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Supertropical Semirings and Supervaluations

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SUPERTROPICAL SEMIRINGS AND SUPERVALUATIONS

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. We interpret a valuation v on a ring R as a map $v : R \to M$ into a so called bipotent semiring M (the usual max-plus setting), and then define a **supervaluation** φ as a suitable map into a supertropical semiring U with ghost ideal M (cf. [IR1], [IR2]) covering v via the ghost map $U \to M$. The set $\mathrm{Cov}(v)$ of all supervaluations covering v has a natural ordering which makes it a complete lattice. In the case that R is a field, hence for v a Krull valuation, we give a complete explicit description of $\mathrm{Cov}(v)$.

The theory of supertropical semirings and supervaluations aims for an algebra fitting the needs of tropical geometry better than the usual max-plus setting. We illustrate this by giving a supertropical version of Kapranov's lemma.

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Introduction

As explained in [IMS] and [G], tropical geometry grew out of a logarithmic correspondence taking a polynomial $f(\lambda_1, \dots, \lambda_n)$ over the ring of Puiseux series to a corresponding polynomial $\bar{f}(\lambda_1, \dots, \lambda_n)$ over the max-plus algebra T. A key observation is Kapranov's Lemma, that this correspondence sends the algebraic variety defined by f into the so-called *corner locus* defined by $\bar{f}$. More precisely, this correspondence involves the negative of a valuation (where the target (T) is an ordered monoid rather than an ordered group), which has led researchers in tropical mathematics to utilize valuation theory. In order to avoid the introduction of the negative, some researchers, such as [SS], have used the min-plus algebra instead of the max-plus algebra.

Note that whereas a valuation v satisfies $v(ab) = v(a) + v(b)$, one only has

$$v(a + b) = \min\{v(a), v(b)\}$$

when $v(a) \neq v(b)$; for the case that $v(a) = v(b)$, $v(a+b)$ could be any element $\geq v(a)$. From this point of view, the max-plus (or, dually, min-plus) algebra does not precisely reflect the tropical mathematics. In order to deal with this issue, as well as to enhance the algebraic structure of the max-plus algebra T, the first author introduced a cover of T, graded by the multiplicative monoid $(\mathbb{Z}_2, \cdot)$, which was dubbed the *extended tropical arithmetic*. Then, in [IR1] and [IR2], this structure has been amplified to the notion of **supertropical semiring**. A supertropical semiring U is equipped with a "**ghost map**" $\nu := \nu_U : U \to U$, which respects addition and multiplication and is idempotent, i.e., $\nu \circ \nu = \nu$. Moreover $a + a = \nu(a)$ for every $a \in U$ (cf. §3). This rule replaces the rule $a + a = a$ in the usual max-plus (or min-plus) arithmetic. We call $\nu(a)$ the "**ghost**" of a (often writing a^ν instead of $\nu(a)$), and we call the elements of U which are not ghost "**tangible**"[1].

The image of the ghost map is a so-called **bipotent semiring**, i.e., a semiring M such that $a + b \in \{a, b\}$ for every $a, b \in M$. So M is a semiring typically occurring in tropical algebra. In this paper supertropical and bipotent semirings are nearly always tacitly assumed to be commutative.

It turns out that supertropical semirings allow a refinement of valuation theory to a theory of "supervaluations". Supervaluations seem to be able to give an enriched version of tropical geometry. In the present paper we illustrate this by giving a refined and generalized version of Kapranov's lemma. (§10, §11). Very roughly one may say that the usual tropical algebra is present in the ghost level of our supertropical setting.

We consider valuations on rings (as defined by Bourbaki [B]) instead of just fields. We mention that these can be understood as families of valuations on fields, cf. e.g. [HK] and [KZ]. We use multiplicative notation, writing a valuation v on a ring R as a map into $\Gamma \cup \{0\}$ with Γ a multiplicative ordered abelian group and $0 < \Gamma$, obeying the rules

$$\begin{gathered} v(0) = 0, \quad v(1) = 1, \quad v(ab) = v(a)v(b), \\ v(a + b) \leq \max(v(a), v(b)). \end{gathered} \tag{$*$}$$

We view the ordered monoid $\Gamma \cup \{0\}$ as a bipotent semiring by introducing the addition $x + y := \max(x, y)$, cf. §1 and §2. It is then very natural to replace $\Gamma \cup \{0\}$ by any bipotent semiring M, and to define an **m-valuation** (= monoid valuation) $v : R \to M$ in the same way $(*)$ as before.

[1] The element 0 may be regarded both as tangible and ghost.

Given an m-valuation $v : R \to M$ there exist multiplicative mappings $\varphi : R \to M$ into various supertropical semirings U, with $\varphi(0) = 0$, $\varphi(1) = 1$, such that M is the ghost ideal of U and $\nu_U \circ \varphi = v$. These are the **supervaluations** covering v, cf. §4.

In §5 we define maps $\alpha : U \to V$ between supertropical semirings, called **transmissions**, which have the property that for a supervaluation $\varphi : R \to U$ the composite $\alpha \circ \varphi : R \to V$ is again a supervaluation. Given two supervaluations $\varphi : R \to U$ and $\psi : R \to V$ (not necessarily covering the same valuation v), we say that φ **dominates** ψ, and write $\varphi \geq \psi$, if there exists a transmission $\alpha : U \to V$, such that $\psi = \alpha \circ \varphi$. {The transmission α then is essentially unique.}

Restricting the dominance relation to the set of supervaluations[2] covering a fixed valuation $v : R \to M$ we obtain a partially ordered set $\mathrm{Cov}(v)$, which turns out to be a complete lattice, as proved in §7. The bottom element of this lattice is the valuation v, viewed as a supervaluation. The top element, denoted $\varphi_v : R \to U(v)$, can be described explicitly in good cases. This description is already given in §4, cf. Example 4.5. The other elements of $\mathrm{Cov}(v)$ are obtained from φ_v by dividing out suitable equivalence relations on the semiring $U(v)$, called MFCE-relations (= multiplicative fiber conserving equivalence relations). They are defined in §6. Finally in §8, we obtain an explicit description of all elements of $\mathrm{Cov}(v)$ in the case that R is a field, hence v is a Krull valuation.

If R is only a ring, our results are far less complete. Nevertheless it seems to be absolutely necessary to work at least in this generality for many reasons, in particular functorial ones, cf. e.g. [HK], [KZ].

If $v : R \to M$ is an m-valuation and $\gamma : M \to N$ is a homomorphism from M to a bipotent semiring N, then $\gamma \circ v$ clearly again is an m-valuation, called a **coarsening** of v. This generalizes the usual notion of coarsening for Krull valuations. It is of interest to look for relations between the lattice $\mathrm{Cov}(v)$ and $\mathrm{Cov}(\gamma \circ v)$. §9 gives a first step in this direction. Given $\gamma : M \to N$ and a supertropical semiring U with ghost ideal M we look for transmissions $\alpha : U \to V$ which **cover** γ, i.e., V has the ghost ideal N and $\alpha(x) = \gamma(x)$ for $x \in M$. Assuming that γ is surjective, we prove that there exists an **initial** such transmission $\alpha = \alpha_{U,\gamma} : U \to U_\gamma$. This means that any other transmission $\alpha' : U \to V'$ covering γ is obtained from α by composition with a transmission $\beta : U_\gamma \to V'$ covering the identity of N. This allows us to define an order preserving map

$$\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma \circ v),$$

sending a supervaluation $\varphi : R \to U$ to $\gamma_*(\varphi) := \alpha_{U,\gamma} \circ \varphi$. (The map γ_* will be only introduced in §12.) In good cases $\alpha_{U,\gamma}$ has a "pushout property" (cf. Definition 9.2), that is even stronger than to be initial, and $\alpha_{U,\gamma}$ can be described explicitly (cf. Theorem 9.11).

In §10 we delve deeper into the supertropical theory to pinpoint a relation, which we call the **ghost surpassing relation**, which seems to be a key for working in supertropical semirings. On the one hand, the ghost surpassing relation restricts to equality on tangible elements, so enables us to specialize to the max-plus theory. On the other hand, the ghost surpassing relation appears in virtually every supertropical theorem proved so far, especially in supertropical matrix theory in [IR2] and [IR3].

In the present paper the ghost surpassing relation is the essential gadget to understand and prove a general version of Kapranov's lemma, valid for any valuation $v : R \to M$ which is "**strong**". This means that $v(a + b) = \max(v(a), v(b))$ whenever $v(a) \neq v(b)$.

[2]More precisely we should consider equivalence classes of supervaluations. We suppress this point here.

If R is a ring, every valuation on R is strong, as is very well known, but if R is only a semiring, this is a restrictive condition. On our way to Kapranov's lemma we employ supervaluations $\varphi \in \mathrm{Cov}(v)$ which are **tangible**, i.e., have only tangible values, and are **tangibly additive**, which means that $\varphi(a+b) = \varphi(a) + \varphi(b)$ whenever $\varphi(a) + \varphi(b)$ is tangible. We apostrophize tangibly additive supervaluations which cover strong m-valuations as **strong supervaluations**.

The strong tangible supervaluations in $\mathrm{Cov}(v)$ seem to be the most suitable ones for applications in tropical geometry also beyond Kapranov's lemma. They form a sublattice $\mathrm{Cov}_{\mathrm{t,s}}(v)$ of $\mathrm{Cov}(v)$. In particular there exists an "initial" tangible strong valuation in $\mathrm{Cov}(v)$, denoted by $\overline{\varphi}_v$, which dominates all others. It gives the "best" supertropical version of Kapranov's lemma, cf. §11. At the end of §11 we make $\overline{\varphi}_v$ explicit in the case that v is the natural valuation of the field of formal Puiseux series in a variable t (with real or with rational exponents). We can interpret the value of $\overline{\varphi}_v(a(t))$ of a Puiseux series $a(t)$ as the leading term of $a(t)$, while $v(a(t))$ can be seen as the t-power contained in the leading term.

Section 12 is devoted to the behavior of tangible strong supervaluations under the map $\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma \circ v)$ given above. It turns out that γ_* maps $\mathrm{Cov}_{\mathrm{t,s}}(v)$ into $\mathrm{Cov}_{\mathrm{t,s}}(\gamma \circ v)$. But usually $\gamma_*(\overline{\varphi}_v)$ is different from the top element $\overline{\varphi}_{\gamma\circ v}$ of $\mathrm{Cov}_{\mathrm{t,s}}(\gamma\circ v)$, while $\gamma_*(\varphi_v) = \varphi_{\gamma\circ v}$. This indicates that it is not advisable to restrict the supervaluation theory from start to the strong supervaluations, even if we are only interested in these.

Strictly speaking, Kapranov's Lemma extends the valuation v to the polynomial ring $R[\lambda_1, \dots, \lambda_n]$ over R, with target in the polynomial ring $M[\lambda_1, \dots, \lambda_n]$, which no longer is bipotent. Thus, the theory in this paper needs to be generalized if we are to deal formally with such notions. This is set forth in the last Section 13, called the epilog, in which the target of a valuation is replaced by a monoid with a binary sup operation. Much of this paper could be formulated in this more general situation, but we only provide the broad outline in the epilog (including the appropriate version of Kapranov's Lemma), since a detailed investigation would carry us too far afield at this stage.

The reader may ask whether valuations and supervaluations on semirings instead of just rings deserve interest apart from formal issues. They do. It is only for not making a long paper even longer that we do not give applications to semirings here.

The semiring $R = \sum A^2$ of sum of squares of a commutative ring (or even a field) A with $-1 \notin R$ is a case in point. R is cancellative, and hence embeds into its Grothendieck ring (which is A if 2 is a unit). But using families of valuations on R, we can pass from R to a semiring R' which is a degeneration, hence simplification, of R of interest. Usually R' will have Grothendieck ring zero, and hence will be completely out of the realm of rings. Real algebra seems to be a fertile ground for studying valuations and supervaluations on semirings. The paper contains only one very small hint pointing in this direction, Example 2.4.

1. Bipotent semirings

Let R be a semiring (always with unit element $1 = 1_R$). Later we will assume that R is commutative, but presently this is not necessary.

Definition 1.1. *We call a pair $(a, b) \in R^2$ **bipotent** if $a + b \in \{a, b\}$. We call the semiring R **bipotent** if every pair $(a, b) \in R^2$ is bipotent.*

Proposition 1.2. *Assume that R is a bipotent semiring. Then the binary relation $(a, b \in R)$*

$$a \le b \quad \textit{iff} \quad a + b = b \tag{1.1}$$

on R is a total ordering on the set R, compatible with addition and multiplication, i.e., for all $a, b, c \in R$

$$\begin{aligned} a \le b \quad &\Rightarrow \quad ac \le bc,\ ca \le cb, \\ a \le b \quad &\Rightarrow \quad a + c \le b + c. \end{aligned}$$

Proof. A straightforward check. ☐

Remark 1.3. *We can define such a binary relation $\le$ by* (1.1) *in any semiring, and then obtain a partial ordering compatible with addition and multiplication. The ordering is total iff R is bipotent. Clearly, $0_R \le x$ for every $x \in R$.*

Definition 1.4. *We call a semiring R a **semidomain**, if R has no zero divisors, i.e., the set $R \setminus \{0\}$ is closed under multiplication. We call R a **semifield**, if R is commutative and every element $x \neq 0$ of R is invertible; hence $R \setminus \{0\}$ is a group under multiplication.*

Given a bipotent semidomain R, the set $G := R \setminus \{0\}$ is a totally ordered monoid under the multiplication of R.

In this way we obtain all (totally) ordered monoids. Indeed, if $G = (G, \cdot)$ is a given ordered monoid, we gain a bipotent semiring R as follows: Adjoin a new element 0 to G and form the set $R := G \cup \{0\}$. Extend the multiplication on G to a multiplication on R by the rules $0 \cdot g = g \cdot 0 = 0$ for any $g \in G$ and $0 \cdot 0 = 0$. Extend the ordering of G to a total ordering on R by the rule $0 < g$ for $g \in G$. Then define an addition on R by the rule

$$x + y := \max(x, y)$$

for any $x, y \in R$. It is easily checked that R is a bipotent semiring, and that the ordering on R by the rule (1.1) is the given one. We denote this semiring R by $T(G)$.

These considerations can be easily amplified to the following theorem.

Theorem 1.5. *The category of (totally) ordered monoids G is isomorphic*[3] *to the category of bipotent semidomains R by the assignments*

$$G \mapsto T(G), \quad R \mapsto R \setminus \{0\}.$$

Here the morphisms in the first category by definition are the order preserving monoid homomorphisms $\gamma : G \to G'$ in the weak sense; i.e., γ is multiplicative, $\gamma(1) = 1$, and $x \le y \Rightarrow \gamma(x) \le \gamma(y)$, while the morphisms in the second category are the semiring homomorphisms (with $1 \mapsto 1$).

In the following we regard an ordered monoid and the associated bipotent semiring as the same entity in a different disguise. Usually we prefer the semiring viewpoint.

Example 1.6. *Starting with the monoid $G = (\mathbb{R}, +)$, i.e., the field of real numbers with the usual addition, we obtain a bipotent semifield*

$$T(\mathbb{R}) := \mathbb{R} \cup \{-\infty\},$$

where addition $\oplus$ and multiplication $\odot$ of $T(\mathbb{R})$ are defined as follows, and the neutral element of addition is denoted by $-\infty$ instead of 0, since our monoid is now given in additive notation.

[3]This is more than equivalent!

For $x, y \in \mathbb{R}$

$$\begin{aligned} x \oplus y &= \max(x, y), \\ x \odot y &= x + y, \\ (-\infty) \oplus x = x \oplus (-\infty) &= x, \\ (-\infty) \odot x = x \odot (-\infty) &= -\infty, \\ (-\infty) \oplus (-\infty) &= -\infty, \\ (-\infty) \odot (-\infty) &= -\infty. \end{aligned}$$

$T(\mathbb{R})$ is the "real tropical semifield" of common tropical algebra, often called the "max-plus" algebra $\mathbb{R} \cup \{-\infty\}$: cf. [IMS], or [SS] (there a "min-plus" algebra is used).

2. m-valuations

In this section we assume that all occurring semirings and monoids are commutative.

Let R be a semiring.

Definition 2.1. *An m-**valuation** (= monoid valuation) on* R *is a map* $v : R \to M$ *into a (commutative) bipotent semiring* $M \neq \{0\}$ *with the following properties:*

$$\begin{aligned} &V1 : v(0) = 0, \\ &V2 : v(1) = 1, \\ &V3 : v(xy) = v(x)v(y) \quad \forall x, y \in R, \\ &V4 : v(x + y) \leq v(x) + v(y) \quad [= \max(v(x), v(y))] \quad \forall x, y \in R. \end{aligned}$$

We call the m-valuation v ***strict****, if instead of V4 the following stronger axiom holds:*

$$V5 : v(x + y) = v(x) + v(y) \quad \forall x, y \in R.$$

We call v ***bipotent*** *if* $\forall x, y \in R$

$$V5' : v(x + y) \in \{v(x), v(y)\}.$$

N.B. $V5'$ is stronger than V4 but weaker than V5. A strict m-valuation $v : R \to M$ is just a semiring homomorphism from R to M.

In the special case that $M = \Gamma \cup \{0\}$ with Γ an ordered abelian group, we call the m-valuation $v : R \to M$ a **valuation**. Notice that in the case that R is a ring (instead of a semiring), this is exactly the notion of a valuation as defined by Bourbaki [B] (Alg. Comm. VI, §3, No.1) and studied, e.g., in [HK] and [KZ, Chap. I], except that for Γ we have chosen the multiplicative notation instead of the additive notation.

If $v : R \to M$ is an m-valuation, we may replace M by the submonoid $v(R)$. We then speak of v as a **surjective** m-**valuation**.

Definition 2.2. *A (commutative) monoid* G *is called* ***cancellative****, if, for any* $a, b, c \in G$*, the equation* $ac = bc$ *implies* $a = b$*.*

Notice that an ordered monoid G is cancellative iff $a < b$ implies $ac < bc$ for any $a, b, c \in G$. An ordered cancellative monoid can be embedded into an ordered abelian group Γ in the well-known way by introducing formal fractions $\frac{a}{b}$ for $a, b \in G$. Then an m-valuation v from R to $T(G) = G \cup \{0\}$ is essentially the same thing as an m-valuation from R to $\Gamma \cup \{0\}$. For this reason, we extend the notion of "valuation" from above as follows.

Definition 2.3. *A* ***valuation*** *on a semiring R is an m-valuation $v : R \to G \cup \{0\}$ with G a cancellative monoid.*

m-valuations on rings have been studied in [HV], and then by D. Zhang [Z].

If R is a **ring**, an m-valuation $v : R \to M$ can **never** be strict, since we have an element $-1 \in R$ with $1 + (-1) = 0$, from which for v strict it would follow that $0_M = v(0) = \max(v(1), v(-1))$; hence $v(1) = 0_M$, a contradiction to axiom V2. But for R a semiring there may exist interesting strict m-valuations, even with values in a group.

Example 2.4. *Let T be a* ***preprime*** *in a ring R, by which we simply mean that T is a sub-semiring of R $(T + T \subset T,\ T \cdot T \subset T,\ 0 \in T, 1 \in T)$. {We do not exclude the case $-1 \in T$ ("improper preprime") but these will not matter.}*

*We say that a valuation $v : R \to M$ is T-**convex** if the restriction $v \mid T : T \to M$ is strict. As is well-known, if $T = \sum R^2$ (and $M \setminus \{0\}$ is a group) the T-convex valuations are just the real valuations on R. (A valuation $v : R \to \Gamma \cup \{0\}$ is called* ***real*** *if the residue class field $k(v)$ is formally real.) See* [KZ1], *§5 for T a preordering, and §2 for $T = \sum R^2$.*

The entire paper [KZ1] witnesses the importance of T-convex valuations for T a preordering.

Bipotent valuations on rings are rare. But for semirings they are rather common. In particular, we have

Example 2.5. *If R is a bipotent semiring, then every multiplicative map $v : R \to M$ to another bipotent semiring, with $v(0) = 0$, $v(1) = 1$, is a bipotent m-valuation; v is strict iff v is a semiring homomorphism.*

In addition to strict and bipotent m-valuations, we introduce two more classes of m-valuations.

Definition 2.6. *We call an m-valuation $v : R \to M$* ***strong*** *if, besides V1–V4, the following holds:*

$$V5'' : \text{If } x, y \in R \text{ and } v(x) \neq v(y), \text{ then } v(x + y) = \max(v(x), v(y)).$$

We call an m-valuation v ***amenable****, if the following condition holds, which is still weaker than both V5′ and V5″.*

$$V5''' : \text{If } x, y \in R \text{ and } v(x) \neq v(y), \text{ then } v(x + y) \in \{v(x), v(y)\}.$$

Note the implications of the following chart:

$$\begin{array}{ccccc} \text{strict} & \Rightarrow & \text{bipotent} & & \\ \Downarrow & & \Downarrow & & \\ \text{strong} & \Rightarrow & \text{amenable} & \Rightarrow & m\text{-valuation.} \end{array}$$

If R is a ring, every m-valuation on R is strong. This can be seen by the same argument as is well-known for valuations on fields. Thus in the bottom line of the chart we are dealing with intricacies which only occur for the semirings.

Semirings, even semifields, may admit valuations which are not even amenable.

Example 2.7. *Let F be a totally ordered field, and $R := \{x \in F | x \geq 0\}$ the subsemifield of nonnegative elements. Further let $\Gamma := \{x \in F | x > 0\}$, viewed as a totally ordered group, and $M := \{0\} \cup \Gamma$ the associated bipotent semifield. The map $v : R \to M$ with $v(0) = 0$, $v(a) = \frac{1}{a}$ for $a \neq 0$, is a valuation on R, which is not amenable.*

Proposition 2.8.

a) *If $v : R \to M$ is an m-valuation and M is a bipotent semidomain, then $v^{-1}(0)$ is a prime ideal of R (i.e., an ideal of R, whose complement in R is closed under multiplication).*

b) *If v is strong, then, for any $x \in R$ and $z \in v^{-1}(0)$,*

$$v(x+z) = v(x). \tag{2.1}$$

Proof. a): If $v(x) = 0$, $v(y) = 0$, then

$$v(x+y) \leq \max(v(x), v(y)) = 0;$$

hence $v(x+y) = 0$. Thus $v^{-1}(0)$ is closed under addition. If $x \in R$, $z \in v^{-1}(0)$, then $v(xz) = v(x)v(z) = 0$. Thus $v^{-1}(0)$ is closed under multiplication by elements in R. If $v(x) > 0$, $v(z) > 0$, then $v(xz) = v(x)v(z) > 0$. Thus $R \setminus v^{-1}(0)$ is closed under multiplication.

b): We have $v(x+z) \leq \max(v(x), v(z)) = v(x)$. Assume that v is strong. If $v(x) \neq 0$ we have

$$v(x+z) = \max(v(x), v(z)) = v(x).$$

□

If $v : R \to M$ is an arbitrary m-valuation, then it is still obvious that $v^{-1}(0)$ is an ideal of R.

Definition 2.9. *We call the ideal $v^{-1}(0)$ the **support** of the m-valuation v, and write $v^{-1}(0) = \operatorname{supp}(v)$. We call the support of v **insensitive**, if the equality* (2.1) *above holds for any $x \in R$ and $z \in \operatorname{supp}(v)$, **sensitive** otherwise.*

Proposition 2.8.b tells us that $\operatorname{supp}(v)$ is insensitive if v is strong. In particular, this holds if R is a ring.

Example 2.10. *Let Γ be an ordered abelian group and H is a convex proper subgroup. Let $\mathfrak{a} := \{g \in \Gamma \mid g > H\} \cup \{0\}$. We regard $\Gamma \cup \{0\}$ as a bipotent semifield (cf. §1), and define a subsemiring M of $\Gamma \cup \{0\}$ by*

$$M := H \cup \mathfrak{a}.$$

Notice that we have $H \cdot \mathfrak{a} \subset \mathfrak{a}$, $\mathfrak{a} \cdot \mathfrak{a} \subset \mathfrak{a}$, and $\mathfrak{a} + \mathfrak{a} \subset \mathfrak{a}$. Thus M is indeed a subsemiring of $\Gamma \cup \{0\}$, and $\mathfrak{a}$ is an ideal of M. We define a map $v : M \to H \cup \{0\}$ by setting $v(x) = x$ if $x \in H$, and $v(x) = 0$ if $x \in \mathfrak{a}$. It is easily checked that v fulfills the axioms V1–V3 and V5′ above. Thus v is a bipotent valuation. But the support $\mathfrak{a}$ of v is sensitive: For $x \in H$, $z \in \mathfrak{a}$ and $z \neq 0$, we have $v(x) > 0$, $v(z) = 0$, $x + z = z$; hence $v(x+z) = 0$.

We switch over to the problem of "comparing" different m-valuations on the same semiring R.

Definition 2.11. *Let $v : R \to M$ and $w : R \to N$ be m-valuations.*

a) *We say that v **dominates** w, if for any $a, b \in R$*

$$v(a) \leq v(b) \Rightarrow w(a) \leq w(b).$$

b) *We say that v **dominates** w **weakly**, if for any $a, b \in R$*

$$v(a) = v(b) \Rightarrow w(a) = w(b).$$

c) *If v dominates w weakly and v is surjective, there clearly exists a unique map $\gamma : M \to N$ with $w = \gamma \circ v$. We denote this map γ by $\gamma_{w,v}$.*

Clearly, $\gamma_{w,v}$ is multiplicative and sends 0 to 0 and 1 to 1. If v dominates w, then $\gamma_{w,v}$ is also order-preserving and hence is a homomorphism from the bipotent semiring M to N.

Proposition 2.12. *Assume that M, N are bipotent semirings and $v : R \to M$ is a surjective m-valuation.*

a) *The m-valuations $w : R \to N$ dominated by v correspond uniquely with the homomorphisms $\gamma : M \to N$ via $w = \gamma \circ v$, $\gamma = \gamma_{w,v}$.*
b) *If v has one of the properties "strict, strong, bipotent, amenable," and dominates w, then w has the same property.*
c) *Assume, in addition, that v is bipotent. Then the m-valuations $w : R \to N$ which are weakly dominated by v are again bipotent. They correspond uniquely with the maps $\gamma : M \to N$ which are multiplicative and send 0 to 0, 1 to 1, via $w = \gamma \circ v$ and $\gamma = \gamma_{w,v}$.*

Proof. It suffices to verify that, given a map $\gamma : M \to N$ of the right kind, the map $\gamma v := \gamma \circ v : R \to N$ is a valuation of the right kind.

a,b): If $\gamma : M \to N$ is a homomorphism, then clearly γv is an m-valuation, and γv inherits from v each of the properties "strict, strong, bipotent, amenable".

c): Assume now that $\gamma : M \to N$ is a multiplicative map with $\gamma(0) = 0$, $\gamma(1) = 1$. Let $a, b \in R$. If v is bipotent, then $v(a + b) \in \{v(a), v(b)\}$. This, of course, implies that $\gamma v(a + b) \in \{\gamma v(a), \gamma v(b)\}$. Thus γ is bipotent. □

3. Supertropical semirings

Definition 3.1. *A **semiring with idempotent** is a pair (R, e) consisting of a semiring R and a **central** idempotent e. {For the moment R is allowed to be noncommutative.}*

We then have an endomorphism $\nu : R \to R$ (which usually does not map 1 to 1) defined by $\nu(a) = ea$. It obeys the rules

$$\nu \circ \nu = \nu, \tag{3.1}$$

$$a\nu(b) = \nu(a)b = \nu(ab). \tag{3.2}$$

Conversely, if a pair (R, ν) is given consisting of a semiring R and an endomorphism ν (not necessarily $\nu(1) = 1$), such that (3.1), (3.2) hold, then $e := \nu(1)$ is a central idempotent of R and $\nu(a) = ea$ for every $a \in R$.

Thus such pairs (R, ν) are the same objects as semirings with idempotents.

Definition 3.2. *A **semiring with ghosts** is a semiring with idempotent (R, e) such that the following axiom holds ($\nu(a) := ea$)*

$$\nu(a) = \nu(b) \quad \Rightarrow \quad a + b = \nu(a). \tag{3.3}$$

Remark 3.3. *This axiom implies that $ea = e(a + b) = ea + eb$ if $\nu(a) = \nu(b)$. We **do not** want to demand that then $eb = 0$. Usually, $(R, +)$ will be a highly non cancellative abelian semigroup.*

Terminology 3.4. *If (R, e) is a semiring with ghosts, then $\nu : x \mapsto ex$, $R \to R$ is called the **ghost map** of (R, e). The idea is that every $x \in R$ has an associated "ghost" $\nu(x)$, which is thought of to be somehow "near" to the zero element 0 of R, without necessarily being 0 itself. {That will happen for all $x \in R$ only if $e = 0$.} We call eR the **ghost ideal** of (R, e).*

Now observe that, if (R, e) is a semiring with ghosts, the idempotent e is determined by the semiring R above, namely

$$e = 1 + 1.$$

Thus we may suppress the idempotent e in the notation of a semiring with ghosts and redefine these objects as follows.

Definition 3.5. *A semiring R is called a* ***semiring with ghosts*** *if*

$$1 + 1 = 1 + 1 + 1 + 1 \tag{3.3$'$}$$

and for all $a, b \in R$

$$a + a = b + b \quad \Rightarrow \quad a + b = a + a. \tag{3.3$''$}$$

Remark 3.6. *If* $(3.3)'$ *holds then $e := 1 + 1$ is a central idempotent of R. Passing from R to $(R, e) = (R, 1+1)$, we see that* $(3.3\,'')$ *is the previous axiom* (3.3)*. Notice also that* $(3.3'')$ *implies that $1+1+1 = 1+1$. (Take $a = 1$, $b = e$.) Thus, $m1 = 1+1$ for all natural numbers $m \geq 2$.*

Terminology 3.7. *If R is a semiring with ghosts, we write $e = e_R$ and $\nu = \nu_R$ if necessary. We also introduce the notation*

$$\mathcal{T} := \mathcal{T}(R) := R \setminus Re,$$
$$\mathcal{G} := \mathcal{G}(R) := Re \setminus \{0\},$$
$$\mathcal{G}_0 := \mathcal{G} \cup \{0\} = Re.$$

We call the elements of $\mathcal{T}$ the ***tangible elements*** *of R and the elements of $\mathcal{G}$ the* ***ghost elements*** *of R. We do not exclude the case that $\mathcal{T}$ is empty, i.e., $e = 1$. In this case R is called a* ***ghost semiring****.*

The ghost ideal $\mathcal{G}_0 = eR$ of R is itself a semiring with ghosts, in fact, a ghost semiring. It has the property $a + a = a$ for every $a \in Re$, as follows from (3.3). {Some people call a semiring T with $a + a = a$ for every $a \in T$ an "idempotent semiring".}

We mention a consequence of axiom (3.3) for the ghost map $\nu : R \to Re$, $\nu(x) := ex$.

Remark 3.8. *If R is a semiring with ghosts, then, for any $x \in R$,*

$$\nu(x) = 0 \quad \Leftrightarrow \quad x = 0.$$

Proof. ($\Leftarrow$): evident.
($\Rightarrow$): We have $\nu(x) = 0 = \nu(0)$; hence by (3.3) $x = x + 0 = \nu(0) = 0$. □

We are ready for the central definition of the section.

Definition 3.9. *A semiring R is called* ***supertropical*** *if R is a semiring with ghosts and*

$$\forall a, b \in R: \;\; a + a \neq b + b \quad \Rightarrow \quad a + b \in \{a, b\}. \tag{3.4}$$

In other terms, every pair (a, b) in R with $ea \neq eb$ is bipotent.

Remarks 3.10.

(i) *It follows that then $\mathcal{G}(R)_0 = Re$ is a bipotent semidomain. Indeed, if a, b are different elements of $\mathcal{G}(R)$, then $a = ea \neq b = eb$; hence $a+b \in \{a, b\}$ by axiom* (3.4)*. If $a = 0$ or $b = 0$, this trivially is also true. If $a = b$ then $a + b = ea = a$. Thus $a + b \in \{a, b\}$ for any $a, b \in \mathcal{G}(R)_0$. The set $\mathcal{G}(R)$ is either empty (the case $1 + 1 = 0$) or $\mathcal{G}(R)$ is an ordered monoid under the multiplication of R, as explained in* §1.

(ii) *The supertropical semirings without tangible elements are just the bipotent semirings.*
(iii) *Every subsemiring of a supertropical semiring is again supertropical.*

Theorem 3.11. *Let R be a supertropical semiring, $e := e_R$, $\mathcal{G} := \mathcal{G}(R)$. Then the addition on R is determined by the multiplication on R and the ordering on the multiplicative submonoid $\mathcal{G}$ of R, in case $\mathcal{G} \neq \emptyset$, as follows. For any $a, b \in R$*

$$a + b = \begin{cases} a & \text{if} \quad ea > eb, \\ b & \text{if} \quad ea < eb, \\ ea & \text{if} \quad ea = eb, \end{cases}$$

If $\mathcal{G} = \emptyset$ then $a + b = 0$ for any $a, b \in R$.

Proof. We may assume that $ea \geq eb$. If $ea = eb$, axiom (3.3) tells us that $a + b = ea$. Assume now that $ea > eb$. By definition of the ordering on eR (cf. §1), we have

$$e(a + b) = ea + eb = ea.$$

By axiom (3.4), $a + b = a$ or $a + b = b$.

Suppose that $a + b = b$. Then $e(a + b) = eb$. Since $ea \neq eb$, this is a contradiction. We conclude that $a + b = a$. □

From now on, **we always assume that our semirings are commutative**.

Remark 3.12. *If R is a supertropical semiring, the ghost map $\nu_R : R \to eR$, $x \mapsto ex$ is a strict m-valuation. Indeed, the axioms V1–V3 and V5 from §2 are clearly valid for ν_R.*

Thus, every supertropical semiring has a natural built-in strict m-valuation.

There are important cases where ν_R is even a valuation (cf. Definition 2.3), as we explicate now.

Proposition 3.13. *Assume that R is a supertropical semiring and $\mathcal{T}(R)$ is closed under multiplication. Then the submonoid $G := e\mathcal{T}(R)$ of $\mathcal{G}(R)$ is cancellative. (N.B. We have $e\mathcal{T}(R) \subset \mathcal{G}(R)$ by Remark 3.8.)*

Proof. Let $a, b, c \in \mathcal{T}(R)$ be given with $(ea)(ec) = (eb)(ec)$, i.e., $eac = ebc$. Suppose that $ea \neq eb$, say $ea < eb$. Then Theorem 3.11 tells us that $a + b = b$ and $ac + bc = ebc$. By assumption, $bc \in \mathcal{T}(R)$; hence $bc \neq ebc$. But the first equation gives $ac + bc = bc$, a contradiction. Thus $ea = eb$. □

In the situation of this proposition we may omit the part $\mathcal{G}(R) \setminus G$, consisting of "useless" ghosts, in the semiring R, and then obtain a "supertropical domain" $U := \mathcal{T}(R) \cup G \cup \{0\}$, as defined below, whose ghost map $\nu_U := U \to G \cup \{0\}$ is a *surjective strict valuation.*

Definition 3.14. *Let M be a bipotent semiring and R a supertropical semiring.*

*a) We say that the semiring M is **cancellative** if for any $x, y, z \in M$*

$$xz = yz, \ z \neq 0 \ \Rightarrow \ x = y.$$

This means that M is a bipotent semidomain (cf. Definition 1.4) and the multiplicative monoid $M \setminus \{0\}$ is cancellative.

*b) We call R a **supertropical predomain**, if $\mathcal{T}(R) = R \setminus eR$ is not empty (i.e., $e \neq 1$) and is closed under multiplication, and moreover eR is a cancellative bipotent semidomain.*

*c) We call R a **supertropical domain**, if $\mathcal{T}(R)$ is not empty and is closed under multiplication, and R maps $\mathcal{T}(R)$ onto $\mathcal{G}(R)$.*

Notice that the last condition in Definition 3.14.c implies that $\mathcal{G}(R)$ is a cancellative monoid (Proposition 3.13). Thus a supertropical domain is a supertropical predomain.

Looking again at Theorem 3.11, we see that a way is opened up to construct supertropical predomains and domains. First notice that the theorem implies the following

Remark 3.15. *If R is a supertropical predomain, we have for every $a \in \mathcal{T}(R)$ and $x \in \mathcal{G}(R)$ the multiplication rule*

$$ax = v(a)x$$

with $v := \nu_R \mid \mathcal{T}(R)$. Thus the multiplication on

$$R = \mathcal{T}(R) \dot\cup \mathcal{G}(R) \dot\cup \{0\}$$

is completely determined by the triple $(\mathcal{T}(R), \mathcal{G}(R), v)$. We write $v = v_R$.

Construction 3.16. *Conversely, let a triple $(\mathcal{T}, \mathcal{G}, v)$ be given with $\mathcal{T}$ a monoid, $\mathcal{G}$ an ordered cancellative monoid and $v : \mathcal{T} \to \mathcal{G}$ a monoid homomorphism. We define a semiring R as follows. As a set*

$$R = \mathcal{T} \dot\cup \mathcal{G} \dot\cup \{0\}.$$

The multiplication on R will extend the given multiplications on $\mathcal{T}$ and $\mathcal{G}$. If $a \in \mathcal{T}$, $x \in \mathcal{G}$, we decree that

$$a \cdot x = x \cdot a := v(a)x.$$

Finally, $0 \cdot z = z \cdot 0 := 0$ for all $z \in R$.

The addition on R extends the addition on $\mathcal{G} \cup \{0\}$ as the bipotent semiring corresponding to the ordered monoid $\mathcal{G}$, as explained in §1. For $x, y \in \mathcal{T}$ we decree

$$x + y := \begin{cases} x & \text{if} \quad v(x) > v(y), \\ y & \text{if} \quad v(x) < v(y), \\ v(x) & \text{if} \quad v(x) = v(y). \end{cases}$$

Finally, for $x \in \mathcal{T}$ and $y \in \mathcal{G} \cup \{0\}$

$$x + y = y + x := \begin{cases} x & \text{if} \quad v(x) > y, \\ y & \text{if} \quad v(x) \le y. \end{cases}$$

It now can be checked in a straightforward way[4] *that R is a supertropical predomain with $\mathcal{T}(R) = \mathcal{T}$, $\mathcal{G}(R) = \mathcal{G}$, $v_R = v$. Thus we have gained a description of all supertropical predomains R by triples $(\mathcal{T}, \mathcal{G}, v)$ as above. We write*

$$R = \mathrm{STR}(\mathcal{T}, \mathcal{G}, v)$$

{STR = *"supertropical"*}. *Notice that in this semiring R every pair $(x, y) \in R^2$ is bipotent except the pairs (a, b) with $a \in \mathcal{T}$, $b \in \mathcal{T}$ and $v(a) = v(b)$. If v is onto, then R is a supertropical domain.*

Definition 3.17. *A semiring R is called a* ***supertropical semifield****, if R is a supertropical domain, and every $x \in \mathcal{T}(R)$ is invertible; hence both $\mathcal{T}(R)$ and $\mathcal{G}(R)$ are groups under multiplication.*

We write down primordial examples of supertropical domains and semifields (cf. [I], [IR1]). Other examples will come up in §4.

[4] Alternatively consult [IKR, §1] (as soon as available), where a detailed proof of a more general statement is given.

Examples 3.18. *Let $\mathcal{G}$ be an ordered cancellative monoid. This given us the supertropical domain (cf. Construction 3.16)*

$$D(\mathcal{G}) := \mathrm{STR}(\mathcal{G}, \mathcal{G}, \mathrm{id}_{\mathcal{G}}).$$

$D(\mathcal{G})$ is a supertropical semifield iff $\mathcal{G}$ is an ordered abelian group.

We come closer to the objects and notations of usual tropical algebra if we take here for $\mathcal{G}$ ordered monoids in ***additive*** *notation, $\mathcal{G} = (\mathcal{G}, +)$, e.g., $\mathcal{G} = \mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ with the usual addition. $D(\mathcal{G})$ contains the set $\mathcal{G}$. For every $a \in \mathcal{G}$ there is an element a^ν in $D(\mathcal{G})$ (read "a-ghost"), and*

$$\mathcal{G}^\nu := \{a^\nu \mid a \in \mathcal{G}\}$$

is a copy of the additive monoid $\mathcal{G}$ disjoint from $\mathcal{G}$. The zero element of the semiring $D(\mathcal{G})$ is now written $-\infty$. Thus

$$D(\mathcal{G}) = \mathcal{G} \dot{\cup} \mathcal{G}^\nu \dot{\cup} \{-\infty\}.$$

Denoting addition and multiplication of the semiring $D(G)$ by $\oplus$ and $\odot$, we have the following rules. For any $x \in D(\mathcal{G})$, $a \in \mathcal{G}$, $b \in \mathcal{G}$,

$$\begin{aligned}
-\infty \oplus x &= x \oplus -\infty = x, \\
a \oplus b &= \max(a, b), \quad \text{if } a \neq b, \\
a \oplus a &= a^\nu, \\
a^\nu \oplus b^\nu &= \max(a, b)^\nu, \\
a \oplus b^\nu &= a, \quad \text{if } a > b, \\
a \oplus b^\nu &= b^\nu, \quad \text{if } a \leq b, \\
-\infty \odot x &= x \odot -\infty = -\infty, \\
a \odot b &= a + b, \\
a^\nu \odot b &= a \odot b^\nu = a^\nu \odot b^\nu = (a + b)^\nu.
\end{aligned}$$

In the case $\mathcal{G} = (\mathbb{R}, +)$ these rules can already be found in [I]. There also motivation is given for their use in tropical algebra and tropical geometry.

We now only say that the semiring $D(\mathcal{G})$ associated to an additive ordered cancellative monoid $\mathcal{G}$ should be compared with the max-plus algebra $T(\mathcal{G}) = \mathcal{G} \cup \{-\infty\}$ introduced in §1. The ghost ideal $\mathcal{G}^\nu \cup \{-\infty\}$ of $D(\mathcal{G})$ is a copy of $T(\mathcal{G})$.

4. Supervaluations

In this section R is always a (commutative) semiring. Usually the letters U, V denote supertropical (commutative) semirings. If U is any such semiring, the idempotent $e_U = 1_U + 1_U$ will be often simply denoted by the letter "e", regardless of which supertropical semiring is under consideration (as we write $0_U = 0$, $1_U = 1$).

Definition 4.1. *a) A* ***supervaluation*** *on R is a map $\varphi : R \to U$ from R to a supertropical semiring U with the following properties.*

$$\begin{aligned}
&SV1 : \varphi(0) = 0, \\
&SV2 : \varphi(1) = 1, \\
&SV3 : \forall a, b \in R : \varphi(ab) = \varphi(a)\varphi(b), \\
&SV4 : \forall a, b \in R : e\varphi(a + b) \leq e(\varphi(a) + \varphi(b)) \quad [= \max(e\varphi(a), e\varphi(b))].
\end{aligned}$$

b) If $\varphi : R \to U$ is a supervaluation, then the map

$$v : R \to eU, \qquad v(a) := e\varphi(a)$$

*is clearly an m-valuation. We denote this m-valuation v by $e_U\varphi$ (or simply by $e\varphi$), and we say that φ **covers** the m-valuation $e_U\varphi = v$.*

*c) We say that a supervaluation $\varphi : R \to U$ is **tangible**, if $\varphi(R) \subset \mathcal{T}(U) \cup \{0\}$, and we say that φ is **ghost** if $\varphi(R) \subset eU$.*

N.B. A ghost supervaluation $\varphi : R \to U$ is nothing other than an m-valuation, after replacing the target U by eU.

Proposition 4.2. *Assume that $\varphi : R \to U$ is a supervaluation and $v : R \to e_U U =: M$ is the m-valuation $e_U\varphi$ covered by φ. Then*

$$U' := \varphi(R) \cup e\varphi(R)$$

is a subsemiring of U. The semiring U' is again supertropical and $e_{U'} = e_U (= e)$.

Proof. The set $v(R)$ is a multiplicative submonoid of the bipotent semiring M; hence is itself a bipotent semiring. In particular, $v(R)$ is closed under addition. If $a, b \in R$ are given with $v(a) \le v(b)$, then either $v(a) < v(b)$, in which case

$$a + b = b, \quad v(a) + b = b, \quad a + v(b) = v(b),$$

or $v(a) = v(b)$, in which case

$$a + b = v(a) + b = a + v(b) = v(a).$$

This proves that $U' + U' \subset U'$. Clearly $0 \in U'$, $1 \in U'$ and $U' \cdot U' \subset U'$. Thus U' is a subsemiring of U. As stated above (Remark 3.10.iii), every subsemiring of a supertropical semiring is again supertropical. Thus U' is supertropical. □

Definition 4.3. *We say that the supervaluation $\varphi : R \to U$ is **surjective** if $U' = U$. We say that φ is **tangibly surjective** if $\varphi(R) \supset \mathcal{T}(U)$.*

Remark 4.4. *If $\varphi : R \to U$ is any supervaluation, then, replacing U by $U' = \varphi(R) \cup e\varphi(R)$, we obtain a surjective supervaluation. If we only replace U by $\varphi(R) \cup (eU)$, which is again a subsemiring of U, we obtain a tangibly surjective supervaluation.*

Thus, whenever necessary we may retreat to tangibly surjective or even surjective supervaluations without loss of generality.

Recall that an m-valuation $v : R \to M$ is called a valuation, if the bipotent semiring M is cancellative (cf. Definition 2.3, Definition 3.14.a). Every valuation can be covered by a tangible supervaluation, as the following easy but important construction shows.

Example 4.5. *Let $v : R \to M$ be a valuation, and let $\mathfrak{q} := v^{-1}(0)$ denote the support of v. We then have a monoid homomorphism*

$$R \setminus \mathfrak{q} \to M \setminus \{0\}, \quad a \mapsto v(a),$$

which we denote again by v. Let

$$U := \mathrm{STR}(R \setminus \mathfrak{q}, M \setminus \{0\}, v),$$

the supertropical predomain given by the triple $(R \setminus \mathfrak{q}, M \setminus \{0\}, v)$, as explained in Construction 3.16. Thus, as a set,

$$U = (R \setminus \mathfrak{q}) \,\dot{\cup}\, M.$$

We have $e = 1_M$, $e \cdot a = v(a)$ for $a \in R \setminus \mathfrak{q}$. The multiplication on U restricts to the given multiplications on $R \setminus \mathfrak{q}$ and on M, and $a \cdot x = x \cdot a = v(a)x$ for $a \in R \setminus \mathfrak{q}$, $x \in M$. The addition on U is determined by e and the multiplication in the usual way (cf. Theorem 3.11). In particular, for $a, b \in R \setminus \mathfrak{q}$, we have

$$a + b = \begin{cases} a & \text{if } v(a) > v(b), \\ b & \text{if } v(a) < v(b), \\ v(a) & \text{if } v(a) = v(b). \end{cases}$$

Now define a map $\varphi : R \to U$ by

$$\varphi(a) := \begin{cases} a & \text{if } a \in R \setminus \mathfrak{q}, \\ 0 & \text{if } a \in \mathfrak{q}. \end{cases}$$

One checks immediately that φ obeys the rules SV1–SV3. If $a \in R \setminus \mathfrak{q}$, then

$$e_U \varphi(a) = 1_M \cdot v(a) = v(a),$$

and for $x \in \mathfrak{q}$, we have

$$e_U \varphi(a) = e_U \cdot 0 = 0 = v(a)$$

also. Thus SV4 holds, and φ is a supervaluation covering v.

By construction φ is tangible and tangibly surjective. If v is surjective then φ is surjective.

Definition 4.6. *We denote the supertropical ring just constructed by $U(v)$ and the supervaluation φ just constructed by φ_v. Later we will call $\varphi_v : R \to U(v)$* ***the initial cover of*** *v, cf. Definition 5.15.*

Notice that $U(v)$ is a supertropical domain iff v is surjective, and that in this case the supervaluation φ_v is surjective.

Remark 4.7. *The supertropical predomain $U(v)$ just constructed deviates strongly in its nature from the supertopical domain $D(\mathcal{G})$ for $\mathcal{G}$ an ordered monoid studied in Examples 3.18. While for $U = D(\mathcal{G})$ the restriction*

$$\nu_U \mid \mathcal{T}(U) : \mathcal{T}(U) \to \mathcal{G}(U)$$

of the ghost map ν_U is bijective, for $U = U(v)$ this map usually has big fibers.

5. Dominance and transmissions

As before now all semirings are assumed to be commutative. R is any semiring, and U, V are bipotent semirings.

Definition 5.1. *If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations, we say that φ* ***dominates*** *ψ, and write $\varphi \geq \psi$, if for any $a, b \in R$ the following holds.*

$$\begin{aligned} &D1. \quad \varphi(a) = \varphi(b) \ \Rightarrow \psi(a) = \psi(b), \\ &D2. \quad e\varphi(a) \leq e\varphi(b) \Rightarrow e\psi(a) \leq e\psi(b), \\ &D3. \quad \varphi(a) \in eU \ \Rightarrow \psi(a) \in eV. \end{aligned}$$

Notice that D3 can be also phrased as follows:

$$\varphi(a) = e\varphi(a) \quad \Rightarrow \quad \psi(a) = e\psi(a).$$

Lemma 5.2. *Let $\varphi : R \to U$ and $\psi : R \to V$ be supervaluations. Assume that φ dominates ψ, and also (without essential loss of generality) that φ is surjective. Then there exists a unique map $\alpha : U \to V$ with $\psi = \alpha \circ \varphi$ and*

$$\forall x \in U : \alpha(e_U x) = e_V \alpha(x)$$

(i.e., $\alpha \circ \nu_U = \nu_V \circ \alpha$).

Proof. By D1 and D2 we have a unique well-defined map $\beta : \varphi(R) \to \psi(R)$ with $\beta(\varphi(a)) = \psi(a)$ for all $a \in R$ and a unique well-defined map $\gamma : e\varphi(R) \to e\psi(R)$ with $\gamma(e\varphi(a)) = e\psi(a)$ for all $a \in R$. Now $U = \varphi(R) \cup e\varphi(R)$, since φ is assumed to be surjective. Suppose that $x \in \varphi(R) \cap e\varphi(R)$. Then $x = \varphi(a)$ for some $a \in R$, and $x = ex = e\varphi(a)$. By axiom D3 we conclude that $\psi(a) = e\psi(a)$. Thus $\beta(x) = \gamma(x)$. This proves that we have a unique well-defined map $\alpha : U \to V$ with $\alpha(x) = \beta(x)$ for $x \in \varphi(R)$ and $\alpha(y) = \gamma(y)$ for $y \in e\varphi(R)$. We have $\alpha(\varphi(a)) = \psi(a)$, i.e., $\psi = \alpha \circ \varphi$. Moreover, for any $a \in R$, $\alpha(e_U\varphi(a)) = \gamma(e_U\varphi(a)) = e_V\psi(a)$. □

We record that in this proof we did not use the full strength of D2 but only the weaker rule that $e\varphi(a) = e\varphi(b)$ implies $e\psi(a) = e\psi(b)$.

Definition 5.3. *Assume that U and V are supertropical semirings.*

a) *If α is a map from U to V with $\alpha(eU) \subset eV$, we say that α **covers** the map $\gamma : eU \to eV$ obtained from α by restriction, and we write $\gamma = \alpha^\nu$. We also say that γ is the **ghost part** of α.*
b) *Assume that $\varphi : R \to U$ is a surjective supervaluation and $\psi : R \to V$ is a supervaluation dominated by φ. Then we call the map α occurring in Lemma 5.2, which is clearly unique, the **transmission from** φ **to** ψ, and we denote this map by $\alpha_{\psi,\varphi}$. Clearly, $\alpha_{\psi,\varphi}$ covers the map $\gamma_{w,v}$ connecting the surjective m-valuation $v := e\varphi : R \to eU$ to the m-valuation $w := e\psi : R \to eV$ introduced in Definition 2.11.*

Theorem 5.4. *Let $\varphi : R \to U$ be a surjective supervaluation and $\psi : R \to V$ a supervaluation dominated by φ. The transmission $\alpha := \alpha_{\psi,\varphi}$ obeys the following rules:*

$$\begin{aligned}
&TM1: \quad \alpha(0) = 0,\\
&TM2: \quad \alpha(1) = 1,\\
&TM3: \quad \forall x, y \in U : \quad \alpha(xy) = \alpha(x)\alpha(y),\\
&TM4: \quad \alpha(e_U) = e_V,\\
&TM5: \quad \forall x, y \in eU : \alpha(x + y) = \alpha(x) + \alpha(y).
\end{aligned}$$

Proof. TM1, TM2, and TM4 are obtained from the construction of α in the proof of Lemma 5.2. This construction tells us also that α sends eU to eV. Using (again) that $U = \varphi(R) \cup e\varphi(R)$, we check easily that $TM3$ holds. The rule D2 (in its full strength) tells us that the map $\gamma : eU \to eV$, obtained from α by restriction, is order preserving. This is TM5. □

Definition 5.5. *If U and V are supertropical semirings, we call any map $\alpha : U \to V$ which the rules TM1–TM5, a **transmissive map** from U to V.*

The axioms TM1-TM5 tell us that a transmissive map $\alpha : U \to V$ is the same thing as a homomorphism from the monoid $(U, \cdot\,)$ to $(V, \cdot\,)$ which restricts to a semiring homomorphism from eU to eV. It is evident that every homomorphism from the semiring U to V is a transmissive map, but there exist quite a few transmissive maps, which are not homomorphisms; cf. §9 below and [IKR].

As a converse to Lemma 5.2 we have the following fact.

Proposition 5.6. *Assume that $\varphi : R \to U$ is a supervaluation and $\alpha : U \to V$ is a transmissive map from U to a supertropical semiring V. Then $\alpha \circ \varphi : R \to V$ is again a supervaluation. If $e\varphi$ has one of the properties "amenable", "bipotent," "strong", "strict", then $e(\alpha \circ \varphi)$ has the same property.*

Proof. Let $\psi := \alpha \circ \varphi$. Clearly ψ inherits the properties SV1–SV3 from φ, since α obeys TM1–TM3. If $a \in R$, then, by TM4,

$$e\psi(a) = e(\alpha(\varphi(a))) = \alpha(e\varphi(a));$$

hence $e\psi = \alpha^\nu \circ (e\varphi)$. Since $e\varphi$ is an m-valuation, and $\alpha^\nu : eU \to eV$ is order preserving by TM5, we conclude that $e\psi$ is again an m-valuation.

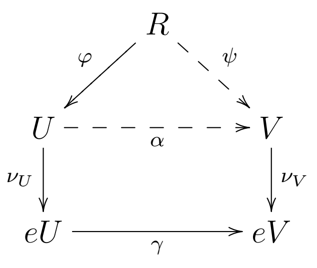
Now assume that $e\varphi$ is amenable. Then we see that $e\psi$ is amenable as follows. Let $a, b \in R$ be given with $e\psi(a) \neq e\psi(b)$, i.e., $\alpha(e\varphi(a)) \neq \alpha(e\varphi(b))$. Then $e\varphi(a) \neq e\varphi(b)$; hence $e\varphi(a + b) \in \{e\varphi(a), e\varphi(b)\}$. Applying the map α^ν we obtain $e\psi(a + b) \in \{e\psi(a), e\psi(b)\}$. In the same way one verifies that $e\psi$ inherits any of the properties "bipotent", "strict," "strong" from $e\varphi$. □

Remark 5.7. *If $\varphi : R \to U$ is a surjective supervaluation (cf. Definition 4.3) and $\alpha : U \to V$ is a surjective transmissive map, then the supervaluation $\alpha \circ \varphi$ is again surjective. Conversely, if $\varphi : R \to U$ and $\psi : R \to V$ are surjective supervaluations, and φ dominates ψ, then the transmission $\alpha_{\psi,\varphi} : U \to V$ is a surjective map.*

Combining Theorem 5.4, Proposition 5.6 and this remark, we read off the following facts.

Scholium 5.8. *Let U, V be supertropical semirings and $\varphi : R \to U$ a surjective supervaluation.*

a) *The supervaluations $\psi : R \to V$ dominated by φ correspond uniquely with the transmissive maps $\alpha : U \to V$ via $\psi = \alpha \circ \varphi$, $\alpha = \alpha_{\psi,\varphi}$.*
b) *If P is one of the properties "strict, strong, bipotent, amenable", and $e\varphi$ has property P, then $e\psi$ has property P.*
c) *The supervaluation ψ is surjective iff the map α is surjective.*
d) *Given a semiring homomorphism $\gamma : eU \to eV$, the supervaluation ψ covers the m-valuation $\gamma \circ (e\varphi)$ iff $\alpha^\nu = \gamma$.*



□

Example 5.9. *Let U be a supertropical semiring with ghost ideal $M := eU$. Then, as we know, the ghost map $\nu_U : U \to M$, $x \mapsto ex$, is a strict m-valuation on the semiring U (Remark 3.12). Clearly, the identity map $\mathrm{id}_U : U \to U$ is a supervaluation covering ν_U. Assume now that $\alpha : U \to V$ is a transmissive map. Let $\gamma := \alpha^\nu$ denote the homomorphism from M to $N := eV$ covered by α. Then $v := \gamma \circ \nu_U = \nu_V \circ \alpha$ is a strict valuation on U with values in N, and $\alpha := \alpha \circ \mathrm{id}_U$ is a supervaluation on U covering v. Thus α is the transmission from the supervaluation $\mathrm{id}_U : U \to U$ to the supervaluation $\alpha : U \to V$ covering v.*

The example tells us in particular that every transmissive map is the transmission between some supervaluations. *Therefore we may and will also use the shorter term "**transmission**" for "transmissive map".*

In general, a transmission does not behave additively; hence is not a homomorphism. We now record cases where nevertheless some additivity holds.

Proposition 5.10. *Let $\alpha : U \to V$ be a transmission and $\gamma : eU \to eV$ denote the ghost part of α, $\gamma = \alpha^\nu$ (which is a semiring homomorphism).*

i) *If $x, y \in U$ and $ex = ey$, then $\alpha(x) + \alpha(y) = \alpha(x + y)$.*
ii) *If $x, y \in U$ and $\alpha(x) + \alpha(y)$ is tangible, then again $\alpha(x) + \alpha(y) = \alpha(x + y)$.*
iii) *If γ is injective, then α is a semiring homomorphism.*

Proof. Let $x, y \in U$ be given, and assume without loss of generality that $ex \le ey$. Notice that this implies

$$e\alpha(x) = \alpha(ex) \le \alpha(ey) = e\alpha(y).$$

i): If $ex = ey$, then $e\alpha(x) = e\alpha(y)$, and we have $x+y = ex$, $\alpha(x)+\alpha(y) = e\alpha(x) = \alpha(ex)$; hence $\alpha(x) + \alpha(y) = \alpha(x + y)$.
ii): If $\alpha(x) + \alpha(y)$ is tangible, then certainly $e\alpha(x) \ne e\alpha(y)$; hence $e\alpha(x) < e\alpha(y)$. This implies $ex < ey$. Thus $x + y = y$, $\alpha(x) + \alpha(y) = \alpha(y)$; hence $\alpha(x) + \alpha(y) = \alpha(x + y)$.
iii): From i) we know that $\alpha(x + y) = \alpha(x) + \alpha(y)$ holds if $ex = ey$. Assume now that $ex < ey$. Since γ is injective this implies $e\alpha(x) < e\alpha(y)$. Thus $x+y = y$, $\alpha(x)+\alpha(y) = \alpha(y)$; hence again $\alpha(x + y) = \alpha(x) + \alpha(y)$.

□

Given an m-valuation $v : R \to M$, we now focus on the supervaluations $\varphi : R \to U$ which cover v, i.e., with $eU = M$ and $e\varphi = \nu_U \circ \varphi = v$. We single out a class of supervaluations which will play a special role.

Definition 5.11. *A supervaluation $\varphi : R \to U$ is called **tangibly injective** if the map φ is injective on the set $\varphi^{-1}(\mathcal{T}(U))$, i.e.,*

$$\forall a, b \in R : \ \varphi(a) = \varphi(b) \in \mathcal{T}(U) \quad \Rightarrow \quad a = b.$$

Example 5.12. *The supervaluation $\varphi_v : R \to U(v)$ constructed in §4 (cf. Example 4.5 and Definition 4.6) is injective on the set $R \setminus v^{-1}(0)$, hence certainly tangibly injective. Notice that $\varphi^{-1}(\mathcal{T}(U(v))) = R \setminus v^{-1}(0)$, i.e., φ is tangible. φ is also surjective.*

Theorem 5.13. *Assume that $\varphi : R \to U$ is a tangibly injective supervaluation covering $v : R \to M$. Let $\psi : R \to V$ be another supervaluation covering v, in particular, $eU = eV = M$.*

a) *φ dominates ψ iff the following holds:*

$$\forall a \in R : \ \varphi(a) = v(a) \quad \Rightarrow \quad \psi(a) = v(a), \tag{5.1}$$

in other terms, $\varphi(a) \in eU \quad \Rightarrow \quad \psi(a) \in eV$.

b) *If, in addition, φ is tangibly surjective (cf. Definition 4.1.c), then φ dominates ψ iff there exists a homomorphism map $\alpha : U \to V$ covering the identity of M such that $\alpha \circ \varphi = \psi$. The supervaluation ψ is tangibly surjective iff α is surjective.*

Proof. a): In the definition of dominance in Definition 5.1, the axiom D2 holds trivially since $e\varphi(a) = e\psi(a) = v(a)$. Axiom D3 is our present condition (5.1). Axiom D1 needs only to be checked in the case $\varphi(a) = \varphi(b) \in \mathcal{T}(U)$, and then holds trivially since this implies $a = b$ by the tangible injectivity of φ.

b): Replacing U by the subsemiring $\mathcal{T}(U) \cup v(R)$ we assume without loss of generality that the supervaluation φ is surjective. A transmission α from φ to ψ is forced to cover the identity of M; hence is a semiring homomorphism, cf. Proposition 5.10.iii. We have $\alpha(U) \supset eV$. Thus α is surjective iff $\alpha(\mathcal{T}(U)) = \mathcal{T}(V)$. This gives us the last claim. □

Corollary 5.14. *Assume that $v : R \to M$ is a valuation. The supervaluation $\varphi_v : R \to U(v)$ dominates every supervaluation $\psi : R \to U$ covering v. Thus these supervaluations ψ correspond uniquely with the transmissive maps $\alpha : U(v) \to U$ covering id_M. They are semiring homomorphisms.*

Proof. φ_v is tangibly injective, and (5.1) holds trivially, since $\varphi_v(a) \in eU$ only if $v(a) = 0$. Theorem 5.13 and Proposition 5.10.iii apply. □

Definition 5.15. *Due to this property of φ_v we call φ_v the **initial supervaluation** covering v (or **initial cover** of v for short).*

Remark 5.16. *We may also regard $v : R \to M$ as a cover of v, viewing M as a ghost supertropical semiring. Clearly every supervaluation $\psi : R \to U$ covering v dominates v with transmission ν_U. Thus we may view $v : R \to M$ as the **terminal supervaluation** covering v (or **terminal cover** of v for short).*

The following proposition gives examples of dominance $\varphi \geq \psi$ where φ is not assumed to be tangibly injective.

Proposition 5.17. *Let U be a supertropical semiring with ghost ideal $M := eU$. Assume that L is a submonoid of $(M, \cdot)$ with $M \cdot (M \setminus L) \subset M \setminus L$.*

a) *The map $\alpha : U \to U$, defined by*

$$\alpha(x) = \begin{cases} x & \text{if } ex \in L, \\ ex & \text{if } ex \in M \setminus L, \end{cases}$$

is an endomorphism of the semiring U.

b) *If $\varphi : R \to U$ is any supervaluation, then the map $\varphi_L := \alpha \circ \varphi$ from R to U is a supervaluation dominated by φ and covering the same m-valuation as φ, i.e. $e\varphi_L = e\varphi$.*

Proof. a): We have $e\alpha(x) = ex$ for every $x \in U$, and $\alpha(x) = x$ for every $x \in M$. One checks in a straightforward way that α is multiplicative, $\alpha(0) = 0$, $\alpha(1) = 1$.

We verify additivity. Let $x, y \in U$ be given, and assume without loss of generality that $ex \leq ey$. We have $e\alpha(x) = \alpha(e)\alpha(x) = \alpha(ex) = ex$ and $e\alpha(y) = ey$. If $ex = ey$ then $x + y = ex$, and $\alpha(x) + \alpha(y) = e\alpha(x) = ex = \alpha(x + y)$. If $ex < ey$ then $x + y = y$ and $\alpha(x) + \alpha(y) = \alpha(y)$; hence again $\alpha(x) + \alpha(y) = \alpha(x + y)$.

b): Now obvious. □

Notice that $\varphi_L = \alpha \circ \varphi$ with a map $\alpha : U \to U$ given by $\alpha(x) = x$ if $ex \in L$, and $\alpha(x) = ex$ if $ex \in M \setminus L$. Thus if φ is surjective, α is the transmission from φ to φ_L.

It is not difficult to find instances where Proposition 5.17 applies.

Example 5.18. *Assume that M is a submonoid of $\Gamma \cup \{0\}$ for Γ an ordered abelian group. Let H be a subgroup of Γ containing the set $\{x \in M \mid x > 1\}$. Then*

$$L = \{x \in M \mid \exists h \in H \quad \text{with} \quad x \geq h\}$$

is a submonoid of $M \setminus \{0\}$. We claim that $M \cdot (M \setminus L) \subset M \setminus L$.

Proof. Let $x \in M$, $y \in M \setminus L$ be given. If $x \leq 1$, then $xy \leq y$; hence, clearly, $xy \in M \setminus L$. Assume now that $x > 1$. Then $x \in H$. Suppose that $xy \in L$; hence $h \leq xy$ for some $h \in H$. Then $x^{-1} \leq y$ and $x^{-1}h \in H$; hence $y \in L$, a contradiction. Thus $xy \in M \setminus L$ again. □

Later we will meet many transmissive maps which are not semiring homomorphisms (cf. Theorem 9.11, [IKR]).

6. Fiber contractions

Before we come to the main theme of this section, we write down functional properties of the class of transmissive maps.

Proposition 6.1. *Let $\alpha : U \to V$ and $\beta : V \to W$ be maps between supertropical semirings.*

i) *If α and β transmissive, then $\beta\alpha$ is transmissive.*
ii) *If α and $\beta\alpha$ are transmissive and α is surjective, then β is transmissive.*

Proof. a) It is evident that analogous statements hold for the class of maps between supertropical semirings obeying the axioms TM1–TM4 in §5. Thus we may assume from the beginning that α, β and (hence) $\beta\alpha$ obey TM1–TM4, and have only to deal with the axiom TM5 (cf. Theorem 5.4, Definition 5.5).

b) We conclude from TM3 and TM4 that α maps eU to eV and β maps eV to eW. TM5 demands that these restricted maps are semiring homomorphisms. Thus it is evident that $\beta\alpha$ obeys TM5 if α and β do. If α is surjective, then also the restriction $\alpha|eU : eU \to eV$ is surjective, since for $x \in U$, $y \in eV$ with $\alpha(x) = y$ we also have $\alpha(ex) = y$. Clearly, TM5 for α and $\beta\alpha$ implies TM5 for β in this case. □

Often we will only need the following special case of Proposition 6.1.

Corollary 6.2. *Let U, V, W be supertropical semirings. Assume that $\alpha : U \to V$ is a surjective semiring homomorphism. Then a map $\beta : V \to W$ is transmissive iff $\beta\alpha$ has this property.* □

In the entire section U is a *supertropical semiring.* We look for equivalence relations on the set U that respect the multiplication on U and the fibers of the ghost map $\gamma_U : U \to eU$.

Definition 6.3. *Let E be an equivalence relation on the set U. We say that E is* ***multiplicative*** *if for any $x_1, x_2, y \in U$,*

$$x_1 \sim_E x_2 \quad \Rightarrow \quad x_1 y \sim_E x_2 y. \tag{6.1}$$

We say that E is ***fiber conserving*** *if for any $x_1, x_2 \in U$,*

$$x_1 \sim_E x_2 \quad \Rightarrow \quad ex_1 = ex_2. \tag{6.2}$$

If E is both multiplicative and fiber conserving, we call E an ***MFCE-relation*** *(multiplicative fiber conserving equivalence relation) for short.*

Examples 6.4. (i) *Assume that* $\alpha : U \to V$ *is a multiplicative map from* U *to a supertropical semiring* V*. Then the equivalence* $E(\alpha)$*, given by*

$$x_1 \sim x_2 \quad \text{iff} \quad \alpha(x_1) = \alpha(x_2),$$

is clearly multiplicative. If in addition $\alpha(e_U) = e_V$*, and if the induced map* $\gamma : eU \to eV$*,* $\gamma(ex) = e\alpha(x)$*, is injective, then* $E(\alpha)$ *is also fiber conserving; hence an MFCE-relation. We usually denote this equivalence* $\sim$ *by* $\sim_\alpha$ *.*

In particular, we have an MFCE-relation $E(\alpha)$ *on* U *for any semiring homomorphism* $\alpha : U \to V$ *which is injective on* eU*.*

(ii) *The ghost map* $\nu = \nu_U : U \to U$ *gives us an MFCE-relation* $E(\nu)$ *on* U*. Clearly*

$$x_1 \sim_\nu x_2 \quad \text{iff} \quad ex_1 = ex_2.$$

$E(\nu)$ *is the coarsest MFCE-relation on* U*.*

(iii) *If* E_1 *and* E_2 *are equivalence relations on the set* U*, then* $E_1 \cap E_2$ *is again an equivalence relation on* U*. {As usual, we regard an equivalence relation on* U *as a subset of* $U \times U$*}. We have*

$$x_1 \sim_{E_1 \cap E_2} x_2 \quad \text{iff} \quad x_1 \sim_{E_1} x_2 \quad \text{and} \quad x_1 \sim_{E_2} x_2.$$

If E_1 *is multiplicative and* E_2 *is an MFCE, then* $E_1 \cap E_2$ *is an MFCE.*

(iv) *In particular, every multiplicative equivalence relation* E *on* U *gives us an MFCE-relation* $E \cap E(\nu)$ *on* U*. This is the coarsest MFCE-relation on* U *which is finer than* E*. We have*

$$x_1 \sim_{E \cap E(\nu)} x_2 \quad \text{iff} \quad x_1 \sim_E x_2 \quad \text{and} \quad ex_1 = ex_2.$$

(v) *We define an equivalence relation* E_{t} *(the "t" alludes to "tangible") on* U *as follows, writing* $\sim_t$ *for* $\sim_{E_{\mathrm{t}}}$*:*

$$\begin{aligned} x_1 \sim_t x_2 \quad \text{iff either} \quad & x_1 = x_2 \\ \text{or} \quad & x_1, x_2 \in \mathcal{T}(U) \quad \text{and} \quad ex_1 = ex_2. \end{aligned}$$

Clearly, this is an MFCE-relation iff for any tangible $x_1, x_2, y \in E$ *with* $ex_1 = ex_2$ *both* $x_1 y$ *and* $x_2 y$ *are tangible or equal. In particular,* E_{t} *is an MFCE if* $\mathcal{T}(U)$ *is closed under multiplication.*

Let F denote the equivalence relation on U which has the equivalence classes $\mathcal{T}(U)$ and eU. It is readily checked that $E_{\mathrm{t}} = F \cap E(\nu)$.

The equivalence classes of E_{t} contained in $\mathcal{T}(U)$ are the sets $\mathcal{T}(U) \cap \nu_U^{-1}(z)$ with $z \in M$, which are not empty. We call them the **tangible fibers** of ν_U.

Our next goal is to prove that, given an MFCE-relation E on U, the set U/E of all E-equivalence classes inherits from U the structure of a supertropical semiring.

Lemma 6.5. *If* E *is a fiber conserving equivalence relation on* U*, then for any* $x_1, x_2, y \in U$

$$x_1 \sim_E x_2 \quad \Rightarrow \quad x_1 + y \sim_E x_2 + y.$$

Proof. $ex_1 = ex_2$. If $ey < ex_1$, we have $x_1 + y = x_1$, $x_2 + y = x_2$. If $ey = ex_1$, we have $x_1 + y = ey = x_2 + y$. If $ey > ex_1$, we have $x_1 + y = y = x_2 + y$. Thus, in all three cases, $x_1 + y \sim_E x_2 + y$. □

Notice that, as a formal consequence of the lemma, more generally

$$x_1 \sim_E x_2,\ y_1 \sim_E y_2 \quad \Rightarrow \quad x_1 + y_1 \sim_E x_2 + y_2.$$

Theorem 6.6. *Let E be an MFCE-relation on a supertropical semiring U. On the set $\overline{U} := U/E$ of equivalence classes $[x]_E$, $x \in U$, we have a unique semiring structure such that the projection map $\pi_E : U \to \overline{U}$, $x \mapsto [x]_E$ is a semiring homomorphism. This semiring $\overline{U}$ is supertropical, and π_E covers a semiring isomorphism $eU \xrightarrow{\sim} \bar{e}\overline{U}$. (Here $\bar{e} := e_{\overline{U}} = \pi_E(e)$.)*

Proof. We write $\bar{x} := [x]_E$ for $x \in U$ and $\pi := \pi_E$. Thus $\pi(x) = \bar{x}$. Due to Lemma 6.5 and condition (6.1), we have a well-defined addition and multiplication on $\overline{U}$, given by the rules $(x, y \in U)$

$$\bar{x} + \bar{y} := \overline{x + y}, \qquad \bar{x} \cdot \bar{y} := \overline{xy}.$$

The axioms of a commutative semiring are valid for these operations, since they hold in U, and the map π is a homomorphism from U onto the semiring $\overline{U}$.

We have $\bar{1} + \bar{1} = \bar{e}$ and $\bar{e}\overline{U} = \pi(eU)$. If $x, y \in eU$ and $x \sim_E y$ then $x = ex = ey = y$, since E is fiber conserving. Thus the restriction $\pi|eU$ is an isomorphism from the bipotent semiring eU onto the semiring $\bar{e}\overline{U}$ (which thus is again bipotent).

We are ready to prove that $\overline{U}$ is supertropical, i.e. that axioms (3.3′), (3.3″), (3.4) from §3 are valid. It is obvious that $\overline{U}$ inherit properties (3.3′) and (3.4) from U. Let $x, y \in E$ be given with $\bar{e}\bar{x} = \bar{e}\bar{y}$, i.e. $\overline{ex} = \overline{ey}$. Then $ex = ey$; hence $x + y = ex$ by axiom (3.3″) for U. Applying the homomorphism π we obtain $\bar{x} + \bar{y} = \bar{e}\bar{x}$. Thus $\overline{U}$ also obeys (3.3″). □

Remark 6.7. *Theorem 6.6 tells us, in particular, that every MFCE-relation E on U is of the form $E(\alpha)$ for some semiring homomorphism $\alpha : U \to V$ with $\alpha|eU$ bijective, namely, $E = E(\pi_E)$.*

Theorem 6.8. *Assume that $\alpha : U \to V$ is a multiplicative map. Let E be an MFCE-relation on U, which is respected by α, i.e., $x \sim_E y$ implies $\alpha(x) = \alpha(y)$. Clearly, we have a unique multiplicative map $\bar{\alpha} : U/E \to V$ with $\bar{\alpha} \circ \pi_E = \alpha$.*

Then, if α is a transmission (a semiring homomorphism), the map $\bar{\alpha}$ is of the same kind.

Proof. Corollary 6.2 gives us all the claims, since π_E is a surjective homomorphism. □

Definition 6.9. *We call a map $\alpha : U \to V$ between supertropical semirings a **fiber contraction**, if α is transmissive and surjective, and the map $\gamma : eU \to eV$ covered by α is strictly order preserving.*

Notice that then α is a semiring homomorphism (cf. Proposition 5.10.iii) (hence α is a transmission), and γ is an isomorphism from eU to eV.

Scholium 6.10.

i) *If E is an MFCE-relation on U, by Theorem 6.6, the map $\pi_E : U \to U/E$ is a fiber contraction. On the other hand, if a surjective fiber contraction $\alpha : U \twoheadrightarrow V$ is given, then clearly $E(\alpha)$ is an MFCE-relation, and, as Theorem 6.8 tells us, α induces a semiring isomorphism $\bar{\alpha} : U/E(\alpha) \xrightarrow{\sim} V$ with $\alpha = \bar{\alpha} \circ \pi_{E(\alpha)}$. In short, every fiber contraction α on U is a map π_E with E an MFCE-relation on U uniquely determined by α, followed by a semiring isomorphism.*

ii) *If the semiring isomorphism $\bar{\alpha}$ is the identity id_M of $M := eU$ (in particular $eU = eV$), we say α is a **fiber contraction over** M.*

If E is an equivalence relation on a set X, and Y is a subset of X, we denote the set of all equivalence classes $[x]_E$ with $x \in Y\}$ by Y/E.

Example 6.11. *Assume that U is a supertropical domain (cf. 3.14). Then the equivalence relation E_{t} introduced in Example 6.4.v is MFCE, and $\mathcal{T}(U)$ is a union of E_{t}-equivalence classes. The ring $\overline{U} = U/E_{\mathrm{t}}$ is a supertropical domain with $\mathcal{T}(\overline{U}) = \mathcal{T}(U)/E_{\mathrm{t}}$ and $\mathcal{G}(\overline{U}) = \mathcal{G}(U)$. The ghost map of $\overline{U}$ maps $\mathcal{T}(\overline{U})$ bijectively to $\mathcal{G}(U)$; hence gives us a monoid isomorphism $v : \mathcal{T}(\overline{U}) \xrightarrow{\sim} \mathcal{G}(U)$. Thus (in notation of Examples 3.18)*

$$U/E_{\mathrm{t}} = D(\mathcal{G}(U)).$$

The map $\pi_{E_{\mathrm{t}}}$ is a fiber contraction over $eU = eU/E_{\mathrm{t}}$.

Example 6.12. *(cf. Proposition 5.17) Let U be a supertropical semiring, $M := eU$, and let L be a submonoid of $(M, \cdot)$ with $M \cdot (M \setminus L) \subset M \setminus L$. Then the map $\alpha : U \to U$ with $\alpha(x) = x$ if $ex \in L$, $\alpha(x) = ex$ if $ex \in M \setminus L$, is a fiber contraction over M. The image of α is the subsemiring $\nu_U^{-1}(L) \cup (M \setminus L)$ of U.*

Example 6.13. *Let again U be a supertropical semiring and $M := eU$. But now assume only that L is a* ***subset*** *of M with $M \cdot (M \setminus L) \subset M \setminus L$. We define an equivalence relation $E(L)$ on U as follows:*

$$x \sim_{E(L)} y \quad \Leftrightarrow \quad \text{either} \quad x = y \quad \text{or} \quad ex = ey \in M \setminus L.$$

One checks easily that $E(L)$ is MFCE. But if L is not a submonoid of $(M, \cdot)$, then in the supertropical semiring $\overline{U} := U/E(L)$ the set $\mathcal{T}(\overline{U})$ of tangible elements is not closed under multiplication. In particular, $\overline{U}$ is not isomorphic to a subsemiring of U.

For later use we introduce one more notation.

Notation 6.14. *If $\varphi : R \to U$ is a supervaluation and E is an MFCE-relation on U, let φ/E denote the supervaluation $\pi_E \circ \varphi : R \to U/E$. Thus, for any $a \in R$,*

$$(\varphi/E)(a) \;:=\; [\varphi(a)]_E \;.$$

7. The lattices $C(\varphi)$ and $\mathrm{Cov}(v)$

Given an m-valuation $v : R \to M$ on a semiring R, we now can say more about the class of all supervaluations φ covering v. Recall that these are the supervaluations $\varphi : R \to U$ with $eU = M$ and $\nu_U \circ \varphi = v$, in other words, $e\varphi = v$. For short, we call these supervaluations φ the **covers** of the m-valuation v. It suffices to focus on covers of v which are tangibly surjective, cf. Remark 4.4. (N.B. Without loss of generality, we could even assume that v is surjective. Then a cover φ of v is tangibly surjective iff φ is surjective.)

Definition 7.1.

a) We call two covers $\varphi_1 : R \to U_1$, $\varphi_2 : R \to U_2$ of v **equivalent**, if $\varphi_1 \geq \varphi_2$ and $\varphi_2 \geq \varphi_1$, i.e., φ_1 dominates φ_2, and φ_2 dominates φ_1. If φ_1 and φ_2 are tangibly surjective (without essential loss of generality, cf. Remark 4.4), this means that $\varphi_2 = \alpha \circ \varphi_1$ with $\alpha : U_1 \to U_2$ a semiring isomorphism over M (i.e., $e\alpha(x) = ex$ for all $x \in U_1$).
b) We denote the equivalence class of a cover $\varphi : R \to U$ of v by $[\varphi]$, and we denote the set of all these equivalence classes by $\mathrm{Cov}(v)$. {Notice that $\mathrm{Cov}(v)$ is really a set, not just a class, since for any tangibly surjective cover $\varphi : R \to U$, we have $U = \varphi(R) \cup M$; hence the cardinality of U is bounded by $\mathrm{Card}\, R + \mathrm{Card}\, M$.} On $\mathrm{Cov}(v)$ we have a partial ordering: $[\varphi_1] \geq [\varphi_2]$ iff φ_1 dominates φ_2. We always regard $\mathrm{Cov}(v)$ as a poset[5] in this way.

[5] = partially ordered set

c) If a covering $\varphi : R \to U$ of v is given, we denote the subposet of $\mathrm{Cov}(\varphi)$ consisting of all $[\psi] \in \mathrm{Cov}(v)$ with $[\varphi] \geq [\psi]$ by $C(\varphi)$. {Notice that this poset is determined by φ alone, since $v = e\varphi$.}

In §5 we have seen that, given a tangibly surjective cover $\varphi : R \to U$ of v, the tangibly surjective covers $\psi : R \to V$ dominated by φ correspond uniquely to the transmissive surjective maps $\alpha : U \to V$ which restrict to the identity on $M = eU = eV$. Scholium 6.10 from the preceding section tell us, in particular, the following

Theorem 7.2. *Suppose that $\varphi : R \to U$ be a tangibly surjective covering of the m-valuation $v : R \to M$.*

a) *The elements $[\psi]$ of $C(\varphi)$ correspond uniquely to the MFCE-relations E on U via $[\psi] = [\varphi/E]$.*
b) *Let $MFC(U)$ denote the set of all MFCE-relations on U, ordered by the coarsening relation: $E_1 \leq E_2$ iff E_2 is coarser than E_1, i.e., $E_1 \subset E_2$, if the E_i are viewed – as customary – as subsets of $U \times U$. The map $E \mapsto [\varphi/E]$ is an anti-isomorphism (i.e., an order reversing bijection) from the poset $MFC(U)$ to the poset $C(\varphi)$.*

If $(E_i \mid i \in I)$ is a family in $MFC(U)$ then the intersection $E := \bigcap_{i\in I} E_i$ is again an MFCE-relation on U, and is the infimum of the family $(E_i \mid i \in I)$ in $MFC(U)$. Since $MFC(U)$ has a biggest and smallest element, namely $E(\nu_U)$ and the diagonal of U in $U \times U$, it is now clear that the poset $MFC(U)$ is a complete lattice. Thus, for any cover $\varphi : R \to U$ of the m-valuation $v : R \to M$, also the poset $C(\varphi)$ is a complete lattice. {We easily retreat to the case that φ is tangibly surjective.}

The supremum of a family $(E_i \mid i \in I)$ in $MFC(U)$ is the following equivalence relation F on U. Two elements x, y of U are F-equivalent iff there exists a finite sequence $x_0 = x, x_1, \dots, x_m = y$ in U such that for each $j \in \{1, \dots, m\}$ the element x_{j-1} is E_k-equivalent to x_j for some $k \in I$.

Construction 7.3. *Assume again that φ is tangibly surjective. The supremum $\bigvee_{i\in I} \xi_i$ of a family $(\xi_i \mid i \in I)$ in $C(\varphi)$ can be described as follows. Choose for each $i \in I$ a tangibly surjective representative $\psi_i : R \to V_i$ of ξ_i. Thus $eV_i = M$, and ψ_i is a cover of v dominated by φ. Let $e_i := e_{V_i}$ $(= 1_M)$, and let V denote the set of all elements $x = (x_i \mid i \in I)$ in the semiring $\prod_{i\in I} V_i$ with $e_i x_i = e_j x_j$ for $i \neq j$. This is a subsemiring of $\prod_{i\in I} V_i$ containing the image M' of M in $\prod V_i$ under the diagonal embedding of M into $\prod V_i$. We identify $M' = M$, and then have*

$$e_U = 1_M = (e_i \mid i \in I) = 1_V + 1_V.$$

It is now a trivial matter to verify that V is a supertropical semiring by checking the axioms in §3. We have $e_V V = eV = M' = M$. The supervaluations $\psi_i : R \to U_i$ combine to a map $\psi : R \to V$, given by

$$\psi(a) := (\psi_i(a) \mid i \in I) \in V$$

for $a \in R$. It is a supervaluation covering v, and $\varphi : R \to U$ dominates ψ (e.g., check the axioms D1–D3 in §5). The class $[\psi]$ is the supremum of the family $(\xi_i \mid i \in I)$ in $C(\varphi)$.

Given again a family $(\xi_i \mid i \in I)$ in $C(\varphi)$ with representatives $\psi_i : R \to V_i$ of the ξ_i, we indicate how the infimum $\wedge \xi_i$ in $C(\varphi)$ can be built, without being as detailed as above for the supremum.

We assume that each supervaluation ψ_i is surjective. The transmission $\delta_i : U \to V_i$ from φ to ψ_i is a surjective semiring homomorphism. We form the categorical direct limit (=

colimit) of the family $(\delta_i \mid i \in I)$ in the category of semirings (cf. [Mit, Chap. II], [ML, III, §3]). Thus we have a semiring V together with a family of semiring homomorphisms $(\alpha_i : V_i \to V \mid i \in I)$ such that $\alpha_i \circ \delta_i = \alpha_j \circ \delta_j$ for $i \neq j$, which is universal. This means that, given a family $(\beta_i : V_i \to W \mid i \in I)$ of homomorphisms with $\beta_i \circ \delta_i = \beta_j \circ \delta_j$ for $i \neq j$, there exists a unique homomorphism $\beta : V \to W$ with $\beta \circ \alpha_i = \beta_i$ for every $i \in I$. Choosing some $i \in I$ let

$$\varepsilon := \alpha_i \circ \delta_i : U \to V.$$

This homomorphism, which is independent of the choice of i, is surjective, due to universality, since all maps $\delta_j : U \to V_j$ are surjective. It turns out that the restriction $\varepsilon|eU$ maps $eU = M$ isomorphically onto eV. We identify M with eV by this isomorphism and then have $\varepsilon|eU = 1_M$.

This can be seen as follows. Let $\nu := \nu_U$ and $\nu_i := \nu_{V_i}$ denote the ghost maps of U and V_i. For every $i \in I$ we have $\nu_i \circ \delta_i = \nu$. By universality we obtain a homomorphism $\mu : V \to M$ with $\mu \circ \alpha_i = \nu_i$ for every i. Let j_i denote the inclusion map from M to V_i. We have $\nu_i \circ j_i = \mathrm{id}_M$; hence

$$\mu \circ \alpha_i \circ j_i = \nu_i \circ j_i = \mathrm{id}_M .$$

The surjective homomorphism α_i maps $M = eV_i$ onto eV. We conclude that the restriction $\alpha_i|M$ gives an isomorphism from M onto eV, the inverse map being given by μ.

We identify M with eV via $\alpha_i|M$. Now $\alpha_i : V_i \to V$ has become a surjective semiring homomorphism over M (for every i). Thus also $\varepsilon : U \to V$ is a surjective homomorphism over M. We conclude, that epsilon gives an MFCE-relation $E(\varepsilon)$ and the semiring V is supertropical. The supervaluation

$$\psi := \varepsilon\varphi = \alpha_i \circ \psi_i \quad \text{is dominated by every} \quad \psi_i \quad \text{and} \quad [\psi] = \bigwedge_i \xi_i.$$

Since $V_i = \psi_i(R) \cup M$ for every i, the semiring V and the α_i can be described completely in terms of the ψ_i without mentioning U and the δ_i. We leave this to the interested reader.

Definition 7.4. *We call a supervaluation φ **initial** if φ dominates every other supervaluation ψ with $e\varphi = e\psi$. We then also say that φ is an **initial cover** of $v := e\varphi$.*

If an m-valuation $v : R \to M$ is given, a supervaluation $\varphi : R \to U$ is an initial cover of v iff $e\varphi = v$ and $[\varphi]$ is the biggest element of the poset $\mathrm{Cov}(v)$.

Such an initial cover had been constructed explicitly in §4 in the case that v is a valuation, namely, the supervaluation $\varphi_v : R \to U(v)$, cf. Definition 4.6 and Corollary 5.14. We now prove that an initial cover always exists, although in general we do not have an explicit description.

Proposition 7.5. *Every m-valuation $v : R \to M$ has an initial cover. The poset* $\mathrm{Cov}(v)$ *is a complete lattice.*

Proof. Let $(\psi_i \mid i \in I)$ be a family of coverings of v which represents every element of the set $\mathrm{Cov}(v)$. Now repeat Construction 7.3 with this family. It gives us a covering $\psi : R \to V$ of v which dominates all ψ_i; hence is an initial covering of v. Of course, $C(\psi) = \mathrm{Cov}(v)$, and thus $\mathrm{Cov}(v)$ is a complete lattice. □

Notation 7.6. *If $v : R \to M$ is any m-valuation, let $\varphi_v : R \to U(v)$, denote a fixed tangibly surjective initial supervaluation covering v. If v is a valuation, we choose for φ_v the supervaluation constructed in Example 4.5.*

Notice that φ_v is unique up to unique isomorphism over M, i.e., if $\psi : R \to V$ is another surjective initial cover of v, there exists a unique semiring isomorphism $\alpha : U(v) \xrightarrow{\sim} V$ which restricts to the identity on M. We call φ_v "**the**" **initial cover** of v. The lattice $\mathrm{Cov}(v)$ coincides with $C(\varphi_v)$.

Given a supervaluation $\varphi : R \to U$ or an m-valuation $v : R \to M$, we view the lattice $C(\varphi)$ and $\mathrm{Cov}(v)$ as a measure of complexity of φ and v, respectively, and thus make the following formal definition.

Definition 7.7. *We call the isomorphism class of the lattice $C(\varphi)$ the **lattice complexity** of the supervaluation φ and denote it by $\mathrm{lc}(\varphi)$. In the same vein we call the isomorphism class of the lattice $\mathrm{Cov}(v)$ the **tropical complexity** of the m-valuation v and denote it by $\mathrm{trc}(v)$. We have $\mathrm{trc}(v) = \mathrm{lc}(\varphi_v)$.*

The word "complexity" in Definition 7.7 should not be taken too seriously. Usually a "measure of complexity" has values in natural numbers or, more generally, in some well understood fixed ordered set. The isomorphism classes of lattices are not values of this kind. Our idea behind the definition is that, if you are given a function m on the class of lattices which measures (part of) their complexity in some way, then $m \circ \mathrm{lc}$, resp. $m \circ \mathrm{trc}$, is such a function on the class of supervaluations, resp. m-valuations.

Theorem 7.2 implies the following remarkable fact.

Theorem 7.8. *If $\varphi : R \to U$ and $\varphi' : R' \to U$ are tangibly surjective supervaluations with values in the same supertropical semiring U, then $\mathrm{lc}(\varphi) = \mathrm{lc}(\varphi')$.*

Proof. Both lattices $C(\varphi)$ and $C(\varphi')$ are anti-isomorphic to $MFC(U)$; hence are isomorphic. □

Example 7.9. *Let $\varphi : R \to U$ be a tangibly surjective supervaluation. The identity $\mathrm{id}_U : U \to U$ is also a supervaluation. It is the initial cover of the ghost map $\nu_U : U \to eU$. We have $\mathrm{lc}(\varphi) = \mathrm{trc}(\nu_U)$.*

8. Orbital equivalence relations

Our main goal in this section is to introduce and study a special kind of MFCE-relations on supertropical semirings, which seems to be more accessible than MFCE-relations in general. But for use in later sections, we will define more generally "orbital" equivalence relations on supertropical semirings. They are multiplicative but not necessarily fiber conserving. The relations we are looking for here then will be the orbital MFCE-relations.

In the following U is a supertropical semiring, and $M := eU$ denotes its ghost ideal. We always assume that $\mathcal{T}(U)$ is not empty, i.e., $e \neq 1$. We introduce the set

$$S(U) := \{x \in U \mid x\mathcal{T}(U) \subset \mathcal{T}(U)\}.$$

This is a subset of $\mathcal{T}(U)$ closed under multiplication and containing the unit element 1_U; hence is a monoid.

The monoid $S(U)$ operates on the sets U and $\mathcal{T}(U)$ by multiplication. If $\mathcal{T}(U)$ itself is closed under multiplication then $S(U) = \mathcal{T}(U)$.

Let G be a submonoid of $S(U)$. Then also G operates on U and on $\mathcal{T}(U)$. For any $x \in U$ we call the set Gx the **orbit** of x under G (as common at least for G a group). We define a binary relation $\sim_G$ on U as follows:

$$x \sim_G y \quad \Leftrightarrow \quad \exists g, h \in G : gx = hy.$$

Thus $x \sim_G y$ iff the orbits Gx and Gy intersect. Clearly this is an equivalence relation on U, which is multiplicative, i.e., obeys the rule (6.1) from §6. We denote this equivalence relation by $E(G)$.

The relation $E(G)$ on U is MFCE, i.e., obeys also the rule (6.2) from §6, iff G is contained in the "unit-fiber"

$$\mathcal{T}_e(U) := \{x \in \mathcal{T}(U) | ex = e\}$$

of $\mathcal{T}(U)$. The biggest such monoid is the unit fiber

$$S_e(U) := \{g \in S(U) \mid eg = e\} = \mathcal{T}_e(U) \cap S(U)$$

of $S(U)$.

Example 8.1. *Assume that R is a field and $v : R \to \Gamma \cup \{0\}$ is a surjective valuation on R. $\{$In classical terms, v is a Krull valuation on R with value group Γ.$\}$ Let*

$$U := U(v) = (R \setminus \{0\}) \dot\cup \Gamma \dot\cup \{0\},$$

cf. Definition 4.6. Then $S(U)$ is the multiplicative group $R^* = R \setminus \{0\}$ of the field R, and $S_e(U)$ is the group $\mathfrak{o}_v^*$ of units of the valuation domain

$$\mathfrak{o}_v := \{x \in R \mid v(x) \le 1\}.$$

Definition 8.2. *We call an equivalence relation E on the supertropical semiring U* ***orbital*** *if $E = E(G)$ for some submonoid G of $S(U)$. We denote the set of all orbital equivalence relations on U by* Orb(U) *and the subset* Orb(U)$\cap$MFC(U), *consisting of the orbital MFCE-relations on U, by* OFC(U). $\{$*"OFC" alludes to "orbital fiber conserving".*$\}$ *Consequently, we call the elements of* OFC(U) *the* ***orbital fiber conserving equivalence relations*** *on U, or* ***OFCE-relations*** *for short.*

Example 8.3. *It is evident that $E(S(U))$ is the coarsest orbital equivalence relation and $F := E(S_e(U))$ is the coarsest OFCE-relation on U. Assume now that U is a supertropical domain. Then $S(U) = \mathcal{T}(U)$, $S_e(U) = \mathcal{T}_e(U)$, and $\mathcal{G}(U) = e\mathcal{T}(U)$. $E(S(U))$ has just 3 equivalence classes, namely, $\mathcal{T}(U)$, $\mathcal{G}(U)$ and $\{0\}$. On the other hand, F is finer than the MFCE-relation E_{t} introduced in Example 6.4.v, whose equivalence classes in $\mathcal{T}(U)$ are the tangible fibers of the ghost map ν_U. Very often E_{t} is not orbital; hence $F \subsetneqq E_{\mathrm{t}}$.*

Subexample 8.4. *Let $R = k[x]$ be the polynomial ring in one variable x over a field k. Choose a real number ϑ with $0 < \vartheta < 1$, and let v be the surjective valuation on R defined by*

$$v(f) = \vartheta^{\deg f}.$$

Thus, $v : R \twoheadrightarrow G \cup \{0\}$ with G the monoid $\{\vartheta^n \mid n \in \mathbb{N}_0\} \subset \mathbb{R}$. Finally, take

$$U := U(v) = (R \setminus \{0\}) \cup G \cup \{0\},$$

cf. Definition 4.6. We have $S(U) = R \setminus \{0\}$ and

$$S_e(U) = \{f \in R \mid \deg f = 0\} = k \setminus \{0\},$$

the set of nonzero constant polynomials. If $f, g \in \mathcal{T}(U)$ are given with $ef = eg$, i.e., $\deg f = \deg g$, then $f \sim_F g$ iff $g = cf$ with c a constant $\neq 0$. Thus, the set of F-equivalence classes in $\mathcal{T}(U)$ can be identified with the set of monic polynomials in $k[x]$, while the E_{t}-equivalence classes are the sets $\{f \in k[x] \mid \deg f = n\}$ with n running thorough $\mathbb{N}_0$. For $n = 0$ this E_{t}-equivalence class is also an F-equivalence class, while for $n > 0$ it decomposes into infinitely many F-equivalence classes if the field k is infinite, and into $|k|^n$ F-equivalence classes if k is finite.

The semiring U/F (cf. §6) can be identified with the subsemiring V of U, which has as tangible elements the monic polynomials in $k[x]$ and has the same ghost ideal $eV = eU$ as U. □

Different submonoids G, H of $S(U)$ may yield the same orbital equivalence relation $E(G) = E(H)$. But this ambiguity can be tamed.

Proposition 8.5. *If G is a submonoid of $S(U)$, then*

$$G' := \{x \in S(U) \mid \exists g \in G : gx \in G\}$$

is a submonoid of $S(U)$ containing G, and $E(G) = E(G')$. If $G \subset S_e(U)$ then $G' \subset S_e(U)$.

Proof. a) It is immediate that G' is a submonoid of $S(U)$ and that $G \subset G'$. Given $x \in G'$ we have elements $g, h \in G$ with $gx = h$. If in addition $G \subset S_e(U)$, then $e = eh = (eg)(ex) = ex$; hence $x \in S_e(U)$. Thus $G' \subset S_e(U)$. It follows from $G \subset G'$ that $E(G) \subset E(G')$.

b) Let $x, y \in U$ be given with $x \sim_{G'} y$. We have elements g_1', g_2' in G' with $g_1'x = g_2'y$. We furthermore have elements h_1, h_2 in G with $h_1 g_1' = g_1 \in G$ and $h_2 g_2' = g_2 \in G$. Now

$$g_1 h_2 x = h_1 h_2 g_1' x = h_1 h_2 g_2' y = h_1 g_2 y.$$

Thus $x \sim_G y$. This proves $E(G') \subset E(G)$; hence $E(G) = E(G')$. □

Definition 8.6. *We call G' the **saturation** of the monoid G (in U), and we say that G is saturated if $G = G'$.*

It is immediate that $(G')' = G'$. Thus G' is always saturated.

Example 8.7. *If $S(U)$ happens to be a group, then the saturation of a submonoid G of $S(U)$ is just the subgroup of $S(U)$ generated by G. Indeed, the elements of G' are the $x \in S(U)$ with $g_1 x = g_2$ for some $g_1, g_2 \in G$, i.e., the elements $g_1^{-1} g_2$ with $g_1, g_2 \in G$.*

Proposition 8.8. *Let E be a multiplicative equivalence relation on U.*

a) *The set*
$$G_E := \{x \in S(U) \mid x \sim_E 1\}$$
is a saturated submonoid of $S(U)$.

b) *If $E = E(H)$ for some submonoid H of $S(U)$, then G_E is the saturation H' of H.*

c) *In general, $E(G_E)$ is the coarsest orbital equivalence relation on U which is finer than E.*

d) *If E is MFCE then $G_E \subset S_e(U)$, and $E(G_E)$ is the coarsest OFCE-relation on U which is finer than E.*

Proof. a): If $x, y \in G_E$ then $x \sim_E 1$, $y \sim_E 1$; hence $xy \sim_E y \sim_E 1$, thus $xy \in G_E$. This proves that G_E is a submonoid of $S(U)$. Let $x \in G_E'$ be given. We have elements $g, h \in G_E$ with $hx = g$. It follows from $g \sim_E 1$, $h \sim_E 1$ that

$$x \sim_E hx = g \sim_E 1.$$

Thus $x \in G_E$. This proves that $G_E' = G_E$.

b): Assume that $E = E(H)$ with H a submonoid of $S(U)$. For $x \in S(U)$ we have

$$x \sim_E 1 \quad \Leftrightarrow \quad \exists h_1, h_2 \in H : h_1 x = h_2 \quad \Leftrightarrow \quad x \in H'.$$

Thus $G_E = H'$.

c): Let $G := G_E$. If $x \sim_G y$ then $g_1 x = g_2 y$ with some $g_1, g_2 \in G$. From $g_1 \sim_E 1$, $g_2 \sim_E 1$, we conclude that

$$x \sim_E g_1 x = g_2 y \sim_E y.$$

Thus $E(G) \subset E$. If H is any submonoid of $S(U)$ with $E(H) \subset E$, then

$$H \subset G_{E(H)} \subset G_E = G.$$

Thus $E(H) \subset E(G)$.

d): Assume that E is MFCE. If $x \in G_E$ then we conclude from $x \sim_E 1$ that $ex = e$. Thus $G_E \subset S_e(U)$. Every multiplicative equivalence relation on U which is finer than E is MFCE. In particular, this holds for orbital relations. We learn from c) that $E(G_E)$ is the coarsest OFCE-relation on U finer than E. □

We denote the set of saturated submonoids of $S(U)$ by $\mathrm{Sat}(S(U))$ and the set of saturated submonoids of $S_e(U)$ by $\mathrm{Sat}(S_e(U))$.

Scholium 8.9. *Propositions 8.5 and 8.8 imply that we have an isomorphism of posets $H \mapsto E(H)$ from $\mathrm{Sat}(S(U))$ to Orb(U), mapping $\mathrm{Sat}(S_e(U))$ onto $\mathrm{OFC}(U)$, with inverse map $E \mapsto G_E$. {Here, of course, both sets $\mathrm{Sat}(S(U))$ and $\mathrm{Orb}(U)$ are ordered by inclusion.}*

It is fairly obvious that $\mathrm{Sat}(S(U))$ is a complete lattice. Indeed, the supremum of a family $(H_i \mid i \in I)$ of saturated submonoids of $S(U)$ is the saturation H' of the submonoid of $S(U)$ generated by the H_i, while the infimum of this family is the saturation $(\bigcap_i H_i)'$ of the intersection of the family. Thus also $\mathrm{Orb}(U)$ is a complete lattice. It follows that $\mathrm{Sat}(S_e(U))$ and $\mathrm{OFC}(U)$ are complete sublattices of $\mathrm{Sat}(S(U))$ and $\mathrm{Orb}(U)$, respectively.

Let $\mathrm{Mult}(U)$ denote the set of all multiplicative equivalence relations on U, partially ordered by inclusion. In §7 we have seen that the subposet $MFC(U)$ of $\mathrm{Mult}(U)$, consisting of the MFCE-relations on U, is a complete lattice. In the same way one proves that $\mathrm{Mult}(U)$ itself is a complete lattice, the supremum and infimum of a family in $\mathrm{Mult}(U)$ being given in exactly the same way as in §7 for MFCE-relations. This makes it also evident that $MFC(U)$ is a complete sublattice of $\mathrm{Mult}(U)$.

We doubt whether $\mathrm{Orb}(U)$ and $\mathrm{OFC}(U)$ are always sublattices of $\mathrm{Mult}(U)$ and $MFC(U)$, respectively. But we have the following partial result.

Proposition 8.10. *Let $(G_i \mid i \in I)$ be a family of submonoids of $S(U)$, and let G denote the monoid generated by this family in $S(U)$. Then, in the lattice $\mathrm{Mult}(U)$,*

$$E(G) = \bigvee_{i \in I} E(G_i).$$

{N.B. Thus the same holds in $MFC(U)$, if every $G_i \subset S_e(U)$.}

Proof. Let $F := \bigvee_i E(G_i)$ in $\mathrm{Mult}(U)$. Of course, $F \subset E(G)$ since each $E(G_i) \subset E(G)$. Let $x, y \in U$ be given with $x \sim_G y$. We want to conclude that $x \sim_F y$, and then will be done.

We have $gx = hy$ with elements g, h of G. Now g and h are products of elements in $\bigcup_i G_i$, and for any $g' \in \bigcup_i G_i$ and $z \in U$, we have $z \sim_F g'z$. It follows that $x \sim_F gx$ and $y \sim_F hy$; hence $x \sim_F y$. □

We present an important case where $\mathrm{OFC}(U)$ and $MFC(U)$ nearly coincide.

Theorem 8.11. *Assume that every $x \in \mathcal{T}(U)$ is invertible; hence $\mathcal{T}(U)$ is a group under multiplication. {The main case is that U is a supertropical semifield.} Let E be an*

MFCE-relation on U. Then either $E = E(\nu)$, i.e., E is the top element of $MFC(U)$ (cf. Example 6.4.ii), or E is orbital.

Proof. a) Assume that there exists some $x_0 \in \mathcal{T}(U)$ with $x_0 \sim_E ex_0$. Multiplying by x_0^{-1} we obtain $1 \sim_E e$, and then obtain $x \sim_E ex$ for every $x \in U$. Thus $E = E(\nu)$.

b) Assume now that $x \not\sim_E ex$ for every $x \in \mathcal{T}(U)$ (i.e., $E \subset E_{\mathrm{t}}$). Clearly $S_e(U) = \mathcal{T}_e(U)$. Let

$$H := G(E) = \{x \in \mathcal{T}(U) \mid x \sim_E 1\}.$$

Then $E(H) \subset E$. Given $x, y \in U$ with $x \sim_E y$, we want to prove that $x \sim_H y$. We have $ex = ey$. If $x \in eU$ or $y \in eU$, we conclude that $x = y$, due to our assumption on E. There remains the case that both x and y are tangible. Then we infer from $x \sim_E y$ that

$$1 = x^{-1}x \sim_E x^{-1}y.$$

Thus $x^{-1}y \in H$, which implies $x \sim_H y$. This completes the proof that $E = E(H)$. □

Corollary 8.12. *If every element of $\mathcal{T}(U)$ is invertible, then the poset $MFC(U) \setminus \{E(\nu)\}$ is isomorphic to the lattice of subgroups of $\mathcal{T}_e(U)$.*

We may apply this to the "initial" supertropical semiring

$$U(v) = (R \setminus \{0\}) \cup \Gamma \cup \{0\}$$

associated to a surjective valuation $v : R \to \Gamma \cup \{0\}$ on a field R. We obtain (cf. Example 8.1)

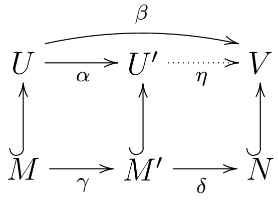
Scholium 8.13. *If v is a Krull valuation on a field R with value group Γ, then the lattice* $\mathrm{Cov}(v)$ *of equivalence classes of supervaluations covering v is isomorphic to the lattice of subgroups of the unit group $\mathfrak{o}_v^*$ of the valuation domain $\mathfrak{o}_v := \{x \in R \mid v(x) \leq 1\}$, augmented by one element at the top.*

9. Initial transmissions and a pushout property

We state the main problem which we address in this section.

Problem 9.1. *Assume that U is a supertropical semiring with ghost ideal $eU = M$, and $\gamma : M \to M'$ is a semiring homomorphism from M to a bipotent semiring M'. Find a supertropical semiring U' with ghost ideal $eU' = M'$ and a transmission $\alpha : U \to U'$ covering γ, i.e., $\alpha^\nu = \gamma$ (cf. Definition 5.3), with the following universal property. Given a transmission $\beta : U \to V$ into a supertropical semiring V, with ghost ideal $N := eV$, and a semiring homomorphism $\delta : M' \to N$, such that $\beta^\nu = \delta\gamma$, there exists a unique transmission $\eta : U' \to V$ such that $\beta = \eta \circ \alpha$ and $\eta^\nu = \delta$.*

We indicate this problem by the following commuting diagram



where the vertical arrows are inclusion mappings.

We call such a map $\alpha : U \to U'$ a *pushout transmission covering* γ. This terminology alludes to the fact that our universal property means that the left square in the diagram above is a pushout (=cocartesian) square in the category STROP, whose objects are the supertropical semirings, and whose morphisms are the transmissions. To see this, just observe that a map $\rho : L \to W$ from a bipotent semiring L to a supertropical semiring W is transmissive iff ρ is a semiring homomorphism from L to eW followed by the inclusion $eW \hookrightarrow W$.

It is now obvious that, for a given homomorphism $\gamma : M \to M'$, Problem 9.1 has at most one solution up to isomorphism over M' and U. More precisely, if both $\alpha : U \to U'$ and $\alpha_1 : U \to U_1$ are solutions, there exists a unique isomorphism $\rho : U' \to U_1$ of semirings over M' with $\alpha_1 = \alpha' \circ \rho$.

We may cast the universal property above in terms of α alone and then arrive at the following formal definition.

Definition 9.2. *We call a map $\alpha : U \to V$ between supertropical semirings a* ***pushout transmission*** *if the following holds:*

1) *α is a transmission.*
2) *If $\beta : V \to W$ is a transmission from U to a supertropical semiring W and $\delta : eV \to eW$ is a semiring homomorphism with $\beta^\nu = \delta \circ \alpha^\nu$, then there exists a unique transmission $\eta : V \to W$ with $\eta^\nu = \delta$ and $\beta = \eta \circ \alpha$.*

We then also say that V is "the" ***pushout of*** *U* ***along*** *γ.*

The notion of a pushout transmission can be weakened by demanding the universal property in Definition 9.2 only for $W = V$ and δ the identity of eV. This is still interesting.

Definition 9.3. *We call a transmission $\alpha : U \to V$ between supertropical semirings an* ***initial transmission****, if, for any transmission $\beta : U \to W$ with $eW = eV$ and $\beta^\nu = \alpha^\nu$, there exists a unique semiring homomorphism $\eta : V \to W$ over $eV = eW$ with $\beta = \eta \circ \alpha$.*

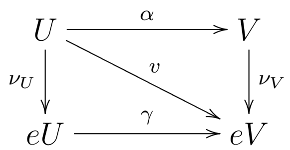
Given a supertropical semiring U and a semiring homomorphism $\gamma : eU \to N$ with N bipotent, it is again clear that there exists at most one initial transmission $\alpha : U \to V$ covering γ (in particular, $eV = N$) up to isomorphism over U and N.

We turn to the problem of existence, first for initial transmissions and then for pushout transmissions. In the first case we can apply results on supervaluations from §4 and §7, due to the following easy but important observation.

Proposition 9.4. *Let $\alpha : U \to V$ be a map between supertropical semirings and $\gamma : eU \to eV$ a semiring homomorphism. The following are equivalent:*

a) *α is a transmission covering γ.*
b) *α is a supervaluation on the semiring U with $\alpha(e_U) = e_V$ covering the strict m-valuation $v := \gamma \circ \nu_U : U \to eV$.*

Diagram



Proof. We have to compare the axioms SV1–SV4 in §4 plus the condition $\alpha(e) = e$ with the axioms TM1–TM5 in §5. The axioms SV1–SV3 say literally the same as TM1–TM3, and the condition $\alpha(e) = e$ is TM4.

We now assume that α fulfills TM1–TM4. For every $x \in U$ we have $\alpha(ex) = \alpha(e)\alpha(x) = e\alpha(x)$. That α is a transmission covering γ means that $\alpha(z) = \gamma(z)$ for all $z \in eU$. This is equivalent to $\alpha(ex) = \gamma(ex)$ for all $x \in U$; hence to the condition $e\alpha(x) = \gamma \circ \nu_U(x)$ for all $x \in U$. But this means that α is a supervaluation covering $\gamma \circ \nu_U$. □

Theorem 9.5. *Given a supertropical semiring U with ghost ideal $M := eU$ and a surjective homomorphism $\gamma : M \to M'$ to a bipotent semiring M', there exists an initial transmission $\alpha : U \to U'$ covering γ.*

Proof. We introduce the strict surjective valuation $v = \gamma \circ \nu_U : U \twoheadrightarrow M'$. By §7 there exists an initial surjective supervaluation $\varphi_v : U \to U(v)$ covering v. (In particular, $eU(v) = M'$.) The other surjective supervaluations $\psi : U \to V$ covering γ are the maps $\pi_T \circ \varphi_v$ with T running through the set of all MFCE-relations on $U(v)$, as explained in §7.

Let $f := \varphi_v(e_U)$ and $e := e_{U(v)} = 1_M$. Proposition 9.4 tells us that $\pi_T \circ \varphi_v$ is the initial transmission covering γ iff $f \sim_T e$ and moreover T is finer than any other MFCE-relation on $U(v)$ with this property. Now we invoke the following easy lemma, to be proved below.

Lemma 9.6. *If W is a supertropical semiring and X is a subset of W, there exists a unique finest MFCE-relation E on W with $x \sim_E e_W x$ for every $x \in X$.*

We apply the lemma to $W = U(v)$ and $X = \{f\}$, and obtain a finest equivalence relation T on $U(v)$ with $f \sim_T ef$. But

$$ef = \nu_{U(v)} \circ \varphi_v(e_U) = v(e_U) = e.$$

Thus, T is the unique finest MFCE-relation on $U(v)$ with $f \sim_T e$, and T gives us the wanted initial transmission $\alpha = \pi_T \circ \varphi_v$. □

Proof of Lemma 9.6. The set $\mathcal{M}$ of all MFCE-relations F on W with $x \sim_F ex$ for all $x \in X$ is not empty, since it contains the relation $E(\nu_W)$. The relation $E := \wedge\mathcal{M}$, i.e., the intersection of all $F \in \mathcal{M}$, has the desired property. □

Notation 9.7. *We denote "the" initial transmission in Theorem 9.5 by $\alpha_{U,\gamma}$, the semiring U' by U_γ, and the equivalence relation $E(\alpha_{U,\gamma})$ by $E(U, \gamma)$.*

This notation is sloppy, since $\alpha_{U,\gamma}$ is determined by U and γ only up to isomorphism. But $E(U, \gamma)$ truly depends only on U and γ. The ambiguity for $\alpha_{U,\gamma}$ can be avoided if γ is surjective, due to the following lemma.

Lemma 9.8. *If $\alpha : U \to V$ is an initial transmission covering a surjective homomorphism $\gamma : M \to M'$, then α itself is a surjective map.*

Proof. $V_1 := \alpha(V)$ is a subsemiring of V and thus a supertropical semiring itself. Replacing V by V_1 we obtain from α a surjective transmission $\alpha_1 : U \to V_1$. Since α is initial there exists a unique transmission $\eta : V \to V_1$ over M' with $\alpha_1 = \eta\alpha$. Also $\alpha = j\alpha_1$ with j the inclusion from V_1 to V. By the universal property of α we conclude from $\alpha = j\eta\alpha$ that $j\eta$ is the identity on V. This forces $V = V_1$. □

Thus, if γ is surjective, we have a canonical choice for U_γ and $\alpha_{U,\gamma}$, namely, $U_\gamma = U/E(U, \gamma)$ and $\alpha_{U,\gamma} = \pi_{U,\gamma}$. Usually we will understand by U_γ and $\alpha_{U,\gamma}$ this semiring and transmission.

In light of Theorem 9.5 our main Problem 9.1 can be posed as follows: Given U and γ, is $\alpha_{U,\gamma} : U \to U_\gamma$ a pushout transmission?

We assume in the following that *$\gamma : M \to M'$ is surjective and M' is a cancellative bipotent domain*; hence $v = \gamma \circ \nu_U$ is a strict surjective valuation. In this case we will obtain a positive solution of the problem. The point here is that we can give an explicit description of U_γ and $\alpha_{U,\gamma}$, which allows us to check the pushout property.

We already have an explicit description of $\varphi_v : U \to U(v)$, given by Example 4.5. Thus all we need is an explicit description of the finest MFCE-relation T on $U(v)$ with $f \sim_T e$. We develop such a description in a more general setting.

Assume that U is a supertropical semiring, $e := e_U$, and f is an idempotent of U. The ideal $L := fU$ of U is again a supertropical semiring with unit element f (under the addition and multiplication of U), since L is a homomorphic image of U. We have $e_L = f + f = ef$.

If F is an equivalence relation on the set L, there is a unique finest equivalence relation E on U extending F. It can be described as follows. Let $x_1, x_2 \in U$. Then $x_1 \sim_E x_2$ iff either $x_1 = x_2$ or $x_1 \in L$, $x_2 \in L$ and $x_1 \sim_F x_2$. We call E the **minimal extension** of the equivalence relation F to U.

Lemma 9.9. *Let F be an equivalence relation on fU, and let E denote the minimal extension of F to U.*

a) If F is multiplicative, then E is multiplicative.

b) If F is fiber conserving, so is E.

Proof. Assume that x_1, x_2 are elements of U with $x_1 \sim_E x_2$. Assume (without loss of generality) that also $x_1 \neq x_2$. Then $x_1, x_2 \in fU$ and $x_1 \sim_F x_2$.

If F is multiplicative then, for any $z \in U$,

$$x_1 z = x_1(fz) \sim_F x_2(fz) = x_2 z;$$

hence $x_1 z \sim_E x_2 z$. Thus E is multiplicative.

If F is fiber conserving, then

$$ex_1 = (ef)x_1 = (ef)x_2 = ex_2.$$

Thus E is fiber conserving. □

Proposition 9.10. *Assume that U is a supertropical semiring, $e := e_U$, and f is an idempotent of U. We define a binary relation E on U be decreeing $(x_1, x_2 \in U)$*

$$x_1 \sim_E x_2 \quad \text{iff either} \quad x_1 = x_2 \quad \text{or} \quad x_1, x_2 \in fU \quad \text{and} \quad ex_1 = ex_2.$$

a) E is an MFCE-relation on U.

b) If $ef = e$, then $e \sim_E f$, and E is finer than any other multiplicative equivalence relation E' on U with $e \sim_{E'} f$.

Proof. a) We apply the preceding lemma with F the relation $E(\nu_L)$ (cf. Examples 6.4.ii) on the supertropical semiring $L := fU$. The minimal extension of F to U is the relation E defined in the proposition. Indeed, for $x_1, x_2 \in L$ we have $x_1 \sim_F x_2$ if $efx_1 = efx_2$. Since $fx_i = x_i$ $(i = 1, 2)$, this means that $ex_1 = ex_2$. By Lemma 9.9 the relation E is MFCE.

b) Assume now that $ef = e$, i.e., $e \in L$. Then $e \sim_E f$ by definition of E. Let E' be any multiplicative equivalence relation on U with $e \sim_{E'} f$. If $x_1, x_2 \in U$ and $x_1 \sim_E x_2$ we want to conclude that $x_1 \sim_{E'} x_2$. We may assume that $x_1 \neq x_2$. Then $x_1, x_2 \in fU$ and $ex_1 = ex_2$. Now $x_i \sim_{E'} ex_i$ $(i = 1, 2)$; hence $x_1 \sim_{E'} x_2$, as desired. □

We are ready for a solution of Problem 9.1 in the case that $\gamma : M \to M'$ is surjective and M' is a cancellative bipotent semidomain; hence $v = \gamma \circ \nu_U$ is a strict surjective valuation. As before, let T denote the finest MFCE-relation on $U(v)$ with $f \sim_T e$ for $e := e_{U(v)}$ and

$f := \varphi_v(e_U)$. Recall from the proof of Theorem 9.5 that $ef = e$. Thus Proposition 9.10 applies. We spell out what the proposition says in the present case.

For that we write the semiring $U(v)$ and the map φ_v in a way different from Definition 4.6. Let $\hat{U}$ denote a copy of U disjoint from U with copying isomorphism $x \mapsto \hat{x}$. We use this to distinguish an element $x \in U \setminus \mathfrak{q}$, with $\mathfrak{q} := \operatorname{supp} v$, from the corresponding element in $\mathcal{T}(U(v))$. Thus we write

$$U(v) = (\hat{U} \setminus \hat{\mathfrak{q}}) \mathbin{\dot\cup} M'$$

with $\hat{\mathfrak{q}} := \{\hat{x} \mid x \in U,\ \gamma(e_U x) = 0\}$, and $\varphi_v(x) = \hat{x}$ for $x \in U \setminus \mathfrak{q}$, $\varphi_v(x) = 0$ for $x \in \mathfrak{q}$. Notice that $fU(v) = (\hat{M} \setminus \hat{\mathfrak{q}}) \cup M'$ with $\hat{M} := \{\hat{x} \mid x \in M\}$.

According to Proposition 9.10 the equivalence relation T has the following description. Let $y_1, y_2 \in U(v)$ be given with $y_1 \neq y_2$. Then $y_1 \sim_T y_2$ iff $y_1 = \hat{x}_1$, $y_2 = \hat{x}_2$, with either $x_1, x_2 \in M$ and $\gamma(e_U x_1) = \gamma(e_U x_2)$ or $x_1, x_2 \in U$ and $\gamma(e_U x_1) = \gamma(e_U x_2) = 0$. We may choose $U_\gamma = U(v)/T$ and $\alpha_{U,\gamma} = \pi_T \circ \varphi_v$. The transmission $\alpha := \alpha_{U,\gamma}$ is a surjective map from U to U_γ, and the equivalence relation $E(\alpha)$ is the relation $E(U,\gamma)$ defined in Notation 9.7. Thus $E := E(U,\gamma)$ has the following description: If $x_1, x_2 \in U$ and $x_1 \neq x_2$ then

$$x_1 \sim_E x_2 \ \Leftrightarrow\ \gamma(e_U x_1) = \gamma(e_U x_2), \text{ and if } x_1 \in \mathcal{T}(U) \text{ or } x_2 \in \mathcal{T}(U),\ \gamma(e_U x_1) = 0.$$

Having found $E(U,\gamma)$ we now redefine $U_\gamma = U/E(U,\gamma)$, $\alpha_{U,\gamma} = \pi_{E(U,\gamma)}$. We arrive at the following theorem.

Theorem 9.11. *Let U be a supertropical semiring, $e := e_U$ (different notation than before!), $M := eU$, and assume that $\gamma : M \to M'$ is a surjective homomorphism from M to a cancellative bipotent semidomain M'. Then $E := E(U,\gamma)$ can be described as follows $(x_1, x_2 \in U)$:*

$$x_1 \sim_E x_2 \ \text{ iff } x_1 = x_2, \quad \text{or} \quad \gamma(ex_1) = \gamma(ex_2),\ ex_1 = x_1,\ ex_2 = x_2, \quad \text{or} \quad \gamma(ex_1) = \gamma(ex_2) = 0.$$

Thus this binary relation on U is a multiplicative equivalence relation, and $\pi_E : U \to U/E$ is the initial transmission covering γ.

N.B. Observe that most often π_E is not a homomorphism.

Theorem 9.12. *If γ is surjective and M' is a cancellative bipotent semidomain, then $\alpha_{U,\gamma}$ is a pushout transmission.*

Proof. Let $\alpha := \alpha_{U,\gamma} = \pi_E : U \to U/E$ with $E := E(U,\gamma)$. Assume that $\delta : M' \to N$ is a homomorphism from M' to a bipotent semiring N and $\beta : U \to V$ is a transmission covering $\delta\gamma : M \to N$, i.e., with $e_V \beta = \delta\gamma e_U$. (In particular $eV = N$.)

We want to verify that β respects the equivalence relation E, i.e., given $x_1, x_2 \in U$, that

$$x_1 \sim_E x_2 \quad \text{implies} \quad \beta(x_1) = \beta(x_2).$$

We may assume that $x_1 \neq x_2$. If x_1 or x_2 is tangible then $\gamma(ex_1) = \gamma(ex_2) = 0$; hence $e_V \beta(x_i) = \delta\gamma(ex_i) = 0$ for $i = 1, 2$. This implies $\beta(x_1) = \beta(x_2) = 0$. Assume now that both x_1 and x_2 are ghost. Then $\gamma(ex_1) = \gamma(ex_2)$; hence $\delta\gamma(ex_1) = \delta\gamma(ex_2)$, i.e., $e_V\beta(x_1) = e_V\beta(x_2)$. But both $\beta(x_1)$ and $\beta(x_2)$ are ghost or zero. Thus $\beta(x_1) = \beta(x_2)$ again.

Since α is surjective, it follows that we have a well-defined map $\rho : U/E \to V$ with $\beta = \rho\alpha$. Now Proposition 6.1 tells us that ρ is a transmission, since both α and β are transmissions and α is surjective. We have

$$\nu_V \rho\alpha = \nu_V \beta = \delta\gamma\nu_U = \delta\nu_{E/U}\alpha.$$

Since α is surjective, this implies that $\nu_V \rho = \delta \nu_{E/U}$, i.e., ρ covers δ. The pushout property of α is verified. □

Assume now that U is **any** supertropical semiring, $M := eU$, and $\gamma : M \to M'$ is an **injective** semiring homomorphism from M to a bipotent semiring M'. Then Problem 9.1 can be solved affirmatively in an easy direct way, as we explicate now.

We may assume, without loss of generality, that M is a subsemiring of M' and γ is the inclusion from M to M'. We define a semiring U' as follows. As a set, U' is the disjoint union of the sets U and $M' \setminus M$. We have $U \subset U'$, $M' \subset U'$, $U \cup M' = U'$, $U \cap M' = M$. Let ν denote the ghost map from U to M, $\nu = \nu_U$. We define addition and multiplication on U by taking the given addition and multiplication on U and on M', and putting

$$x \cdot z = z \cdot x = \nu(x) \cdot z$$

$$x + z = z + x = \nu(x) + z$$

for $x \in U$, $z \in M'$. In the cases that $x \in M$ and $z \in M'$, or $x \in U$ and $z \in M$, these new products are the same as the ones in M' or U, respectively. Thus we have well-defined operations $\cdot$ and $+$ on U'. One checks in any easy and straightforward way that they obey all semiring axioms. Thus U' is now a commutative semiring with $1_{U'} = 1_U$. It clearly obeys the axioms (3.3$'$), (3.3$''$), (3.4). Thus U' is supertropical. We have $e_{U'} = e_U$, $eU' = M'$, $\mathcal{T}(U') = \mathcal{T}(U)$.

Definition 9.13. *We call U' the supertropical semiring* ***obtained from*** U ***by extension of the ghost ideal*** M ***to*** M'. *We also say, more briefly, that U' is a* ***ghost extension of*** U.

Let α denote the inclusion $U \hookrightarrow U'$. It is obvious that α is a transmission covering the inclusion $\gamma : U \hookrightarrow U'$. We verify that α is a pushout transmission.

Let $\delta : M' \to N$ be a homomorphism from M' to a bipotent semiring N and $\beta : U \to V$ a transmission covering $\delta\gamma$. This means that $eV = N$, and

(1) $\beta(x) = \delta(x)$ for $x \in M$.

Clearly, we have a unique well-defined map $\rho : U' \to V$ with $\rho|U = \beta$ and

(2) $\rho(x) = \delta(x)$ for $x \in M'$.

We have $\rho(0) = 0$, $\rho(1) = 1$, $\rho(e_{U'}) = e_V$. One checks easily that ρ is multiplicative.

We now know that ρ is a transmission covering δ. We have proved the following theorem.

Theorem 9.14. *Assume that M' is a bipotent semiring and M is a subring of M'. Assume further that U is a supertropical semiring with ghost ideal M, and U' is the supertropical semiring obtained from U by extension of the ghost ideal M to M'. Then the inclusion mapping $U \to U'$ is a pushout tranmission covering the inclusion mapping $M \hookrightarrow M'$.*

Combining Theorems 9.12 and 9.14, we obtain the most comprehensive solution of Problem 9.1 that we can offer here.

Theorem 9.15. [6] *Let $\gamma : M \to M'$ be a homomorphism between bipotent semirings, and assume that the bipotent semiring $\gamma(M)$ is a cancellative. {N.B. This holds if M' is a cancellative.} Let U be a supertropical semiring with $eU = M$. Then $\alpha_{U,\gamma} : U \to U_\gamma$ is a pushout transmission.*

[6] There exist pushout transmissions, which are not covered by this theorem, cf. [IKR, Proposition 3.17].

Proof. We have a factorization $\gamma = i \circ \bar{\gamma}$, with $\bar{\gamma}$ the map $x \mapsto \gamma(x)$ from M to the subsemiring $\gamma(M)$ of M', and i the inclusion from $\gamma(M)$ to M'. By Theorems 9.12 and 9.14 there exist pushout transmissions $\alpha : U \to \overline{U}$ and $\beta : \overline{U} \to U'$ covering $\bar{\gamma}$ and i, respectively. Now look at the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow[\alpha]{} & \overline{U} & \xrightarrow[\beta]{} & U' \\ \uparrow & & \uparrow & & \uparrow \\ M & \underset{\bar{\gamma}}{\twoheadrightarrow} & \gamma(M) & \underset{i}{\hookrightarrow} & M' \end{array}$$

where the vertical arrows denote inclusions. Here the left and the right square are pushout diagrams in the category STROP of supertropical semirings and transmissions. Thus also the outer rectangle is a pushout in this category (cf., e.g., [ML, §7]), i.e., $\beta\alpha$ is a pushout transmission. If $\alpha_{U,\gamma} : U \to U_\gamma$ is any prechosen initial covering of γ, there exists an isomorphism $\rho : U' \to U_\gamma$ over M' with $\rho\beta\alpha = \alpha_{U,\gamma}$. Thus also $\alpha_{U,\gamma}$ is a pushout transmission. □

10. The ghost surpassing relation; strong supervaluations

Let U be any supertropical semiring. If $x, y \in U$, it has become customary to write

$$x = y + \text{ghost}$$

if x equals y plus an unspecified ghost element (including zero). In more formal terms we have a binary relation $\underset{\text{gs}}{\models}$ on U defined as follows:

Definition 10.1.

$$x \underset{\text{gs}}{\models} y \ \Leftrightarrow \ \exists z \in eU \ \textit{with } x = y + z.$$

We call $\underset{\text{gs}}{\models}$ *the* ***ghost surpassing relation*** *on* U *or* ***GS-relation****, for short.*

The GS-relation seems to be at the heart of many supertropical arguments. Intuitively $x \underset{\text{gs}}{\models} y$ means that x coincides with y up to some "negligible" or "near-zero" element, namely a ghost element. But we have to handle the GS-relation with care, since it is not symmetric. In fact it is antisymmetric, see below.

The GS-relation is clearly transitive:

$$x \underset{\text{gs}}{\models} y, \ y \underset{\text{gs}}{\models} z \ \Rightarrow \ x \underset{\text{gs}}{\models} z.$$

It is also compatible with addition and multiplication: For any $z \in U$, $x \underset{\text{gs}}{\models} y$ implies $x + z \underset{\text{gs}}{\models} y + z$, and $xz \underset{\text{gs}}{\models} yz$.

We observe the following further properties of this subtle binary relation.

Remark 10.2. *Let* $x, y \in U$.

(i) $x = y \ \Rightarrow \ x \underset{\text{gs}}{\models} y \ \Rightarrow \ \nu(x) \geq \nu(y)$.

(ii) *If* $x \in \mathcal{T}(U) \cup \{0\}$, *then* $x \underset{\text{gs}}{\models} y \ \Leftrightarrow \ x = y$.

(iii) *If* $x \in \mathcal{G}(U) \cup \{0\}$, *then* $x \underset{\text{gs}}{\models} y \ \Leftrightarrow \ \nu(x) \geq \nu(y)$.

(iv) $x \underset{\text{gs}}{\models} 0$ *iff* $x = eU$.

Lemma 10.3. *The GS-relation is antisymmetric, i.e.;*

$$x \underset{\text{gs}}{\models} y,\ y \underset{\text{gs}}{\models} x \ \Rightarrow\ x = y.$$

Proof. If $x \in \mathcal{T}(U)$ or $y \in \mathcal{T}(U)$ this is clear by Remark 10.2.ii. Assume now that both $x, y \in eU$. Then $\nu(x) \geq \nu(y)$ and $\nu(y) \geq \nu(x)$ by Remark 10.2.iii; hence $\nu(x) = \nu(y)$, i.e., $x = y$. ☐

Proposition 10.4.

(i) *Assume that $\alpha : U \to V$ is a transmission. Then, for any $x, y \in U$,*

$$x \underset{\text{gs}}{\models} y \ \Rightarrow\ \alpha(x) \underset{\text{gs}}{\models} \alpha(y).$$

(ii) *Assume that $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations with $\varphi \geq \psi$. Then for any $a, b \in R$*

$$\varphi(a) \underset{\text{gs}}{\models} \varphi(b) \ \Rightarrow\ \psi(a) \underset{\text{gs}}{\models} \psi(b).$$

Proof. i): Let $x \underset{\text{gs}}{\models} y$. If x is tangible or zero, then $x = y$; hence $\alpha(x) = \alpha(y)$. If x is ghost, then $\nu(x) \geq \nu(y)$; hence

$$\nu(\alpha(x)) = \alpha(\nu(x)) \geq \alpha(\nu(y)) = \nu(\alpha(y))$$

by rule TM5 in §5. Since $\alpha(x)$ is ghost, this means $\alpha(x) \underset{\text{gs}}{\models} \alpha(y)$, cf. Remark 10.2.iii above.

ii): We may assume that the supervaluation φ is surjective. By §5 we have a (unique) transmission $\alpha : U \to V$ with $\alpha \circ \varphi = \psi$. Thus the claim follows from part i). ☐

We cannot resist giving a second proof of part ii) of the proposition relying only on Definition 5.1 of dominance (conditions D1-D3).

Second proof of Proposition 10.4.ii. Assume that $\varphi(a) \underset{\text{gs}}{\models} \varphi(b)$. If $\varphi(a)$ is tangible or zero, then $\varphi(a) = \varphi(b)$; hence $\psi(a) = \psi(b)$ by D1; hence $\psi(a) \underset{\text{gs}}{\models} \psi(b)$. If $\varphi(a)$ is ghost then $e\varphi(a) \geq e\varphi(b)$; hence $e\psi(a) \geq e\psi(b)$ by D2. By D3 the element $\psi(a)$ is ghost. Thus $\psi(a) \underset{\text{gs}}{\models} \psi(b)$ again, ☐

The GS-relation seems to be helpful for analyzing additivity properties of supervaluations.

Lemma 10.5. *If $\varphi : R \to U$ is a supervaluation on a semiring R with $\varphi(a) + \varphi(b) \in eU$, then*

$$\varphi(a) + \varphi(b) \ \underset{\text{gs}}{\models}\ \varphi(a + b). \qquad (*)$$

Proof. Let $v : R \to eU$ denote the m-valuation covered by φ, $v = e\varphi$. We have $v(a + b) \leq v(a) + v(b)$; hence $e\varphi(a + b) \leq e(\varphi(a) + \varphi(b))$. If $\varphi(a) + \varphi(b) \in eU$, this shows that $\varphi(a) + \varphi(b) \underset{\text{gs}}{\models} \varphi(a + b)$. ☐

It will turn out to be desirable to have supervaluations on R at hand, where the property $(*)$ holds for **all** elements a, b of R.

Definition 10.6. *We call a supervaluation $\varphi : R \to U$* ***tangibly additive****, if in addition to the rules SV1-SV4 from §4 the following axiom holds:*

$$SV5: \text{If } a,b \in R \text{ and } \varphi(a)+\varphi(b) \in \mathcal{T}(U), \text{ then } \varphi(a)+\varphi(b) = \varphi(a+b).$$

Proposition 10.7. *A supervaluation $\varphi : R \to U$ is tangibly additive iff for any $a,b \in R$*

$$\varphi(a)+\varphi(b) \underset{\text{gs}}{\models} \varphi(a+b).$$

Proof. This is clear by Lemma 10.5 and Remark 10.2.ii above. □

Corollary 10.8. *If $\varphi : R \to U$ is tangibly additive, then for every finite sequence $a_1,\dots,a_m$ of elements of R*

$$\sum_{i=1}^{m} \varphi(a_i) \underset{\text{gs}}{\models} \varphi\left(\sum_{i=1}^{m} a_i\right).$$

Proof. This holds for $m = 2$ by Proposition 10.7. The general case follows by an easy induction using the transitivity of the GS-relation. □

Comment: We elaborate what it means that a given supervaluation $\varphi : R \to U$ is tangibly additive in the case that the underlying m-valuation $v = e\varphi : R \to eU$ is strong.

Let $a,b \in R$ be given with $\varphi(a)+\varphi(b) \in \mathcal{T}(U)$, i.e., $v(a) \neq v(b)$, and assume without loss of generality that $v(a) < v(b)$. Then $v(a+b) = v(b)$. Hence, $\varphi(a+b)$ is some element of the fiber $\nu_U^{-1}(v(b))$; but the axioms SV1-SV4 say little about the position of $\varphi(a+b)$ in this fiber. SV5 demands that $\varphi(a+b)$ has the "correct" value $\varphi(a)+\varphi(b) = \varphi(b)$. □

Concerning applications the strong m-valuations seem to be more important than the others. (Recall that any m-valuation on a ring is strong.) Thus the tangibly additive supervaluations covering strong m-valuations deserve a name on their own.

Definition 10.9. *We call a* ***supervaluation*** *$\varphi : R \to U$* ***strong*** *if φ is tangibly additive and the covered m-valuation $e\varphi : R \to eU$ is strong.*

We exhibit an important case where a tangibly additive supervaluation is automatically strong.

Proposition 10.10. *Assume that $\varphi : R \to U$ is a tangible (cf. Definition 4.1) and tangibly additive supervaluation. Then φ is strong.*

Proof. We have to verify that $v := e\varphi$ is strong. Let $a,b \in R$ be given with $v(a) \neq v(b)$. Suppose without loss of generality that $v(a) < v(b)$. Then $\varphi(a),\varphi(b) \in U$ and $\varphi(b) \neq 0$. Since φ is tangible, $\varphi(b) \in \mathcal{T}(U)$. It follows that $\varphi(a)+\varphi(b) \in \mathcal{T}(U)$; hence

$$\varphi(a)+\varphi(b) = \varphi(a+b),$$

because φ is tangibly additive. Multiplying by e we obtain

$$v(a)+v(b) = v(a+b).$$

□

We now are ready to aim at an application of the supervaluation theory developed so far. We start with the polynomial semiring $R[\lambda] = R[\lambda_1,\dots,\lambda_n]$ in a sequence $\lambda = (\lambda_1,\dots,\lambda_n)$ of n variables over a semiring R. Let $\varphi : R \to U$ be a tangibly additive valuation with underlying m-valuation $v : R \to M$, $M := eU$.

Given a polynomial

$$f = \sum_i c_i \lambda^i \in R[\lambda] \tag{10.1}$$

in the usual multimonomial notation (i runs though the multi-indices $i = (i_1, \dots, i_n) \in \mathbb{N}_0^n$, $\lambda^i = \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, only finite many $c_i \neq 0$), we obtain from f polynomials

$$\widetilde{\varphi}(f) \ := \ \sum_i \varphi(c_i)\lambda^i \in U[\lambda],$$

$$\tilde{v}(f) \ := \ \sum_i v(c_i)\lambda^i \in M[\lambda],$$

by applying φ and v to the coefficients of f. This gives us maps

$$\widetilde{\varphi}: \ R[\lambda] \ \to \ U[\lambda], \quad \tilde{v}: \ R[\lambda] \ \to \ M[\lambda].$$

Let $a = (a_1, \dots, a_n \in R^n)$ be an n-tuple of elements of R. It gives us n-tuples

$$\varphi(a) = (\varphi(a_1), \dots, \varphi(a_n)), \quad v(a) = (v(a_1), \dots, v(a_n))$$

in U^n and M^n, respectively. We have an evaluation map $\varepsilon_a : R[\lambda] \to R$, which sends the polynomial f (notation as in (10.1)) to

$$\varepsilon_a(f) \ = f(a) \ = \ \sum_i c_i a^i \tag{10.2}$$

and analogous evaluation maps

$$\varepsilon_{\varphi(a)}(f): \ U[\lambda] \to U, \quad \varepsilon_{v(a)}(f): \ M[\lambda] \to M.$$

These evaluation maps are semiring homomorphisms. We have a diagram

$$\begin{array}{ccc} R[\lambda] & \xrightarrow{\varepsilon_a} & R \\ {\scriptstyle\widetilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ U[\lambda] & \xrightarrow{\varepsilon_{\varphi(a)}} & U \end{array}$$

(and an analogous diagram with v instead of φ) which usually does not commute. But it commutes "nearly".

Theorem 10.11. *For $f \in R[\lambda]$*

$$\varepsilon_{\varphi(a)}(\widetilde{\varphi}(f)) \ \underset{\text{gs}}{\models} \ \varphi(\varepsilon_a(f)).$$

Proof. Let again $f = \sum_i c_i \lambda^i$. Now $\varphi(\varepsilon_a(f)) = \varphi(\sum_i c_i a^i)$, while

$$\varepsilon_{\varphi(a)(\widetilde{\varphi}(f))} = \sum_i \varphi(c_i)\varphi(a)^i = \sum_i \varphi(c_i a^i).$$

Thus the claim is that

$$\sum_i \varphi(c_i a^i) \ \underset{\text{gs}}{\models} \ \varphi(\sum_i c_i a^i). \tag{$*$}$$

This follows from Corollary 10.8 above. $\square$

We draw a consequence of this theorem. Let

$$Z(f) \ := \ \{a \in R^n \mid f(a) = 0\},$$

the zero set of f. Let further

$$Z_0(\widetilde{\varphi}(f)) \ := \ \{b \in U^n \mid \widetilde{\varphi}(f)(b) \in eU\},$$

which we call the **root set** of $\widetilde{\varphi}(f)$. For $a \in Z(f)$ we have $\varphi\left(\sum_i c_i a^i\right) = 0$. It follows by Theorem 10.11 that $\widetilde{\varphi}(f)(\varphi(a)) \underset{\rm gs}{\models} 0$, i.e., $\widetilde{\varphi}(f)(\varphi(a))$ is ghost.

We have proved

Corollary 10.12. *If $\varphi : R \to U$ is tangibly additive, then, for any $f \in R[\lambda]$,*

$$\varphi(Z(f)) \subset Z_0(\widetilde{\varphi}(f)).$$

□

Assume now that φ is *tangible* and tangibly additive; hence strong (cf. Proposition 10.10). Then, of course, $\varphi(Z(f)) \subset \mathcal{T}(U)_0^n$ with $\mathcal{T}(U)_0 := \mathcal{T}(U) \cup \{0\}$. Thus we have

$$\varphi(Z(f)) \ \subset \ Z_0(\widetilde{\varphi}(f))_{\rm tan} \qquad (**)$$

with

$$Z_0(\widetilde{\varphi}(f))_{\rm tan} \ := \ Z_0(\widetilde{\varphi}(f)) \cap \mathcal{T}(U)_0^n,$$

which we call **tangible root set** of $\widetilde{\varphi}(f)$. We want to translate $(**)$ into a statement about the relation between $Z(f)$ and the so called "corner locus", of the polynomial $\tilde{v}(f) \in M[\lambda]$, to be defined.

We call a polynomial $g = \sum_i d_i \lambda^i \in M[\lambda]$ a **tropical polynomial**, and define the **corner-locus** $\mathrm{Corn}(g)$ of g as the set of all $b \in M^n$ such that there exists two different multi-indices $j, k \in \mathbb{N}_0^n$ with

$$d_j b^j = d_k b^k \geq d_i b^i$$

for all $i \neq j, k$. We also say that $\mathrm{Corn}(g)$ is the **tropical hypersurface** defined by the tropical polynomial g.

This is well established terminology at least in the "classical case" that M is the bipotent semiring $T(\mathbb{R})$ given by the order monoid $(\mathbb{R}, +\ )$, the so called max-plus algebra of $\mathbb{R}$ (cf. §1, [IMS, §1.5]). A small point here is, that we admit coordinates with value $0_M =: -\infty$, which usually is not done in tropical geometry. On the other hand we could work as well with Laurent polynomials. Then of course we would have to discard the zero element.

Returning to our tangible strong supervaluation $\varphi : R \to U$ and the m-valuation

$$v = e\varphi : R \to M,$$

we look at the tropical polynomial

$$\tilde{v}(f) = \sum_i v(c_i) \lambda^i$$

from above. Let $a \in R^n$. Then

$$\widetilde{\varphi}(f)(\varphi(a)) \ = \ \sum \varphi(c_i a^i),$$

and all summands are the right side are in $\mathcal{T}(U)_0$. Thus the sum is ghost iff the maximum of the ν-values

$$\nu(\varphi(c_i a^i)) = v(c_i) v(a^i) \qquad (i \in \mathbb{N}_0^n)$$

is attained for at least two multi-indices. This means that $v(a) \in \mathrm{Corn}(\tilde{v}(f))$.

Thus $(**)$ has the following consequence

Corollary 10.13. *Let $v : R \to M$ be a strong m-valuation on a semiring R. Assume that there exists a tangible supervaluation $\varphi : R \to U$ covering v. Then for any polynomial $f \in R[\lambda]$,*

$$v(Z(f)) \subset \mathrm{Corn}(\tilde{v}(f)).$$

We have arrived at a very general version of the Lemma of Kapranov ([EKL, Lemma 2.1.4]), as soon as we find a tangible cover $\varphi : R \to U$ of the given m-valuation $v : R \to M$. This turns out to be easy in the case that M is cancellative (i.e., v is a strong *valuation*).

Lemma 10.14. *Suppose there is given a* ***tangible multiplicative section*** *of the ghost map $\nu : U \to M$, i.e., a map $s : M \to \mathcal{T}(U)_0$ with $s(0) = 0$, $s(1) = 1$, $s(xy) = s(x)s(y)$, and $\nu(s(x)) = x$ for any $x, y \in M$. Let $v : R \to M$ be a strong m-valuation. Then $s \circ v : R \to U$ is a tangible strong supervaluation covering v.*

Proof. Clearly $\varphi = sv$ obeys SV1-SV4. Let $a, b \in R$ be given with $v(a) < v(b)$. Then $v(a+b) = v(b)$; hence $sv(a+b) = sv(b)$. Thus SV5 holds true. We have $e\varphi = \nu \circ \varphi = v$. □

Example 10.15. *If U is a supertropical semifield, it is known that such a section s always exists (*[IR3, Proposition 1.6]*).*

Example 10.16. *Assume that M is a cancellative bipotent semiring, and $v : R \to M$ is a strong valuation. We take $U := D(M \setminus \{0\})$ (Example 3.18), for which we write more briefly $D(M)$. For every $z \in M$ there exists a unique $x \in \mathcal{T}(U)_0$ with $\nu(x) = z$. We write $x = \hat{z}$. Clearly $z \mapsto \hat{z}$ is a tangible multiplicative section of the ghost map, in fact the only one. By the lemma we obtain a tangible supervaluation*

$$\hat{v} : R \to U, \quad \hat{v}(z) := \widehat{v(z)},$$

which covers v, in fact the only such supervaluation.

Looking again at Corollary 10.13 we now know that

$$v(Z(f)) \subset \mathrm{Corn}(\tilde{v}(f)),$$

whenever $v : R \to M$ is a strong valuation and $f \in R[\lambda]$.

11. The tangible strong supervaluations in $\mathrm{Cov}(v)$

Given an m-valuation $v : R \to M$, recall from §7 that the equivalence classes $[\varphi]$ of supervaluations φ covering v form a complete lattice $\mathrm{Cov}(v)$. Abusing notation, we usually will not distinguish between a supervaluation φ and its class $[\varphi]$, thus writing $\varphi \in \mathrm{Cov}(v)$ if φ covers v. This will cause no harm in the present section. {N.B If you are sceptical about this, you may always assume that φ is surjective, more specially, that $\varphi = \varphi_v/E$ with φ_v the initial covering of v and E an MFCE-relation on $U(v)$ (cf. Notation 6.14). These supervaluations φ are canonical representatives of their classes $[\varphi]$.}

Lemma 11.1. *Assume that $\varphi : R \to U$ and $\psi : R \to V$ are supervaluation with $\varphi \geq \psi$.*

(i) *If ψ is tangible, then φ is tangible.*
(ii) *It φ is tangibly additive, then ψ is tangibly additive.*

Proof. i): is clear from the axiom D3 in the definition of dominance (cf. Definition 5.1).

ii): follows from Propositions 10.7 and 10.4.ii. □

Starting from now we assume that v is a strong valuation (which means in particular that M is cancellative). Let $\mathfrak{q}$ denotes the support of v, i.e., $\mathfrak{q} = v^{-1}(0)$.

Notation 11.2. $\mathrm{Cov}_{\mathrm{t}}(v)$ *denotes the set of tangible supervaluations in* $\mathrm{Cov}(v)$*, and* $\mathrm{Cov}_{\mathrm{s}}(v)$ *denotes the set of strong (= tangibly additive) supervaluations in* $\mathrm{Cov}(v)$*. Finally, let*

$$\mathrm{Cov}_{\mathrm{ts}}(v) \ := \ \mathrm{Cov}_{\mathrm{t}}(v) \cap \mathrm{Cov}_{\mathrm{s}}(v),$$

be the set of tangible strong supervaluations covering v.

We already know by Example 10.16 that the set $\mathrm{Cov}_{\mathrm{ts}}(v)$ is not empty. Lemma 11.1 tells us in particular that $\mathrm{Cov}_{\mathrm{t}}(v)$ is an upper set and $\mathrm{Cov}_{\mathrm{s}}(v)$ is a lower set in the poset $\mathrm{Cov}(v)$.

Let us study these sets more closely. We start with $\mathrm{Cov}_{\mathrm{t}}(v)$. The initial supervaluation $\varphi_v : R \to U(v)$ (cf. Definition 5.15) is the top (= biggest) element of $\mathrm{Cov}(v)$, and thus is also the top element of $\mathrm{Cov}_{\mathrm{t}}(v)$. This can also be read off from the explicit description of φ_v in Example 4.5. The other elements of $\mathrm{Cov}(v)$ are the supervaluations $\varphi_v/E : R \to U(v)/E$, with E running through the MFCE-relations on $U(v)$. We have to find out which MFCE-relations E on $U(v)$ give tangible supervaluations φ_v/E.

Here is a definition which - for later use - is slightly more general than what we need now:

Definition 11.3. *We call an equivalence relation E on a supertropical semiring U* ***ghost separating*** *if for all $x \in \mathcal{T}(U)$, $y \in U$,*

$$x \sim_E y \quad \Rightarrow \quad y \in \mathcal{T}(U) \ \text{or}\ x \sim_E 0.$$

If E is an MFCE-relation on U, then $x \sim_E 0$ only if $x = 0$. Thus, E is ghost separating iff $\mathcal{T}(U)$ is a union of E-equivalence classes. This means that E is finer than the MFCE-relation E_{t} introduced in Examples 6.4.v, whose equivalence classes are the tangible fibers of ν_U and the one-point sets in eU.

If $\varphi : R \to Y$ is a surjective tangible supervaluation and E is an MFCE-relation on U, then it is obvious that $\varphi/E : R \to U/E$ is again tangible iff E is ghost separating. Thus we see that φ_v/E_{t} is the bottom (= smallest) element of $\mathrm{Cov}_{\mathrm{t}}(v)$.

Now recall from Example 6.11 that, in the notion at the end of §10 (Example 10.16),

$$U(v)/E_{\mathrm{t}} \ = \ D(M);$$

hence φ_v/E_{t} coincides with the only tangible cover $\hat{v}$ of v with values in $D(M)$, cf. Example 10.16. We conclude that

$$\mathrm{Cov}_{\mathrm{t}}(\varphi) = \{\psi \in \mathrm{Cov}(v) \mid \psi \geq \hat{v}\}.$$

Again by Example 10.16 we know that $\hat{v}$ is strong. This $\hat{v}$ is also the bottom of the poset $\mathrm{Cov}_{\mathrm{ts}}(\varphi)$.

We turn to $\mathrm{Cov}_{\mathrm{s}}(v)$. We will construct a new element of this poset in a direct way. For that reason we introduce an equivalence relation on R.

Definition 11.4. *Let $S(v)$ denote the equivalence relation on the set R defined as follows.* {*We write $\sim_v$ for $\sim_{S(v)}$.*}

If $a_1, a_2 \in R$ then

$$a_1 \sim_v a_2 \iff \begin{array}{l} \text{either } v(a_1) = v(a_2) = 0 \\ \text{or } \exists c_1, c_2 \in R, \ \text{with } v(c_1) < v(a_1), \ v(c_2) < v(a_2), \\ a_1 + c_1 = a_2 + c_2. \end{array}$$

It is easily checked that $S(v)$ is indeed an equivalence relation on the set R, by making strong use if the assumption that the valuation v is strong. This is the finest equivalence relation E on U such that $a \sim_E a + c$ if $v(c) < v(a)$. Observe also that

$$a_1 \sim_v a_2 \implies v(a_1) = v(a_2).$$

We claim that $S(v)$ is compatible with multiplication, i.e.,

$$a_1 \sim_v a_2 \implies a_1 b \sim_v a_2 b$$

for every $b \in R$. This is obvious if $a_1 \in \mathfrak{q}$ or $a_2 \in \mathfrak{q}$, or $b \in \mathfrak{q}$. Otherwise $v(b) > 0$, and we have elements $c_1, c_2 \in R$ with $v(c_1) < v(a_1)$, $v(c_2) < v(a_2)$, $a_1 + c_1 = a_2 + c_2$. Then $a_1 b + c_1 b = a_2 b + c_2 b$ and

$$v(c_i b) \ = \ v(c_i)v(b) \ < \ v(a_i)v(b) \ = \ v(a_i b)$$

for $i = 1, 2$, since by assumption M is cancellative. Thus indeed $a_1 b \sim_v a_2 b$.

We denote the $S(v)$-equivalence class of an element a of R by $[a]_v$. The set $\overline{R} := R/S(v)$ is a monoid under the well defined multiplication

$$[a]_v \cdot [b]_v \ = \ [ab]_v$$

for $a, b \in R$. The subset $R \setminus \mathfrak{q}$ of R is a union of $S(v)$-equivalence classes and the subset $\overline{R \setminus \mathfrak{q}} := (R \setminus \mathfrak{q})/S(v)$ of $\overline{R}$ is a submonoid of R. We have

$$\overline{R} = \overline{R \setminus \mathfrak{q}} \ \cup \ \{\bar{0}\}$$

with $\bar{0} = [0]_v = \mathfrak{q}$.

Since $a_1 \sim_v a_2$ implies $v(a_1) = v(a_2)$, we have a well defined monoid homomorphism $\overline{R} \to M$, $[a]_v \mapsto v(a)$, which restricts to a monoid homomorphism

$$\bar{v} : \overline{R \setminus \mathfrak{q}} \ \to \ M \setminus \{0\}.$$

This map $\bar{v}$ gives us a supertropical semiring

$$U := \mathrm{STR}(\overline{R \setminus \mathfrak{q}}, M \setminus \{0\}, \bar{v}),$$

cf. Construction 3.16. Notice that $\mathcal{T}(U) = \overline{R \setminus \mathfrak{q}}$ and $eU = M$. We identify $\mathcal{T}(U)_0 = \overline{R}$.

Proposition 11.5. *The map $\chi : R \to U$ given by*

$$\chi(a) := 0 \quad \text{if} \quad a \in \mathfrak{q}, \qquad \chi(a) := [a]_v \in \mathcal{T}(U) = \overline{R \setminus \mathfrak{q}} \quad \text{if} \quad a \notin \mathfrak{q},$$

is a tangible strong supervaluation covering v.

Proof. It is obvious that χ obeys the rules SV1-SV3 in the definition of supervaluations (Definition 4.1). Due to our construction of U we have $\nu_U \circ \chi = v$. Thus χ also obeys SV4, and hence is a supervaluation covering the strong valuation v. It is clearly tangible.

It remains to verify that χ is tangibly additive. Let $a, b \in R$ be given with $\chi(a) + \chi(b) \in \mathcal{T}(U)$, i.e., $v(a) \neq v(b)$. Assume without loss of generality that $v(a) < v(b)$. Then $a + b \sim_v b$. This means that $\chi(a + b) = \chi(b)$, as desired. □

We strive for an understanding of the set of all $\psi \in \mathrm{Cov}(v)$ which are dominated by this supervaluation χ. We need a new definition.

Definition 11.6. *We call a supervaluation $\varphi : R \to V$ **very strong**, if*

$$SV5^* : \forall a, b \in R : e\varphi(a) < e\varphi(b) \implies \varphi(a+b) = \varphi(b).$$

Clearly SV5* implies that the m-valuation v is strong. If we require this property only for $a, b \in R$ with $e\varphi(a) < e\varphi(b)$ and $\varphi(b)$ tangible, we are back to condition SV5 given above (Definition 10.6). Thus, a very strong supervaluation is certainly strong. On the other hand, every **tangible** strong supervaluation is very strong.

Lemma 11.7. *If $\varphi : R \to V$ is very strong, then any supervaluation $\psi : R \to W$ dominated by φ is again very strong.*

Proof. Let $a, b \in R$ be given with $e\psi(a) < e\psi(b)$. It follows from axiom D2 that $e\varphi(a) < e\varphi(b)$, since $e\varphi(a) \geq e\varphi(b)$ would imply $e\psi(a) \geq e\psi(b)$. Thus $\varphi(a+b) = \varphi(b)$, and we obtain by D1 that $\psi(a+b) = \psi(b)$. □

Returning to our given strong valuation $v : R \to M$, let $\mathrm{Cov}^*_\mathrm{s}(v)$ denote the subset of all $\varphi \in \mathrm{Cov}(v)$ which are very strong. Lemma 11.7 tells us in particular that $\mathrm{Cov}^*_\mathrm{s}(v)$ is a lower set in the poset $\mathrm{Cov}(v)$, and hence is $\mathrm{Cov}_\mathrm{s}(v)$. We have

$$\mathrm{Cov}_\mathrm{t}(v) \cap \mathrm{Cov}^*_\mathrm{s}(v) \;=\; \mathrm{Cov}_\mathrm{t}(v) \cap \mathrm{Cov}_\mathrm{s}(v) \;=\; \mathrm{Cov}_{\mathrm{t,s}}(v).$$

Theorem 11.8. *The tangible strong supervaluation $\chi : R \to U$ from above (Proposition 11.5) dominates every very strong supervaluation covering v, and hence is the top element of both $\mathrm{Cov}^*_\mathrm{s}(v)$ and $\mathrm{Cov}_{\mathrm{t,s}}(v)$.*

Proof. Let $\psi : R \to V$ be a very strong supervaluation covering v (in particular $eV = M$). We verify axioms D1-D3 for the pair χ, ψ, and then will be done. D2 is obvious, and D3 holds trivially since χ is tangible. Concerning D1, assume that $\chi(a_1) = \chi(a_2)$. By definition of χ this means that $a_1 \sim_v a_2$.

We have to prove that $\psi(a_1) = \psi(a_2)$. Either $a_1, a_2 \in \mathfrak{q}$, or there exist $c_1, c_2 \in R$ with $v(c_1) < v(a_1)$, $v(c_2) < v(a_2)$, $c_1 + a_1 = c_2 + a_2$. In the first case $e\psi(a_1) = e\psi(a_2) = 0$ hence $\psi(a_1) = \psi(a_2) = 0$. In the second case we have

$$\psi(a_1) = \psi(a_1 + c_1) = \psi(a_2 + c_2) = \psi(a_2)$$

since ψ is very strong. Thus $\psi(a_1) = \psi(a_2)$ in both cases. □

Notation 11.9. *We denote the semiring U given above by $\overline{U(v)}$ and the supervaluation χ given above by $\overline{\varphi}_v$. We call*

$$\overline{\varphi}_v : R \;\to\; \overline{U(v)} = \mathrm{STR}(\overline{R \setminus \mathfrak{q}}, M \setminus \{0\}, \bar{v})$$

*the **initial very strong supervaluation** covering v.*

In this notation

$$\begin{aligned} \mathrm{Cov}^*_\mathrm{s}(v) \;&=\; \{\psi \in \mathrm{Cov}(v) \mid \overline{\varphi}_v \geq \psi\}, \\ \mathrm{Cov}_{\mathrm{t,s}}(v) \;&=\; \{\psi \in \mathrm{Cov}(v) \mid \overline{\varphi}_v \geq \psi \geq \hat{v}\}. \end{aligned}$$

Let $E(v)$ denote the equivalence relation on $U(v)$ whose equivalence classes are the sets $[a]_v$ with $a \in R \setminus \mathfrak{q} = \mathcal{T}(U(v))$ and the one point set $\{x\}$ with $x \in M$. In other terms, the restriction $E(v)|\mathcal{T}(U)$ coincides with $S(v)|R\setminus\mathfrak{q}$, while $E(v)|M$ is the diagonal $\mathrm{diag}(M)$ of M. We identify

$$U(v)/E(v) = \overline{U(v)}$$

in the obvious way.

Proposition 11.10. *$E(v)$ is a ghost separating MFCE-relation and*

$$\overline{\varphi}_v = \varphi_v/E(v).$$

Proof. It is immediate that $E(v)$ is MFCE and ghost separating. For a in $R\setminus\mathfrak{q}$ we have

$$\pi_{E(v)}(\varphi_v(a)) = \pi_{S(v)}(a) = [a]_v = \overline{\varphi}_v(a)$$

and for $a\in\mathfrak{q}$

$$\pi_{E(v)}(\varphi_v(a)) = \pi_{E(v)}(a) = 0 = \overline{\varphi}_v(a),$$

again, Thus $\pi_{E(v)}$ is the transmission from φ_v to $\overline{\varphi}_v$. □

Corollary 11.11. *The MFCE-relations E on $U(v)$ such that φ_v/E is very strong are precisely all $E\in MFC(U(v))$ with $E\supset E(v)$.*

Proof. This is a consequence of our observations above (Lemma 11.7, Theorem 11.8, Proposition 11.10) and the theory in §7, cf. Theorem 7.2. □

We now focus on the special case that R is a semifield. Slightly more generally, we assume that every element of $R\setminus\mathfrak{q}$ is invertible, while $\mathfrak{q}$ may be different from $\{0\}$.

$\mathcal{T}(U(v)) = R\setminus\mathfrak{q}$ is a group under multiplication. Thus the results from the end of §8 apply. We have

$$\mathcal{T}_e(U(v)) = \{a\in R \mid v(a) = 1_M\} = \mathfrak{o}_v^*,$$

with $\mathfrak{o}_v^*$ the unit group of the subsemiring

$$\mathfrak{o}_v := \{a\in R \mid v(a)\le 1_M\}$$

of R. Notice that the set

$$\mathfrak{m}_v := \mathfrak{o}_v\setminus\mathfrak{o}_v^* = \{a\in R \mid v(a) < 1_M\}$$

is an ideal of $\mathfrak{o}_v$, just as in the classical (and perhaps most important) case, where R is a field and v is a Krull valuation on R.

By Theorem 8.11 and Corollary 8.12 we know that every MFCE-relation on $U(v)$ except $E(\nu)$ is orbital, hence ghost separating. We have

$$\varphi_v/E(\nu) = v,$$

viewed as a supervaluation. The other supervaluations φ covering v correspond uniquely with the subgroups H of $\mathfrak{o}_v^*$ via $\varphi = \varphi_v/E(H)$; cf. Scholium 8.9.

Instead of $U(v)/E(H)$ and $\varphi_v/E(H)$ we now write $U(v)/H$ and φ_v/H respectively. In this notation

$$\mathcal{T}(U(v)/H) = (R\setminus\mathfrak{q})/H,$$

and $\varphi_v/H : R\to U(v)/H$ is given by

$$(\varphi_v/H)(a) = \begin{cases} aH & \text{if } a\in R\setminus\mathfrak{q},\\ 0 & \text{if } a\in\mathfrak{q}.\end{cases}$$

Theorem 11.12. *Assume that every element of $R\setminus\mathfrak{q}$ is invertible (e.g. R is semifield).*

(i) *Every strong supervaluation covering v is very strong. Except v itself, viewed as a supervaluation, all these supervaluations are tangible. In other terms,*

$$\mathrm{Cov_s}(v) = \mathrm{Cov_s^*}(v) = \mathrm{Cov_{t,s}}(v) \cup \{v\}.$$

(ii) $\overline{\varphi}_v = \varphi_v / 1 + \mathfrak{m}_v$.

(iii) *The tangible strong supervaluations φ covering v correspond uniquely with the subgroup H of $\mathfrak{o}_v^*$ containing the group $1 + \mathfrak{m}_v$ via $\varphi = \varphi_v/H$. Thus we have an anti-isomorphism $H \mapsto \varphi_v/H$ from the lattice of all subgroups H of $\mathfrak{o}_v^*$ containing $1+\mathfrak{m}_v$ to the lattice* $\mathrm{Cov_{t,s}}(v)$.

Proof. i): This follows from the discussion above, since every tangible strong supervaluation is very strong and, of course, the supervaluation $v : R \to M$ is also very strong.

ii): We know that $\overline{\varphi}_v = \varphi_v/E(v)$ (Proposition 11.10). $E(v)$ is ghost separating, hence orbital. The subgroup H of $\mathfrak{o}_v^*$ with $E(H) = E(v)$ has the following description (cf. Proposition 8.8): If $a \in R \setminus \mathfrak{q} = \mathcal{T}(U(v))$, then $a \in H$ iff $a \sim_v 1$. This means that there exist elements $c_1, c_2 \in \mathfrak{m}_v$ with $a + c_1 = 1 + c_2$. Now $a + c_1 = a(1 + d_1)$ with $d_1 = \frac{c_1}{a} \in \mathfrak{m}_v$. Thus $a \sim_v 1$ iff $a \in 1 + \mathfrak{m}_v$.

iii): Now obvious, since $\overline{\varphi}_v$ is the top element of $\mathrm{Cov_{t,s}}(v)$.

□

We look again at the GS-sentence

$$\varepsilon_{\varphi(a)}(\widetilde{\varphi}(f)) \underset{\mathrm{gs}}{\models} \varphi(\varepsilon_a(f)) \qquad (*)$$

from §10, valid for any $\varphi \in \mathrm{Cov_s}(v)$, $f \in R[\lambda]$, $a \in R^n$, cf. Theorem 10.11. Choosing here any $\varphi \in \mathrm{Cov_{t,s}}(v)$, we learned that $(*)$ implies Kapranov's lemma (Corollary 10.13). But the statement $(*)$ itself has a different content for different $\varphi \in \mathrm{Cov_{t,s}}(v)$. If also $\psi \in \mathrm{Cov_{t,s}}(v)$ and $\varphi \geq \psi$, then we obtain statement $(*)$ for ψ from the statement $(*)$ for φ, leaving f and the tuple a fixed, by applying the transmission $\alpha_{\psi,\varphi}$. Thus it seems that $(*)$ has the most content if we choose for φ the initial strong supervaluation $\overline{\varphi}_v : R \to \overline{U(v)}$.

We close this section by an explicit description of $\overline{U(v)}$ and $\overline{\varphi}_v$ in a situation typically met in tropical geometry. Let $R := F\{t\}$ be the field of formal **Puiseux series with real powers** over any field[7] F, cf. [IMS, p.6]. The elements of R are the formal series

$$a(t) = \sum_{j \in I} c_j t^j$$

with $c_j \in F^*$ and $I \subset \mathbb{R}$ a well ordered set, in set theoretic sense, (including $I = \emptyset$). Let further M be the bipotent semifield $T(\mathbb{R}_{>0})$ (cf. Theorem 1.5), i.e.,

$$M = \mathbb{R}_{>0} \cup \{0\} = \mathbb{R}_{\geq 0},$$

with the max-plus structure.

We define a (automatically strong) valuation

$$v : F\{t\} \to M$$

by putting

$$v(a(t)) := \vartheta^{\min(I)}$$

[7]For the matter of geometric applications, one usually needs F to be algebraically closed, but here we can omit this restriction.

if $a(t) \neq 0$, written as above, and $v(0) := 0$. Here ϑ is a fixed real number with $0 < \vartheta < 1$ (cf. [IMS]) loc. cit, but we use a multiplicative notation). Now $\mathfrak{o}_v^*$ is the group consisting of all series

$$a(t) = c_0 + \sum_{j>0} c_j t^j, \qquad c_0 \neq 0,$$

in $F\{t\}$, and $1 + \mathfrak{m}_v$ is the subgroup of these series with $c_0 = 1$.

The equivalence relation $S(v)$ on $R^* = \mathcal{T}(U(v))$ is given by

$$a(t) \sim_v b(t) \iff \frac{a(t)}{b(t)} \in 1 + \mathfrak{m}_v.$$

This means that the series $a(t)$ and $b(t)$ have the same leading term $\ell(a(t)) = \ell(b(t))$. Thus the group of monomials

$$G := \{ct^j \mid c \in F^*,\ j \in \mathbb{R}\}$$

is a system of representatives of the equivalence classes of $S(v)$. We identify

$$G = R^*/S(v) = \mathcal{T}(U(v))/E(v).$$

Then $\overline{U(v)} = \mathrm{STR}(G, \mathbb{R}_{>0}, v|G) = G\dot{\cup}M$ in the notation of Construction 3.16, and our supervaluation $\overline{\varphi}_v : R \to \overline{U(v)}$ is the map $a(t) \mapsto \ell(a(t))$, which sends each formal series $a(t)$ to its leading term. {We read $\ell(0) = 0$, of course.}

In short, applying v to a series $a(t)$ means taking its leading t-power and replacing t by ϑ, while applying $\overline{\varphi}_v$ means taking its leading term.

Similarly we can interpret the bottom supervaluation $\hat{v} \in \mathrm{Cov}_{\mathrm{t,s}}(v)$. The t-powers t^j, $j \in \mathbb{R}$, are a multiplicative set of representatives of the E_{t}-equivalence classes. Identifying

$$U(v)/E_{\mathrm{t}} = \{t^j \mid j \in \mathbb{R}\},$$

we can say that $\hat{v}(a(t))$ is the leading t-power of the series $a(t)$. The ghost map from $U(v)/E_{\mathrm{t}} = D(M)$ to M sends t to ϑ.

12. Pushouts of tangible supervaluations

If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluation on a semiring R, and φ dominates ψ, then we also say that ψ is a **coarsening** of φ . Recall that this happens iff there is a transmission $\alpha : U \to V$ with $\psi = \alpha \circ \varphi$. If in addition φ is surjective, i.e., $U = \varphi(R) \cup e\varphi(R)$, which is no essential loss of generality, then α is uniquely determined by φ and ψ, and we write $\alpha = \alpha_{\psi,\varphi}$ (cf. §5).

Assume now that $v : R \to M$ is a surjective m-valuation and $\varphi : R \to U$ is a surjective supervaluation covering v (in particular $M = eU$). Moreover, let $\gamma : M \to N$ be a surjective homomorphism to another (bipotent) semiring N.

Definition 12.1. *We say that a surjective supervaluation $\psi : R \to V$ is the* ***initial coarsening of*** φ ***along*** γ*, if ψ is a coarsening of φ and $\alpha_{\psi,\varphi}$ is the initial transmission covering γ (cf. Definition 9.3). In the notation 9.7; which we will obey in the following, this means that $V = U_\gamma$ and $\alpha_{\psi,\varphi} = \alpha_{U,\gamma}$. We then write $\psi = \gamma_*(\varphi)$.*

In this way we obtain a map

$$\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma v)$$

between complete lattices.

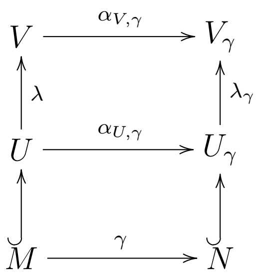
[We could define such a map γ_* also if $\gamma : M \to N$ is not necessarily surjective. But in the present section this will give no additional insight.]

In the following, we will tacitly assume that all occurring supervaluations are surjective, again without essential loss of generality.

We write down a functional property of the initial transmissions $\alpha_{U,\gamma}$, which will give us simple properties of the maps γ_*. The map $\gamma : M \to N$ is always assumed to be a surjective homomorphism between bipotent semirings (as before).

Proposition 12.2. *Let U and V be supertropical semirings with $eU = eV = M$ and let $\lambda : U \to V$ be a transmission over M (hence a homomorphism, cf. Proposition 5.10.iii).*

(a) *Then there exists a unique transmission from U_γ to V_γ over N, denoted by λ_γ, such that*
$$\lambda_\gamma \circ \alpha_{U,\gamma} \;=\; \alpha_{V,\gamma} \circ \lambda.$$
We thus have a commuting diagram



with inclusion mappings $M \hookrightarrow U$ and $N \hookrightarrow U_\gamma$.

(b) *If $\xi : V \to W$ is a second homomorphism over M then*
$$\xi_\gamma \lambda_\gamma \;=\; (\xi\lambda)_\gamma.$$

Proof. a): $\alpha_{V,\gamma}\lambda : U \to V_\gamma$ is a transmission covering γ. Now use the universal property of the initial transmission $\alpha_{U,\gamma}$.

b): $\xi_\gamma\lambda_\gamma : U_\gamma \to W_\gamma$ is a transmission over N such that
$$\xi_\gamma\lambda_\gamma\alpha_{U,\gamma} = \xi_\gamma\alpha_{V,\gamma}\lambda = \alpha_{W,\gamma}\xi\lambda.$$
By the uniqueness part in a) we conclude that $\xi_\gamma\lambda_\gamma = (\xi\lambda)_\gamma$. □

As an immediate consequence of part b) we have

Corollary 12.3. *The map $\gamma_* : \mathrm{Cov}(v) \to \mathrm{Cov}(\gamma v)$ is order preserving in the weak sense, i.e., $\varphi \geq \psi$ implies $\gamma_*(\varphi) \geq \gamma_*(\psi)$.* □

Corollary 12.4. *If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations covering v (in particular $eU = eV = M$) with $\varphi \geq \psi$ then*
$$\alpha_{\gamma_*(\psi),\gamma_*(\varphi)} \;=\; (\alpha_{\psi,\varphi})_\gamma.$$

Proof. We have $\psi = \lambda\varphi$ with $\lambda := \alpha_{\psi,\varphi}$. From this we conclude that
$$\gamma_*(\psi) \;=\; \alpha_{V,\gamma}\lambda\varphi \;=\; \lambda_\gamma\alpha_{U,\gamma}\varphi \;=\; \lambda_\gamma\gamma_*(\varphi).$$
Thus λ_γ is the transmission from $\gamma_*(\varphi)$ to $\gamma_*(\psi)$. □

Starting from now *we assume that the bipotent semirings M and N are cancellative*; hence $v : R \to M$ and $\gamma v : R \to N$ are valuations. We define

$$\mathfrak{p} := \gamma^{-1}(0), \qquad \mathfrak{q} := v^{-1}(0) = \operatorname{supp}(v), \qquad \mathfrak{q}' := v^{-1}(\mathfrak{p}) = \operatorname{supp}(\gamma v).$$

Notice that $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{q}'$ are prime ideals of M and R, respectively.

Given any supertropical semiring U with $eU = M$, we now know that $\alpha_{U,\gamma} : U \to U_\gamma$ is a pushout transmission (Theorem 9.12). Consequently, if $\varphi \in \operatorname{Cov}(v)$, we now call $\gamma_*(\varphi)$ the **pushout of φ along** γ (instead of "initial coarsening of φ along γ").

The good thing is that we now have an explicit descriptions of U_γ and $\alpha_{U,\gamma}$ which we recall from §9, cf. Theorem 9.11.

We start with a multiplicative equivalence relation $E(U,\gamma)$ on U defined as follows. For x, y in U

$$x \sim_{E(U,\gamma)} y \quad \Longleftrightarrow \quad \begin{array}{l} \text{either } x = y, \\ \text{or both } x, y \in M \text{ and } \gamma(x) = \gamma(y), \\ \text{or } ex \in \mathfrak{p},\ ey \in \mathfrak{p}. \end{array}$$

The restriction $E(U,\gamma)|M$ is the equivalence relation $E(\gamma)$ given by $\gamma : M \twoheadrightarrow N$. We identify every class $[x]_{E(U,\gamma)}$, $x \in M$, with the image $\gamma(x) \in N$ and then have

$$M/E(U,\gamma) = N.$$

As proved in §9, we may choose[8] $U_\gamma = U/E(U,\gamma)$ and then have

$$\alpha_{U,\gamma} = \pi_{E(U,\gamma)} : x \longmapsto [x]_{E(U,\gamma)}.$$

Let $x \in \mathcal{T}(U)$. If $ex \notin y$, then $[x]_{E(U,\gamma)} = \{x\}$, but if $ex \in y$, then $[x]_{E(U,\gamma)} = 0 \in N$. Thus we see that $\mathcal{T}(U_\gamma) = U_\gamma \setminus N$ is the bijective image of $\{x \in \mathcal{T}(U) \mid ex \notin y\}$. We identify $[x]_{E(U,\gamma)}$ with x, if x lies in this set, and then have

$$\mathcal{T}(U_\gamma) = \{x \in \mathcal{T}(U) \mid ex \notin \mathfrak{p}\}, \qquad U_\gamma = \{x \in \mathcal{T}(U) \mid ex \notin \mathfrak{p}\} \,\dot\cup\, N.$$

Notice that the multiplicative monoid $\mathcal{T}(U_\gamma)$ has become a submonoid of $\mathcal{T}(U)$, since $E(U,\gamma)$ is multiplicative, but the sum of two elements of $\mathcal{T}(U_\gamma)$, computed in the semiring U_γ, can be very different from their sum in U.

After all these identifications we have

Lemma 12.5. *For any $x \in U$,*

$$\alpha_{U,\gamma}(x) \;=\; \begin{cases} x & \text{if } x \in \mathcal{T}(U), ex \notin \mathfrak{p}, \\ 0 & \text{if } x \in \mathcal{T}(U), ex \in \mathfrak{p}, \\ \gamma(x) & \text{if } x \in M. \end{cases}$$

□

Clearly $\alpha_{U,\gamma}$ maps $\mathcal{T}(U)_0$ into (in fact onto) $\mathcal{T}(U_\gamma)_0$. In other words, $E(U,\gamma)$ is ghost separating (cf. Definition 11.3). This implies

Proposition 12.6. $\gamma_*(\operatorname{Cov}_{\mathrm{t}}(v)) \subset \operatorname{Cov}_{\mathrm{t}}(\gamma v)$.

We further have the following important fact.

Theorem 12.7. *The pushout of the initial covering $\varphi_v : R \to U(v)$ of v is the initial covering $\varphi_{\gamma v} : R \to U(\gamma v)$ of γv. In particular $U(\gamma v) = U(v)_\gamma$.*

---

[8]Recall that $\alpha_{U,\gamma} : U \to U_\gamma$ is the solution of a universal problem.

Proof. Recall that $\mathcal{T}(U(v)) = R \setminus \mathfrak{q}$ and $\mathcal{T}(U(\gamma v)) = R \setminus \mathfrak{q}'$ with $\mathfrak{q} = \operatorname{supp}(v)$ and $\mathfrak{q}' = \operatorname{supp}(\gamma v) = v^{-1}(\mathfrak{p})$. Thus it is pretty obvious that $U(\gamma v) = U(v)_\gamma$. If $a \in R$, we have

$$\gamma_*(\varphi_v)(a) \;=\; \alpha_{U,\gamma}(\varphi_v(a));$$

hence, by Lemma 12.5, $\gamma_*(\varphi_v)(a) = \varphi(a)$ if $v(a) = e\varphi_v(a) \notin \mathfrak{p}$, while $\gamma_*(\varphi_v)(a) = 0$ if $v(a) \in \mathfrak{p}$. These are precisely the values attained by $\varphi_{\gamma v}$. $\square$

We now focus on the restriction

$$\gamma_{*,\mathrm{t}} : \mathrm{Cov}_\mathrm{t}(v) \to \mathrm{Cov}_\mathrm{t}(\gamma v)$$

of γ_* to tangible supervaluations. It maps the top element φ_v of $\mathrm{Cov}_\mathrm{t}(\gamma)$ to the top element $\varphi_{\gamma v}$ of $\mathrm{Cov}_\mathrm{t}(\gamma v)$. But it almost never maps the bottom element $\hat{v}$ of $\mathrm{Cov}_\mathrm{t}(v)$ to the bottom element $\widehat{\gamma v}$ of $\mathrm{Cov}_\mathrm{t}(\gamma v)$, as we will see below.

Our goal now is to exhibit a sublattice of $\mathrm{Cov}_\mathrm{t}(v)$ which maps bijectively onto $\gamma_{*,\mathrm{t}}(\mathrm{Cov}_\mathrm{t}(v))$ under the pushout map $\gamma_{*,\mathrm{t}}$. For that we need a construction of general interest.

In the following *we always assume that $eU = M$ and $\mathcal{T}(U)$ is closed under multiplication.*

Given an ideal $\mathfrak{a}$ of M we introduce the equivalence relation

$$E_{\mathrm{t},\mathfrak{a}} \;:=\; E_{\mathrm{t},\mathfrak{a}}(U) \;:=\; E_\mathrm{t} \cap E(M \setminus \mathfrak{a}),$$

with E_t and $E(M \setminus \mathfrak{a})$ the MFCE-relations defined in Examples 6.4.v and 6.12. Clearly $E_{\mathrm{t},\mathfrak{a}}$ is a ghost separating equivalence relation.

$E := E_{\mathrm{t},\mathfrak{a}}(U)$ has the follows explicit description. Let $x, y \in U$ be given, If $x \in M$, or if $x \in \mathcal{T}(U)$, but $ex \notin \mathfrak{a}$, then $x \sim_E y$ iff $x = y$. If $x \in \mathcal{T}(U)$ and $ex \in \mathfrak{a}$, then $x \sim_E y$ iff $y \in \mathcal{T}(U)$ and $ex = ey$.

Definition 12.8. (a) *We call the supertropical semiring $U/E_{\mathrm{t},\mathfrak{a}}(U)$ consisting of the $E_{\mathrm{t},\mathfrak{a}}(U)$-equivalence classes the **t-collapse (= tangible collapse)** of U over $\mathfrak{a}$ and we denote this semiring by $c_{\mathrm{t},\mathfrak{a}}(U)$.*

(b) *We call the natural semiring homomorphism*

$$\pi_{E_{\mathrm{t},\mathfrak{a}}(U)} : U \to c_{\mathrm{t},\mathfrak{a}}(U)$$

*the **t-collapsing map of** U **over** $\mathfrak{a}$, and we denote this map by $\pi_{\mathrm{t},\mathfrak{a}}$, or $\pi_{\mathrm{t},\mathfrak{a},U}$ if necessary.*

(c) *If $\varphi : R \to U$ is a tangible supervaluation covering v, we call the supervaluation*

$$\varphi/E_{\mathrm{t},\mathfrak{a}}(U) = \pi_{\mathrm{t},\mathfrak{a}} \circ \varphi$$

*the **t-collapse of** φ **over** $\mathfrak{a}$, and we denote this supervaluation by $c_{\mathrm{t},\mathfrak{a}}(\varphi)$.*

(d) *Finally, we say that U is **t-collapsed over** $\mathfrak{a}$, if $\pi_{\mathrm{t},\mathfrak{a}}$ is an isomorphism, for which we abusively write $c_{\mathrm{t},\mathfrak{a}}(U) = U$, and we say that φ is **t-collapsed over** $\mathfrak{a}$ if $c_{\mathrm{t},\mathfrak{a}}(\varphi)) = \varphi$ (which happens iff $c_{\mathrm{t},\mathfrak{a}}(U) = U$, since our supervalutions are assumed to be surjective).*

We describe the semiring $c_{\mathrm{t},\mathfrak{a}}(U)$ more explicitly. Without essential loss of generality we assume that $e\mathcal{T}(U)_0 = M$.

If Z is any subset of M, let $U|_Z$ denote the preimage of Z under the ghost map ν_U,

$$U|_Z \;:=\; \{x \in U \mid ex \in Z\}.$$

Now, if U is t-collapsed over $\mathfrak{a}$, every $z \in U$ has a unique tangible preimage under ν_U. We denote this preimage by $\hat{z}$, and then have

$$U|_\mathfrak{a} \;=\; \mathfrak{a} \,\dot{\cup}\, \hat{\mathfrak{a}}$$

with $\hat{\mathfrak{a}} = \{\hat{z} | z \in \mathfrak{a}\}$.

In general we identify

$$c_{\mathrm{t},\mathfrak{a}}(U)|_{M\setminus\mathfrak{a}} \ = \ U|_{M\setminus\mathfrak{a}}.$$

This makes sense since $[x]_{E_{\mathrm{t},\mathfrak{a}}} = \{x\}$ for any $x \in U|_{M\setminus\mathfrak{a}}$. We then have

$$c_{\mathrm{t},\mathfrak{a}}(U) \ = \ (U|_{M\setminus\mathfrak{a}} \ \cup \ M) \ \dot{\cup} \ \hat{\mathfrak{a}}$$

and

$$U|_{M\setminus\mathfrak{a}} \ \cap \ M \ = \ M \setminus \mathfrak{a}.$$

After these identifications the following is obvious.

Lemma 12.9. (i) *If $x \in U$ then*

$$\pi_{\mathrm{t},\mathfrak{a}}(x) \ = \ \begin{cases} x & \text{if } x \in M \text{ or } ex \notin \mathfrak{a}, \\ (ex)^{\wedge} & \text{if } ex \in \mathfrak{a} \end{cases}$$

(ii) *If $\varphi \in \mathrm{Cov}_{\mathrm{t}}(U)$ and $a \in R$, then*

$$c_{\mathrm{t},\mathfrak{a}}(\varphi)(a) \ = \ \begin{cases} \varphi(a) & \text{if } v(a) \notin \mathfrak{a}, \\ \widehat{v(a)} & \text{if } v(a) \in \mathfrak{a}. \end{cases}$$

□

We now look at the map

$$c_{\mathrm{t},\mathfrak{a}} : \mathrm{Cov}_{\mathrm{t}}(v) \to \mathrm{Cov}_{\mathrm{t}}(v)$$

which sends each $\varphi \in \mathrm{Cov}_{\mathrm{t}}(v)$ to its t-collapse $c_{\mathrm{t},\mathfrak{a}}(\varphi)$ over $\mathfrak{a}$. It is clearly order preserving, and is idempotent, i.e., $(c_{\mathrm{t},\mathfrak{a}})^2 = c_{\mathrm{t},\mathfrak{a}}$. We denote its image by $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$. Its elements are the t-collapsed tangible supervaluation over $\mathfrak{a}$ which cover v.

Using the description of suprema and infima in the complete lattice $\mathrm{Cov}(v)$ in §7, it is an easy matter to verify the following

Proposition 12.10. $\mathrm{Cov}_{\mathrm{t}}(v)$ *is a complete sublattice of* $\mathrm{Cov}(v)$, *and*

$$c_{\mathrm{t},\mathfrak{a}} : \mathrm{Cov}_{\mathrm{t}}(v) \to \mathrm{Cov}_{\mathrm{t}}(v)$$

respects suprema and infima in $\mathrm{Cov}_{\mathrm{t}}(v)$. *Thus, also* $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$ *is a complete sublattice of* $\mathrm{Cov}(v)$.

Remark 12.11. *Independently of this proposition it is clear that* $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$ *is a lower set in* $\mathrm{Cov}_{\mathrm{t}}(v)$ *with top element* $c_{\mathrm{t},\mathfrak{a}}(\varphi_v)$. *It follows that*

$$\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v) \ = \ \{\psi \in \mathrm{Cov}(v) \mid c_{\mathrm{t},\mathfrak{a}}(\varphi_v) \geq \psi \geq \hat{v} \ \}.$$

This proves again that $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{a}}(v)$ *is a complete sublattice of* $\mathrm{Cov}(v)$.

We return to the surjective homomorphism $\gamma : M \to N$ and now choose for $\mathfrak{a}$ the prime ideal $\mathfrak{p} = \gamma^{-1}(0)$ of M.

Proposition 12.12. *Let* $V := c_{\mathrm{t},\mathfrak{p}}(U)$.

(i) *The homomorphism $c : U \to V$ induces an isomorphism $(\pi_{\mathrm{t},\mathfrak{p}})_\gamma : U_\gamma \tilde{\to} V_\gamma$ over N. More precisely, using the identifications from above we have $U_\gamma = V_\gamma$, and then $(\pi_{\mathrm{t},\mathfrak{p}})_\gamma$ is the identity of U_γ.*

(ii) $\alpha_{U,\gamma} = \alpha_{V,\gamma} \circ \pi_{\mathrm{t},\mathfrak{p}}$.

(iii) *If $\varphi \in \mathrm{Cov}_{\mathrm{t}}(v)$ then $\gamma_*(\varphi) = \gamma_*(c_{\mathrm{t},\mathfrak{p}}(\varphi))$.*

Proof. We have the identification

$$\mathcal{T}(U|_{M\setminus\mathfrak{p}}) \ = \ \mathcal{T}(V|_{M\setminus\mathfrak{p}})$$

(see above). On the other hand, $\alpha_{U,\gamma}$ maps $U|_{\mathfrak{p}}$ to $\{0_N\}$, and $\alpha_{V,\gamma}$ maps $V|_{\mathfrak{p}}$ to $\{0_N\}$. Finally

$$\alpha_{U,\gamma}|_M \ = \ \alpha_{V,\gamma}|_M \ = \gamma.$$

Thus it is evident that, under our identifications, $U_\gamma = V_\gamma$ and then $\alpha_{U,\gamma} = \alpha_{V,\gamma} \circ \pi_{\mathrm{t},\mathfrak{p}}$. Reading this equality as

$$\mathrm{id}_{U_\gamma} \circ \alpha_{U,\gamma} \ = \ \alpha_{V,\gamma} \circ \ \pi_{\mathrm{t},\mathfrak{p}}$$

we conclude by Proposition 12.2.a that $(\pi_{\mathrm{t},\mathfrak{p}})_\gamma = \mathrm{id}_{U_\gamma}$. Finally, if $\varphi \in \mathrm{Cov}_{\mathrm{t}}(v)$, then

$$\gamma_*(c_{\mathrm{t},\mathfrak{p}}(\varphi)) \ = \ \alpha_{V,\gamma} \circ c_{\mathrm{t},\mathfrak{p}}(\varphi) \ = \ \alpha_{V,\gamma}(\pi_{\mathrm{t},\mathfrak{p}}(\varphi)) \ = \ \alpha_{U,\gamma}(\varphi) \ = \ \gamma_*(\varphi).$$

□

Lemma 12.13. *Let U, V be supertropical semirings with $eU = eV = M$, and $\lambda : U \to V$ a transmission over M with $\lambda(\mathcal{T}(U)) \subset \mathcal{T}(V)$. Assume further that U is t-collapsed over $\mathfrak{p}$. Finally assume that $\lambda_\gamma : U_\gamma \to V_\gamma$ is injective. Then $\lambda : U \to V$ is injective.*

Proof. The upper square of the of the diagram in Proposition 12.2.a restricts to a commuting square

$$\begin{array}{ccc} \mathcal{T}(U|_{M\setminus\mathfrak{p}}) & \xrightarrow[\mathrm{id}]{\sim} & \mathcal{T}(U_\gamma) \\ {\scriptstyle ``\lambda"}\downarrow & & \downarrow{\scriptstyle ``\lambda_\gamma"} \\ \mathcal{T}(V|_{M\setminus\mathfrak{p}}) & \xrightarrow[\mathrm{id}]{\sim} & \mathcal{T}(V_\gamma) \end{array}$$

Here the vertical arrows are restrictions of the maps λ and λ_γ. The vertical arrow on the right is an injective map by assumption. Thus, also the left vertical arrow is an injective map. The restriction $\lambda|\mathcal{T}(U|_{\mathfrak{p}})$ is a priori forced to be injective, since U is t-collapsed over $\mathfrak{p}$. Finally λ restricts to the identity on M. Thus, λ is injective. □

We now are ready for the main result of this section

Theorem 12.14. *As before assume that $\mathcal{T}(U)$ is closed under multiplication.*

(a) *The pushout map*

$$\gamma_{*,\mathrm{t}} : \mathrm{Cov}_{\mathrm{t}}(v) \ \to \ \mathrm{Cov}_{\mathrm{t}}(\gamma v)$$

restricts to a bijection from $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{p}}(v)$ to $\gamma_*(\mathrm{Cov}_{\mathrm{t}}(\gamma v))$. Consequently $\gamma_*(\mathrm{Cov}_{\mathrm{t}}(\gamma v))$ is a sublattice of $\mathrm{Cov}_{\mathrm{t}}(\gamma v)$ isomorphic to $\mathrm{Cov}_{\mathrm{t},c,\mathfrak{p}}(v)$.

(b) *If $\varphi, \psi \in \mathrm{Cov}_{\mathrm{t}}(v)$ then $\gamma_*(\varphi) = \gamma_*(\psi)$ iff φ and ψ have the same t-collapse over $\mathfrak{p}$.*

Proof. a): Since we know already that $\gamma_*|\,\mathrm{Cov}_{\mathrm{t},c,\mathfrak{p}}(v)$ is a lattice homomorphism (Proposition 12.10), it suffices to verify the following: If $\varphi, \psi \in \mathrm{Cov}_{\mathrm{t}}(v)$ are t-collapsed over $\mathfrak{p}$ and $\varphi \geq \psi$, but $\varphi \neq \psi$, then $\gamma_*(\varphi) \neq \gamma_*(\psi)$.

We have a unique surjective transmission $\lambda := \alpha_{\psi,\varphi} : U \to V$ with $\psi = \lambda\varphi$. This implies $\gamma_*(\psi) = \lambda_\gamma \gamma_*(\varphi)$ by Corollary 12.4. If λ_γ would be an isomorphism then also λ would be an isomorphism by Lemma 12.13 above. But this is not true. Thus λ_γ is not an isomorphism, and this means that $\gamma_*(\psi) \neq \gamma_*(\varphi)$.

b): We know by Proposition 12.12 that $\gamma_*(\varphi) = \gamma_*(c_{\mathrm{t},\mathfrak{p}}(\psi))$. Thus $\gamma_*(\varphi) = \gamma_*(\psi)$ iff $\gamma_*(c_{\mathrm{t},\mathfrak{p}}(\varphi)) = \gamma_*(c_{\mathrm{t},\mathfrak{p}}(\psi))$. By part a) this happens iff $c_{\mathrm{t},\mathfrak{p}}(\varphi) = c_{\mathrm{t},\mathfrak{p}}(\psi)$. □

We turn to the image of the map $\gamma_{*,\mathrm{t}} : \mathrm{Cov_t}(v) \to \mathrm{Cov_t}(\gamma v)$. Here we will put emphasis on strong supervaluations. Thus we now assume in addition that the surjective valuation $v : R \to M$ is strong.

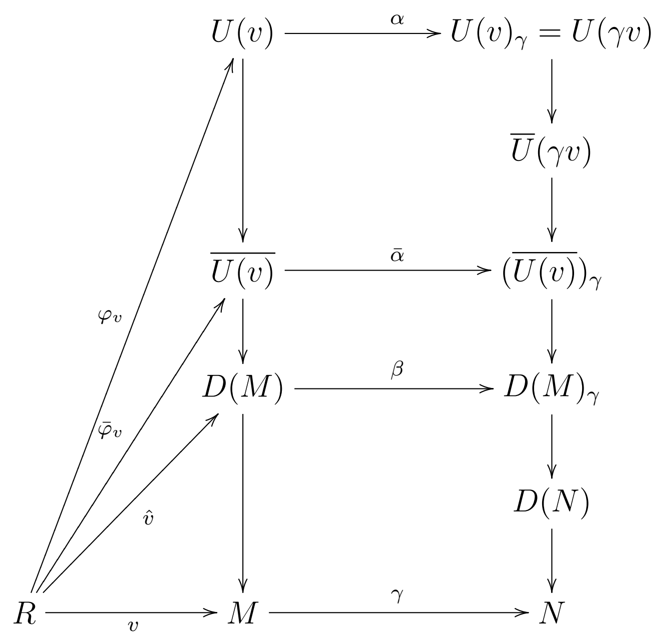
If $\varphi : R \to U$ is a strong supervaluation covering v, then $\gamma_*(\varphi) = \alpha_{U,\gamma} \circ \varphi$ is again a strong supervaluation, as follows from Lemma 11.1.ii and the definition of "strong" (Definition 10.9). Thus

$$\gamma_*(\mathrm{Cov_{t,s}}(\varphi)) \ \subset \mathrm{Cov_{t,s}}(\gamma v).$$

We have seen that $\gamma_*(\varphi_v) = \varphi_{\gamma v}$, but we can only state that the pushout $\gamma_*(\overline{\varphi}_v)$ of the initial strong supervaluation $\overline{\varphi} : R \to \overline{U(v)}$ is dominated by $\overline{\varphi}_{\gamma v} : R \to \overline{U(\gamma v)}$. On the other side, the pushout $\gamma_*(\hat{v})$ of the bottom element $\hat{v} : R \to D(M)$ of both $\mathrm{Cov_{t,s}}(\varphi)$ and $\mathrm{Cov_t}(v)$ dominates $\widehat{\gamma v} : R \to D(N)$. Using the abbreviations

$$\alpha := \alpha_{U(v),\gamma}, \qquad \bar{\alpha} := \alpha_{\overline{U(v)},\gamma}, \qquad \beta := \alpha_{D(M),\gamma},$$

we thus have a commuting diagram



with surjective transmissions over M and N respectively as vertical arrows.

The following questions immediately come to mind.

Questions 12.15.

(1) *Can we expect that* $\overline{\varphi}_{\gamma v} = \gamma_*(\overline{\varphi}_v)$*?*
(2) *Can we expect that* $\widehat{\gamma v} = \gamma_*(\hat{v})$*?*
(3) *Is* $\gamma_*(\mathrm{Cov_t}(v))$ *convex*[9] *in* $\mathrm{Cov_t}(\gamma v)$*?*
(4) *Is* $\gamma_*(\mathrm{Cov_{t,s}}(v))$ *convex in* $\mathrm{Cov_{t,s}}(\gamma v)$*?*

Recall that $\mathrm{Cov_{t,s}}(\gamma v)$ is convex in $\mathrm{Cov_t}(\gamma v)$, and $\mathrm{Cov_t}(\gamma v)$ is convex in $\mathrm{Cov}(\gamma v)$, as we have seen in §11.

Question (2) has a negative answer: If $z \in N \setminus \{0\}$, then the tangible fiber of $\{x \in D(M)_\gamma | ex = z\}$ is the union of the tangible fibers of $D(M)$ over the points of $\gamma^{-1}(z)$,

[9]A subset Y of a poset X is called convex in X if $y \leq x \leq z$ for $y, z \in Y$, $x \in X$ implies that $x \in Y$.

and thus will quite often contain more than one point. The other questions will be answered here completely only in a special case to which we turn now.

Assume that $R \setminus \mathfrak{q}$ is a group under multiplication. Then we can give a very explicit description of the map $\gamma_{*,\mathrm{t}}$, and even γ_*.

Now $M \setminus \{0\} = v(R \setminus \mathfrak{q})$ and $N \setminus \{0\} = \gamma(M \setminus \{0\})$ are groups, i.e., M and N are bipotent semifields. This forces $\mathfrak{p} = 0$ and $\mathfrak{q} = \mathfrak{q}'$.

Since $\mathfrak{p} = 0$ we conclude from Theorem 12.14 and Proposition 12.12 that γ_* is an isomorphism of the lattice $\mathrm{Cov}_{\mathrm{t}}(v)$ onto its image $\gamma_*(\mathrm{Cov}_{\mathrm{t}}(v))$, By §8 the MFCE-relations on $U(v)$ except $E(\nu_U)$ are orbital, hence ghost separating. Thus $\mathrm{Cov}(v) = \mathrm{Cov}_{\mathrm{t}}(v) \cup \{v\}$ (as essentially observed in §11). We have $\gamma_*(v) = \gamma v$, and we conclude that γ_* *is an isomorphism from* $\mathrm{Cov}(v)$ *onto its image.*

We have $M = \Gamma \cup \{0\}$ with Γ an ordered abelian group. Let $\Delta := \gamma^{-1}(1_N)$. This is a convex subgroup of Γ, since $\gamma : M \to N$ is an order preserving monoid homomorphism. The map γ induces an isomorphism from $M/\Delta = \Gamma/\Delta \cup \{0\}$ onto N. In the following we assume without loss of generality that $N = M/\Delta$ and γ is the map $x \mapsto \Delta x$ from M to N. Excluding a trivial case we assume that $\Delta \neq 1$.

Returning to the notation from the end of §11 we have $\mathfrak{o}_v^* = \{a \in R \mid v(a) \in 1_M\}$ and $\mathfrak{o}_{\gamma v}^* = \{a \in R \mid v(a) \in \Delta\}$, further $\mathfrak{m}_v = \{a \in R \mid v(a) < 1_M\}$ and $\mathfrak{m}_{\gamma v} = \{a \in R \mid v(a) < \Delta\}$. $\{v(a) < \Delta$ means $v(a) < \delta$ for every $\delta \in \Delta$.$\}$

If H is a subgroup of $\mathfrak{o}_v^*$ then H is also a subgroup of $\mathfrak{o}_{\gamma v}^*$, since $\mathfrak{o}_v^*$ is a subgroup of $\mathfrak{o}_{\gamma v}^*$. Thus H gives us a transmission

$$\pi_{H,U(v)} : U(v) \ \to \ U(v)/E(H)$$

over M and a transmission

$$\pi_{H,U(\gamma v)} : U(\gamma v) \ \to \ U(\gamma v)/E(H)$$

over N. {Previously both maps sloppily had been denoted by π_H.}

Theorem 12.16. *If H is any subgroup of $\mathfrak{o}_v^*$, then*

(a) $(\pi_{H,U(v)})_\gamma = \pi_{H,U(\gamma v)}$,

(b) $\gamma_*(\varphi_v/H) = \varphi_{\gamma v}/H$.

Proof. a): Let $V := U(v)/E(H)$. We are done by Proposition 12.2.a if we verify that

$$\pi_{H,U(\gamma v)} \circ \alpha_{U(v),\gamma} \ = \ \alpha_{V,\gamma} \circ \pi_{H,U(v)}.$$

This is easily verified by use of Lemma 12.5.

b): We know (Theorem 12.7) that

$$\varphi_{\gamma v} \ = \ \gamma_*(\varphi_v) \ = \ \alpha_{U(v),\gamma} \circ \varphi_v$$

Thus

$$\varphi_{\gamma v}/H \ = \ \pi_{H,U(\gamma v)} \circ \alpha_{U(v),\gamma} \circ \varphi_v.$$

On the other hand

$$\gamma_*(\varphi_v/H) \ = \ \alpha_{V,\gamma}(\varphi_v/H) \ = \ \alpha_{V,\gamma} \circ \pi_{H,U(v)} \circ \varphi_v.$$

By step a) we conclude that indeed

$$\gamma_*(\varphi_v/H) = \varphi_{\gamma v}/H.$$

□

We learned before (§8, §11) that the elements φ of $\mathrm{Cov_t}$ correspond uniquely with the subgroups H of $\mathfrak{o}_v^*$ via $\varphi = \varphi_v/H$, and now conclude by Theorem 12.16 that

$$\gamma_*(\mathrm{Cov_t}(v)) \ = \ \{\varphi_{\gamma v}/H \mid H \leq \mathfrak{o}_v^*\}.$$

(" $\leq$ " means subgroup). On the other hand

$$\mathrm{Cov_t}(\gamma v) \ = \ \{\varphi_{\gamma v}/H \mid H \leq \mathfrak{o}_{\gamma v}^*\}.$$

Thus, $\gamma_*(\mathrm{Cov_t}(v))$ is an upper set of the complete lattice $\mathrm{Cov_t}(\gamma v)$ with bottom element

$$\gamma_*(\hat{v}) \ = \ \varphi_{\gamma v}/\mathfrak{o}_v^*.$$

This element is definitely different from

$$\widehat{\gamma v} \ = \ \varphi_{\gamma v}/\mathfrak{o}_{\gamma v}^*,$$

since $\mathfrak{o}_{\gamma v}^*/\mathfrak{o}_v^* \cong \Delta$. Thus question 12.15.(2) has a negative answer (which we know already), while question 12.15.(3) has a positive answer.

How about question 12.15.(1)? The top element of $\mathrm{Cov_{t,s}}(v)$ is $\overline{\varphi}_v$. We saw in §11 that $\overline{\varphi}_v = \varphi_v/1 + \mathfrak{m}_v$, and now conclude by Theorem 12.16 that

$$\gamma_*(\overline{\varphi}_v) \ = \ \varphi_{\gamma v}/1 + \mathfrak{m}_v.$$

But

$$\overline{\varphi}_{\gamma v} \ = \ \varphi_{\gamma v}/1 + \mathfrak{m}_{\gamma v},$$

and $\mathfrak{m}_{\gamma v}$ is definitely smaller than $\mathfrak{m}_v$. Thus $\overline{\varphi}_{\gamma v} \gneqq \gamma_*(\overline{\varphi}_v)$. Question 12.15.(1) has a negative answer.

Returning to the general situation, but still with $v : R \to M$ strong, we should expect that $\overline{\varphi}_{\gamma v} \gneqq \gamma_*(\overline{\varphi}_v)$ except in rather pathological cases. Indeed, it seems often possible to pass from $v : R \to M$ to a strong valuation $\tilde{v} : \widetilde{R} \to \widetilde{M}$, with $\widetilde{R}$ a semifield by a localization process (which we did not discuss), and to argue in $\mathrm{Cov_t}(\tilde{v})$. We leave this matter to a future publication [IKR].

Concerning applications, the strong supervaluations seem to be more important than the others. But the fact that $\gamma_*(\overline{\varphi}_v)$ differs from $\overline{\varphi}_{\gamma v}$, while $\gamma_*(\varphi_v) = \varphi_{\gamma v}$, indicates that it would not be advisable in supervaluation theory to restrict the study to strong supervaluations from start, as said already in the Introduction.

13. Epilog: iq-valuations on polynomial semirings and related supervaluations.

Since the semiring of polynomials over a supertropical domain is no longer supertropical (or analogously, the semiring of polynomials over a bipotent semiring is no longer bipotent), we would like a theory generalizing valuations to maps with values in these polynomial semirings. Unfortunately, the target is no longer an ordered group (and is not even an ordered monoid), so the theory must be framed in a somewhat broader context.

In this - compare to its goals- short section, we formulate some concepts of this paper in the more general context of monoids with a supremum, instead of ordered monoids, and show for example how this encompasses Kapranov's Lemma. Recall that an operation $a \vee b$ on a set S is called a **sup** if it has a distinguished element 0 and satisfies the following properties for all $a, b, c \in S$:

(1) $0 \vee a = a$;
(2) $a \vee b = b \vee a$;

(3) $a \vee a = a$;
(4) $a \vee (b \vee c) = (a \vee b) \vee c$.

In this case, we can define a partial order on S by defining $a \leq b$ when $a \vee b = b$. Then the following properties are immediate for all $a, b, c \in S$:

(a) $0 \leq a$;
(b) $a \vee b \geq a$ and $a \vee b \geq b$;
(c) if $a \leq c$ and $b \leq c$, then $a \vee b \leq c$. (Indeed, if $a \vee c = c$ and $b \vee c = c$, then $(a \vee b) \vee c = (a \vee c) \vee (b \vee c) = c \vee c = c$.)

We also say that a given sup $x \vee y$ on a monoid M is **compatible** with M if $a(x \vee y) = ax \vee ay$ for all $a, x, y \in M$.

In order to axiomatize this in the language of semirings, we recall that an **idempotent semiring** R satisfies the property that $x + x = x$ for all $x \in R$.

Proposition 13.1.

(i) *Every idempotent semiring R can be viewed as a multiplicative monoid with a compatible sup $\vee$ defined by*
$$x \vee y := x + y.$$
(ii) *Conversely, given a monoid M with a compatible sup, we can define an idempotent semiring structure on M, with the same multiplication, and with addition given by $x + y := x \vee y$.*

Proof. All of the other verifications are immediate. □

Remark 13.2. *If R is an idempotent semiring, then so is the polynomial semiring $R[\lambda]$ as well as the matrix semiring $M_n(R)$.*

Both of these assertions fail when we substitute "bipotent" for "idempotent." Thus, it makes sense to pass to idempotent semirings when studying polynomials and matrices. In the case of semifields, we actually have a lattice structure.

Proposition 13.3. *If R is a semifield, where $\vee$ is given by addition (as in Proposition 13.1), then there is a compatible inf relation $\wedge$ given by $x \wedge y := \frac{xy}{x+y}$ (taking $0 \wedge 0 = 0$), thereby making $(R, \vee, \wedge)$ a distributive lattice satisfying*
$$(x \vee y)(x \wedge y) = xy, \quad \forall x, y \in R. \tag{13.1}$$

Proof. Property (13.1) follows at once from the definitions, and implies that $a(x \wedge y) = ax \wedge ay$, as well as associativity of $\wedge$. To check distributivity, we need to check
$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$
Since $\leq$ is clear, we only check $\geq$, and also may assume $x, y, z \neq 0$. Now
$$\begin{aligned} (x \wedge y) \vee z &= \frac{xy}{x+y} + z \\ &\geq \frac{xy}{x+y+z} + z\frac{x+y+z}{x+y+z} \\ &= \frac{(x+z)(y+z)}{x+y+z} = \frac{(x+z)(y+z)}{(x+z)+(y+z)} \\ &= (x \vee z) \wedge (y \vee z). \end{aligned} \tag{13.2}$$
□

Having the translation of the sup relation to semirings at hand, we are ready to reformulate some of the results of this paper. But first it is instructive to introduce a parallel of the ghost surpassing relation.

Definition 13.4. $y \models x \;\Leftrightarrow\; \exists a \in R$ *with* $y = x + a$.

Clearly, $\models$ is a transitive binary relation on R.

Definition 13.5. *R is an **upper-bound** semiring, written ub-semiring, if the relation $\models$ is anti-symmetric; i.e.,*

$$x \models y \text{ and } y \models x \quad \Leftrightarrow \quad x = y.$$

The reason for this terminology is that now the relation $\models$ gives a partial ordering on the set R ($a \leq b$ iff $b \models a$), and $x + y$ is an upper bound of x, y in this ordering.

Remark 13.6.

(i) *The condition that a semiring R is ub can be rephrased as follows:*
For any $a, b, x \in R$, if $x + a + b = x$, then $x + a = x$.

(ii) *Any ub-semiring R has the property that $a + b = 0$ implies $a = b = 0$, by (i). (Take $x = 0$.)*

Proposition 13.7. *Any idempotent semiring is an ub-semiring.*

Proof. If $x + a + b = x$, then

$$x + a = (x + a + b) + a = x + a + b = x.$$

□

Lemma 13.8. *Any ub-semiring satisfies the following properties:*

(i) *If $x + a + b = x$, then $x + a = x$.*

(ii) *If $a + b = 0$, then $a = b = 0$.*

Proof. (i): Obviously $x + a \models x$, and the hypothesis implies $x = (x + a) + b \models x + a$, so $x + a = x$ by anti-symmetry.

(ii): Take $x = 0$ in (i). □

If R is any semiring, let $R[\lambda] = R[\lambda_1, \dots, \lambda_n]$ denote the polynomial semiring over R in a set of variables $\lambda = (\lambda_1, \dots, \lambda_n)$.

Proposition 13.9. *Every supertropical semiring U is upper bound, and $U[\lambda_1, \dots, \lambda_n]$ is upper bound for every n.*

Proof. We have to check the condition in Remark 13.6.i. Let $x, a, b \in U$ be given with $x+a+b = x$. We have to verify that $x+a = x$. Multiplying by e we obtain $ex+ea+eb = ex$, hence $ea \leq ex$ and $eb \leq ex$. If $ea < ex$, then $x + a = x$ right away.If $eb < ex$, then $x+b = x$, hence $x = x + a + b = x + a$ again. There remains the case that $ea = eb = ex$. Now $x + a + b = ex$, hence x is ghost, and $x + a = ex = x$ again. This proves that U is ub.

Let now $f, g, h \in U[\lambda_1, \dots, \lambda_n]$ be given with $f + g + h = f$. We write $f = \sum \alpha_i \lambda^i$, $g = \sum \beta_i \lambda^i$, $h = \sum \gamma_i \lambda^i$. Then $\alpha_i + \beta_i + \gamma_i = \alpha_i$ for every i, and we conclude that $\alpha_i + \beta_i = \alpha_i$ for every i, hence $f + g = f$, as desired. □

The reason we want to consider the idempotent semiring $M[\lambda]$ is that we want to extend any m-valuation $v : R \to M$ to the map $\tilde{v} : R[\lambda] \to M[\lambda]$, where we define

$$\tilde{v}\left(\sum_i \alpha_i \lambda_1^{i_1} \dots \lambda_n^{i_n}\right) = \sum_i v(\alpha_i)\lambda_1^{i_1} \dots \lambda_n^{i_n}. \tag{13.3}$$

Since $M[\lambda]$ is no longer bipotent in the natural way, we would like to generalize Definition 2.1 to permit valuations to idempotent semirings.

Unfortunately, $\tilde{v}$ as defined in (13.3) need not satisfy property V3 of Definition 2.1, since $\tilde{v}(fg)$ could differ from $\tilde{v}(f)\tilde{v}(g)$. Indeed, if $f = \sum_i \alpha_i \lambda^i$ and $g = \sum_j \beta_j \lambda^j$, with $i = (i_1, \dots, i_n)$ and $j = (j_1, \dots, j_n)$, then writing $fg = \sum_k \left(\sum_{i+j=k} \alpha_i \beta_j\right) \lambda^k$, we have

$$\begin{aligned}\tilde{v}(fg) &= \sum_k v\Big(\sum_{i+j=k} \alpha_i\beta_j\Big)\lambda^k \\ &\leq \sum_k \sum_{i+j=k} v(\alpha_i)v(\beta_j)\lambda^k \\ &= \left(\sum v(\alpha_i)\lambda^i\right)\left(\sum v(\beta_j)\lambda^j\right),\end{aligned}$$

where there could be strict inequality. Accordingly, we need a weaker notion:

Definition 13.10. *An iq-**valuation** (= idempotent monoid quasi-valuation) on a semiring R is a map $v : R \to M$ into a (commutative) idempotent semiring $M \neq \{0\}$ with the following properties:*

$$\begin{aligned}&IQV1 : v(0) = 0,\\ &IQV2 : v(1) = 1,\\ &IQV3 : v(xy) \leq v(x)v(y) \quad \forall x, y \in R,\\ &IQV4 : v(x + y) \leq v(x) + v(y) \quad \forall x, y \in R.\end{aligned}$$

{NB:Here we use the partial order introduced above following Definition 13.5.}

Proposition 13.11. *Suppose M is a bipotent semiring and $v : R \to M$ is an m-valuation.*

(i) *Then the map $\tilde{v} : R[\lambda] \to M[\lambda]$ given above is an iq-valuation.*
(ii) *For any given $a \in M^n$, the map $\varepsilon_a \circ \tilde{v} : R[\lambda] \to M$ is again an iq-valuation. {Here ε_a denotes the evaluation map $f(\lambda) \mapsto f(a)$, as in the previous sections.}*

□

If v is strong we can do better.

Theorem 13.12. *Assume that $v : R \to M$ is a surjective strong m-valuation. Then, for any $a \in M^n$, $\varepsilon_a \circ \tilde{v} : R[\lambda] \to M$ is again a strong m-valuation.*

Proof. By an easy induction we restrict to the case of $n = 1$. Given $f = \sum_i \alpha_i \lambda^i$, $g = \sum_j \beta_i \lambda^i$ in $R[\lambda]$ we have to verify the following:

(1) $\varepsilon_a \tilde{v}(fg) = \varepsilon_a \tilde{v}(f) \cdot \varepsilon_a \tilde{v}(g)$;
(2) If $\varepsilon_a \tilde{v}(f) < \varepsilon_a \tilde{v}(g)$, then $\varepsilon_a \tilde{v}(f + g) = \varepsilon_a \tilde{v}(g)$.

(1): We know already that

$$\varepsilon_a\tilde{v}(fg) \le \varepsilon_a\tilde{v}(f)\cdot\varepsilon_a\tilde{v}(g).$$

Due to the bipotence of M we have smallest indices k and ℓ such that

$$\varepsilon_a\tilde{v}(f) = \sum_i v(\alpha_i)a^i = v(\alpha_k)a^k,$$
$$\varepsilon_a\tilde{v}(g) = \sum_j v(\beta_i)a^j = v(\beta_\ell)a^\ell.$$

We chose some $c \in R$ with $v(c) = a$. Since v is strong and k,ℓ have been chosen minimally we have

$$v\left(\sum_{i+j=k+\ell}\alpha_i c^i\beta_j c^j\right) = v(\alpha_k c^k\beta_\ell c^\ell) = \varepsilon_a\tilde{v}(f)\cdot\varepsilon_a\tilde{v}(g).$$

Thus

$$\begin{aligned}\varepsilon_a\tilde{v}(fg) &= \sum_r v\left(\sum_{i+j=r}\alpha_i c^i\beta_j c^j\right)\\ &\ge \sum_{i+j=k+\ell} v\left(\sum_{i+j=k}\alpha_i c^i\beta_j c^j\right)\\ &= \varepsilon_a\tilde{v}(f)\cdot\varepsilon_a\tilde{v}(g).\end{aligned}$$

We conclude that

$$\varepsilon_a\tilde{v}(fg) = \varepsilon_a\tilde{v}(f)\cdot\varepsilon_a\tilde{v}(g).$$

(2): Assume that $\varepsilon_a\tilde{v}(f) < \varepsilon_a\tilde{v}(g)$. Using the same k,ℓ, and c as before we have for all i

$$v(\alpha_i c^i) < v(\beta_\ell c^\ell),$$
$$v(\beta_i c^i) \le v(\beta_\ell c^\ell).$$

Now

$$\varepsilon_a\tilde{v}(f+g) = \sum_i v[(\alpha_i+\beta_i)c^i],$$

and $v[(\alpha_i+\beta_i)c^i] \le v(\beta_\ell c^\ell)$ for all i, with

$$v[(\alpha_\ell+\beta_\ell)c^\ell] = v(\beta_\ell c^\ell).$$

Thus,

$$\varepsilon_a\tilde{v}(f+g) = v(\beta_\ell c^\ell) = \varepsilon_a\tilde{v}(g).$$

□

In particular, we could take v to be the natural valuation on the field of Puiseux series with rational exponents, as used in [G], or with real exponents as introduced above in §11.

Let us formulate the analogue of Definition 4.1 in the realm of semirings with ghosts.

Definition 13.13. *An iq-**supervaluation** on a semiring R is a map $\varphi : R \to U$ from R to a ub-semiring U with ghosts, satisfying the following properties.*

$$IQSV1 : \varphi(0) = 0,$$
$$IQSV2 : \varphi(1) = 1,$$
$$IQSV3 : \forall a, b \in R : \varphi(ab) \leq \varphi(a)\varphi(b),$$
$$IQSV4 : \forall a, b \in R : e\varphi(a + b) \leq e(\varphi(a) + \varphi(b)).$$

Here again we use the ordering given by the relation $\underset{\text{gs}}{\models}$. This is justified by Proposition 13.9.

We are ready for the main purpose of this epilog.

Theorem 13.14. *Assume that $\varphi : R \to U$ is a surjective strong supervaluation, and*

$$v : R \to eU = M$$

is the strong m-valuation covered by φ. Let $a = (a_1, \dots, a_n) \in U^n$ be given, and let $b := (ea_1, \dots, ea_n) \in M^n$.

(i) *φ can be extended to an iq-supervaluation $\widetilde{\varphi} : R[\lambda] \to U[\lambda]$ by the formula*

$$\widetilde{\varphi}\left(\sum_i \alpha_i \lambda^i\right) = \sum_i \varphi(\alpha_i)\lambda^i.$$

(ii) *$\varepsilon_a \circ \widetilde{\varphi} : R[\lambda] \to U$ is a strong supervaluation. It covers the (strong) valuation $\varepsilon_b \circ \tilde{v} : R[\lambda] \to M$.*

Proof. (i): If $a, b \in R$ then we know from §10 that $\varphi(a) + \varphi(b) \underset{\text{gs}}{\models} \varphi(a + b)$. This implies $\varphi(a) + \varphi(b) \models \varphi(a + b)$, i.e.

$$\varphi(a + b) \leq \varphi(a) + \varphi(b). \qquad (*)$$

An argument parallel to the one before Definition 13.10 now tells us that for $f, g \in R[\lambda]$ we have

$$\widetilde{\varphi}(fg) \leq \widetilde{\varphi}(f) \cdot \widetilde{\varphi}(g).$$

Clearly $\widetilde{\varphi}$ extends φ, in particular $\widetilde{\varphi}(0) = 0$, $\widetilde{\varphi}(1) = 1$. From $(*)$ it is also obvious that $\widetilde{\varphi}(f + g) \leq \widetilde{\varphi}(f) + \widetilde{\varphi}(g)$, hence

$$e\widetilde{\varphi}(f + g) \leq e\widetilde{\varphi}(f) + e\widetilde{\varphi}(g).$$

Thus, $\widetilde{\varphi}$ is an iq-supervaluation. Clearly $e\widetilde{\varphi}(f) = \tilde{v}(f)$ for all $f \in R[\lambda]$. {By the way, this gives us again that $e\widetilde{\varphi}(f + g) \leq e\widetilde{\varphi}(f) + e\widetilde{\varphi}(g)$.}

(ii): Again we restrict to the case of $n = 1$ by an easy induction. It is pretty obvious that $\varepsilon_a\widetilde{\varphi} : R[\lambda] \to U$ obeys the rules SV1, SV2, SV4 from §4 (Definition 4.1), and $e \cdot \varepsilon_a\widetilde{\varphi}(f) = \varepsilon_b\tilde{v}(f)$ for every $f \in R[\lambda]$. Given $f = \sum_i \alpha_i\lambda^i$, $g = \sum_i \beta_i\lambda^i$ in $R[\lambda]$ it remains to prove the following:

(1) $\varepsilon_a\widetilde{\varphi}(fg) = \varepsilon_a\widetilde{\varphi}(f) \cdot \varepsilon_a\widetilde{\varphi}(g)$,
(2) If $\varepsilon_a\widetilde{\varphi}(f) \leq \varepsilon_a\widetilde{\varphi}(g)$ then $\varepsilon_a\widetilde{\varphi}(f + g) = \varepsilon_a\widetilde{\varphi}(g)$.

(1): Let k, ℓ be the minimal indices such that

$$e\sum_i \varphi(\alpha_i)a^i = e\varphi(\alpha_k)a^k = e\varepsilon_a\widetilde{\varphi}(f), \qquad (**)$$

$$e\sum_i \varphi(\beta_i)a^i = e\varphi(\beta_\ell)a^\ell = e\varepsilon_a\widetilde{\varphi}(g), \qquad (***)$$

(as in the proof of Theorem 13.12). We know by Theorem 13.12 that

$$e(\varepsilon_a \circ \widetilde{\varphi})(fg) = e\varphi(\alpha_k)a^k \cdot e\varphi(\beta_\ell)a^\ell = e(\varepsilon_a \circ \widetilde{\varphi})(f) \cdot e(\varepsilon_a \circ \widetilde{\varphi})(g).$$

We chose some $c \in R$ with $\varphi(c) = a$. Using $(*)$ we obtain

$$\begin{aligned}(\varepsilon_a \circ \widetilde{\varphi})(fg) &= \sum_r \varphi\left(\sum_{i+j=r} \alpha_i\beta_j\right) a^r \\ &= \sum_r \varphi\left(\sum_{i+j=r} \alpha_i c^i \cdot \beta_j c^j\right) \\ &\leq \sum_r \sum_{i+j=r} \varphi(\alpha_i c^i) \cdot \varphi(\beta_j c^j) \\ &= \sum_{i,j} \varphi(\alpha_i)a^i \cdot \varphi(\beta_j)a^j.\end{aligned}$$

There is a single ν-dominating term in this sum iff there is a single ν-dominating term on the left of $(**)$ and of $(***)$, so we conclude that

$$\varepsilon_a\widetilde{\varphi}(fg) = \varepsilon_a\widetilde{\varphi}(f) \cdot \varepsilon_a\widetilde{\varphi}(g)$$

in all cases, using the fact that tangible elements x, y of U with $x \leq y$, $ex = ey$ are equal.

(2): This can be proved in the way analogous to claim (2) in the proof of Theorem 13.12. □

Thus, for U a supertropical semiring, the evaluation map returns us from iq-supervaluations with values in $U[\lambda]$ to the firmer ground of supervaluations.

Looking again at Theorem 10.11 we realize now that the theorem gives pleasant examples of pairs of supervaluations which obey a "GS-relation" in the following sense.

Definition 13.15. *If $\rho : A \to V$ and $\sigma : A \to V$ are supervaluations on a semiring A with values in the same supertropical semiring V, then we say that ρ* ***surpasses*** *σ* ***by ghost****, and write $\rho \underset{\text{gs}}{\models} \sigma$, if $\rho(a) \underset{\text{gs}}{\models} \sigma(a)$ for every $a \in A$.*

In this terminology Theorem 10.11 reads as follows:

Theorem 13.16. *Let $\varphi : R \to U$ be a strong supervaluation. Then for any $a \in R^n$ the supervaluation $\varepsilon_{\varphi(a)} \circ \tilde{\varphi} : R[\lambda_1, \dots, \lambda_n] \to U$ surpasses the supervaluation $\varphi \circ \varepsilon_a : R[\lambda_1, \dots, \lambda_n] \to U$ by ghost.*

Of course, we should look for other examples of pairs of supervaluations $\rho : A \to V$ and $\sigma : A \to V$ with $\rho \underset{\text{gs}}{\models} \sigma$. Here the "classical" case that A is a semifield, or even a field, and eV is cancellative, is perhaps not the most interesting one. Indeed, for such pairs ρ, σ we have $e\rho(a) \geq e\sigma(a)$ for every $a \in A$, and this forces $e\rho(a) = e\sigma(a)$ since for $a \neq 0$ also $e\rho(a^{-1}) \geq e\sigma(a^{-1})$. Thus ρ and σ cover the same valuation $e\rho = e\sigma : A \to eV$. But for the pairs occurring in Theorem 13.16, where A is a polynomial semiring, the valuation $e\rho$ and $e\sigma$ usually will have even different support, and ρ can be a very interesting "perturbation" of σ by ghosts.

The phenomenon of "surpassing by ghost" for supervaluations shows clearly the importance of studying valuations and supervaluations on semirings instead of just semifields. We defer a thorough study of the GS-relation for supervaluations to a future occasion.

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