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ZUR IZHAKIAN AND JOHN RHODES

New Representations of Matroids and Generalizations

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NEW REPRESENTATIONS OF MATROIDS AND GENERALIZATIONS

ZUR IZHAKIAN AND JOHN RHODES

Abstract. We extend the notion of matroid representations by matrices over fields by considering new representations of matroids by matrices over finite semirings, more precisely over the boolean and the superboolean semirings. This idea of representations is naturally generalized to include hereditary collections (also known as abstract simplicial complexes). We show that a matroid that can be directly decomposed as matroids, each of which is representable over a field, has a boolean representation, and more generally that any arbitrary hereditary collection is superboolean-representable.

1. Introduction

Traditionally, matroids have been represented by using matrices defined over fields [3, 19, 21, 27, 31, 33, 34], mainly finite fields [25, 35], or partial fields [28]; matroids that have such a representation are called field-representable. Over the years much work has been done to specify families of matroids that are field-representable, especially with respect to specifying the characteristic field used for constructing a matroid representation. It is well known, however, that not every matroid is field-representable. One of the most celebrated examples of such a non-representable matroid over any field is the direct sum of the Fano and the non-Fano matroids (see [20, Corollary 5.4.]).

In this paper we introduce the idea of replacing the customary field structure used for representations of matroids, and consider instead representations of matroids by matrices over semirings; in particular over a certain 3-element supertropical semiring [10], we call the *superboolean semiring* $\mathbb{SB}$. This semiring is a "cover" of the boolean semiring, defined over the element set $\mathbb{SB} := \{1, 1^\nu, 0\}$, and its arithmetic is a modification of the familiar boolean algebra (see §3). Despite the absence of subtraction, the superboolean structure allows natural algebraic analogs of classical notions (such as dependence of vectors and singularity of matrices) that are very important for a representation theory. These notions lead naturally to the key setting of *vector hereditary collections* (cf. Definition 4.3) which are at the heart of our representation approach.

A matroid that has a representation by a superboolean matrix (i.e., is isomorphic to a vector hereditary collection) is said to be superboolean-representable. Using this concept, we show that all matroids are superboolean-representable and, more generally, that any finite hereditary collection, also known as abstract simplicial complex [22, 23, 29], is superboolean-representable.

Theorem 4.6. *Any hereditary collection is superboolean-representable.*

The proof of this theorem shows an explicit simple construction of such superboolean representations.

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Focusing on boolean representations, which are, by definition, those representations determined by matrices having only entries $1, 0$, we prove:

Theorem 5.4. *Any field-representable matroid is also boolean-representable.*

We also provide an explicit algorithm to produce the matroid's boolean-representation from its field representation. More generally, we extend this result to obtain the following:

Theorem 5.5. *Matroids that are directly decomposable into field-representable matroids also have boolean representation.*

By using this representation approach, matroids that are not representable over fields, for example, as mentioned above, the direct sum of the Fano matroid and the non-Fano matroid, can be represented by boolean matrices, cf. §5.3.

As expected, these new representation ideas and their development in this paper pave the way to a new set of open questions (stated in §6).

The superboolean semiring is perhaps the simplest instance of a supertropical semiring. The appendixes indicates the generalization of these representation ideas to matroid representations by matrices that take place over an arbitrary supertropical semiring [10]. A major example for such a semiring is the extended tropical semiring [8], which is a "cover" of the standard (max-plus) tropical semiring [1, 5, 7, 26], having a much richer algebraic structure that allows the carry of systematic theory of tropical linear algebra [8, 11, 12, 13, 14].

Using a trivial embedding of the superboolean semiring in any supertropical semiring, we conclude that any hereditary collection is "super-regular":

Corollary. *Every hereditary collection is R-representable, for any supertropical semiring R.*

(See Appendix B.)

This paper presents only the preliminary results on boolean and superboolean representations of matroids and finite abstract simplicial complexes. Further results will be developed in future.

2. Hereditary collections and simplicial complexes

2.1. **Hereditary collections.** Throughout this paper we always assume that the **ground set**, denoted by E, is a fixed finite set. We write $|E|$ for the cardinality of E and $\mathrm{Pw}(E)$ for the **power set** of E, i.e., the set of all subsets (including the empty set $\emptyset$) of E. In what follows, unless otherwise is specified, we always assume that $|E| = n$, and thus have $|\mathrm{Pw}(E)| = 2^n$. Subsets of E of cardinality k are termed k-sets.

We use [19], [20], and [21] as general references, especially in regard to matroid theory (matroids are presented in §2.3 below).

Definition 2.1. *Let E be a set and let $\mathcal{H} \subseteq \mathrm{Pw}(E)$ be an nonempty collection of subsets J of E. The nonempty collection $\mathcal{H}$ is called **hereditary** if every subset J' of any $J \in \mathcal{H}$ is also in $\mathcal{H}$, more precisely:*

HT1:	*$\mathcal{H}$ is nonempty,*
HT2:	$J' \subseteq J,\ J \in \mathcal{H} \ \Rightarrow\ J' \in \mathcal{H}.$

*(Hence, the empty set $\emptyset$ is also in $\mathcal{H}$.) The pair $\mathscr{H} := (E, \mathcal{H})$, with $\mathcal{H}$ hereditary over E, is called a **hereditary collection**.*

Hereditary collections are also known in the literature as **abstract simplicial complexes** [22, 23, 29].

The members of the collection $\mathcal{H}$ are called the **independent subsets** of E, and therefore the empty set is considered independent. A subset $J \subseteq E$ which is not contained in $\mathcal{H}$ is called **dependent**. We denote the collection of dependent subsets of E by

$$\mathcal{H}^{c} := \{X \subseteq E : X \notin \mathcal{H}\},$$

i.e., $\mathcal{H}^{c} = \mathrm{Pw}(E) \setminus \mathcal{H}$. (Clearly, $\emptyset \notin \mathcal{H}^{c}$.)

A maximal independent subset (with respect to inclusion) of $\mathscr{H}$ is called a **basis** of the hereditary collection $\mathscr{H}$. The set of all bases of $\mathscr{H}$ is denoted as $\mathcal{B}(\mathscr{H}) \subseteq \mathcal{H}$ and termed the **basis set** of the hereditary collection $\mathscr{H}$. Clearly, $\mathcal{B}(\mathscr{H})$ is canonically defined and by Axiom HT2 determines the hereditary collection $\mathscr{H}$ uniquely. Note that the family $\mathscr{H}_t := (E, \mathcal{H}_t)$ of hereditary collections with fixed ground set E and $\mathcal{H}_t$ varying is in 1:1 correspondence with the anti-chains of the lattice $(\mathrm{Pw}(E), \subseteq)$, cf. [2], given by $\mathscr{H}_t \to \mathcal{B}(\mathscr{H}_t)$.

A minimal subset (with respect to inclusion) of the collection $\mathcal{H}^{\mathrm{c}}$ of the dependent subsets of E is called a **circuit**. We denote the collection of all circuits of a hereditary collection $\mathscr{H}$ by $\mathcal{C}(\mathscr{H})$, i.e., $\mathcal{C}(\mathscr{H}) \subseteq \mathcal{H}^{\mathrm{c}}$. Then, the family $\mathscr{H}_t := (E, \mathcal{H}_t)$ with fixed ground set E and $\mathcal{H}_t$ varying is in 1:1 correspondence with the anti-chains of the lattice $(\mathrm{Pw}(E), \subseteq)$ given now by $\mathscr{H}_t \to \mathcal{C}(\mathscr{H}_t)$.

The **rank** $\mathrm{rk}(\mathscr{H})$ of a hereditary collection $\mathscr{H}$ is defined to be the cardinality of the largest member of the basis set $\mathcal{B}(\mathscr{H})$ of $\mathscr{H}$:

$$\mathrm{rk}(\mathscr{H}) := \max\{|B| \ : \ B \in \mathcal{B}(\mathscr{H})\}.$$

In particular, one always has $0 \leq \mathrm{rk}(\mathscr{H}) \leq n$, and $\mathrm{rk}(\mathscr{H}) = 0$ iff $\mathcal{H} = \{\emptyset\}$.

Example 2.2. *Let us start with some elementary examples of hereditary collections.*

(a) $\mathscr{H} = (E, \{\emptyset\})$ *is a hereditary collection of rank* 0. *(This is a matroid, to be defined below in Definition 2.8.)*

(b) $\mathscr{H} = (E, \mathrm{Pw}(E))$ *is a hereditary collection (also a matroid) whose basis set contains only the set* E, *i.e.,* $\mathcal{B}(\mathscr{H}) = \{E\}$, *and thus has rank* n.

(c) *The **uniform hereditary collection** (also a matroid)* $U_{m,n} := (E, \mathcal{H}_{m,n})$, *with* $0 \leq m \leq n$, $|E| = n$, *is defined to have the collection of independent subsets*

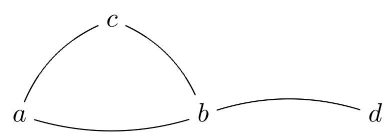
$$\mathcal{H}_{m,n} := \{X \subseteq E : |X| \leq m\},$$

and has rank m.

Notice that the hereditary collections in (a) and (b) above can be written in this notation as $U_{0,n} = (E, \{\emptyset\})$ *and* $U_{n,n} = (E, \mathrm{Pw}(E))$, *respectively. We also have* $U_{n-1,n} = (E, \mathrm{Pw}(E) \setminus \{E\})$.

(d) *Let* $E = \{1, 2, 3, 4\}$ *and let* $\mathscr{H}$ *be the hereditary collection having the bases* $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{1, 4\}$. *Hence, all the 2-subsets of* E *are independent and are members of* $\mathcal{H}$. *(This example is not a matroid.)*

(e) *Consider the hereditary collection over the ground set* $E = \{a, b, c, d\}$ *with the bases* $\{a, b\}$, $\{b, c\}$,$\{a, c\}$, *and* $\{b, d\}$, *corresponding to the edges of the diagram*



(This example is not a matroid.)

The above examples provide some typical cases of hereditary collections satisfying additional properties, to be discussed later.

Definition 2.3. *Hereditary collections* $\mathscr{H}_1 = (E_1, \mathcal{H}_1)$ *and* $\mathscr{H}_2 = (E_2, \mathcal{H}_2)$ *are said to be **isomorphic** if there exits a bijective map* $\varphi : E_1 \to E_2$ *that respects dependence; that is*

$$\varphi(X_1) \in \mathcal{H}_2 \ \Leftrightarrow \ X_1 \in \mathcal{H}_1,$$

for any $X_1 \subseteq E_1$.

Definition 2.4. *The **direct sum** of two hereditary collections* $\mathscr{H}_1 = (E_1, \mathcal{H}_1)$ *and* $\mathscr{H}_2 = (E_2, \mathcal{H}_2)$, *with disjoint nonempty ground sets* E_1 *and* E_2, *is*

$$\mathscr{H}_1 \oplus \mathscr{H}_2 := (E_1 \cup E_2, \{J_1 \dot{\cup} J_2 : J_1 \in \mathcal{H}_1, J_2 \in \mathcal{H}_2\}).$$

A hereditary collection $\mathscr{H}$ is ***decomposable*** *if it can be written as a direct sum $\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_\ell$ of some hereditary collections $\mathscr{H}_i$ with disjoint nonempty ground sets E_i's, otherwise $\mathscr{H}$ is said to be* ***indecomposable****.*

2.2. **Point replacement.** We will impose various additional axioms on hereditary collections.

In what follows, to simplify notation, given a subset $X \subseteq E$, and elements $x \in X$ and $p \in E$, we write $X - x$ and $X + p$ for $X \setminus \{x\}$ and $X \cup \{p\}$, respectively; accordingly we write $X - x + p$ for $(X \setminus \{x\}) \cup \{p\}$. Abusing the terminology, we sometimes say that an element $p \in E$ is independent iff $\{p\}$ is independent, i.e., $\{p\} \in \mathcal{H}$.

Definition 2.5. *We say that a hereditary collection $\mathscr{H} = (E, \mathcal{H})$ satisfies the* ***point replacement property*** *iff*

> PR: *For every $\{p\} \in \mathcal{H}$ and every nonempty subset $J \in \mathcal{H}$ there exists $x \in J$ such that $J - x + p \in \mathcal{H}$.*

Notice that Examples 2.2.(a)–(d) satisfy PR, while example (e) does not. One also observes by example (d) that PR does not imply that all the bases have the same cardinality.

Proposition 2.6. *The following are all equivalent for a hereditary collection $\mathscr{H}$.*

(i) *Point replacement;*

(ii) *$\{p\}, X \in \mathcal{H}$ with $p \notin X \neq \emptyset$ implies $\exists x \in X$ such that $X - x + p \in \mathcal{H}$;*

(iii) *$\{p\}, B \in \mathcal{H}$ with $B \in \mathcal{B}(\mathscr{H})$ implies $\exists b \in B$ such that $B - b + p \in \mathcal{H}$;*

(iv) *$\{p\}, B \in \mathcal{H}$ with $B \in \mathcal{B}(\mathscr{H})$ and $p \notin B$ implies $\exists b \in B$ such that $B - b + p \in \mathcal{H}$.*

Proof. $(i) \Leftrightarrow (ii)$ $((iii) \Leftrightarrow (iv))$ since if $p \in X$ $(b \in B)$ chose $x = p$ (b=p).

$(i) \Rightarrow (iii)$ is trivial; so it suffices to prove $(iii) \Rightarrow (i)$. Assume $\{p\} \in \mathcal{H}$ and let $X \neq \emptyset$ be a member of $\mathcal{H}$. Thus, there exists a basis $B \in \mathcal{B}(\mathscr{H})$ that contains X. Then, by (iii), $\exists b \in B$, such that $B - b + p \in \mathcal{H}$. If $b \in X$, then $X - b + p \in \mathcal{H}$ and we are done. Otherwise, if $b \notin X$ then $X + p \subseteq B - b + p$ which is a member of $\mathcal{H}$, and thus, for any $x \in X$, $X - x + p \in \mathcal{H}$ also lies in $\mathcal{H}$. This implies $(iii) \Rightarrow (i)$. $\square$

Remark 2.7. *Define the* ***basis replacement*** *condition as follows:*

> BR: *If $\{p\}$ is independent and $B \in \mathcal{B}(\mathscr{H})$ then $\exists b \in B$ such that $B - b + p$ is a basis.*

By Proposition 2.6.(iii), BR implies PR. However BR is not equivalent to point replacement; since Example 2.2.(d) fulfills PR, but does not satisfy BR. (It fails for $p = 3$ and the basis $B = \{1, 4\}$, since neither $\{3, 4\}$ nor $\{1, 3\}$ is a basis.)

2.3. **Matroids.** We now turn to the classical notion of matroids, cf. [4, 20, 21].

Definition 2.8. *A* ***matroid*** *$\mathscr{M}$ is a pair $(E, \mathcal{H})$ with $\mathcal{H}$ hereditary over the ground set E that satisfies the following axiom:*

> MT: *If I and J are in $\mathcal{H}$ and $|I| = |J| + 1$, then there exists $i \in I \setminus J$ such that $J + i$ is in $\mathcal{H}$.*

Proposition 2.9. *The following properties, cf. [4], are equivalent for a hereditary collection $\mathscr{M} = (E, \mathcal{H})$ to be a matroid.*

(i) ***Exchange property (EP)****: $\forall A, B \in \mathcal{B}(\mathscr{M})$ and $\forall a \in A \setminus B$, $\exists b \in B \setminus A$ such that $A - a + b$ is a basis of $\mathscr{M}$, i.e., it is an element of $\mathcal{B}(\mathscr{M})$.*

(ii) ***Dual exchange property (DEP)****: $\forall A, B \in \mathcal{B}(\mathscr{M})$ and $\forall a \in A \setminus B$, $\exists b \in B \setminus A$ such that $B - b + a \in \mathcal{B}(\mathscr{M})$.*

(iii) ***Symmetric exchange property (SEP)****:* $\forall A, B \in \mathcal{B}(\mathscr{M})$ *and* $\forall a \in A \setminus B$, $\exists b \in B \setminus A$ *such that* $B - b + a \in \mathcal{B}(\mathscr{M})$ *and* $A - a + b \in \mathcal{B}(\mathscr{M})$.

The proof of these equivalences, as well as the next lemma, are standard in matroid theory, see [21] and [4].

Lemma 2.10 ([21, Lemma 1.2.4])**.** *In a matroid* $\mathscr{M}$ *all the bases have the same cardinality, which is then equal the rank of* $\mathscr{M}$*.*

Example 2.11. *Consider the hereditary collection* $\mathscr{H}$ *of Example 2.2.(d). The* 2*-subset* $\{1, 4\}$ *is maximal in* $\mathcal{H}$ *with respect to inclusion, and thus is a basis of* $\mathscr{H}$*. Therefore, since* $\mathscr{H}$ *has rank* 3*,* $\mathscr{H}$ *is not a matroid (recall that it does not satisfy BR) but it satisfies PR.*

Proposition 2.12. *Any matroid satisfies the point replacement property PR (cf. Definition 2.5) as well as the base replacement property BR (cf. Remak 2.7).*

Proof. We assume the dual exchange property, cf. Proposition 2.9.(ii), and the hypothesis of Proposition 2.6.(iv). Then we need to prove that for a given basis $B \in \mathcal{B}(\mathscr{M})$ and a point $p \in E$ there is an element $b \in B$ such that $B - b + p$ is independent.

Pick a basis $A \in \mathcal{B}(\mathscr{M})$ containing p, set $a = p \in A \setminus B$, and apply the dual exchange property, yielding that there is $b \in B$ so that $B - b + p$ is a basis, hence independent. □

Example 2.13. *Consider the elementary examples of Example 2.2.*

(a) *Example 2.2.(d) satisfies PR is of rank* 3*, and is not a matroid.*

(b) *Example 2.2.(e) does not satisfy PR and thus is not a matroid. (Take the element* d *with respect to the 2-subset* $\{a, c\}$*.)*

Proposition 2.14. *Let* $\mathscr{H} = (E, \mathcal{H})$ *be a hereditary collection of rank at most* 2*. Then* $\mathscr{H}$ *satisfies PR iff* $\mathscr{H}$ *is a matroid.*

(Note that Example 2.2.(e) shows that the proposition fails for rank ≥ 3.)

Proof. ($\Rightarrow$) Assuming that $\mathscr{H}$ is of rank 2 and satisfies PR, we show that $\mathscr{H}$ satisfies the dual exchange property.

First, we claim that all bases of $\mathscr{H}$ have cardinality 2. Indeed, pick $\{p\} \in \mathcal{H}$ and take $X \in \mathcal{H}$ with $|X| = 2$, which exists by assumption. Suppose $X = \{x_1, x_2\}$. If $p \in X$ we are done. Otherwise, $p \notin X$ and the set $\{p, x_1\}$ or $\{p, x_2\}$ is independent by PR, and thus $\{p\}$ is not basis of $\mathscr{H}$ by maximality.

We next need to verify that the dual exchange propriety (Proposition 2.9.(ii)) is satisfied. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, where $A \neq B$ are two bases of $\mathscr{H}$. If $A \cap B = \emptyset$ then PR implies the dual exchange property for A and B. On the other hand, if $A \cap B \neq \emptyset$, say $A = \{a, c\}$ and $B = \{b, c\}$, then the dual exchange property is trivial by taking $a = b$.

($\Leftarrow$) By Proposition 2.12. □

The answer to the next question is known to be false.

Question. *Is the BR condition equivalent to the MT axiom?*

By Proposition 2.12 shows that MT implies BR, the next section shows that the converse is false. (We know by Example 2.2.(d) that BR is a stronger condition than PR.)

In the next couple of sections we consider operations on hereditary collections, resulting in new hereditary collections. These operations are standard for the case of matroids but are somewhat less obvious for hereditary collections.

2.4. **Duality.**

Definition 2.15. *We define the* ***dual hereditary collection*** $\mathscr{H}^*$ *of a hereditary collection* $\mathscr{H} = (E, \mathcal{H})$ *in terms of its bases as:*

$$\mathscr{H}^* := (E, \mathcal{H}^*), \qquad B^* := E \setminus B \in \mathcal{B}(\mathscr{H}^*) \;\Leftrightarrow\; B \in \mathcal{B}(\mathscr{H}).$$

Clearly, we have $(\mathscr{H}^*)^* = \mathscr{H}$, for any hereditary collection $\mathscr{H}$.

Example 2.16. *Consider Example 2.2.(d), which satisfies PR but is not a matroid. The bases of the dual hereditary collection $\mathscr{H}^*$ are $\{4\}$, $\{1\}$, and $\{2,3\}$ taken over the same ground set $E = \{1,2,3,4\}$. Thus $\mathscr{H}^*$ which does not satisfy PR, and therefore point replacement is not preserved under duality.*

Proposition 2.17 ([21, Theorem 2.1.1])**.** *The dual of a matroid is a matroid.*

Proof. Immediate by the exchange property and the dual exchange property, Proposition 2.9.(i) and Proposition 2.9.(ii), respectively. □

Example 2.18. *This is an example from James Oxley (private communication) showing that a hereditary collection $\mathscr{H}$ and its dual $\mathscr{H}^*$ can both satisfy PR (also BR) without $\mathscr{H}$ being a matroid.*

Let $E = \{1,\dots,6\}$ and consider the hereditary collection $\mathscr{H} = (E, \mathcal{H})$ whose bases are all 3-subsets of E except the 3-subsets $A = \{1,2,3\}$ and $B = \{1,3,4\}$. These are not bases of a matroid, since A and B are circuits, so $\{1,2,4\}$ must contain a circuit which it does not. (See [21, Chapter 1] *the weak circuit elimination axiom.) On the other hand, BR is true for $\mathscr{H}$, since every 4-subset contains at least two bases.*

Similarly, BR is true for $\mathscr{H}^*$ since every 2-subset is contained in at least two bases of $\mathscr{H}$.

Proposition 2.19. *Condition BR for $\mathscr{H}$ is implied by PR together with the assumption that all the bases have the same cardinality k.*

Proof. This is true since from Proposition 2.6.(iii) if all the bases have cardinality k, then any independent subset with k elements is a basis. □

Question 2.20. *Is the converse of Proposition 2.19 true?*

Remark 2.21.

(1) *When all the bases of a hereditary collection $\mathscr{H}$ have the same cardinality k (e.g., when $\mathscr{H}$ is a matroid), then all the bases of $\mathscr{H}^*$ are of cardinality $|E| - k$.*

(2) *If $\mathscr{H}$ satisfies PR and has rank 2, then by Proposition 2.17 $\mathscr{H}^*$ is a matroid of rank $|E| - 2$.*

Given a hereditary collection $\mathscr{H} == (E, \mathcal{H})$ of rank 2 we associate to $\mathscr{H}$ the graph $G := (V_G, E_G,)$ with vertex set $V_G = W$, such that the bases of $\mathscr{H}$ are 2-subsets corresponding to the edges of G. See Example 2.2.(e). The circuits of $\mathscr{H}$ are 2-subsets corresponding to the missing edges of G, and all subsets of E_G that give complete subgraphs on 3 vertices.

2.5. **Deletion, contraction, and minors.**

Definition 2.22. *The **deletion** of a subset $X \subseteq E$ from a hereditary collection $\mathscr{H} = (E, \mathcal{H})$ is defined as*

$$\mathscr{H} \setminus X := (E \setminus X, \mathcal{H} \setminus X),$$

where $\mathcal{H} \setminus X := \{Y \in \mathcal{H} : Y \subseteq E \setminus X\}$.

*The **contraction**, denoted $\mathscr{H}/X$, of X is defined as $(E \setminus X, \mathcal{H}/X)$, where $\mathcal{H}/X$ is given by:*

$$Y \in \mathcal{H}/X \quad \Leftrightarrow \quad Y \cup B_X \in \mathcal{H} \text{ for some } \mathscr{H} \text{ maximal independent subset } B_X \text{ of } X.$$

One sees that the empty set is contained in $\mathscr{H}/X$ and the contraction of any basis $B \in \mathcal{B}(\mathscr{H})$ gives the hereditary collection $(E \setminus B, \{\emptyset\})$; while on the other hand $\mathscr{H}/\emptyset = \mathscr{H}$.

Remark 2.23.

(i) *In many applications the subset X is assumed to be independent in $\mathscr{H}$, i.e., $X \in \mathcal{H}$, so in this case $Y \in \mathcal{H}/X$ if $X \cup Y$ is independent in $\mathscr{H}$.*

(ii) *The definition of contraction does **not** satisfy $\mathscr{H}/X = (\mathscr{H}^* \setminus X)^*$ for hereditary collections in general, but does for matroids.*

(iii) *For disjoint subsets X and Y of E, one has*

$$\mathscr{H} \ / \ Y \setminus X = \mathscr{H} \setminus X \ / \ Y,$$

as is easy to verify.

Definition 2.24. *A* ***minor*** *$\mathscr{H}' \subseteq \mathscr{H}$ of a hereditary collection $\mathscr{H} = (E, \mathcal{H})$ is a hereditary collection which is obtained from $\mathscr{H}$ by a sequence of deletions and contractions, which is equivalent to $\mathscr{H}'$ being*

$$\mathscr{H}' = \mathscr{H}/X \setminus Y = \mathscr{H} \setminus Y/X$$

for some disjoint subsets $X, Y \subseteq E$, where $\mathscr{H} = \mathscr{H} \setminus \emptyset = \mathscr{H}/\emptyset$.

A minor $\mathscr{H}'$ of $\mathscr{H}$ is said to be a ***proper minor*** *if $\mathscr{H}' \neq \mathscr{H}$.*

Proposition 2.25. *Given a hereditary collection $\mathscr{H} = (E, \mathcal{H})$. Let $Y \subseteq E$ be a subset of E, then for any $X \subseteq E \setminus Y$:*

(i) *X is independent in $\mathscr{H} \setminus Y$ iff X is independent in $\mathscr{H}$.*
(ii) *X is a circuit of $\mathscr{H} \setminus Y$ iff X is a circuit of $\mathscr{H}$.*
(iii) *X is a basis of $\mathscr{H} \setminus Y$ iff X is a maximal subset of $E \setminus Y$ that is independent in $\mathscr{H}$.*
(iv) *X is independent in $\mathscr{H}/Y$ iff $X \cup B_Y$ is independent in $\mathscr{H}$ for some subset B_Y of Y that is independent in $\mathscr{H}$.*
(v) *X is a basis of $\mathscr{H}/Y$ iff $X \cup B_Y$ is a basis of $\mathscr{H}$ for some maximal subset B_Y of Y that is independent in $\mathscr{H}$.*

Proof. Straightforward. □

Proposition 2.26.

(i) *Matroids are closed under taking duals, deletions and contractions, and hence under minors.*
(ii) *The class of hereditary collections satisfying PR is closed under deletion, but is not closed under contractions and duals, and hence not under minors.*
(iii) *A hereditary collection $\mathscr{H}$ is a matroid iff all minors of $\mathscr{H}$ satisfy PR.*

Proof. (i): Standard, see [20] or [21].

(ii): That PR is preserved under deletion is clear. To see that is not closed under taking duals, consider the hereditary collection $\mathscr{H}$ given in Example 2.2.(d), which satisfies PR. Its dual, given in Example 2.16, shows that $\mathscr{H}^*$ does not satisfy PR. Similarly, since $\mathscr{H}$ has bases $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{1, 4\}$, then $\mathscr{H}/\{1\}$ has the bases $\{2, 3\}$, $\{4\}$, which clearly do not satisfy PR.

(iii): We must show that if all the minors of $\mathscr{H}$ satisfy PR, then $\mathscr{H}$ satisfies the dual exchange property, cf. Proposition 2.9(ii). Given bases A, B, let $C = A \cap B$, C is an independent set of $\mathscr{H}$. Consider $\mathscr{H}/C$, which must satisfy PR. Now, by Proposition 2.25 and Remark 2.23.(i), we see that PR for $\mathscr{H}/C$ implies $\forall a \in A - B$, $\exists b \in B - A$ such that $(B - A) - b + a$ is independent in $\mathscr{H}/C$. So $((B - A) - b + a) \cup C = B - b + a$ is independent in $\mathscr{H}$.

Therefore, we have proved the following condition:

$$\forall A, B \in \mathcal{B}(\mathscr{H}),\ \forall a \in A - B,\ \exists b \in B - A \text{ such that } B - b + a \text{ is independent.} \qquad (*)$$

We will use $(*)$ to show that all the bases of $\mathscr{H}$ have the same cardinality.

If A, B are different bases of $\mathscr{H}$, with $|A| < |B|$, then applying $(*)$ inductively $|A|$ times and extending $B - b + a$ to a basis each time would imply $A \subsetneqq B$, with B a basis – a contradiction.

But if all the bases of $\mathscr{H}$ have the same cardinality, then condition $(*)$ is the same as the dual exchange property. □

3. Boolean and superboolean algebras

In this section all the proofs will be self-contained but see the references for further exposition (and generalizations).

The very well known **boolean semiring** is the two element idempotent semiring (see Appendix A for the formal definition)

$$\mathbb{B} := (\{0,1\}, +, \cdot\ ),$$

whose addition and multiplication are given by the following tables:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}\ .$$

The **superboolean semiring** $\mathbb{SB}$ is the finite supertropical semiring [10], a "cover" of the boolean semiring, with the three elements

$$\mathbb{SB} := (\{1, 0, 1^\nu\}, +, \cdot\ )$$

endowed with the two binary operations:

$$\begin{array}{c|ccc} + & 0 & 1 & 1^\nu \\ \hline 0 & 0 & 1 & 1^\nu \\ 1 & 1 & 1^\nu & 1^\nu \\ 1^\nu & 1^\nu & 1^\nu & 1^\nu \end{array} \qquad \begin{array}{c|ccc} \cdot & 0 & 1 & 1^\nu \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1^\nu \\ 1^\nu & 0 & 1^\nu & 1^\nu \end{array}$$

addition and multiplication, respectively. The superboolean semiring $\mathbb{SB}$ is totally ordered by

$$1^\nu \ > \ 1 \ > \ 0.$$

Note that $\mathbb{SB}$ is **not** an idempotent semiring, since $1+1 = 1^\nu$, and thus $\mathbb{B}$ is **not** a subsemiring of $\mathbb{SB}$.

The element 1^ν is called the **ghost** element, where $\mathcal{G}_0 := \{0, 1^\nu\}$ is the **ghost ideal**[1] of $\mathbb{SB}$. Two elements a and b of $\mathbb{SB}$ are **ν-equivalent**, written $a \cong_\nu b$, if $a = b$ or $a, b \in \{1, 1^\nu\}$.

(Further details on supertropical semiring structures are given in Appendix A below. Full details can be found in [10].)

In some sense the superboolean semiring could be understood as an extreme degeneration of a fuzzy ring [6, 32]. However, as shown in the next sections, in compare to fuzzy ring, the superboolean matrix algebra is established on the "weak" semiring operations, leading to a different approach to algebraic notions, which allows a capturing of representations of wider families of objects.

3.1. Superboolean matrix algebra.

The semiring $M_n(\mathbb{SB})$ of $n \times n$ superboolean matrices with entries in $\mathbb{SB}$ is defined in the standard way, where addition and multiplication are induced from the operations of $\mathbb{SB}$ as in the familiar matrix construction. The unit element I of $M_n(\mathbb{SB})$, is the matrix with 1 on the main diagonal and whose off-diagonal entries are all 0.

A typical matrix is often denoted as $A = (a_{i,j})$, and the zero matrix is written as (0). A matrix is said to be a **ghost matrix** if all of its entries are in $\mathcal{G}_0$. A boolean matrix is a matrix with coefficients in $\{0,1\}$, the subset of boolean matrices is denoted by $M_n(\mathbb{B})$.

The following discussion is presented for superboolean matrices, where boolean matrices are considered as superboolean matrices with entries in $\{0,1\}$. Note that boolean matrices $M_n(\mathbb{B})$ are **not** a sub-semiring of the semiring of superboolean matrices $M_n(\mathbb{SB})$.

In the standard way, for any matrix $A \in M_n(\mathbb{SB})$, we define the **permanent** of $A = (a_{i,j})$ as:

$$\operatorname{per}(A) := \sum_{\pi \in S_n} a_{\pi(1),1} \cdots a_{\pi(n),n} \tag{3.1}$$

where S_n stands for the group of permutations of $\{1, \dots, n\}$. Note that the permanent of a boolean matrix can be 1^ν. We say that a matrix A is **nonsingular** if $\operatorname{per}(A) = 1$, otherwise A is said to be **singular**.

[1] In the supertropical setting, the elements of the complement of $\mathcal{G}_0$ are called **tangibles**.

Remark 3.1. *One major computational tool in tropical matrix theory is the digraph of a matrix, we recall some basic definitions from* [11, §3.2]*, restricted here for the case of superboolean semiring.*

Given an $n \times n$ superboolean matrix $A = (a_{i,j})$, we associate the matrix A with the ***digraph*** *$G_A = (V_G, E_G)$ defined to have vertex set $V_G = \{1, \dots, n\}$, and an edge (i, j) from i to j, labeled $a_{i,j}$, whenever $a_{i,j} \neq 0$.*

The ***length*** *$\ell(p)$ of a path p is the number of edges of the path. An edge (i, i) is called a* ***self loop****. A path is* ***simple*** *if each vertex appears only once. A* ***simple cycle*** *is a simple path except that the starting vertex and the terminating vertex are the same. We define a k-****multicycle*** *σ in a digraph to be the union of vertex disjoint simple cycles, the sum of whose lengths is k; a k-multicycle σ is labeled 1^ν if one of its edges is labeled 1^ν, otherwise σ is labeled 1.*

From this graph view, each nonzero summand $a_{\pi(1),1} \cdots a_{\pi(n),n}$ in Formula (3.1) *corresponds to the n-multicycle*

$$\sigma = (\pi(1), 1), (\pi(2), 2), \dots, (\pi(n), n) \tag{3.2}$$

in the digraph G_A of A. Conversely, any digraph $G = (V_G, E_G)$ with n vertices and edges labeled 1 or 1^ν corresponds to the $n \times n$ ***adjacency matrix*** *$A_{\mathrm{adj}}(G)$ over the semiring $\mathbb{SB}$.*

A matrix $A \in M_n(\mathbb{SB})$ is nonsingular iff there is exactly one nonzero summand in (3.1) *equals 1, in particular no summand is 1^ν. This summand corresponds to a unique n-multicycle of G_A with all edges labeled 1 and G_A has no other n-multicycle, otherwise the matrix A is singular.*

As in the case of determinants, the permanent of a matrix $A \in M_n(\mathbb{SB})$ can written in terms of its minors. Denoting by $A_{i,j}$ the minor of A obtained by deleting the i'th row and the j'th column, the permanent in Formula (3.1) can be written equivalently as

$$\mathrm{per}(A) := \sum_j a_{i,j} \,\mathrm{per}(A_{i,j}), \tag{3.3}$$

for a fixed $i = 1, \dots, n$.

It easy to verify that the permanent has the following properties:

(1) Permuting rows or columns of a superboolean matrix leave the permanent unchanged;

(2) A matrix and its transpose have the same permanent;

(3) Multiplication of any given row or column of a superboolean matrix by 1^ν or 0 makes it singular.

Lemma 3.2. *A matrix $A \in M_n(\mathbb{SB})$ is nonsingular iff by independently permuting columns and rows it has the triangular form*

$$A' := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ * & \cdots & * & 1 \end{pmatrix}, \tag{3.4}$$

with all diagonal entries 1, all entries above the diagonal are 0, and the entries below the diagonal belong to $\{1, 1^\nu, 0\}$.

Such reordering of A is equivalent to multiplying the matrix A by two permutation matrices Π_1 and Π_2 on the right and on the left, respectively, i.e., $A' := \Pi_1 A \Pi_2$.

Proof. Any matrix A' of the triangular form (3.4) is nonsingular since the only permutation whose evaluation is not equal to zero in (3.1) is the identity permutation, which has value 1 by construction, and therefore $\mathrm{per}(A') = 1$.

($\Rightarrow$): Assume that $A = (a_{i,j})$ is nonsingular, then Equation (3.1) has a unique summand (corresponding to unique permutation $\pi_0 \in S_n$) of value 1 and all other summands (corresponding to permutations $\neq \pi_0$ in S_n) are of value 0. Permuting rows (columns) of A, we may assume that π_0 is the identity permutation, i.e., $a_{i,i} = 1$ for any $i = 1, \dots, n$. Then, each of the vertices of the digraph G_A of A has a self loop $\sigma_i = (i, i)$. Then, by Remark 3.1, since A is nonsingular, G_A has the unique n-multicycle σ consisting of these n self-loops.

Let $\widetilde{G}_A$ be the digraph obtained from G_A by deleting all the self loops σ_i. $\widetilde{G}_A$ is then an acyclic digraph, i.e., has nos cycles, since otherwise $\widetilde{G}_A$ would have a cycle which together with some other self-loops σ_i of σ in G_A composes another n-multicycle (in G_A) which would contradict the nonsingularity of A, since then G_A would have more then one n-multicycle, cf. Remark 3.1. Thus, the digraph $\widetilde{G}_A$ can be reordered such that $i > j$ for any edge (i, j); in other words $a_{i,j} = 0$ for any $j \geq i$. This reordering is equivalent to independently permuting columns and rows of the associated matrix. Joining back the self-loops σ_i that were omitted to the vertices of $\widetilde{G}_A$, corresponding to the diagonal entries of the adjacency matrix $A_{\text{adj}}(\widetilde{G}_A)$, we get the desired matrix $A' = A_{\text{adj}}(\widetilde{G}_A) + I$, which is of the Form (3.4).

($\Leftarrow$): This can be seen directly since multiplying the matrix A by a permutation matrix on left (resp. right) is equivalent to permuting rows (resp. columns) of A. But, as known, permuting rows or columns of a superboolean matrix leaves the permanent unchanged. $\square$

Lat A be an $m \times n$ superboolean matrix. We say that an $k \times \ell$ matrix B, with $k \leq m$ and $\ell \leq n$, is a **submatrix** of A if B can be obtained by deleting rows and columns of A. In particular, a **row** of a matrix A is an $1 \times n$ submatrix of A, where a **subrow** of A is an $1 \times \ell$ submatrix of A, with $\ell \leq n$. A **minor** is a submatrix obtained by deleting exactly one row and one column of a square matrix.

Definition 3.3. *A **marker** ρ in a matrix is a subrow having a single 1-entry and all whose other entries are 0; the length of ρ is the number of its entries. A marker of length k is written k-marker.*

For example the nonsingular matrix A' in (3.4) has a k-marker for each $k = 1, \dots, n$, appearing in this order from bottom to top. (Note that in general markers need not be disjoint.)

Corollary 3.4. *If a matrix $A \in M_n(\mathbb{SB})$ is a nonsingular matrix, then A has an n-marker.*

Proof. Since A is nonsingular, by Lemma 3.2, it can be reordered to the From (3.4). Then it is easy to see that the top row is an n-marker. $\square$

Note that if A is $n \times n$ nonsingular matrix then it has a k-marker for any $k = 1, \dots, n$, and by Lemma 3.2 we have such (disjoint) markers with each lies in a different row. On the other hand, a ghost matrix has no markers at all.

Example 3.5. *The following are all the possible nonsingular 2×2 superboolean matrices, up to reordering of columns and rows:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1^\nu \\ 0 & 1 \end{pmatrix},$$

each has a 2-marker.

We define the superboolean n-space $\mathbb{SB}^{(n)} = \mathbb{SB} \times \cdots \times \mathbb{SB}$ as the direct product of n copies of $\mathbb{SB}$. The elements of $\mathbb{SB}^{(n)}$ are the n tuples $(a_1, \dots, a_n)$ with entries a_i in $\mathbb{SB}$, which we call superboolean vectors. A vector v in $\mathbb{SB}^{(n)}$ is boolean if all of whose entries are in $\{0, 1\}$. A vector whose entries are all in $\mathcal{G}_0$ is called a **ghost vector**.

Definition 3.6 ([9, Definition 1.2])**.** *A collection of vectors $v_1, \dots, v_m \in \mathbb{SB}^{(n)}$ is said to be **dependent** if there exist $\alpha_1, \dots, \alpha_m \in \{0, 1\}$, not all of them 0, for which*

$$\alpha_1 v_1 + \cdots + \alpha_m v_m \in {\mathcal{G}_0}^{(n)}.$$

*Otherwise the vectors are said to be **independent**.*

Note that when one of the v_i's is ghost, or $v_i = v_j$ for some $i \neq j$, then the vectors are dependent. A set of nonzero boolean vectors can also be dependent; for example the vectors

$$(0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0)$$

are dependent, since their sum is $(1^\nu, 1^\nu, 1^\nu)$. (This example also shows that the notions of dependence and spanning do not coincide in this framework, since none of these vectors can be written in terms of the others.)

The following results can be found in [11] and [14] for the general supertropical setting, however, to make this paper self-contained we bring the superboolean versions of these results with easier proofs.

Definition 3.7. *A set $v_1, \dots, v_k$ of vectors has* ***rank defect*** *ℓ if there are ℓ columns, denoted $j_1, \dots, j_\ell$, such that $v_{i,j_u} = 0$ for all $1 \le i \le k$ and $1 \le u \le \ell$.*

For example, the vectors $v_1 = (1,0,1,0), v_2 = (0,0,0,1), v_3 = (1,0,0,0)$ have rank defect 1, since they are all 0 in the second column; v_1 and v_2 have rank defect 2.

Proposition 3.8 ([11, Proposition 2.10])**.** *An $n \times n$ matrix A has permanent 0, iff, for some $1 \le k \le n$, A has k rows having rank defect $n+1-k$.*

Proof. ($\Leftarrow$) The case of $k = n$ is obvious, since some column is entirely 0. The case of $k = 1$ is clear, since A is the zero matrix. If $n > k$, we take one of the columns j other than $j_1, \dots, j_k$ of Definition 3.7. Then for each i, the minor $A_{i,j}$ has at least $k-1$ rows with rank defect $(n-1)+1-(k-1)$, so has permanent 0 by induction; hence $\mathrm{per}(A) = 0$, by Formula (3.3).

($\Rightarrow$) We are done if all entries of A are 0, so assume for convenience that $a_{n,n} \neq 0$. Then, $\mathrm{per}(A_{n,n}) = 0$ and, by induction, $A_{n,n}$ has $k \ge 1$ rows of rank defect

$$(n-1)+1-k = n-k.$$

For notational convenience, we assume that $a_{i,j} = 0$ with $1 \le i \le k$ and $1 \le j \le n-k$. Thus, A has the partition

$$A = \left(\begin{array}{c|c} (0) & B' \\ \hline B'' & C \end{array}\right),$$

where (0) stands for the $k \times (n-k)$ zero matrix, B' is a $k \times k$ matrix, B'' is an $(n-k) \times (n-k)$ matrix, and C is an $(n-k) \times k$ matrix.

By inspection, $\mathrm{per}(B')\,\mathrm{per}(B'') = \mathrm{per}(A) = 0$; hence $\mathrm{per}(B') = 0$ or $\mathrm{per}(B'') = 0$. If $\mathrm{per}(B') = 0$, then, by induction, B' has k' rows of rank defect $k+1-k'$, so altogether, the same k' rows in A have rank defect $(n-k)+k+1-k' = n+1-k'$, and we are done by taking k' instead of k.

If $\mathrm{per}(B'') = 0$, then, by induction, B'' has k'' rows of rank defect $(n-k)+1-k''$, so altogether, these $k+k''$ rows in A have rank defect $n+1-(k+k'')$, and we are done, taking $k+k''$ instead of k. ☐

Example 3.9. *Suppose the rows of $A \in M_2(\mathbb{SB})$, $A = (a_{i,j})$, are dependent. Then there are $\alpha_1, \alpha_2 \in \mathcal{G}_0$ such that $\alpha_1(a_{1,1}, a_{1,2}) + \alpha_2(a_{2,1}, a_{2,2}) \in {\mathcal{G}_0}^{(2)}$. If $\alpha_1 = 0$, then $\alpha_2 = 1$ and $(a_{2,1}, a_{2,2}) \in {\mathcal{G}_0}^{(2)}$, implying A is singular. By the same argument if $\alpha_2 = 0$ then A is again singular.*

Assume that $\alpha_1 = \alpha_2 = 1$, then $a_{1,1} + a_{2,1} \in \mathcal{G}_0$ and $a_{1,2} + a_{2,2} \in \mathcal{G}_0$, which implies that $\mathrm{per}(A) = a_{1,1}a_{2,2} + a_{1,2}a_{2,1} \in \mathcal{G}_0$, i.e., A is singular.

Lemma 3.10. *The rows of any singular $n \times n$ matrix are dependent.*

Proof. We induct on n, the case $n = 1$ is obvious. (The case $n = 2$ is provided in Example 3.9.) Permuting independently rows and columns, we may assume that the value of (3.1), up to ν-equivalence, is the attained by identity, i.e.,

$$\mathrm{per}(A) \cong_\nu a_{1,1} \cdots a_{n,n}.$$

Let $v_1, \dots, v_n$ denote the rows of A.

Case I: Assume that $\mathrm{per}(A) = 1^\nu$. Let A' be an $m \times m$ singular submatrix of A with $\mathrm{per}(A') \cong_\nu 1$ whose diagonal lies on the diagonal of A. For notational convenience, we assume that such a singular submatrix A' with m minimal is the upper left submatrix of A; in particular if $a_{i,i} = 1^\nu$, for some i, renumbering the indices we may assume that $a_{1,1} = 1^\nu$.

Let

$$\alpha_i = \begin{cases} 0 & \text{if} \quad \mathrm{per}(A_{i,1}) = 0, \\ 1 & \text{if} \quad \mathrm{per}(A_{i,1}) \cong_\nu 1. \end{cases} \tag{3.5}$$

By assumption, some $\alpha_i = 1$. We claim that $\sum_i \alpha_i v_i \in {\mathcal{G}_0}^{(n)}$, that is,

$$\sum_i \alpha_i a_{i,j} \in \mathcal{G}_0, \qquad \text{for each } j = 1, \dots, n. \tag{3.6}$$

When $j = 1$, Formula (3.6) is just the expansion of $\operatorname{per}(A)$, up to ν-equivalence, along the first column of A, which we claim is 1^ν. Indeed, when $m = 1$, i.e., $a_{1,1} = 1^\nu$, we are done since $\operatorname{per}(A) \cong_\nu a_{1,1} \operatorname{per}(A_{1,1})$. Otherwise, since $m > 1$ is minimal, there is some other permutation besides the identity that also attains $\operatorname{per}(A')$; that is $a_{1,1} \operatorname{per}(A'_{1,1}) \cong_\nu a_{1,i} \operatorname{per}(A'_{1,i}) \cong_\nu 1$ for some $1 < i \leq m$. Thus,

$$a_{1,1} \operatorname{per}(A'_{1,1}) a_{m+1,m+1} \cdots a_{n,n} \cong_\nu a_{1,i} \operatorname{per}(A'_{1,i}) a_{m+1,m+1} \cdots a_{n,n},$$

and therefore $\alpha_1 a_{1,1} \cong_\nu \alpha_i a_{i,1}$.

Suppose $j > 1$, if $\sum_i \alpha_i a_{i,j} = 0$ we are done. So, assume that $\alpha_\ell a_{\ell,j} \neq 0$ for some ℓ. Then $a_{\ell,j} \cong_\nu 1$ and

$$\alpha_\ell \cong_\nu \operatorname{per}(A_{\ell,1}) \cong_\nu \prod_{i \neq \ell} a_{i,\sigma(i)} \cong_\nu 1,$$

for some $\sigma \in S_n$ with $\sigma(\ell) = 1$. Let u be the index for which $\sigma(u) = j$; in particular $u \neq \ell$. Let $\sigma' \in S_n$ be the permutation with $\sigma'(u) = 1$, $\sigma'(\ell) = j$, and $\sigma'(i) = \sigma(i)$ for each $i \neq u, \ell$. Then we have

$$\alpha_u \cong_\nu \operatorname{per}(A_{u,1}) \cong_\nu \prod_{i \neq u} a_{i,\sigma'(i)} \cong_\nu 1.$$

Thus, $\alpha_u a_{u,j}$ and $\alpha_\ell a_{\ell,j}$ are two different summands in Formula (3.6) with $\alpha_u a_{u,j} \cong_\nu \alpha_\ell a_{\ell,j} \cong_\nu 1$, as desired.

Case II: Suppose that $\operatorname{per}(A) = 0$ and A has a minor $A_{i,j}$ with $\operatorname{per}(A_{i,j}) \neq 0$. Permuting independently rows and columns we may assume that $i = j = 1$. We define the α_i's as in (3.5) and claim that Equation (3.6) is true for these α_i's. When $j = 1$, Formula (3.6) is just the expansion of $\operatorname{per}(A)$ along the first column of A, which we know is 0 since $\operatorname{per}(A) = 0$. For $j > 1$ we apply the same argument as in Case I.

Case III: Assume that $\operatorname{per}(A) = 0$ with all $\operatorname{per}(A_{i,j})$ are 0. We take m maximal such that A' is an $m \times m$ submatrix with a minor of permanent $\neq 0$. By induction, we may assume that $m = n - 1$. Furthermore, it is enough to find a dependence among the k rows obtained in Proposition 3.8, so, again, by induction, we may assume that $k = n - 1$, and the entries in the first column are all 0. Since $a_{1,1} = 0$ and $\operatorname{per}(A') \neq 0$, the proof is then completed by the argument of Case II. $\square$

Theorem 3.11 ([9, Theorem 2.10]). *The rows (columns) of a matrix $A \in M_n(\mathbb{SB})$ are independent iff A is nonsingular, i.e., $\operatorname{per}(A) = 1$.*

Proof. $(\Rightarrow)$: By Lemma 3.10.

$(\Leftarrow)$: Suppose A is nonsingular and assume by contradiction that the rows $v_1, \dots, v_n$ of A are dependent. Permuting independently rows and columns do not change the dependence relations of $v_1, \dots, v_n$, so by Lemma 3.2 we may assume that A is of the Form (3.4) and there are $\alpha_1, \dots, \alpha_n \in \{0, 1\}$, not all of them 0, such that $\sum_i \alpha_i v_i \in \mathcal{G}_0{}^{(n)}$. Let $\sum_i \alpha_i v_i = w$, where $w = (w_1, \dots, w_n)$. Suppose that i is the largest index for which $\alpha_i = 1$, then it easy to see that $w_i = 1$ – a contradiction. Thus, the rows of A are independent. $\square$

Corollary 3.12 ([14, Corollary 2.13]). *Any $k > n$ vectors in $\mathbb{SB}^{(n)}$ are dependent.*

Proof. Assume $v_1, \dots, v_{n+1}$ are vectors in $\mathbb{SB}^{(n)}$ and consider the $(n+1) \times n$ matrix whose rows are these vectors. Extend this matrix by duplicating one of the columns to get a singular matrix, whose rows are dependent by Theorem 3.11. $\square$

Theorem 3.13 ([14, Theorem 3.6]). *Let $A = (a_{i,j})$ be an $m \times n$ matrix with $n \geq m$, and suppose that each of whose $m \times m$ submatrices is singular. Then the rows $v_1, \dots, v_m$ of A are dependent.*

Proof. We induct on n, having proved the theorem for $m = n$ in Theorem 3.11. Thus, we may assume that $m < n$.

For each $j = 1, \dots, n$ we define $v_i^{(j)}$ to be the vector obtained by deleting the j entry, and $A^{(j)}$ to be the submatrix of A obtained by deleting the j column of A. Namely, the vectors $v_1^{(j)}, \dots, v_m^{(j)}$ are the rows of $A^{(j)}$ and by induction are dependent, i.e., there are $\alpha_{i,j} \in \{0, 1\}$ such that $\sum_{i=1}^n \alpha_{i,j} v_i^{(j)} \in \mathcal{G}_0{}^{(n-1)}$.

We are done if $\sum_i \alpha_{i,j} a_{i,j} \in \mathcal{G}_0$ for some j, since then $\sum_i \alpha_{i,j} v_i \in \mathcal{G}_0{}^{(n)}$. So, we may assume for each j that $\sum_i \alpha_{i,j} a_{i,j} = 1$. Pick i_j such that $\sum_i \alpha_{i,j} a_{i,j} = \alpha_{i_j,j} a_{i_j,j} = 1$.

Since there are at least $m+1$ values of i_j, and by pigeonhole principle two are the same, say $i_{j'} = i_{j''}$. To ease notation, we assume that $i_{j'} = i_{j''} = 1$. Thus, $\alpha_{1,j'} a_{1,j'} = 1$ and $\alpha_{1,j''} a_{1,j''} = 1$, and in particular $\alpha_{1,j'} = 1$ and $\alpha_{1,j''} = 1$. Let

$$\alpha_i = \begin{cases} \alpha_{1,j''}\alpha_{i,j'} & \text{if} \quad \alpha_{1,j''}\alpha_{i,j'} = \alpha_{1,j'}\alpha_{i,j''}, \\ \alpha_{1,j''}\alpha_{i,j'} + \alpha_{1,j'}\alpha_{i,j''} & \text{else.} \end{cases}$$

We need to show that for each j

$$\sum_i \alpha_i a_{i,j} \in \mathcal{G}_0. \tag{3.7}$$

The case of $j \neq j', j''$ is immediate, since we are given $\sum_i \alpha_{i,j'} a_{i,j} \in \mathcal{G}_0$ and $\sum_i \alpha_{i,j''} a_{i,j} \in \mathcal{G}_0$, implying at once that $\sum_i \alpha_i a_{i,j} \in \mathcal{G}_0$. Thus, we need to verify (3.7) for $j = j'$ and $j = j''$; by symmetry, we assume that $j = j'$. By assumption, $\alpha_{i,j'} a_{i,j'} = 0$ for each $i \neq 1$. Thus, $\alpha_1 a_{1,j'} = \alpha_{1,j'}\alpha_{1,j''} a_{1,j'} > \alpha_{1,j''}\alpha_{i,j'} a_{i,j'} = 0$. On the other hand,

$$\alpha_{1,j'}\alpha_{1,j''} a_{1,j'} \leq \sum_i \alpha_{1,j'}\alpha_{i,j''} a_{i,j'} = \alpha_{1,j'} \sum_i \alpha_{i,j''} a_{i,j'} \in \mathcal{G}_0,$$

by the dependence of $v_1^{(j'')}, \dots, v_m^{(j'')}$; so we conclude that

$$\sum_i \alpha_i a_{i,j} = \alpha_{1,j'} \sum_i \alpha_{i,j''} a_{i,j'} \in \mathcal{G}_0,$$

as desired. □

Corollary 3.14. *The columns (resp. rows) of an $m \times n$ matrix A, with $n \leq m$ (resp. $n \geq m$), are independent iff A contains an $n \times n$ (resp. $m \times m$) nonsingular submatrix.*

Proof. $(\Rightarrow)$: If all the $n \times n$ submatrices of A are singular then the columns of A are dependent by Theorem 3.13.

$(\Leftarrow)$: Let A' be an $n \times n$ nonsingular submatrix of A, then its columns are independent by Theorem 3.11. Since the columns of A' are subcolumns of A, then the columns of A are also independent. □

The **column rank** of a superboolean matrix A is defined to be the maximal number of independent columns of A. The **row rank** is defined similarly with respect to the rows of A.

We denote the rank of a superboolean matrix A by $\mathrm{rk}_{\mathbb{SB}}(A)$, or simply by $\mathrm{rk}(A)$, when it is clear form the context. Note that an $n \times n$ nonzero matrix has rank 0 if all of its entries are in $\mathcal{G}_0$, i.e., when A is a ghost matrix.

Corollary 3.15 ([9, Corollary 3.7])**.** *A matrix in $M_n(\mathbb{SB})$ is of rank n iff it is nonsingular.*

Proof. Immediate by 3.11. □

The rank of a superboolean matrix is then invariant under the following operations:

(i) permuting of rows (columns);

(ii) deletion of a row (column) whose entries are all in $\mathcal{G}_0$;

(iii) deletion of a repeated row or column.

Theorem 3.16 ([9, Theorem 3.11])**.** *For any matrix A the row rank and the column rank are the same, and this rank is equal to the size of the maximal nonsingular submatrix of A.*

Proof. Let k be the row rank of A, and let ℓ be the rank of the maximal nonsingular matrix. Clearly $k \geq \ell$, since any $\ell \times \ell$ nonsingular matrix has independent rows by Theorem 3.11. On the other hand, Theorem 3.13 shows that $k \leq \ell$, so $k = \ell$. The assertion for columns follows by considering the transpose matrix, since obviously the submatrix rank of a matrix and of its transpose are the same, both being equal to the size of a maximal nonsingular square submatrix. □

Corollary 3.17. *The rank of a superboolean matrix is invariant under transposition.*

3.2. **Ranks of matrices.** A boolean matrix $A \in M_n(\mathbb{B})$ can be formally considered as a matrix over a field $\mathbb{F}$, i.e., a member of the ring of matrices $M_n(\mathbb{F})$, where 1 and 0 are respectively the multiplicative unit and the zero of $\mathbb{F}$. In this view, the **field rank** of A is defined to be the standard matrix rank of A in $M_n(\mathbb{F})$; this rank is denoted by $\mathrm{rk}_{\mathbb{F}}(A)$.

Proposition 3.18. $\mathrm{rk}_{\mathbb{F}}(A) \geq \mathrm{rk}_{\mathbb{SB}}(A)$ *for any* $A \in M_n(\mathbb{B})$ *and over any field* $\mathbb{F}$.

Proof. Suppose $\mathrm{rk}_{\mathbb{SB}}(A) = k$, where $0 \leq k \leq n$. If $k = 0$ we are done, since A is a boolean matrix and thus $A = (0)$. Otherwise, by Corollary 3.14, A has a $k \times k$ nonsingular submatrix B, which by Lemma 3.2 can be permuted to the triangular form (3.4), for which we clearly have $\mathrm{rk}_{\mathbb{F}}(B) = k$. Therefore, $\mathrm{rk}_{\mathbb{F}}(A) \geq k$. □

Example 3.19. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

If $\mathbb{F}$ *is a field of characteristic* $\neq 2$*, then* $\mathrm{rk}_{\mathbb{F}}(A) = 3$*, while* $\mathrm{rk}_{\mathbb{SB}}(A) = 2$.

4. Representations of hereditary collections

4.1. **Classical representations of matroids over fields.** The traditional approach to represent a matroid uses matrices defined over fields, often finite fields, which in their turn generate vector matroids as explained below. This will be generalized later in §4.2 to hereditary collection with respect to the superboolean semiring.

In the sequel, we write $A_{\mathbb{K}}$ to indicate that a given matrix A is considered as a matrix over the ground structure $\mathbb{K}$ – either a field or a semiring.

Notation 4.1. *Given a matrix* $A_{\mathbb{K}}$ *and a subset* $X \subseteq \mathrm{Col}(A_{\mathbb{K}})$ *of columns of* $A_{\mathbb{K}}$*, we write* $A_{\mathbb{K}}[\,*\,, X]$ *for the submatrix of* $A_{\mathbb{K}}$ *having the columns* X*. Sometimes we refer to* $\mathrm{Col}(A_{\mathbb{K}})$ *as a collection of vectors, but no confusion should arise. Given also a subset* $Y \subseteq \mathrm{Row}(A_{\mathbb{K}})$ *of rows of* $A_{\mathbb{K}}$*, we define* $A_{\mathbb{K}}[Y, X]$ *to be the submatrix of* $A_{\mathbb{K}}$ *having the intersection of columns* X *and the rows* Y*, often also referred to as a collection of sub-vectors.*

Any $m \times n$ matrix $A_{\mathbb{F}}$ over a field $\mathbb{F}$ gives rise to a matroid $\mathscr{M}(A_{\mathbb{F}})$ constructed in the following classical way [34]. We label uniquely the columns of $A_{\mathbb{F}}$ (realized as vectors in $\mathbb{F}^{(m)}$) by a set $E := E(A_{\mathbb{F}})$, $|E| = |\mathrm{Col}(A_{\mathbb{F}})|$. The independent subsets $\mathcal{H} := \mathcal{H}(A_{\mathbb{F}})$ of $\mathscr{M}(A_{\mathbb{F}})$ are subsets of E corresponding to column subsets of $A_{\mathbb{F}}$ that are linearly independent in $\mathbb{F}^{(m)}$. This construction is well known, cf. [21], and $\mathscr{M}(A_{\mathbb{F}}) := (E(A_{\mathbb{F}}), \mathcal{H}(A_{\mathbb{F}}))$ is called a **vector matroid**.

Proposition 4.2. $\mathscr{M}(A_{\mathbb{F}})$ *is a matroid.*

An equivalent way to describe the independent subsets of a vector matroid $\mathscr{M}(A_{\mathbb{F}})$, using Notation 4.1, is as follows (WT stands for "witness"):

$$\boxed{\begin{aligned} \text{WT:} \quad X \in \mathcal{H}(A_{\mathbb{K}}) \quad \Leftrightarrow \quad & \exists Y \subseteq \mathrm{Row}(A_{\mathbb{K}}) \text{ with } |X| = |Y| \\ & \text{such that } A_{\mathbb{K}}[Y, X] \text{ is nonsingular over } \mathbb{K}, \end{aligned}}$$

where here we take $\mathbb{K} = \mathbb{F}$ to be a field. (This condition is central in our development, to be used later for semirings as well.)

A matroid $\mathscr{M}'$ that is isomorphic (cf. Definition 2.3) to a vector matroid $\mathscr{M}(A_{\mathbb{F}})$ for some matrix $A_{\mathbb{F}}$ over a field $\mathbb{F}$, is said to be **field-representable**, written $\mathbb{F}$**-representable**, and the matrix $A_{\mathbb{F}}$ is called a **field-representation**, written $\mathbb{F}$**-representation**, of $\mathscr{M}'$. We write $A_{\mathbb{F}}(\mathscr{M}')$ for an $\mathbb{F}$-representation of $\mathscr{M}'$, which need not be unique. Given a subset $X \in \mathcal{H}(\mathscr{M}')$, with $|X| = k$, a nonsingular $k \times k$ minor $A_{\mathbb{F}}[Y, X]$, $Y \subseteq \mathrm{Row}(A_{\mathbb{F}})$, of the $\mathbb{F}$-representation $A_{\mathbb{F}} := A_{\mathbb{F}}(\mathscr{M}')$ of $\mathscr{M}'$ is termed a **witness** of X in $A_{\mathbb{F}}$. (In particular the columns of $A_{\mathbb{F}}[Y, X]$, and thus the columns of $A_{\mathbb{F}}[\,*\,, X]$, are independent.)

The *new simple idea of this paper* is to replace the role of the field $\mathbb{F}$, used for classical matroid representations, by some commutative semiring; this allows the representation of any matroid, and moreover of any hereditary collection, as will be described next. In this paper we take this commutative semiring to be the superboolean semiring and show that for some cases the use of the boolean semiring is sufficient.

4.2. **$\mathbb{SB}$-vector hereditary collection.** Given a matrix $A_{\mathbb{SB}}$ over the superboolean semiring $\mathbb{SB}$, by the same construction as explained above for vector matroids, using condition WT with $\mathbb{K} = \mathbb{SB}$, we define the hereditary collection $\mathscr{H}(A_{\mathbb{SB}})$, where now dependence of columns and nonsingularity of submatrices are taken in the superboolean sense, cf. Definition 3.6. Formally, we have the following important key definition:

(Key) Definition 4.3. *Given an $m \times n$ superboolean matrix $A_{\mathbb{SB}}$, we define $\mathscr{H}(A_{\mathbb{SB}}) := (E, \mathcal{H})$ to be the hereditary collection with $E := E(\mathrm{Col}(A_{\mathbb{SB}}))$ corresponds uniquely to the columns of $A_{\mathbb{SB}}$, i.e., $|E| = |\mathrm{Col}(A_{\mathbb{SB}})|$, and whose independent subsets $\mathcal{H} := \mathcal{H}(A_{\mathbb{SB}})$ are column subsets of $A_{\mathbb{SB}}$ that are independent in the m-space $\mathbb{SB}^{(m)}$, namely, satisfying condition WT above for $\mathbb{K} = \mathbb{SB}$.*

*We call $\mathscr{H}(A_{\mathbb{SB}})$ an $\mathbb{SB}$-**vector hereditary collection**, and say that it is a $\mathbb{B}$-vector hereditary collection when $A_{\mathbb{SB}}$ is a boolean matrix.*

Having this notion of $\mathbb{SB}$-vector hereditary collections, we say that a hereditary collection $\mathscr{H}'$ is **superboolean-representable**, written $\mathbb{SB}$**-representable**, if it is isomorphic (cf. Definition 2.3) to an $\mathbb{SB}$-vector hereditary collection $\mathscr{H}(A_{\mathbb{SB}})$ for some superboolean matrix $A_{\mathbb{SB}}$ and write $A_{\mathbb{SB}}(\mathscr{H})$ for an $\mathbb{SB}$**-representation** of $\mathscr{H}$. When the matrix $A_{\mathbb{SB}}(\mathscr{H})$ is a boolean matrix, i.e., with 1, 0 entries, we call this representation a **boolean representation**, written $\mathbb{B}$**-representation**, and say that $\mathscr{H}$ is $\mathbb{B}$-**representable**. We use the same terminology as before and called a $k \times k$ nonsingular minor $A_{\mathbb{SB}}[Y, X]$ of $A_{\mathbb{SB}}$ the **witness** of the independent subset $X \subseteq E$, $|X| = k$, in the $\mathbb{SB}$-representation $A_{\mathbb{SB}} := A_{\mathbb{SB}}(\mathscr{H})$ of the hereditary collection $\mathscr{H} = (E, \mathcal{H})$.

Remark 4.4. *Given a superboolean matrix $A_{\mathbb{SB}}$ the $\mathbb{SB}$-vector hereditary collection $\mathscr{H}(A_{\mathbb{SB}})$ needs not be a matroid. For example the vector hereditary collection $\mathscr{H}(A_{\mathbb{SB}})$ of the matrix*

$$A_{\mathbb{SB}} = \begin{pmatrix} 1 & 0 & 1^{\nu} & 1 \\ 0 & 1 & 1 & 1^{\nu} \end{pmatrix}$$

is not a matroid. The independent subsets of $E = \{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{1, 3\}$, $\{2, 4\}$, all the singletons of E, and the empty set. Therefore, Axiom MT is not satisfied for the subset $\{4\}$ with respect to $\{1, 3\}$.

When a matrix $A_{\mathbb{B}}$ is a nonzero boolean matrix, yet the vector hereditary collection $\mathscr{H}(A_{\mathbb{B}})$ needs not be a matroid. For example consider the $\mathbb{SB}$-vector hereditary collection $\mathscr{H}(A_{\mathbb{B}})$ of the matrix

$$A_{\mathbb{B}} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

whose bases are $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{3, 4\}$. Thus, $\mathscr{H}(A_{\mathbb{B}})$ is not a matroid.

Example 4.5. *The uniform matroid $U_{2,n}$ (cf. Example 2.2.(c)) is $\mathbb{B}$-representable by the $(n-1) \times n$ boolean matrix*

$$A_{\mathbb{B}}(U_{2,n}) = \begin{pmatrix} 0 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 1 \end{pmatrix}.$$

One sees that any pair of columns of $A_{\mathbb{B}}(U_{2,n})$ are independent since they contain either one 0-entry or two 0-entries in different positions, and thus a 2×2 witness of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, respectively. On the other hand, any column has at most one 0-entry, and therefore any 3×3 submatrix is singular. Thus, any subset of more than 2 columns is dependent.

4.3. **Superboolean representations of hereditary collections.** We are now ready for one of our main theorems of this paper.

Theorem 4.6. *Every hereditary collection $\mathscr{H} = (E, \mathcal{H})$ over a ground set E of n elements is $\mathbb{SB}$-representable by an $m \times n$ superboolean matrix.*

Proof. We prove the theorem by constructing an explicit $\mathbb{SB}$-representation $A_{\mathbb{SB}}(\mathscr{H})$ for a given hereditary collection $\mathscr{H}$. The columns of $A_{\mathbb{SB}}(\mathscr{H})$ will be labeled by the ground set E and each independent subset $X \subseteq E$, with $|X| = k$, will correspond to a column subset labeled by X and containing a witness, i.e., a $k \times k$ nonsingular minor, cf. WT above.

When E is empty, then $\mathscr{H}$ is represented by the formal 0×0 matrix, i.e., by the empty matrix. So, throughout we assume that $|E| > 0$. In the case that $\mathcal{H} = \{\emptyset\}$, $\mathscr{H}$ can be $\mathbb{SB}$-represented by any $m \times n$ ghost matrix, and in particular by any $1 \times n$ ghost matrix.

Suppose that $\mathcal{H}$ contains a nonempty subset of E, and let $\mathcal{B}(\mathscr{H}) = \{J_1, \dots, J_\ell\}$ be the set of bases of $\mathscr{H}$. Given a basis $J_i \in \mathcal{B}(\mathscr{H})$, $J_i = \{b_{i_1}, b_{i_2}, \dots, b_{i_{m_i}}\}$, with m_i elements, we define the $m_i \times n$ matrix $C^{(i)}_{\mathbb{SB}} := A_{\mathbb{SB}}(J_i)$ having the $m_i \times m_i$ nonsingular minor, whose columns correspond to the elements $b_{i_1}, b_{i_2}, \dots, b_{i_{m_i}}$ of J_i assuming $b_{i_1} \leq i_2 \leq \dots \leq i_{m_i}$, to be of the form (we use Notation 4.1 for submatrices)

$$C^{(i)}_{\mathbb{SB}}[\,*\,, J_i] = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1^\nu & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 1^\nu & \cdots & 1^\nu & 1 \end{pmatrix}, \tag{4.1}$$

i.e., $C^{(i)}_{\mathbb{SB}}[\,*\,, J_i]$ has 1 on the main diagonal, 0 strictly above the diagonal, whence 1^ν strictly below the diagonal; all the other entries of $C^{(i)}_{\mathbb{SB}}$ are 1^ν. Namely, after permuting the columns of $C^{(i)}_{\mathbb{SB}}$, we have the form

$$A_{\mathbb{SB}}(J_i) = C^{(i)}_{\mathbb{SB}} := \left(\begin{array}{c|ccc} & 1^\nu & \cdots & 1^\nu \\ \quad C^{(i)}_{\mathbb{SB}}[\,*\,, J_i] \quad & \vdots & \ddots & \vdots \\ & 1^\nu & \cdots & 1^\nu \end{array} \right), \tag{4.2}$$

with $i_1 = 1, i_2 = 2, \dots, i_{m_i} = m_i$. Clearly $C^{(i)}_{\mathbb{SB}}$ has rank m_i, since it contains the $m_i \times m_i$ witness $C^{(i)}_{\mathbb{SB}}[\,*\,, J_i]$, whence its rows are linearly independent by Corollary 3.14. Accordingly, $A_{\mathbb{SB}}(J_i)$ is an $\mathbb{SB}$-representation of the hereditary collection $\mathscr{H}_i = (E, \mathrm{Pw}(J_i))$, $J_i \subseteq E$.

Having the matrices $C^{(i)}_{\mathbb{SB}} = A_{\mathbb{SB}}(J_i)$ at hand, for each basis J_i of $\mathscr{H}$, $i = 1, \dots, \ell$, we construct the matrix

$$B_{\mathbb{SB}} := \left(\begin{array}{rl} A_{\mathbb{SB}}(J_\ell) := & \boxed{\quad (1^\nu) \quad \Big| \quad C^{(\ell)}_{\mathbb{SB}}[\,*\,, J_\ell] \quad \Big| \quad (1^\nu) \quad} \\ & \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ A_{\mathbb{SB}}(J_2) := & \boxed{\quad (1^\nu) \qquad\quad \Big| \qquad C^{(2)}_{\mathbb{SB}}[\,*\,, J_2] \quad} \\ A_{\mathbb{SB}}(J_1) := & \boxed{\quad C^{(2)}_{\mathbb{SB}}[\,*\,, J_1] \qquad \Big| \qquad (1^\nu) \qquad} \end{array} \right) \tag{4.3}$$

by *stacking* the matrices $A_{\mathbb{SB}}(J_i)$ one over the other with respect to their columns labeling; (1^ν) stands for a matrix all of its entries are 1^ν. (Note that Form (4.3) only illustrates the construction of $B_{\mathbb{SB}}$, the columns of $C^{(i)}_{\mathbb{SB}}[\,*\,, J_i]$ need not be consecutive in $A_{\mathbb{SB}}(J_i)$, and these blocks need not overlap each other.)

The matrix $B_{\mathbb{SB}}$ has the following properties:

(a) every row has exactly one 1-entry;

(b) each k-marker, cf. Definition 3.3, is contained in a $k \times k$ witness which is a $k \times k$ submatrix of $A_{\mathbb{SB}}(J_i)$, for some i.

To prove that $B_{\mathbb{SB}}$ is a proper $\mathbb{SB}$-representation of $\mathscr{H}$, we need to verify that the dependence and independence relations satisfied by the columns of $B_{\mathbb{SB}}$ are exactly those of $\mathscr{H}$. These relations have been recorded separately by the matrices $A_{\mathbb{B}}(J_i)$ for the bases J_i of $\mathscr{H}$.

Let $Y_i \subseteq \mathrm{Row}(B_{\mathbb{SB}})$ denote the subset of rows of $B_{\mathbb{SB}}$ corresponding to the matrix $A_{\mathbb{B}}(J_i)$. Given a k-subset $X \subseteq E$, i.e., $|X| = k$, we claim that $\mathrm{Col}(B_{\mathbb{SB}}[\,*\,, X])$ are independent iff $\mathrm{Col}(B_{\mathbb{SB}}[Y_i, X])$ are independent for some $i = 1, \dots, \ell$.

$(\Leftarrow)$: If $\mathrm{Col}(B_{\mathbb{SB}}[Y_i, X])$ are independent, then $B_{\mathbb{SB}}[Y_i, X]$ contains a $k \times k$ witness, where $k \leq m_i$, which is also contained in $B_{\mathbb{SB}}[\,*\,, X]$, and we are done by Corollary 3.14.

$(\Rightarrow)$: Suppose $\mathrm{Col}(B_{\mathbb{SB}}[\,*\,, X])$ are independent then it contains a $k \times k$ witness, which by Corollary 3.4 contains a k-marker ρ. But, by construction, ρ belongs to a $k \times k$ witness that is contained in some $Y_i \subseteq \mathrm{Row}(A_{\mathbb{B}}(J_i))$; therefore $\mathrm{Col}(B_{\mathbb{SB}}[Y_i, X])$ is independent again by Corollary 3.14. □

Corollary 4.7. *Any matroid is $\mathbb{SB}$-representable.*

We aim for an $\mathbb{SB}$-representation of minimal size, i.e., has a minimal number of rows. Let us start with a naive upper bound obtained directly from our construction in the proof of Theorem 4.6, that is

$$m \leq \sum_{J \in \mathcal{B}(\mathscr{H})} |J|, \tag{4.4}$$

where we recall that $|J|$ stands for the cardinality of a basis J of $\mathscr{H}$. (Clearly, by Corollary 3.12, the lower bound is determined by the rank of the hereditary collection $\mathscr{H}$.)

Remark 4.8. *Given an $\mathbb{SB}$-representation $A_{\mathbb{B}}(\mathscr{H})$ of a hereditary collection $\mathscr{H}$, one can reduce $A_{\mathbb{B}}(\mathscr{H})$ by erasing repeated rows, leaving a single representative for each subset of identical rows; ghost rows can also be omitted, as long as $A_{\mathbb{B}}(\mathscr{H})$ remains with at least one row.*

Example 4.9. *Let $E = \{1,2,3\}$, and let $\mathcal{H} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$ be the independent subsets of the hereditary collection $\mathscr{H} = (E, \mathcal{H})$. Thus, $\{1,2\}, \{1,3\}$ are the bases of $\mathscr{H}$. Using the construction of the proof of Theorem 4.6 we obtain the $\mathbb{SB}$-representation*

$$A_{\mathbb{SB}}(\mathscr{H}) = \begin{pmatrix} 1 & 1^\nu & 1 \\ 0 & 1^\nu & 1 \\ 1 & 1 & 1^\nu \\ 0 & 1 & 1^\nu \end{pmatrix}.$$

However, this hereditary collection $\mathscr{H}$ can also be represented by the smaller matrix

$$A'_{\mathbb{SB}}(\mathscr{H}) = \begin{pmatrix} 1 & 0 & 1^\nu \\ 0 & 1 & 1 \end{pmatrix}.$$

This is this a minimal possible $\mathbb{SB}$-representation of $\mathscr{H}$, i.e., $m = 1$.

Example 4.10. *The uniform matroid $U_{m-1,m}$ can be $\mathbb{SB}$-represented by the $m \times m$ matrix $A_{\mathbb{SB}} = (a_{i,j})$ of the form:*

$$A_{\mathbb{SB}} = \begin{pmatrix} 1 & 1^\nu & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & \ddots & 0 \\ 0 & & \ddots & 1 & 1^\nu \\ 1^\nu & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{4.5}$$

which has 1 on the main diagonal, $a_{1,2} = \cdots = a_{i,i+1} = \cdots = a_{m-1,m} = 1^\nu$, $a_{m,1} = 1^\nu$, and all other entries are 0.

$A_{\mathbb{SB}}$ is a singular matrix; this seen easily by taking the sum of its columns, which are dependent and thus $A_{\mathbb{SB}}$ is singular, cf. Corollary 3.15). On the other hand, each of whose diagonal minor $M_i = A_{i,i}$

(obtained by deleting the column i and the row i) is nonsingular. To see the latter, consider the digraph G_A of $A_{\mathbb{SB}}$ which has two m-multicycles. One m-multicycle is given by the m self loops $\sigma_i := (i,i)$, all labeled 1, while the other is given by the sequence

$$\sigma := (1,2),(2,3),\dots,(i,i+1),\dots,(m-1,m),(m,1)$$

of edges, all labeled 1^ν. The subgraph G_{M_i} of G_A corresponding to a minor M_i is obtained by deleting the vertex i from G_A, and thus deleting the single self loop (i,i) and two edges of σ emerging and terminating at i. Accordingly, G_{M_i} has a unique $(m-1)$-multicycle, composed from $(m-1)$ self loops σ_i, all labeled 1, and thus the minor M_i is nonsingular by Remark 3.1.

The above examples show that an $\mathbb{SB}$-representation can often be reduced further.

Theorem 4.11. *When each basis of a hereditary collection $\mathscr{H} = (E, \mathcal{H})$ is contained in a circuit, $\mathscr{H}$ can be represented in terms of its circuits.*

Proof. Let $C_1,\dots,C_\ell \in \mathcal{C}(\mathscr{H})$ be the circuits of $\mathscr{H}$. Given a circuit C_i with $|C_i| = m_i$ we construct the $m_i \times n$ matrix $D^{(i)}_{\mathbb{SB}} := A_{\mathbb{SB}}(C_i)$ whose submatrix $D^{(i)}_{\mathbb{SB}}[\,*\,,C_i]$ is of the Form (4.5) and all its other entries are 1^ν. As shown above, $D^{(i)}_{\mathbb{SB}}[\,*\,,C_i]$ is an $\mathbb{SB}$-representation of the uniform hereditary collection U_{m_i-1,m_i} and thus, by Example 4.10, also of the circuit C_i, which in the view of Example 2.2.(c) is just $\mathscr{H}_i := (C_i, \mathrm{Pw}(C_i) \setminus \{C_i\})$.

The proof is then completed by stacking these matrices $A_{\mathbb{SB}}(C_i)$ (each corresponds to a different circuit C_i of $\mathscr{H}$) one over the other, with respect to their columns labeling, and applying the same argument as in the proof of Theorem 4.6. □

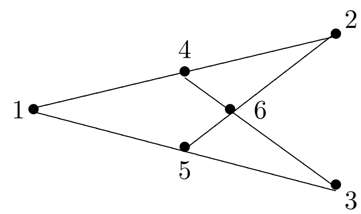
Under the condition of Theorem 4.11, we have the following upper bound for the size of $\mathbb{SB}$-representations of a hereditary collection:

$$m \le \min\left\{ \sum_{J\in\mathcal{B}(\mathscr{H})} |J|, \sum_{C\in\mathcal{C}(\mathscr{H})} |C| \right\}. \tag{4.6}$$

4.4. **Examples of $\mathbb{SB}$-representations for matroids.** In this section we provide examples of $\mathbb{SB}$-representations for some well known matroids. (See [21] for further explanation of the notation.)

Example 4.12. *Let $\mathscr{M}$ be the matroid $M(K_4)$; that is the matroid corresponding to the following diagram (dependent 3-subsets correspond to 3 colinear points in the diagram):*

$M(K_4)$:



i.e., the matroid over 6 elements where all the 3-subsets are independent expect:

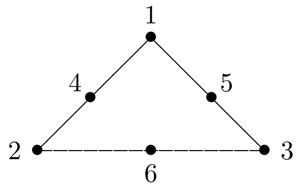
$$\{1,2,4\},\{1,3,5\},\{2,5,6\},\{3,4,6\}.$$

$\mathscr{M} = M(K_4)$ is $\mathbb{SB}$-representable by the matrix

$$A_{\mathbb{B}}(\mathscr{M}) = \begin{pmatrix} 1^\nu & 1^\nu & 1^\nu & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Example 4.13. *The matroid $\mathcal{W}^3$, corresponding to the diagram:*

$\mathcal{W}^3$:



is $\mathbb{SB}$*-represented by the matrix*

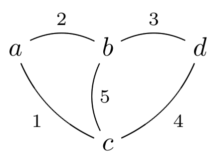
$$A_{\mathbb{B}}(\mathcal{W}^3) = \begin{pmatrix} 1^\nu & 1^\nu & 1^\nu & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

5. Boolean representations

In this section we study boolean representations; these representations are a special case of $\mathbb{SB}$-representations provided by boolean matrices. Recall that we write $\mathbb{B}$-representations for the boolean representations, and say that a hereditary collection $\mathscr{H}$ is $\mathbb{B}$-representable if it has a $\mathbb{B}$-representation.

5.1. **Graphic matroids.** We begin with the classical concept of Whitney for a connection between matroids and graphs, see [20, 34].

Given a finite graph $G := (V_G, E_G)$ with vertex set V_G and set of edges E_G (G might have multiple edges), we consider the $|V_G| \times |E_G|$ incidence matrix $A_{\mathrm{inc}}(G) := (a_{i,j})$ with entry $a_{i,j} = 1$ if the vertex v_i is an end point of the edge e_j and e_j is not a self-loop, otherwise we set $a_{i,j} = 0$. For example, the incidence matrix of the graph



is

$$A_{\mathrm{inc}}(G) = \begin{array}{c|ccccc} a & 1 & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 & 1 \\ c & 1 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 1 & 1 & 0 \\ \hline & 1 & 2 & 3 & 4 & 5 \end{array}.$$

For ease of exposition, throughout we assume that G is a connected graph. Note that now the rows of the matrix $A_{\mathrm{inc}}(G)$ are labeled by the vertices V_G, and columns are labeled by the edges E_G of the graph G.

By this construction we see that each column of the matrix $A_{\mathrm{inc}}(G)$ has either two or no 1-entries, and multiple edges introduce identical columns. Accordingly, without loss of generality, we may consider the ground set E_G as a collection of 2-subsets of V_G. A matroid constructed by this way is called **graphic matroid** and we denote it $\mathscr{M}_{\mathrm{inc}}(G)$.

Let $A_{\mathbb{F}_2} := A_{\mathrm{inc}}(G)$ be the incidence matrix $A_{\mathrm{inc}}(G)$ of the graph G, considered as a matrix over the field $\mathbb{F}_2$ of characteristic 2. Recall that $\mathscr{M}(A_{\mathbb{F}_2})$ denotes the vector matroid of the matrix $A_{\mathbb{F}_2}$ over the field $\mathbb{F}_2$.

Theorem 5.1 ([20, Theorem 2.16])**.** *The independent subsets of the vector matroid $\mathscr{M}(A_{\mathbb{F}_2})$ correspond to subsets of edges of G that do not contain a cycle and $\mathscr{M}(A_{\mathbb{F}_2}) = \mathscr{M}_{\mathrm{inc}}(G)$.*

Accordingly, if G is a connected graph, then the bases of $\mathscr{M}_{\mathrm{inc}}(G)$ are precisely the edge subsets of the spanning trees of G and, if G has ℓ vertices, each spanning tree has exactly $\ell - 1$ edges, so $\mathrm{rk}(\mathscr{M}_{\mathrm{inc}}(G)) = \ell - 1$.

Proposition 5.2. *Let $G := (E_G, V_G)$ be a connected graph and let $A_{\mathrm{inc}}(G)$ be its adjacency matrix as described above. Considering $A_{\mathrm{inc}}(G)$ as a matrix over $\mathbb{F}_2$ (the finite field of 2-elements) or over the boolean semiring $\mathbb{B}$ gives the same matroid of rank $|V_G| - 1$ whose collection of bases correspond to the edges of the spanning trees of G.*

Proof. Write $A_{\mathbb{F}_2}$ and $A_{\mathbb{B}}$ for the incidence matrix $A_{\mathrm{inc}}(G)$ considered as a matrix over $\mathbb{F}_2$ and $\mathbb{B}$, respectively, and let $\mathscr{M}(A_{\mathbb{F}_2})$ and $\mathscr{M}(A_{\mathbb{B}})$ be the corresponding vector matroids. Theorem 5.1 gives us the correspondence between the bases of $\mathscr{M}(A_{\mathbb{F}_2})$ and the spanning trees of G, so we need to prove that the bases of $\mathscr{M}(A_{\mathbb{B}})$ are in one-to-one correspondence with the spanning trees of G. (This will also prove that $\mathscr{M}(A_{\mathbb{B}})$ is indeed a matroid.)

Suppose $|V_G| = \ell$ and let $T_1 \subseteq E_G$ be a spanning tree of G. Then T_1 has $\ell - 1$ edges. Consider the $\ell \times (\ell - 1)$ submatrix $B_1 := A_{\mathbb{B}}[\,*\,, T_1]$ of $A_{\mathrm{inc}}(G)$, cf. Notation 4.1, corresponding to T_1 and having two 1-entries in each column by construction. Pick a leaf vertex $i_1 \in V_G$, which belongs to a unique edge (i_1, j_1) of spanning tree T_1, and erase i_1 and its connecting edge (i_1, j_1) from T_1 to obtain the subtree T_2 of T_1 (which clearly is connected, and has $\ell - 2$ edges). This deletion is expressed by erasing the i_1'th row of B_1, which is an $(\ell - 1)$-marker, and the column j_1 (corresponding to the edge (i_1, j_1)) of B_1. Denote this matrix corresponding to subtree T_2 by B_2 and let D_1 be the matrix composed of the i_1'th row of $A_{\mathbb{B}}[\,*\,, T_1]$.

We repeat this process recursively, erasing at each step a new leaf vertex i_k from the tree T_k having $\ell - k$ edges, expressed as a deletion of the row and the column corresponding to vertex the i_k and its connecting edge (i_k, j_k) in the matrix B_k, and joining the i_k'th row of $A_{\mathbb{B}}[\,*\,, T_1]$ to the matrix D_{k-1} from below. At the end of this process, after $\ell - 1$ steps, we obtain the triangular $(\ell - 1) \times (\ell - 1)$ matrix $D_{\ell-1}$, which by construction is of the Form (3.4) – a nonsingular matrix by Lemma 3.2. $D_{\ell-1}$ is a submatrix of $A_{\mathbb{B}}[\,*\,, T_1]$, up to permuting of rows, and thus an $(\ell - 1) \times (\ell - 1)$ witness. Therefore, the columns of $A_{\mathbb{B}}[\,*\,, T_1]$ are independent.

Conversely, suppose $X \subseteq E_G$ with $|X| = k$ and assume that the columns of $A_{\mathbb{B}}[\,*\,, X]$ are independent, namely $\mathrm{rk}_{\mathbb{SB}}(A_{\mathbb{B}}[\,*\,, X]) = k$. But then, by Proposition 3.18, $\mathrm{rk}_{\mathbb{F}_2}(A_{\mathbb{F}_2}[\,*\,, X]) \geq k$ which implies $\mathrm{rk}_{\mathbb{F}_2}(A_{\mathbb{F}_2}[\,*\,, X]) = k$, since $|X| = k$. Hence, by Theorem 5.1, $A_{\mathbb{F}_2}[\,*\,, X]$ corresponds to a spanning tree of G and so does $A_{\mathbb{B}}[\,*\,, X]$. □

5.2. **Boolean representations of hereditary collections.** Our first result in this section ties boolean representations to the point replacement property, PR, (cf. Definition 2.5) for an arbitrary hereditary collection.

Theorem 5.3. *If a hereditary collection $\mathscr{H} = (E, \mathcal{H})$ has a $\mathbb{B}$-representation, then $\mathscr{H}$ satisfies PR.*

Proof. The case of $|E| = 0$ is obvious, so throughout we assume that $|E| > 0$. Suppose $\mathscr{H}$ is $\mathbb{B}$-representable by the matrix $A_{\mathbb{B}} := A_{\mathbb{B}}(\mathscr{H})$, we need to verify Proposition 2.6.(ii), that is

$$\{p\}, X \in \mathcal{H}, \text{ with } p \notin X \neq \emptyset, \text{ implies } \exists x \in X, \text{ such that } X - x + p \in \mathcal{H}.$$

Let $P = \{p\}$ and suppose $|X| = k$. Reordering independently the columns and rows of $A_{\mathbb{B}}$ we may assume that $A_{\mathbb{B}}$ has a $k \times k$ witness $A_{\mathbb{B}}[Y, X]$ of the triangular form (3.4) for some $Y \subseteq \mathrm{Row}(A_{\mathbb{B}}(\mathscr{H}))$ with $|Y| = k$. (We use again Notation 4.1 for submatrices.) Clearly $A_{\mathbb{B}}[\,*\,, P]$ has a nonzero entry, since $P \in \mathcal{H}$ and $P \neq \emptyset$.

Assume first that $A_{\mathbb{B}}[Y, P]$ has a nonzero entry, and let ℓ be the smallest index entry of the column $A_{\mathbb{B}}[Y, P]$ which is not zero. Then, we are done by interchanging $A_{\mathbb{B}}[\,*\,, P]$ with the ℓ'th column of $A_{\mathbb{B}}[\,*\,, X]$, since we have preserved the triangular form (3.4). Otherwise, let $Z \supset Y$ be a subset of rows with $|Z| = |Y| + 1$ such that $A_{\mathbb{B}}[Z, P]$ has a single nonzero entry. Then $A_{\mathbb{B}}[Z, X \cup P]$ is a $(k+1) \times (k+1)$ witness, and thus any subset of k columns of $A_{\mathbb{B}}[\,*\,, X \cup P]$ is independent by Theorem 3.11. Namely, any column of $A_{\mathbb{B}}[\,*\,, X]$ can be replaced by $A_{\mathbb{B}}[\,*\,, P]$, preserving the independence relations. □

We are now ready for another main result of this paper relating to matroids.

Theorem 5.4. *If a matroid $\mathscr{M}$ is $\mathbb{F}$-representable, for some field $\mathbb{F}$, then $\mathscr{M}$ is also $\mathbb{B}$-representable.*

Proof. Let $\mathscr{M}$ be a matroid of rank m and suppose it is $\mathbb{F}$-representable by the matrix $A_{\mathbb{F}} := A_{\mathbb{F}}(\mathscr{M})$. We may assume that $A_{\mathbb{F}}$ has rank m since otherwise by row operations (including subtraction, since $\mathbb{F}$ is a field) we can bring $A_{\mathbb{F}}$ to have exactly m nonzero rows. If $m = 0$ we are done, so throughout we assume that $m > 0$.

Let $J_1, \dots, J_\ell$ be the bases of $\mathscr{M}$. Given a basis J_i, then $A_{\mathbb{F}}$ has an $m \times m$ witness $A_{\mathbb{F}}[\,*\,, J_i]$ (see Notation 4.1). Applying classical row operations to $A_{\mathbb{F}}$, including subtraction, we can reduce $A_{\mathbb{F}}$ so that the submatrix $A_{\mathbb{F}}[\,*\,, J_i]$ is a triangular matrix, i.e., 1 over all the main diagonal and 0 above the diagonal; the entries of $A_{\mathbb{F}}[\,*\,, E \setminus J_i]$ can take arbitrary values. We denote this matrix by $A_{\mathbb{F}}^{(i)}$. We repeat the same process with respect to each basis J_i, $i = 1, \dots, \ell$, to obtain the $m \times n$ matrices $A_{\mathbb{F}}^{(i)}$ over $\mathbb{F}$.

We construct the $m\ell \times n$ matrix $B_{\mathbb{F}}$ by *stacking* the ℓ matrices $A_{\mathbb{F}}^{(i)}$ by the indexing order $i = 1, \dots, \ell$. Note that, since we have used only row operations to obtain the matrices $A_{\mathbb{F}}^{(i)}$'s, as well as duplications of rows, the columns of $B_{\mathbb{F}}$ satisfy exactly the same linear dependence relations which were satisfied by the columns of $A_{\mathbb{F}}$. Thus $B_{\mathbb{F}}$ is also an $\mathbb{F}$-representation of $\mathscr{M}$.

We introduce the boolean matrix $B_{\mathbb{B}}$ obtained from $B_{\mathbb{F}}$ by setting all the nonzero entries of $B_{\mathbb{F}}$ to 1 and leaving the 0's as they were. In the same way, we obtain the boolean matrices $B_{\mathbb{B}}^{(i)}$ from $A_{\mathbb{F}}^{(i)}$. (Of course stacking the boolean matrices $B_{\mathbb{B}}^{(i)}$ by the indexing order yields $B_{\mathbb{B}}$ again.)

Let $Y_i \subseteq \mathrm{Row}(B_{\mathbb{F}})$ be the rows of $B_{\mathbb{F}}$ corresponding to the matrices $A_{\mathbb{F}}^{(i)}$. Therefore, $A_{\mathbb{F}}^{(i)}[Y_i, J_i]$ is an $m \times m$ witness and so does $B_{\mathbb{B}}^{(i)}[Y_i, J_i]$, since is of the Form (3.4). Using Corollary 3.14, it easy to see that by this construction the columns of $B_{\mathbb{B}}[\,*\,, J_i]$ are independent since $B_{\mathbb{B}}^{(i)}[Y_i, J_i]$ is an $m \times m$ witness contained in $B_{\mathbb{B}}[\,*\,, J_i]$.

To complete the proof we need to show that we have not introduced new independent column subsets other than the ones we had in $A_{\mathbb{F}}$. Suppose $X \subseteq E$, with $|X| = k$, and assume that the columns of $B_{\mathbb{B}}[\,*\,, X]$ are independent. Thus, $B_{\mathbb{B}}[\,*\,, X]$ contains a $k \times k$ witness $D_{\mathbb{B}} := B_{\mathbb{B}}[Y, X]$, for some $Y \subseteq \mathrm{Row}(B_{\mathbb{B}})$ with $|Y| = k$, which by Lemma 3.2 is of the Form (3.4), up to permutation. But the witness $D_{\mathbb{B}}$ was obtained from a submatrix $D_{\mathbb{F}}$ of $B_{\mathbb{F}}$ by changing every nonzero entry to be 1. Thus, $D_{\mathbb{F}}$ is also of the Form (3.4), up to permeating of rows and columns, where the elements below the diagonal are now elements of $\mathbb{F}$. This means that the matrix $D_{\mathbb{F}}$ is a $k \times k$ (field) witness, also a submatrix of $B_{\mathbb{F}}$. Namely the columns of $B_{\mathbb{F}}[\,*\,, X]$ were independent, and thus also in the initial $\mathbb{F}$-representation $A_{\mathbb{F}}(\mathscr{M})$ of $\mathscr{M}$.

Therefore this shows that $B_{\mathbb{B}}$ is a proper boolean representation of the matroid $\mathscr{M}$. □

Having Theorem 5.4 at hand, we can generalize it much further.

Theorem 5.5. *Suppose $\mathscr{M} = \mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_\ell$, where $\mathscr{M}_i := (E_i, \mathcal{H}_i)$, is a matroid, with disjoint E_i's and each $\mathscr{M}_i$ is $\mathbb{F}_i$-representable matroid for some field $\mathbb{F}_i$. Then $\mathscr{M}$ has a boolean representation.*

Proof. As proved in Theorem 5.4, every $\mathscr{M}_i$ is $\mathbb{B}$-representable, let $A_{\mathbb{B}}(\mathscr{M}_i)$ be its $\mathbb{B}$-representation. Then we claim that the matroid $\mathscr{M}$ has the $\mathbb{B}$-representation

$$A_{\mathbb{B}}(\mathscr{M}) := \begin{pmatrix} A_{\mathbb{B}}(\mathscr{M}_1) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{\mathbb{B}}(\mathscr{M}_\ell) \end{pmatrix}.$$

Given any subsets $X_1 \in \mathcal{H}_1, \dots, X_\ell \in \mathcal{H}_\ell$, where X_i can be empty, clearly the submatrix $A_{\mathbb{B}}[\,*\,, \bigcup_i X_i]$ is of rank $\sum_i |X_i|$ by construction. On the other hand, suppose that the columns of the submatrix $A_{\mathbb{B}}[\,*\,, X]$ are independent and write $X = X_1 \,\dot\cup\, \cdots \,\dot\cup\, X_\ell$, with $X_i \subseteq E_i$ (could be $\emptyset$). Let $Y_i \subset \mathrm{Row}(A_{\mathbb{B}}(\mathscr{M}))$ be the subset of rows corresponding to $A_{\mathbb{B}}(\mathscr{M}_i)$ in $A_{\mathbb{B}}(\mathscr{M})$. Then, since $A_{\mathbb{B}}[\,*\,, X]$ is independent, any of its column subsets is also independent and in particular each $A_{\mathbb{B}}[\,*\,, X_i]$. Since we have not introduced any new 1-entries in $A_{\mathbb{B}}(\mathscr{M})$, expect those of the matrices $A_{\mathbb{B}}^{(i)} := A_{\mathbb{B}}(\mathscr{M}_i)$, the columns of $A_{\mathbb{B}}[\,*\,, X_i]$ are independent, as well as $A_{\mathbb{B}}[Y_i, X_i] = A_{\mathbb{B}}^{(i)}[\,*\,, X_i]$, and thus $X_i \in \mathcal{H}_i$. □

We can conclude the following immediately:

Corollary 5.6. *There are hereditary collections (and in particular matroids) which are* ***not*** *$\mathbb{F}$-representable over any field $\mathbb{F}$ but do have a $\mathbb{B}$-representation.*

As an example for the corollary consider the well known matroids, the Fano matroid F_7 and the non-Fano matroid F_7^- (see Section 5.3 below for explicit description). It is known that F_7 is $\mathbb{F}$-repressible iff

$\mathbb{F}$ is a field of characteristic 2, while F_7^- has a field representation iff $\mathbb{F}$ is of characteristic $\neq 2$, cf. [20, Proposition 5.3]. Accordingly, the direct sum $F_7 \oplus F_7^-$ of these matroids is not representable over any field, cf. [20, Corollary 5.4], but it is $\mathbb{B}$-representable by Theorem 5.4.

5.3. **Fano and non-Fano matroids.** Let

$$A_7 := \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

be the matrix whose columns are labeled by the ground set $E = \{1, \dots, 7\}$.

Considering A_7 as a boolean matrix, written $\mathbb{B}(A_7)$, the columns of $\mathbb{B}(A_7)$ are all the possible nonzero boolean 3-tuples of the 3-space $\mathbb{B}^{(3)}$. The independent column subsets of A_7 correspond to the independent subsets of vectors of $\mathbb{B}^{(3)}$, and thus introduce a $\mathbb{B}$-vector hereditary collection (cf. Definition 4.3), denoted by $\mathscr{H}(\mathbb{B}^{(3)})$ and identified with $\mathbb{B}(A_7)$. Abusing notation we write $\mathbb{B}(A_7)$ for this hereditary collection, but no confusion should arise.

A direct computation shows that the independent subsets of E, determined by $\mathbb{B}(A_7)$, are, the empty set, all the subsets with 1 or 2 elements, and all the 3-subsets of E except the following ten 3-subsets:

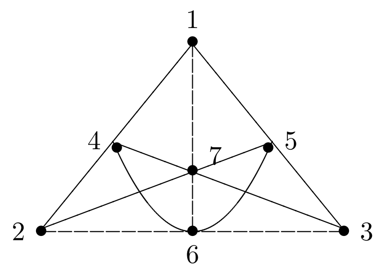
$$\{1,2,4\}, \quad \{1,3,5\}, \quad \{1,6,7\}, \quad \{2,3,6\}, \quad \{2,5,7\},$$
$$\{3,4,7\}, \quad \{4,5,6\}, \quad \{4,5,7\}, \quad \{4,6,7\}, \quad \{5,6,7\}.$$

$\mathbb{B}(A_7)$ satisfies PR but is not a matroid, since considering the two bases

$$B_1 = \{1,5,7\} \qquad \text{and} \qquad B_2 = \{2,4,7\}$$

we see that the element 5 from B_1 can replace neither 2 nor 4 and preserve BR. The hereditary collection $\mathscr{H}(\mathbb{B}^{(3)})$ has $35 - 10 = 25$ bases.

Next, consider A_7 as a matrix over a field $\mathbb{F}_2$ of characteristic 2 to obtain the Fano matroid $F_7 := \mathbb{F}_2(A_7)$, described by the diagram



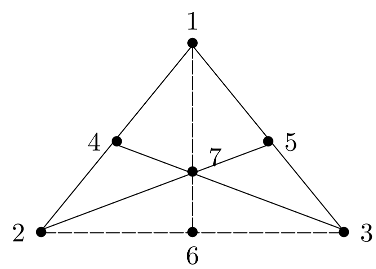
(See [21] for more explanation of the notation.)

The bases of the matroid F_7 are all the 3-subsets of E except those 3-subsets which lie on a same line (could also be a curved line); these 3-subsets are:

$$\{1,2,4\}, \quad \{1,3,5\}, \quad \{1,6,7\}, \quad \{2,3,6\}, \quad \{2,5,7\},$$
$$\{3,4,7\}, \quad \{4,5,6\}.$$

So, we have joined the three independent 3-subsets $\{4,5,7\}, \{4,6,7\}, \{5,6,7\}$ to those of $\mathbb{B}(A_7)$.

The non-Fano matroid, denoted $F_7^- := \mathbb{F}_3(A_7)$, is given by the diagram



with A_7 considered as a matrix over a field $\mathbb{F}_3$ of characteristic 3. The bases of F_7^- are then all the 3-subsets of E except:

$$\begin{aligned}&\{1,2,4\}, \quad \{1,3,5\}, \quad \{1,6,7\}, \quad \{2,3,6\}, \quad \{2,5,7\},\\ &\{3,4,7\}.\end{aligned}$$

The boolean representation $A_{\mathbb{B}}(F_7)$ of F_7, obtained from $\mathbb{B}(A_7)$, is given by the matrix

$$A_{\mathbb{B}}(F_7) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

One sees that the restriction of $A_{\mathbb{B}}(F_7)$ to the three upper rows is the matrix $\mathbb{B}(A_7)$. It easy to verify that the two bottom lines of $A_{\mathbb{B}}(F_7)$ provides the independence of the 3-subsets $\{4,5,7\}, \{4,6,7\}, \{5,6,7\}$ and has no influence on the other dependent 3-subsets.

By the same argument we obtain from the matrix $A_{\mathbb{B}}(F_7)$, the $\mathbb{B}$-representation $A_{\mathbb{B}}(F_7^-)$ of F_7^-:

$$A_{\mathbb{B}}(F_7^-) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

where now we had to add an additional row to make the column set $\{4,5,6\}$ independent without changing the existing dependence relations of the columns of $A_{\mathbb{B}}(F_7)$.

The matrix

$$A(F_7 \oplus F_7^-) = \left(\begin{array}{c|c} A_{\mathbb{B}}(F_7) & 0 \\ \hline 0 & A_{\mathbb{B}}(F_7^-) \end{array}\right)$$

gives a boolean representation of the direct sum $F_7 \oplus F_7^-$, which is a matroid that is known not to be representable over any field, cf. [20, Corollary 5.4].

5.4. **Boolean representations and matching in bipartite graphs.** Given a hereditary collection $\mathscr{H} = (E, \mathcal{H})$, with $|E| = n$, that has a boolean-representation by an $m \times n$ matrix $A_{\mathbb{B}} := A_{\mathbb{B}}(\mathscr{H})$, we associate the matrix $A_{\mathbb{B}} = (a_{i,j})$ with the bipartite graph $G := (V'_G \cup V''_G, E_G)$ having $m+n$ vertices $V'_G \cup V''_G$, i.e., $|V'_G| = m$ and $|V''_G| = n$, and edges $(i', j'') \in E_G$, where $i' \in V'_G$ and $j'' \in V''_G$, iff $a_{i',j''} = 1$ in the matrix $A_{\mathbb{B}}$. (See [19, Chapter 2], and also [15], [21], for more details.)

Recall that by condition WT (cf. Definition 2.5) a subset $X \subseteq E$ (realized also as a column subset $X \subseteq \mathrm{Col}(A_{\mathbb{B}})$) is independent iff there exists a row subset $Y \subseteq \mathrm{Row}(A_{\mathbb{B}})$ with $|X| = |Y|$ such that the submatrix $A_{\mathbb{B}}[Y, X]$ is a witness (cf. Notation 4.1). Abusing the notation and considering respectively X and Y also as vertex subsets of V''_G and V'_G in the graph G, we see that X is independent in $\mathscr{H}$ iff there exists a vertex subset $Y \subseteq V'_G$ so that G has a **unique matching** of X onto Y. That is, there is one and only one matching of X onto Y in the graph G of $A_{\mathbb{B}}$.

If we consider the vertex subsets of V''_G which have some matching (not necessarily unique) to some vertex subsets Y of V'_G with respect to G, we obtain the usual transversal matroid of G (see [21]). When we restrict to those subsets of V''_G having unique onto matchings in the above sense, we obtain, in general, a hereditary collection over the same ground structure with less independent subsets of V''_G which satisfy PR (cf. Definition 2.5) but is not necessarily a matroid. However, since every traversal matroid is representable by some field (cf. [21]), every transversal matroid corresponds to the unique onto matchings of some, in general different, bipartite graph by Theorem 5.4.

Thus, boolean representations of hereditary collection (which must satisfy PR) have a strong connection to classical matching theory.

6. Open questions

We open with the main questions first.

Question 6.1. *Do all the matroids have to have a $\mathbb{B}$-representation?*
If not, which matroids do have $\mathbb{B}$-representations?

Question 6.2. *Do all the hereditary collections satisfying PR have $\mathbb{B}$-representations?*
If not, which such hereditary collections have $\mathbb{B}$-representations?

Question 6.3. *Which hereditary collections (matroids) with $|E| = n$ are $\mathbb{SB}$-representable by $m \times n$ matrices where $m \leq n$?*

Let $\mathbb{K}$ be a commutative semiring, for a given hereditary collection $\mathscr{H}$ we define the function

$$s_{\mathbb{K}} : \ \mathscr{H} \ \to \mathbb{N} \cup \{\infty\}$$

whose value in $\mathbb{N}$ is the minimal number of rows of any $\mathbb{K}$-representation of $\mathscr{H}$ and is ∞ when $\mathscr{H}$ is not $\mathbb{K}$-representable. Clearly $s_{\mathbb{K}}(\mathscr{H}) \geq \operatorname{rk}(\mathscr{H})$ for any $\mathscr{H}$. For example, we showed in Example 4.5 that $s_{\mathbb{B}}(U_{2,n}) \leq n-1$ over the boolean semiring.

Question 6.4. *Compute $s_{\mathbb{K}}(\mathscr{H})$.*

Question 6.5. *Given a hereditary collection, is $s_{\mathbb{K}}(\mathscr{H})$ computable in the computer science sense for $\mathbb{K}$ boolean, superboolean, max plus, etc.?*

When $\mathbb{K}$ is a field and $\mathscr{M}$ is a matroid $s_{\mathbb{K}}(\mathscr{M})$ is either ∞ or equals to the rank of $\mathscr{M}$. So the main questions are the Rota's conjectures [20, 21].

Question 6.6. *What are the possible values of $s_{\mathbb{K}}(\mathscr{H})$ when $\mathscr{H}$ is hereditary collection which also satisfies PR and $\mathbb{K}$ is the boolean or the superboolean semiring? What are the lower and upper bounds for $\mathbb{SB}$-representations?*

Theorem 4.6 shows that $s_{\mathbb{SB}}(\mathscr{H}) < \infty$ for any hereditary collection $\mathscr{H}$ and a bound on $s_{\mathbb{SB}}(\mathscr{H})$ is given in Equation (4.4).

Question 6.7. *When is $s_{\mathbb{SB}}(\mathscr{H}) \leq |E|$, or $s_{\mathbb{SB}}(\mathscr{H}) = \operatorname{rk}(\mathscr{H})$? Find a lower bound for $\mathbb{SB}$-representations with respect to the rank of $\mathscr{H}$.*

Appendix A. Tropical and supertropical algebra

A **semiring** $(R, +, \cdot\,, \mathbb{0}, \mathbb{1})$, written $(R, +, \cdot\,)$ for short, is a set R endowed with two binary operations $+$ and $\cdot$, addition and multiplication, respectively, and distinguished elements $\mathbb{0}$ and $\mathbb{1}$, such that $(R, \cdot\,, \mathbb{1})$ is a monoid and $(R, +, \mathbb{0})$ is an commutative monoid satisfying distributivity of multiplication over addition on both sides, and such that $\mathbb{0} \cdot a = a \cdot \mathbb{0} = \mathbb{0}$ for every $a \in R$ [27, §8-§9]. A (two sided) semiring **ideal** $\mathfrak{a}$ of $R := (R, +, \cdot\,, \mathbb{0}, \mathbb{1})$ is an additive sub-semigroup of $(R, +, \mathbb{0})$, i.e., $a, b \in \mathfrak{a}$ implies $a + b \in \mathfrak{a}$, for which $xa \in \mathfrak{a}$ and $ax \in \mathfrak{a}$ for every $x \in R$ and $a \in \mathfrak{a}$.

A semiring R is additively **idempotent** if $a + a = a$ for every $a \in R$. Letting $R^{\times} := R \setminus \{\mathbb{0}\}$, when $(R^{\times}, \cdot\,, \mathbb{1})$ is an Abelian group, we say that R is a **semifield**. The notion of semifield does not have a formal consistent definition in the literature, for that reason we preserve the terminology of semirings along this paper.

A.1. Tropical structures. Traditionally, tropical algebra takes place over the tropical (max-plus) semiring $\mathbb{R}_{(\max,+)} := (\mathbb{R} \cup \{-\infty\}, \max, +)$, the real numbers together with the formal element $-\infty$ equipped with the operations of maximum and summation, providing respectively the semiring addition and the multiplication. Over this setting $\mathbb{0} := -\infty$ is the zero element of the semiring and the number 0 is the multiplicative unit, denoted $\mathbb{1}$. Dually, one has the min-plus semiring $\mathbb{R}_{(\min,+)} := (\mathbb{R} \cup \{\infty\}, \min, +)$, where now $\mathbb{0} := \infty$. (Both structures are semifields according to the above definition.)

Remark. *The boolean semiring is embedded naturally in the tropical semiring $\mathbb{R}_{(\max,+)}$, the embedding $\varphi : \mathbb{B} \hookrightarrow \mathbb{R}_{(\max,+)}$ is given by*

$$\varphi : 1 \mapsto 0, \qquad \varphi : 0 \mapsto -\infty.$$

The max-plus semiring $\mathbb{R}_{(\max,+)}$ is a special case of an (additive) idempotent semiring [18], i.e., a semiring in which $a + a = a$ for any $a \in \mathbb{R}$. In general, one may replace the semiring $\mathbb{R}_{(\max,+)}$ by an idempotent semiring $R := (R, +, \cdot\,)$ satisfying the **bipotence property**

$$a + b \in \{a, b\}, \qquad \text{for any } a, b \in R.$$

(Note that R is then ordered by the rule $a \geq b \Leftrightarrow a + b = a$.) We call such a semiring a **bipotent semiring**; for example the boolean semiring, as well as the tropical semiring, is a bipotent semiring.

Bipotent semirings arisen naturally from (totally) ordered cancellative monoids in the following way. Given an ordered monoid $(M, \cdot\,)$, we adjoin M with the formal element $-\infty$, declaring $-\infty < a$ for any $a \in M$. Then, the addition of $M \cup \{-\infty\}$ is defined as

$$a + b = \max\{a, b\} \qquad \text{for any } a, b \in M,$$

where the multiplication is given by the original monoid operation of M, extended with $a(-\infty) = (-\infty)a = -\infty$. By this construction, when the monoid M is an Abelian group, the obtained semiring is a semifield.

A.2. Supertropical structures. A **supertropical semiring** is a semiring $R := (R, +, \cdot\,, \mathcal{G}_0, \nu)$ with a distinguished ideal $\mathcal{G}_0$, called the **ghost ideal**, and a semiring projection $\nu : R \to \mathcal{G}_0$, satisfying the axiom (writing a^ν for $\nu(a)$):

$$\begin{array}{lll} \textit{Supertropicality}: & a + b = a^\nu & \text{if} \quad a^\nu = b^\nu. \\ \textit{Bipotence}: & a + b \in \{a, b\} & \text{if} \quad a^\nu \neq b^\nu. \end{array}$$

Thus $\mathcal{G}_0$ is equipped with the natural partial order $a^\nu \geq b^\nu$ iff $a^\nu + b^\nu = a^\nu$, which is incorporated into the semiring structure, written

$$a \geq_\nu b \qquad \text{iff} \qquad \nu(a) \geq \nu(b),$$

Note that, by definition, in the supertropical arithmetics we have

$$\mathbb{1} + \mathbb{1} = \mathbb{1} + \mathbb{1} + \mathbb{1} = \ \cdots \ = \mathbb{1} + \mathbb{1} + \cdots + \mathbb{1} = \mathbb{1}^\nu$$

for any arbitrary number of summands greater than two, and furthermore

$$a + a = a + a + a = \ \cdots \ = a + a + \cdots + a = a^\nu, \qquad \text{for any } a \in R;$$

therefore a supertropical semiring is not idempotent. Accordingly we also have

$$\mathbb{1} + \mathbb{1}^\nu = \mathbb{1}^\nu + \mathbb{1}^\nu = \ \cdots \ = \mathbb{1}^\nu + \cdots + \mathbb{1}^\nu = \mathbb{1}^\nu,$$

and the same for $a + a^\nu = a^\nu$.

A **supertropical semifield** $F := (F, +, \cdot\,, \mathcal{G}_0, \nu)$ is a supertropical semiring with a totally ordered ghost ideal $\mathcal{G}_0$, such that $\mathcal{T} := F \setminus \mathcal{G}_0$ is an Ablian group, called the group of **tangible elements**, for which the restriction $\nu|_\mathcal{T} : \mathcal{T} \to \mathcal{G}$, $\mathcal{G} := \mathcal{G}_0 \setminus \{\mathbb{0}\}$, is onto.

A.3. Supertropicalization. Any bipotent semiring $R = (R, +, \cdot\,)$ can be "supertropicalized" as following, cf. [10]. Consider the disjoint union

$$T(R) := \mathcal{T} \ \dot\cup \ \{\mathbb{0}\} \ \dot\cup \ \mathcal{G},$$

with $\mathcal{T} = \mathcal{G} = R \setminus \{\mathbb{0}\}$. Denote the members of $\mathcal{G}$ by a^ν, for each $a \in \mathcal{T}$, and let $\nu : T(R) \to \mathcal{G}$ be the map sending $a \mapsto a^\nu$ and be the identity on $\mathcal{G} \cup \{\mathbb{0}\}$. Writing x, y for general elements in R, the new semiring operations $\tilde{+}$ and $\tilde{\cdot}$, addition and multiplication respectively, are then defined as:

$$x \,\tilde{+}\, y = \begin{cases} x & \nu(x) > \nu(y), \\ y & \nu(x) < \nu(y), \\ \nu(x) & \nu(x) = \nu(y), \end{cases}$$

for any $x, y \in R$, and

$$\begin{array}{ll} a \,\tilde{\cdot}\, b = ab & a, b \in \mathcal{T}, \\ a \,\tilde{\cdot}\, b^\nu = b^\nu \,\tilde{\cdot}\, a = (ab)^\nu & a \in \mathcal{T}, b^\nu \in \mathcal{G}, \\ x \,\tilde{\cdot}\, \mathbb{0} = \mathbb{0} \,\tilde{\cdot}\, x = \mathbb{0} & \forall x \in T(R). \end{array}$$

Then, $T(R) := (T(R), \widetilde{+}, \widetilde{\cdot}, \mathcal{G}_0, \nu)$, with $\mathcal{G}_0$ and the ghost map $\nu : T(R) \to \mathcal{G}$ as defined above, is a supertropical semiring.

The ghost ideal in this construction is the copy R^ν of R and the **tangible** elements are $\mathcal{T} = R \setminus \{\mathbb{0}\}$. Moreover, the semiring ideal $\mathcal{G} := R^\nu$ is a semiring by itself isomorphic to R, therefore ν composed with this isomorphism provides an epimorphism $T(R) \to R$. When the initial semiring R is a semifield, then supertropicalization $T(R)$ is a supertropical semifield.

A major example of the above construction is provided by starting with the familiar max-plus algebra. Taking the supertropicalization of the standard tropical semiring $\mathbb{R}_{(\max,+)}$ we obtained the extended tropical semiring [8]

$$\mathbb{T} := T(\mathbb{R}_{(\max,+)}) = \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^\nu,$$

having the tangibles $\mathcal{T} := \mathbb{R}$ and ghosts $\mathcal{G} := \mathbb{R}^\nu$, the ghost map is given by $a \mapsto a^\nu$ for any $a \in \mathbb{R}$, where the semiring operations of $\mathbb{T}$ are as described above. Therefore, $\mathbb{T}$ can be thought of as the super-max-plus algebra.

Using the same construction, one sees that the superboolean semiring is a supertropicalization of the boolean semiring, in other words $\mathbb{SB} = T(\mathbb{B})$.

A.4. Supertropical matrix algebra. Given a supertropical semifield R, the algebra of matrices over R is developed exactly along the same line of §3.1, see [11, 12], using similar definitions which are now taken with respect to the larger ghost ideal of the ground supertropical semifield. The results from matrix algebra presented in our exhibition in §3.1, Theorem 3.11, Corollary 3.12, Corollary 3.15, and Theorem 3.16, are all valid in general for matrices taking place over any supertropical semifield, [11, 14].

Appendix B. Tropical representations of hereditary collections

Superboolean representations of hereditary collections can be performed in a much wider context obtained by replacing the ground superboolean semiring $\mathbb{SB}$ by a supertropical semifield F, for example by $\mathbb{T} := T(\mathbb{R}_{(\max,+)})$. Namely, given an $m \times n$ matrix A_F over a supertropical semifield F, we associate the ground set $E := E(A_F)$ to the set of columns $\mathrm{Col}(A_F)$ of A_F, which as usual are realized as vectors in $F^{(n)}$. The independent subsets $\mathcal{H} := \mathcal{H}(A_F)$ of E are subsets corresponding to column subsets that are tropically independent of the n-space $F^{(n)}$, cf. [11, Definition 6.3]. The F-vector hereditary collection $(E(A_F), \mathcal{H}(A_F))$ is denoted $\mathscr{H}(A_F)$. A hereditary collection $\mathscr{H}'$ that is isomorphic to $\mathscr{H}(A_F)$ for some matrix A_F over a supertropical semifield F is called F-representable; the matrix A_F is called an F-representation of $\mathscr{H}'$.

There is a natural semiring embedding $\varphi : \mathbb{SB} \hookrightarrow F$, given by $\varphi : 1 \mapsto \mathbb{1}$, $\varphi : 1^\nu \mapsto \mathbb{1}^\nu$, $\varphi : 0 \mapsto \mathbb{0}$, of the superboolean semiring $\mathbb{SB}$ into an arbitrary supertropical semifield F. Since $\{\mathbb{1}, \mathbb{1}^\nu, \mathbb{0}\} \subseteq F$ is a subsemiring of F, this embedding induces a natural matrix embedding $\widetilde{\varphi} : M_n(\mathbb{SB}) \hookrightarrow M_n(F)$, and thus an embedding of representations. Therefore, $\mathbb{SB}$-representations can be viewed as F-representations, which in a sense are more comprehensive than $\mathbb{SB}$-representations, and more generally as R-representations, for R a (commutative) supertropical semiring. Then, by Theorem 4.6, we immediately conclude the following.

Corollary. *Every hereditary collection is R-representable, over any supertropical semiring R.*

Of course one can construct "richer" F-representations of hereditary collections by involving elements of F other than $\mathbb{0}$, $\mathbb{1}$, or $\mathbb{1}^\nu$.

In general, all the results within this paper can be stated in the context of supertropical semifields. However, to make the exposition clearer, in this paper we have used the simpler structure of matrices over the superboolean semiring $\mathbb{SB}$, aiming to introduce the idea of representing hereditary collections by considering matrices over semirings. As have been shown these matrices are suitable enough for this purpose.

F-representations of matroids, and more genrally of hereditary collections, will be discussed in details in a future paper.

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