ is abelian and the groups $\widetilde{G}_{2,i}/G_{2,i}$ are cyclic, the characters λ, η_i can be extended to $\widehat{\lambda} \in \mathrm{Irr}\left(\widetilde{T}_1\right)$, $\widehat{\eta}_i \in \mathrm{Irr}\left(\widetilde{G}_{2,i}\right)$ ($i \in \{1, 2, 3\}$) respectively. The character $\widehat{\lambda}.\widehat{\eta}_1.\widehat{\eta}_2.\widehat{\eta}_3$ is by definition an extension of χ_0. As $\widetilde{L}_0/L_0$ is abelian, the theorem of Gallagher [14, 6.17] implies

$$\left\{ \psi\rceil_{L_{\chi_0}} \;\middle|\; \psi \in \mathrm{Irr}\left(\widetilde{L}_0 \;\middle|\; \chi_0\right) \right\} = \mathrm{Irr}\left(L_{\chi_0} \mid \chi_0\right).$$

According to the Clifford correspondence [14, 6.11] there exists $\widehat{\chi}_0 \in \mathrm{Irr}\left(\widetilde{L}_0 \;\middle|\; \chi_0\right)$ with $\chi = \left(\widehat{\chi}\rceil_{L_{\chi_0}}\right)^L$.

The group $\widetilde{L}_0$ is a normal subgroup of $\widetilde{N}_0 = \widetilde{N}_1 \circ \widetilde{G}_{2,1} \circ \widetilde{G}_{2,2} \circ \widetilde{G}_{2,3}$ with $U_0 := \mathrm{C}_U(\mathbf{G}_2)$ and $\widetilde{N}_1 := \left\langle \widetilde{T}_1, U_0 \right\rangle$. The group $\mathrm{I}_{\widetilde{N}_1}(\widehat{\chi}_0)$ coincides with $\left\langle \mathrm{I}_{U_0}(\chi), \widetilde{L}_0 \right\rangle$. The characters of $\mathrm{Irr}(L_0 \mid \chi)$ differ in their restriction to G_2. Because of $[U_0, \mathbf{G}_2] = 1$ the character χ_0 is invariant in $\mathrm{I}_{U_0}(\chi)$. As elements of U_0 also act trivially on L/L_0, each element $u \in \mathrm{I}_{U_0}(\chi)$ stabilises $\widehat{\chi}_0\rceil_{L_{\chi_0}}$ and this implies $\widehat{\chi}_0^u = \widehat{\chi}_0$.

Now we construct a maximal extension of $\widehat{\chi}_0$ in $\widetilde{N}_0$. Computer calculations prove that one can choose a subgroup $U \leq V$ such that $\langle L, U\rangle = N$, $L \cap U \leq T_1$ and that maximal extensibility holds with respect to $U \cap T_1 \lhd U$.

Let $\widetilde{\lambda}_0$ be a maximal extension of $\lambda_0 := \widehat{\lambda}\Big|_{U \cap T_1}$ in U. With the help of $\widetilde{\lambda}_0$ we may define $\widetilde{\lambda}$, a maximal extension of $\widehat{\lambda}$ in $\widetilde{N}_1$ according to the formula in the proof of 4.2. The character $\widetilde{\lambda}.\widehat{\eta}_1.\widehat{\eta}_2.\widehat{\eta}_3$ is a maximal extension of $\widehat{\chi}_0$ in $\widetilde{N}_0$. Analogously the character $\widetilde{\chi}_0' := \left(\widetilde{\lambda}.\widehat{\eta}_1.\widehat{\eta}_2.\widehat{\eta}_3\right)\Big|_{\mathrm{I}_{\widetilde{N}_0}(\widehat{\chi}_0) \cap N}$ is a maximal extension of χ_0 in $\langle U_0, L\rangle$.

By definition the character $\widetilde{\chi}_0'$ is invariant in $\mathrm{I}_U(\chi_0) \cap \mathrm{I}_U(\chi)$. As all Sylow subgroups of U/U_0 are cyclic, lemma [14, 11.22] implies that $\widetilde{\chi}_0'$ is extendible to $\langle \mathrm{I}_U(\chi_0) \cap \mathrm{I}_U(\chi), L_{\chi_0}\rangle$. For a maximal extension $\widetilde{\chi}_0$ of $\widehat{\chi}\rceil_{L_{\chi_0}}$ in N the character $(\widetilde{\chi}_0)^{\mathrm{I}_N(\chi)}$ is a maximal extension of χ in N.

A close look at the action of L/L_0 and U/U_0 on $\eta_1.\eta_2.\eta_3$ shows that after conjugation with a suitable element $l \in \langle x \in \langle L, \mathbf{G}_2\rangle \mid \mathcal{L}(x) \in \mathrm{Z}(N)\rangle$ the equation

$\langle \mathrm{I}_U(\chi_0^l) \cap \mathrm{I}_U(\chi^l), L_{\chi_0} \rangle = \mathrm{I}_N(\chi^l)$ holds. A key step in the proof of the statement is to choose a transversal in $\mathrm{Irr}\,(G_{2,i})$ with respect to the action of $\widetilde{G}_{2,i}$. As χ_0^l is maximal extendible in N and l stabilises N, the character χ must also be maximal extendible in N, which proves the statement. $\square$

A more detailed proof of the above lemma can also be found in [21, 5.3]. In all the remaining cases the statement of theorem A is clear, as the associated relative Weyl group is cyclic. The results about the inner structure of the other Sylow Levi subgroups will be needed in section 9, where we prove an equivariance statement for nonregular Sylow tori.

8 The action of outer automorphisms and regular Sylow tori

In this section we describe the action of graph, field and diagonal automorphisms on $\mathrm{Irr}\,(N)$, respectively on maximal extensions of characters on L. For this purpose we introduce a parameterisation of $\mathrm{Irr}\,(N)$.

Lemma 8.1 *Assume setting 2.2, where R is a root system of exceptional type. Let d be a regular number of $(\mathbf{G}, F)$ and κ the field automorphism of $\mathbf{G}$ acting via $x_\alpha(t) \mapsto x_\alpha(t^p)$. Then there exist a Sylow d-torus $\mathbf{S}$, $L := \mathrm{C}_{\mathbf{G}^F}(\mathbf{S})$, $N := \mathrm{N}_{\mathbf{G}^F}(\mathbf{S})$ and a map*

$$\Lambda : \mathrm{Irr}\,(L) \to \bigcup_{\chi \in \mathrm{Irr}(N)} \mathrm{Irr}\,(\mathrm{I}_N(\chi) \mid \chi),$$

such that

- *$\Lambda(\chi)$ is a maximal extension for every $\chi \in \mathrm{Irr}\,(N)$,*

- *$\Lambda(\chi)^n = \Lambda(\chi^n)$ for every $n \in N$ and*

- *$\Lambda(\chi)^\kappa = \Lambda(\chi^\kappa)$.*

Proof. According to theorem D there exists a very good Sylow d-twist $v\Gamma$, i.e., $v\Gamma$ is a good Sylow d-twist and maximal extensibility holds with respect to $U := \mathrm{C}_H(v\Gamma) \lhd K := \mathrm{C}_V(v\Gamma)$. In $\mathrm{Irr}\,(U)$ we choose a transversal $\mathcal{T}$ under the action of N. This is also a transversal of $\mathrm{Irr}\,(N)$ under the action of $\langle K, \kappa \rangle$, as κ acts trivially on V. We construct an 'extension map'

$$\Lambda' : \mathrm{Irr}\,(K) \to \bigcup_{\lambda \in \mathrm{Irr}(K)} \mathrm{Irr}\,(\mathrm{I}_U(\lambda) \mid \lambda)$$

mapping a character to one of its maximal extensions in U in the following way. First we choose possible values on $\mathcal{T}$. The other values are then determined uniquely by $\Lambda'(\lambda^n) = \Lambda'(\lambda)^n$ for all $n \in U$. By definition this map is equivariant under the action of $\langle U, \kappa \rangle$. The map Λ fulfilling $\Lambda(\chi)(u) = \Lambda'(\chi\rceil_K)(u)$ for all $u \in \mathrm{I}_U(\chi)$ has the desired properties. $\square$

In the case where R is of type B_2, E_6, F_4 or G_2 the above statement can be generalised.

Lemma 8.2 *Assume setting of lemma 8.1, where R is a root system of type B_2, E_6, F_4 or G_2 and q a prime power, such that there exists an automorphism Γ_0 of $\mathbf{G}^F$ induced from the nontrivial automorphism of the Coxeter diagram stabilising N and L. Let $n \in V$ be a element such that $L^{n\Gamma_0} = L$. Then the map Λ in 8.1 can be chosen $n\Gamma_0$-equivariant.*

Proof. We first concentrate on the case where R is of type E_6. Let v be a very good Sylow d-twist with $v^{\widetilde{\mathbf{w}}_0} = v$, $U := \mathrm{C}_V(v)$ and $K := \mathrm{C}_H(v)$. We choose a transversal $\mathcal{T}$ in $\mathrm{Irr}\,(K)$ under the action of $\langle U, \widetilde{\mathbf{w}}_0\Gamma_0, \sigma \rangle$. The automorphisms $\widetilde{\mathbf{w}}_0\Gamma_0$ and σ act trivially on U. With $\mathcal{T}$ we construct Λ as in the proof above and get a map with the wanted properties.

In the remaining cases the field automorphism is a power of the graph automorphism and it suffices to check $\Lambda(\lambda^{\Gamma_0}) = \Lambda(\lambda)^{\Gamma_0}$ and the N-equivariance. Let v be a very good Sylow d-twist with $v^{\widetilde{\mathbf{w}}_0} = v$, $U := \mathrm{C}_V(v)$ and $K := \mathrm{C}_H(v)$. If the root system is of type B_2 or F_4 the underlying characteristic is even, K trivial and v can be chosen such that U is Γ_0-stable. There exists an $\langle U, \Gamma_0 \rangle$-equivariant map $\Lambda' : \mathrm{Irr}\,(K) \to \mathrm{Irr}\,(U)$ mapping the trivial character on K onto the trivial character on U. With this map we obtain the $\langle N, \Gamma_0, \sigma \rangle$-equivariant map with the construction of lemma 8.1.

In the case where R is a root system of type G_2, every regular number is a divisor of 6. We have constructed maximal extensions for every $\lambda \in \mathrm{Irr}\,(H)$ in section 5. This uniquely defines a map $\Lambda' : \mathrm{Irr}\,(H) \to \bigcup_{\lambda \in \mathrm{Irr}(H)} \mathrm{Irr}\,(\mathrm{I}_V(\lambda))$, mapping each character on its maximal extension, which has been constructed there. This map is by definition $\langle V, \Gamma_0 \rangle$-equivariant. Let v be a very good Sylow d-twist, constructed with good roots of $\mathbf{w}_0^2$ in the associated braid group, and the groups K and U defined as above. For $d \in \{1, 2\}$ the groups coincide with H and V respectively and we can deduce from Λ' the map Λ with the construction used above. For $d \in \{3, 6\}$ we construct a map $\Lambda'' : \mathrm{Irr}\,(K) \to \bigcup_{\lambda \in \mathrm{Irr}(K)} \mathrm{Irr}\,(\mathrm{I}_U(\lambda))$, mapping each character on its maximal extension in U, which is constructed with the method introduced in 4.6 and extensions obtained from the map Λ'. According to this construction the map Λ'' is equivariant with respect to all automorphisms induced from $\langle V, \Gamma_0 \rangle$ stabilising U and K. ¿From this we deduce the map Λ with the wanted properties.$\square$

It remains to show how diagonal automorphisms act on Λ. Unfortunately in several cases Λ is not equivariant under the action of diagonal automorphisms.

Lemma 8.3 *Assume the setting of lemma 8.1. Let $t \in \mathbf{T}$ be an element with $t^{vF}t^{-1} \in \mathrm{Z}(\mathbf{G})$, $\chi \in \mathrm{Irr}\,\left(\mathbf{T}^{vF}\right)$, χ' an extension of χ to $\langle L, t \rangle$ and $\lambda \in \mathrm{Irr}\,(\mathrm{I}_N(\chi))$ with $\Lambda(\chi) = \Lambda(\chi)^t \lambda$. Then $\lambda \in \mathrm{Irr}\,(\mathrm{I}_N(\chi))$ is faithful on $\mathrm{I}_N(\chi)/\mathrm{I}_N(\chi')$.*

Proof. As χ is linear, λ fulfils $\lambda(n) = \Lambda(\chi)([n, t^{-1}]) = \chi([n, t^{-1}])$. For $n \in \mathrm{I}_N(\chi')$ we know $1 = \chi'^{n^{-1}}(t)\chi'(t^{-1}) = \chi([n, t^{-1}])$ and analogous $1 \neq \chi([n, t^{-1}])$ for $n \in \mathrm{I}_N(\chi') \setminus \mathrm{I}_N(\chi)$. This proves the statement.$\square$

Altogether we have the following action on $\mathrm{Irr}\,(N)$.

Remark 8.4 *Let $\psi \in \mathrm{Irr}\,(N)$, $\chi \in \mathrm{Irr}\,(L \mid \psi)$ and $\theta \in \mathrm{Irr}\,(\mathrm{I}_N(\chi)/L)$ with $(\Lambda(\chi)\theta)^N = \psi$. Then $\psi^{\Gamma_0^j \sigma^i t} = \psi$ $(i, j \in \mathbb{Z})$, if and only if there exists $n \in N$ with $\chi^{n\Gamma_0^j \sigma^i} = \chi$ and $\theta^{n\Gamma_0^j}\lambda = \theta$, where the linear character $\lambda \in \mathrm{Irr}\,(\mathrm{I}_N(\chi))$ is defined by $\Lambda(\chi)^t = \Lambda(\chi)\lambda$.*

9 The action of outer automorphisms and nonregular Sylow tori

In this section we calculate how outer automorphisms of $\mathbf{G}^F$, stabilising some Sylow torus $\mathbf{S}$ of $(\mathbf{G}, F)$, act on $\mathrm{Irr}\,(N)$, where N is the Sylow normaliser of $(\mathbf{G}, F)$ associated to $\mathbf{S}$. We mainly concentrate on how these automorphisms act on maximal extensions of $\chi \in \mathrm{Irr}\,(L)$, where L is the Sylow Levi subgroup associated to $\mathbf{S}$.

As a main tool in the proof we use the properties of L, determined in section 7. We especially use the results about certain subgroups which are summarised in table 3. The maximal extensions are constructed with the following procedure. In all succeeding proofs we will look at maximal extensions constructed this way and with these notations.

9.1 (Construction of $\widetilde{\chi}$) *Let L be a Sylow Levi subgroup constructed as in 7.1, N its Sylow normaliser and $\chi \in \mathrm{Irr}\,(L)$. We also use the groups L_0 and U from remark 7.1. Let $U_0 := \mathrm{C}_U(G_2)$.*

Let $\lambda_0 \in \mathrm{Irr}\,(U \cap L \mid \chi)$ and $\widetilde{\lambda}_0$ its maximal extension in U, which exists as maximal extensibility holds with respect to $U \cap L \lhd U$. We define a maximal extension $\widetilde{\lambda}$ of $\lambda \in \mathrm{Irr}\,(T_1 \mid \chi)$ to $\langle \mathrm{I}_{U_0}(\chi), T_1 \rangle$ by

$$\widetilde{\lambda}(ut) = \widetilde{\lambda}_0(u)\lambda(t) \text{ for all } u \in \mathrm{I}_{U_0}(\chi) \text{ and } t \in T_1.$$

Because of $[L, T_1] = 1$ the character λ is uniquely defined and $\mathrm{I}_{U_0}(\chi) = \mathrm{I}_{U_0}(\lambda)$.

According to 7.2 each character $\chi_0 \in \mathrm{Irr}\,(L_0 \mid \chi)$ is of the form $\lambda.\eta$ with a unique character $\eta \in \mathrm{Irr}\,(G_2)$, as L_0 is the central product of T_1 and G_2. Analogous $\widetilde{\lambda}.\eta$ is a irreducible character of $\langle \mathrm{I}_{U_0}(\chi), T_1 \rangle \circ G_2$ and an extension of χ_0.

Assume that $\widetilde{L}/L_0$ is cyclic then one of the following cases holds.

- *If $\chi_0 = \chi \rceil_{L_0}$, then an element $l \in L \setminus L_0$ with $\chi(l) \neq 0$ exists, as the $|L/L_0|$ different extensions of χ_0 are the products of χ and the linear character of L/L_0 according to the theorem of Gallagher [14, 6.17].*

  The equation $\chi(l) = \chi(l^u)$ for $u \in \mathrm{I}_{U_0}(\chi)$ implies $\chi(u) = \lambda([l, u])\chi(l)$ and $\lambda([l, u]) = 1$ because of $[l, u] \in T_1 \leq \mathrm{Z}(L)$. This shows $\mathrm{I}_L(\widetilde{\lambda}.\eta) = L$. One extension ψ of $\widetilde{\lambda}.\eta$ to $\langle \mathrm{I}_{U_0}(\chi), L \rangle$ is at the same time an extension of χ. This character ψ is by definition invariant in $\mathrm{I}_N(\chi)$ and can be extended to this group.

- *Otherwise we have $\chi = \chi_0^L$. The character $\widetilde{\lambda}.\eta$ is extendible to $\widetilde{\chi}_0 \in \mathrm{Irr}\,(\mathrm{I}_N(\chi_0))$ as the factor group $\mathrm{I}_N(\chi_0)/\langle \mathrm{I}_{U_0}(\chi), T_1 \rangle \circ G_2$ is a subgroup of U/U_0 and hence cyclic. The character $\widetilde{\chi}_0^{\mathrm{I}_N(\chi)}$ is then a maximal extension of χ according to Mackey's lemma.*

In general we always get a maximal extension of χ as we extend $\chi_0 \in \text{Irr}(L_0 \mid \chi)$ to $\langle I_{U_0}(\chi), L \rangle$ by using the character $\widetilde{\lambda}_0$.

For the extensions constructed in this way we prove the following proposition. All automorphisms of $\mathbf{G}^F$ are restrictions of automorphisms of $\mathbf{G}$.

Proposition 9.2 *Let $\chi \in \text{Irr}(L)$, $\mathbf{S}$ be the non-regular Sylow torus of $(\mathbf{G}, F)$, such that L is the corresponding Sylow Levi subgroup, and κ an automorphism of $\mathbf{G}$ with $\mathbf{S}^\kappa = \mathbf{S}$ and $\chi^\kappa = \chi$, which is as $\mathbf{G}^F$-automorphism is a product of an inner, a graph and a field automorphism of $\mathbf{G}^F$. Then every maximal extension $\widetilde{\chi}$ constructed like in 9.1 is κ-invariant.*

Further let κ be an automorphism of $\mathbf{G}$ with $\mathbf{S}^\kappa = \mathbf{S}$ and $\chi^\kappa = \chi$, which is as $\mathbf{G}^F$-automorphism a product of an inner, a diagonal, a graph and a field automorphism of $\mathbf{G}^F$ and acts trivially on N/L. Then every maximal extension $\widetilde{\chi}$ constructed like in 9.1 fulfils $\widetilde{\chi}^\kappa = \widetilde{\chi}$, if and only if $I_N(\chi') = I_N(\chi)$ for an extension χ' of χ to $\langle L, \kappa \rangle$.

To prove 9.2 we use the following facts, which can also be used in more general settings.

Remark 9.3 (Some remarks on extensions) *(a) Let $\mathcal{L} \lhd \mathcal{G}$, $\psi \in \text{Irr}(\mathcal{L})$, $\widetilde{\psi}, \widetilde{\psi}' \in \text{Irr}(\mathcal{G})$ with $\widetilde{\psi}\big|_{\mathcal{L}} = \widetilde{\psi}'\big|_{\mathcal{L}} = \psi$ and $\mathcal{T}$ a subset of $\mathcal{G}$ fulfilling both $\langle \mathcal{T}, \mathcal{L} \rangle = \mathcal{G}$ and $\widetilde{\psi}(u) = \widetilde{\psi}'(u) \neq 0$ for every $u \in T$. Then $\widetilde{\psi} = \widetilde{\psi}'$.*

(b) Let $\mathcal{L} \lhd \mathcal{G}$, $\psi \in \text{Irr}(\mathcal{L})$, $\widetilde{\psi} \in \text{Irr}(\mathcal{G})$ with $\widetilde{\psi}\big|_{\mathcal{L}} = \psi$. Further let κ be an automorphism stabilising $\mathcal{L}$, $\mathcal{G}$ and ψ with $(o(\kappa), |\mathcal{G}/\mathcal{L}|) = 1$, where $o(\kappa)$ denotes the order of κ as an automorphism. If κ acts trivially on $\mathcal{G}/\mathcal{L}$, then $\widetilde{\psi}^\kappa = \widetilde{\psi}$.

(c) Let $\mathcal{L} \lhd \mathcal{G}$, $\psi \in \text{Irr}(\mathcal{L})$, $u \in C_{\mathcal{G}}(\mathcal{L})$ and κ an automorphism of $\mathcal{G}$ with $\mathcal{N}^\kappa = \mathcal{N}$ and $u^\kappa = u$. If an extension ψ' of ψ to $\langle \mathcal{L}, \kappa \rangle$ fulfils $\psi'(\kappa) \neq 0$, then $\psi''(\kappa u) \neq 0$ holds for every extension ψ'' of ψ to $\langle \mathcal{L}, \kappa u \rangle$.

Proof. The first part is a direct consequence of the theorem of Gallagher. In the situation of part (b) the theorem of Gallagher implies $\widetilde{\psi}^\kappa = \widetilde{\psi}\alpha$ for some linear character $\alpha \in \text{Irr}(\mathcal{G}/\mathcal{L})$. The equation $\widetilde{\psi} = \widetilde{\psi}^{\kappa^{o(\kappa)}} = \widetilde{\psi}\left(\alpha^{o(\kappa)}\right)$ shows $\alpha = 1$, as the order of α has to divide $(o(\kappa), |\mathcal{G}/\mathcal{L}|)$.

For the proof of the third statement we extend ψ to $\psi' \in \text{Irr}(\langle \mathcal{L}, \kappa \rangle)$ and this character again to $\langle \mathcal{L}, \kappa, u \rangle$, which is possible because of $\psi'(\kappa) \neq 0$. Because of $u \in Z(\langle \mathcal{L}, \kappa, u \rangle)$ the statement follows. $\square$

The following remark describes some specific properties, which hold in this situation

Remark 9.4 *(a) We use the description of automorphisms of $\mathbf{G}^F$ introduced in the previous section. For proving this lemma we may concentrate on the Sylow Levi subgroups and Sylow normalisers constructed in 7.1 and $\kappa \in \left\langle U, \sigma, \Gamma_0 \widetilde{\mathbf{w}}_0, \widetilde{T} \right\rangle$ with $\widetilde{T} := \left\langle t \in \mathbf{T} \mid F(t^v)t^{-1} \in Z(\mathbf{G}^{vF}) \right\rangle$. The automorphisms in $\left\langle \sigma, \Gamma_0 \widetilde{\mathbf{w}}_0, \widetilde{T} \right\rangle$ act trivially on W and hence on N/L.*

(b) Let $\kappa \in \langle U, \sigma, \Gamma_0 \widetilde{\mathbf{w}_0} \rangle$ and $\chi \in \mathrm{Irr}\,(L)$ with $\chi^\kappa = \chi$. Then σ and $\Gamma_0 \widetilde{\mathbf{w}_0}$ act trivially on U and an $\mathrm{Irr}\,(U \cap L \mid \chi)$. If $\chi_0^\kappa = \chi_0$, then $(\widetilde{\lambda}.\eta)^\kappa = \widetilde{\lambda}.\eta$.

These statements already enable us to prove the following lemma.

Lemma 9.5 *Proposition 9.2 holds, if* $\mathbf{G}$ *has a root system of type* E_6.

Proof. In this situation the group $\mathrm{Out}(\mathbf{G}^F)$ is generated by a field automorphism σ, the graph automorphism Γ_0 and a diagonal automorphism, whose order as outer automorphism is $(3, q-1)$, if F acts as a standard Frobenius endomorphism, and $(3, q+1)$ otherwise.

As $\mathbf{S}$ is a nonregular Sylow torus of $(\mathbf{G}, F)$, we know $N/L \cong \mathrm{C}_5$ from table 2, where N and L are the associated Sylow normalizer and the associated Levi subgroup respectively. By the remarks 9.3 (b) and 9.4 (a) we may assume $\kappa \in \left\langle U, \sigma, \Gamma_0 \widetilde{\mathbf{w}_0}, \widetilde{T} \right\rangle$ and that κ induces an automorphism of order 5^i $(0 \le i \in \mathbb{Z})$ on $\mathbf{G}^F$ because of $N/L \cong \mathrm{C}_5$, hence $\kappa \in \langle U, \sigma \rangle$.

Therefore $\chi_0 \in \mathrm{Irr}\,(L_0 \mid \chi)$ is κ-invariant. Now remark 9.4 (b) implies $(\widetilde{\lambda}.\eta)^\kappa = \widetilde{\lambda}.\eta$. With this κ-invariant extension of χ_0 we also get a κ-invariant extension of χ by remark 9.3 (a).$\square$

In order to prove the analogous statement in the remaining cases we need more informations which automorphisms stabilise characters $\chi \in \mathrm{Irr}\,(G_2)$. The following remark generalises lemma 15.1 of [15].

Remark 9.6 *Let* δ *be a diagonal automorphism of* $\mathrm{SL}_3(q)$, Γ_0 *the graph automorphism and* $\eta \in \mathrm{Irr}\,(\mathrm{SL}_3(q))$ *with* $\eta^\delta \ne \eta$. *Then one character of* $\left\{ \eta, \eta^\delta, \eta^{\delta^2} \right\}$ *is invariant under* $\langle \sigma, \Gamma_0 \rangle$, *if* $\ker(\eta) \ne 1$ *or* $[\mathrm{Z}(\mathrm{SL}_3(q)), \sigma] = 1$. *Otherwise one character of* $\left\{ \eta, \eta^\delta, \eta^{\delta^2} \right\}$ *is* $\sigma\Gamma_0$-*invariant.*

Proof. This follows from the character table of $\mathrm{SL}_3(q)$, which can be found in [20]. In order to see the results from the character table one should use

$$\begin{pmatrix} \omega^k & & \\ \xi^l & \omega^k & \\ & 1 & \omega^k \end{pmatrix}$$

as canonical representative of the conjugacy class $C_2^{(k,l)}$ $(k, l \in \{0, \dots, |\mathrm{Z}(\mathrm{SL}_3(q))| - 1\})$, where $\omega \in \mathbb{F}_q^*$ is an element of order $|\mathrm{Z}(\mathrm{SL}_3(q))|$ in $\mathbb{F}_q^*$ and $\langle \xi \rangle \in \mathbb{F}_q^*$. These are the only $\mathrm{SL}_3(q)$-conjugacy classes which are not stable under the action of $\mathrm{GL}_3(q)$. $\square$

For succeeding proofs we need the following statement.

Lemma 9.7 *Let* $\mathcal{G} := \mathrm{SL}_3(q)$, Γ_0 *be the graph automorphism of* $\mathcal{G}$, σ *the field automorphism and* $\eta \in \mathrm{Irr}\,(\mathcal{G})$ *with* $\eta^{\Gamma_0} = \eta^\sigma = \eta$. *Then* η *is extendible to* $\mathcal{G} \rtimes \langle \sigma, \Gamma_0 \rangle$.

For the proof we need the following fact from [2, 4.3]: Let σ' be a field automorphism σ' of $\mathrm{GL}_3(q)$, $\eta' \in \mathrm{Irr}\,(\mathrm{GL}_3(q))$ with $\eta'^{\sigma'} = \eta'$ and $\widetilde{\eta}'$ an extension of η' to $\langle \mathrm{GL}_3(q), \sigma' \rangle$. Then $\widetilde{\eta}'(\sigma') \neq 0$.

Proof. First we assume that η can be extended to $\eta' \in \mathrm{Irr}\,(\mathrm{GL}_3(q))$. According to [15, 14.1] we may choose η' to be σ-invariant. We can extend η' to a character $\eta'' \in \mathrm{Irr}\,(\mathrm{GL}_3(q) \rtimes \langle \sigma \rangle)$. This character fulfils $\eta''(\sigma) \neq 0$. By remark 9.3 (a) the character $\eta''|_{\mathrm{SL}_3(q) \rtimes \langle \sigma \rangle}$ can be extended to $\mathrm{SL}_3(q) \rtimes \langle \sigma, \Gamma_0 \rangle$.

Otherwise η has an extension η' to $\langle \mathrm{Z}(\mathrm{GL}_3(q)), \mathrm{SL}_3(q) \rangle$ with $\eta'(z) = \eta(1)$ for every element $z \in \mathrm{Z}(\mathrm{GL}_3(q))$, as $\mathrm{Z}(\mathrm{SL}_3(q)) \leq \ker(\eta)$. The characters η' and $\alpha := \eta'^{\mathrm{GL}_3(q)}$ are hence $\langle \sigma, \Gamma_0 \rangle$-invariant. As the extension $\widetilde{\alpha}$ of α to $\mathrm{GL}_3(q) \rtimes \langle \sigma \rangle$ fulfils $\widetilde{\alpha}(\sigma) \neq 0$, the character $\widetilde{\alpha}$ is Γ_0-invariant and has an extension $\widehat{\alpha} \in \mathrm{Irr}\,(\mathrm{GL}_3(q) \rtimes \langle \sigma, \Gamma_0 \rangle)$. Because of the Clifford correspondence there exists a character $\widetilde{\chi} \in \mathrm{Irr}\,\big(\mathrm{I}_{\mathrm{GL}_3(q) \rtimes \langle \sigma, \Gamma_0 \rangle}(\chi) \mid \chi \big)$ with $\widetilde{\chi}^{\mathrm{I}_{\mathrm{GL}_3(q) \rtimes \langle \sigma, \Gamma_0 \rangle}(\chi)} = \widehat{\alpha}$. Because of $\widehat{\alpha}(1) = 3\chi(1)$ this is an extension of χ. $\square$

The following remark is relevant for proving proposition 9.2, in situations where $G_2 \cong \mathrm{SL}_2(q)$.

Remark 9.8 *Let σ be a field automorphism of $\mathrm{SL}_2(q)$ of odd order, $\eta \in \mathrm{Irr}\,(\mathrm{SL}_2(q))$ with $\eta^\sigma = \eta$. Then $\widetilde{\eta}(\sigma) \neq 0$ for any extension $\widetilde{\eta}$ of η to $\langle \mathrm{SL}_2(q), \sigma \rangle$.*

Proof. If η has an extension of $\mathrm{GL}_2(q)$ the statement is an easy consequence of [2, 4.3] and [15, 14.1].

Otherwise η has an extension η' to $\langle \mathrm{SL}_2(q), \mathrm{Z}(\mathrm{GL}_2(q)) \rangle$ with $\eta'(z) = 1$ for every $z \in \mathrm{Z}(\mathrm{GL}_2(q))$ of odd order. According to [15, 14.1], the automorphism σ of odd order acts trivially on η'. The character $(\eta')^{\mathrm{GL}_2(q)}$ has an extension ψ to $\langle \mathrm{GL}_2(q), \sigma \rangle$, which fulfils $\psi(\sigma) \neq 0$ because of [2]. The character $\psi|_{\langle \mathrm{SL}_2(q), \sigma \rangle}$ is the sum of an extension $\widetilde{\psi}$ of ψ to $\langle \mathrm{SL}_2(q), \sigma \rangle$ and $\widetilde{\psi}^\delta$ with $\delta = \begin{pmatrix} 1 & \\ & \xi \end{pmatrix}$, where $\xi \in \mathbb{F}_q^*$ is an element whose multiplicative order is the 2-part of $q - 1$. Because of $[\delta, \sigma] = 1$ this proves the remark. $\square$

With these statements we can prove the following result.

Lemma 9.9 *Proposition 9.2 holds, if $\mathbf{G}$ has a root system of type E_8.*

Proof. The outer automorphism group of $\mathbf{G}^F := \mathsf{E}_8(q)$ is generated by the field automorphism $\sigma : \mathbf{G}^F \to \mathbf{G}^F$ with $x_\alpha(t) \mapsto x_\alpha(t^p)$.

First we assume $\mathbf{G}_2 \cong \mathrm{SL}_2(\overline{\mathbb{F}}_q)$. Without loss of generality we may assume $\kappa \in \langle U, \sigma \rangle$. The equation $\chi^\kappa = \chi$ implies $\chi_0^{\kappa l} = \chi_0$ for some $\chi_0 \in \mathrm{Irr}\,(L_0 \mid \chi)$ and $l \in L$. The group L_0 is the central product of $G_2 \cong \mathrm{SL}_2(q)$ and an abelian group T_1. According to remark 7.2 we know $\chi_0 = \lambda.\eta$ for some $\lambda \in \mathrm{Irr}\,(T_1)$ and $\eta \in \mathrm{Irr}\,(G_2)$. Furthermore the equation $\eta^{\kappa l} = \eta$ implies $\eta^\kappa = \eta$ and $\eta^l = \eta$ according to [15, 15.1]. Because of $T_1 \leq \mathrm{Z}(L)$ this shows $\chi_0^\kappa = \chi_0$. The maximal extension of χ_0 in N from remark 9.1 is by definition

invariant under κ according to remarks 9.3 and 9.4 (b), hence this also holds for the constructed maximal extension of χ.

By table 2 we have $\mathbf{G}_2 = \mathrm{SL}_3(\overline{\mathbb{F}}_p)$ in the remaining cases and may assume $\kappa \in \langle U, \sigma \rangle$. From $\chi^\kappa = \chi$ we obtain the equation $\chi_0^{\kappa l} = \chi_0$ for some $\chi_0 \in \mathrm{Irr}\,(L_0 \mid \chi)$ and $l \in L$, where we may write $\chi_0 = \lambda.\eta$ with certain characters $\lambda \in \mathrm{Irr}\,(T_1)$ and $\eta \in \mathrm{Irr}\,(G_2)$. Remark 9.6 shows that one can choose $\chi_0 \in \mathrm{Irr}\,(L_0 \mid \chi)$ with $\chi_0^\kappa = \chi_0$. The arguments used in the case $\mathbf{G}_2 \cong \mathrm{SL}_2(\overline{\mathbb{F}}_q)$ prove that the maximal extension of χ in $\langle U_0, L \rangle$ constructed in remark 9.1 is κ-invariant. This is a maximal extension of χ in N if $\mathrm{I}_N(\chi) \le \langle U_0, L \rangle$.

Otherwise we may assume $\kappa \in \langle U_0, \sigma \rangle$. We consider the cases $\chi\rceil_{L_0} = \chi_0$ and $\chi\rceil_{L_0} \ne \chi_0$ separately. In the first case we extend χ to a character ψ of $\langle \mathrm{I}_{U_0}(\chi), L, \kappa \rangle$. According to remark 9.3 (c) and statement [2, 4.3] this character fulfils $\psi(\kappa) \ne 0$. For $u_\kappa \sigma^i = \kappa$ $(u_\kappa \in U)$ and $u \in \mathrm{I}_U(\chi) \setminus U_0$ we know $\psi^{u^{-1}}(\kappa) = \psi(u_\kappa^u \sigma^i) = \psi([u, u_\kappa^{-1}] u_\kappa \sigma^i) = \widetilde{\lambda}_0([u, u_\kappa^{-1}]) \psi(\kappa)$. As $\widetilde{\lambda}_0$ is a linear character of $\mathrm{I}_U(\lambda_0)$ and $u, u_\kappa \in \mathrm{I}_U(\lambda_0)$ this shows that ψ is u-invariant. Hence ψ has an extension to $\langle L, \mathrm{I}_U(\chi), \kappa \rangle$. The same holds for every other maximal extension of χ in N, because the factor group $^N/_L$ is abelian and κ acts trivially on $^N/_L$.

Assume $\chi_0(1) \ne \chi(1)$. We first extend $\lambda_0.\eta \in \mathrm{Irr}\,(\langle U \cap L, G_2 \rangle \mid \chi_0)$. The character $\lambda'.\eta$ with $\lambda' := \widetilde{\lambda}_0\Big\rvert_{\mathrm{I}_{U_0}(\lambda_0)}$ is a σ^i-invariant extension to $\langle \mathrm{I}_{U_0}(\lambda_0), G_2 \rangle$, if $u_\kappa \sigma^i = \kappa$ for some $u_\kappa \in U_0$ and $i \in \mathbb{Z}$. This character can be extended to ψ on $\mathrm{I}_{U_0}(\lambda_0) \circ \langle G_2, \sigma^i \rangle$. According to 9.7 we can thereby choose ψ such that $\psi\rceil_{\langle G_2, \sigma^i \rangle}$ is invariant under the graph automorphism of G_2 acting trivially on σ^i. Hence ψ can be extended to $\widetilde{\psi}$ on $\widetilde{G} := \langle \mathrm{I}_U(\lambda_0.\eta), G_2, \sigma^i \rangle$. Let $\mathcal{D}_1$ be a representation of $\widetilde{\psi}$. The equation

$$\mathcal{D}(tg) = \lambda(t)\mathcal{D}_1(g) \text{ for } g \in \mathrm{I}_{\widetilde{G}}(\chi_0),\, t \in T_1$$

defines a representation $\mathcal{D}$ on $\langle \mathrm{I}_N(\chi_0), \kappa \rangle$. The associated character is a maximal extension of χ_0 in $\left\langle \widetilde{G}, T_1 \right\rangle$. Every other maximal extension of χ in N is also κ-invariant, as $^N/_L$ is abelian and κ acts trivially on $^N/_L$. $\square$

Similar arguments and some more in-depth calculations prove proposition 9.2 for groups with a root system of type E_7.

10 Proof of Theorem C

As a consequence of theorem A and [17] we can verify the McKay-conjecture for exceptional groups and odd primes different from the defining characteristic. Two occurring exceptions have already been dealt with in [1] and [16]. Furthermore the McKay-conjecture has already been verified for $^2\mathsf{G}_2(3^{2f+1})$ and $^2\mathsf{B}_2(3^{2f+1})$ in [15, Theorem A].

Proof of theorem C. Let $\mathbf{G}$ be a simply-connected algebraic group and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map, such that the root system of $\mathbf{G}$ is exceptional or $\mathbf{G}^F$ is a Suzuki-group or Steinberg's triality group. Furthermore let $\ell \ne 2$ be a prime with $\ell \big| \big| \mathbf{G}^F \big|$ and $\ell \ne 3$, if $\mathbf{G}^F \in \left\{ \mathsf{G}_2(q), {}^2\mathsf{F}_4(2^{2f+1}) \right\}$, as we use results from [17]. Let $\nu \in \mathrm{Irr}\,(\mathrm{Z}(\mathbf{G}^F))$.

First assume F to be a Frobenius endomorphism. Let d be the order of q in $(\mathbb{Z}/\ell\mathbb{Z})^*$ and $\mathbf{S}$ a Sylow d-torus of $(\mathbf{G}, F)$. We know from [17, Theorem 5.14] that there exists a Sylow ℓ-subgroup P of $\mathbf{G}^F$ with $N_{\mathbf{G}^F}(P) \leq N$ where N is the Sylow d-normaliser associated to $\mathbf{S}$. By [17, Theorem 7.8] and theorem A the equation $\left|\mathrm{Irr}_{\ell'}\left(\mathbf{G}^F \mid \nu\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N \mid \nu\right)\right|$ holds.

For $\ell \geq 5$ and $L := C_{\mathbf{G}^F}(\mathbf{S})$ we know by [7, Theorem 4.4] that $N \neq N_{\mathbf{G}^F}(P)$ implies $\ell \,\big|\, |N/L|$. Calculating the primes dividing $|W|$, where W is the Weyl group of $\mathbf{G}$, shows $\ell \in \{3, 5, 7\}$. By definition d divides $\ell - 1$, hence $d \in \{1, 2, 3, 4, 6\}$. By table 3 the number d is regular for $(\mathbf{G}, F)$, whenever $N \neq N_{\mathbf{G}^F}(P)$. As $\ell = 3$ implies $d \in \{1, 2\}$, we may assume d to be a regular number of $(\mathbf{G}, F)$.

If F is not a Frobenius endomorphism and hence $\mathbf{G}^F$ is a Suzuki- or Ree-group, theorems 8.4 and 8.5 of [17] show that there exists a regular Sylow torus $\mathbf{S}$ of $(\mathbf{G}, F)$, such that $N \geq N_{\mathbf{G}^F}(P)$ for some Sylow ℓ-subgroup P of $\mathbf{G}^F$ and $N := N_{\mathbf{G}^F}(\mathbf{S})$, and the equation $\left|\mathrm{Irr}_{\ell'}\left(\mathbf{G}^F \mid \nu\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N \mid \nu\right)\right|$ holds.

We verify the statement by proving $\left|\mathrm{Irr}_{\ell'}\left(N \mid \nu\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \nu\right)\right|$ and $\left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \nu\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_0 \mid \nu\right)\right|$ with $N_0 := N_N(P)$, $N_1 := \langle N_0, L\rangle$ and $L := C_{\mathbf{G}^F}(\mathbf{S})$.

Let $\mathcal{T}$ be a set of representatives of characters $\chi \in \mathrm{Irr}\left(L \mid \nu\right)$, which fulfil $P \leq I_N(\chi)$, up to N_0-conjugacy. Then $\mathcal{T}$ is also a set of representatives of characters $\chi \in \mathrm{Irr}\left(L \mid \nu\right)$, which fulfil $\ell \nmid \frac{|N|}{I_N(\chi)}$, up to N-conjugacy. By Clifford theory

$$\mathrm{Irr}_{\ell'}\left(N_1 \mid \nu\right) = \dot{\bigcup_{\chi \in \mathcal{T}}} \mathrm{Irr}_{\ell'}\left(N_1 \mid \chi\right) \text{ and } \mathrm{Irr}_{\ell'}\left(N \mid \nu\right) = \dot{\bigcup_{\chi \in \mathcal{T}}} \mathrm{Irr}_{\ell'}\left(N \mid \chi\right).$$

In the following we prove $\left|\mathrm{Irr}_{\ell'}\left(N \mid \chi\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \chi\right)\right|$ for $\chi \in \mathcal{T}$. As χ extends to $I_N(\chi)$ by theorem A, we know $\left|\mathrm{Irr}_{\ell'}\left(N \mid \chi\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(I_N(\chi)/L\right)\right|$. Computer calculations show that the McKay-conjecture is valid for all $\mathcal{U} \leq W$, hence $\left|\mathrm{Irr}_{\ell'}\left(I\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_I(\overline{P})\right)\right|$ for $I := I_N(\chi)/L$ and $\overline{P} := \langle P, L\rangle/L$.

Similar arguments show $\left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \chi\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(I_{N_1}(\chi)/L\right)\right|$. By the Sylow theorems $I_{N_1}(\chi)/L = N_I(\overline{P})$, hence $\left|\mathrm{Irr}_{\ell'}\left(N \mid \chi\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \chi\right)\right|$.

In the next step of the proof we verify $\left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \nu\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_0 \mid \nu\right)\right|$. We obtain $L = L' \times (P \cap L)$, where L' is the subgroup of all ℓ'-elements in the abelian group L. As P acts on L' with $(|P|, |L'|) = 1$. the equation $L' = [L', P] \times C_{L'}(P)$ holds by [13, 14.5(c)]. The fact $P \leq I_{N_1}(\chi)$ implies $\ker(\chi) \leq [L', P]$, hence

$$\mathrm{Irr}_{\ell'}\left(N_0 \mid \nu\right) = \dot{\bigcup_{\chi \in \mathcal{T}}} \mathrm{Irr}_{\ell'}\left(N_0 \mid \chi\right]_{L_1}\right)$$

for $L_1 := C_{L'}(P) \times (P \cap L)$.

As the character $\chi' := \chi]_{L_1}$ $(\chi \in \mathcal{T})$ extends to $I_{N_0}(\chi') = I_{N_1}(\chi) \cap N_0$, we rewrite $\left|\mathrm{Irr}_{\ell'}\left(N_0 \mid \chi'\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(I_{N_0}(\chi')/L_1\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(I_{N_1}(\chi)/L\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_1 \mid \chi\right)\right|$. This proves $\left|\mathrm{Irr}_{\ell'}\left(N \mid \nu\right)\right| = \left|\mathrm{Irr}_{\ell'}\left(N_0 \mid \nu\right)\right|$.

The remaining cases where $\ell = 3$ and $\mathbf{G}^F \in \left\{\mathsf{G}_2(q), {}^2\mathsf{F}_4(2^{2i+1})\right\}$ have been dealt with in [1] and [16]. $\square$

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