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The Cleavage Operad and String Topology of
Higher Dimension

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The Cleavage Operad and String Topology of Higher Dimension

November 26, 2011

1 Introduction

In [Vor05], Voronov gave life to the Cacti operad: An operad whose homology acts on the shifted homology of the free loop space over a compact, smooth, orientable manifold $M - \mathbb{H}_*(M^{S^1})$ – giving it the structure of a Batalin-Vilkovisky algebra; hereby recovering the Chas-Sullivan product of [CS99]. As an intermediate operad, Kaufmann in [Kau05] gave an E_2-operad – the spineless cacti operad – whose homology acts to give a Gerstenhaber structure, underlying the Batalin-Vilkovisky structure of Chas and Sullivans String Topology; all reflected in the fact that taking the semi-direct product of the spineless cacti with SO(2) yields the Cacti operad.

We follow the same general string of ideas, but generalize them by replacing S^1 with a manifold $N \subseteq \mathbb{R}^{n+1}$ – of arbitrary dimension – embedded in euclidean space. Our methods will involve certain decompositions of N, and for convexly embedded spheres these decompositions are simple enough to obtain results within topology, our focus will thus take a shift towards $N := S^n \subset \mathbb{R}^{n+1}$ the unit-sphere.

What we construct is a coloured operad that acts on M^N – the space of maps from N to M – in a related manner to how the Cacti operad acts on M^{S^1}. As revealed by the previous sentence, we found it necessary to broaden the scope of the use of the word 'operad' – and enter the realm of coloured operads; coloured over topological spaces. As we describe in section 2, this colouring is similar to picking a category internal to topological spaces, with traditional operads being one-object gadgets. We show in 5.22 that for $N = S^n$, the homotopy type of this operad is computable, using combinatorial methods of [Ber97], as a coloured E_{n+1}-operad. We then show how to form a semidirect product of this operad with SO($n+1$), providing a $(n+1)$-Batalin-Vilkovisky structure on the homology of M^{S^n}.

In [CV06][Ch. 5], an outline is given for a generalisation of the Cacti operad to the n-dimensional Cacti operad, by replacing lobes with copies of S^n floating in $\mathbb{R}^{n+1}$. Our original motivation was to explicitly compute the structure of this operad; attempting to construct homotopies equivalating the little $(n+1)$-disks operad and the n-dimensional

cacti operad, as was done in [Bar08] for the 1-dimensional case. However, such attempts did not seem to have a shortcut, bypassing the structure of the diffeomorphism group $\mathrm{Diff}(S^n)$.

Although we are working with coloured operads, and so give an operad different from the Cacti operad, morally we take the stance of [Kau05] – starting from a more rigid structure, where diffeomorphisms have no influence until the twisting of section 6 can be inferred. The coloured operad we define have an operadic structure that is basically given by cleaving N into smaller submanifolds – timber – and therefore we dub the operad, the *Cleavage Operad over N*, $\mathcal{Cleav}_N$.
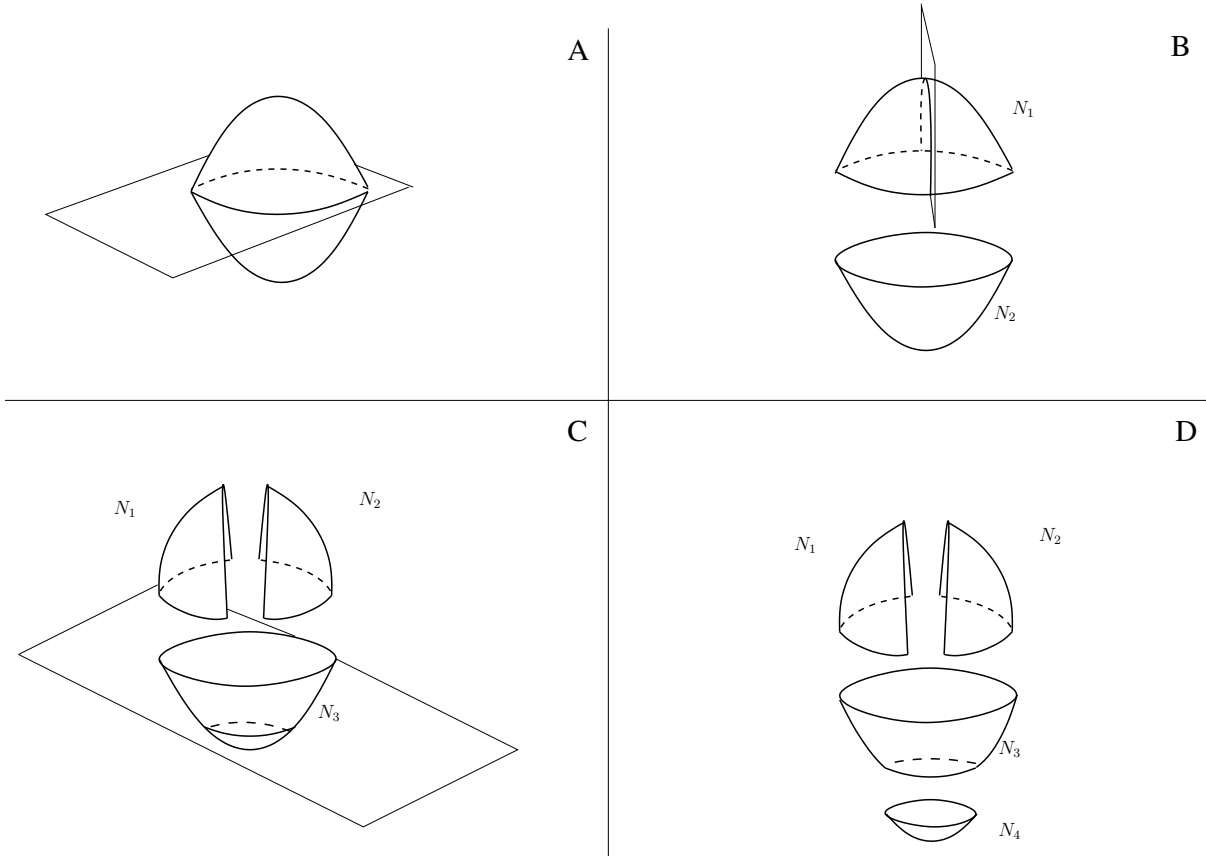

Figure 1: Recursive procedure cleaving a sphere into four pieces of timber

An example of such a cleavage is given in Figure 1, starting in picture A, by cleaving N by a single hyperplane chopping N into two pieces of timber N_1 and N_2 in picture B; and hence succesively cleaving the timber produced into smaller subsets of N. Note that the non-linear ordering in which the cleaving is done constitutes part of the data; for instance, interchanging the cuts made in A and B would yield a different cleavage.

However, in order to have an interesting topology, we need to forget as much of the ordering dictated by the indexing trees as possible. In 3.8, we forget what needs to be forgotten by defining $\mathcal{Cleav}_N$ as a quotient of an operad with k-ary structure given as pairs $(T, \underline{P})$ where T is a k-ary binary tree expressing the ordering with which N

2

should be cleaved, and $\underline{P} = \{P_1, \ldots, P_{k-1}\}$ a tuppel of affine hyperplanes – conveying information on where to cleave by decorating the internal vertices of T.

In terms of Figure 1, this quotient would allow for an interchange of the ordering of hyperplanes, such that the hyperplane of C cleaves before A and B, as well as allowing an interchange of the hyperplanes of B and C.

Summary of Results

As usual in String Topology, fix a dimension d of the manifold M, and let $\mathbb{H}_*(-) := \mathrm{H}_{*+d}(-)$.

Recall that the correspondence category $\mathrm{Corr}(\mathcal{C})$ over a co-complete category $\mathcal{C}$ is given by letting $\mathrm{Ob}(\mathrm{Corr}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$, and the set of morphisms from objects X to Y, $\mathrm{Corr}(\mathcal{C})(X, Y)$ be given as diagrams $X \longleftarrow Z \longrightarrow Y$ where Z is an arbitrary object, and the arrows are morphisms in $\mathcal{C}$. Composition is given by taking pull-backs.

Theorem A The operad $\mathcal{Cleav}_N$ acts on M^N in the category of correspondences over topological space.

We furthermore indicate how this gives rise to the statement that letting $N := S^n \subseteq \mathbb{R}^{n+1}$ the unit sphere, and M a compact, smooth, orientable d-manifold, $\mathrm{H}_*(\mathcal{Cleav}_{S^n})$ acts on $\mathbb{H}_*(M^{S^n})$.

Producing this homological action requires new techniques, and these are the focus of the upcoming [Bar11]

To examplify the action on correspondences, made precise in section 4, take the cleavage in picture D of Figure 1, note first that the collective boundary of the submanifolds N_1, N_2, N_3 and N_4 of N has two components C_1 and C_2. Consider the space

$$M^N_{C_1, C_2} := \{f \in M^N \mid f(C_1) = \{k_1\} \subseteq M, f(C_2) = \{k_2\} \subseteq M\}$$

– i.e. the space of maps from N to M that are constant on all of C_1 and all of C_2. The correspondence $M^N \xleftarrow{\iota} M^N_{C_1, C_2} \xrightarrow{\varphi} (M^N)^k$ is given with ι the inclusion. The other map φ maps into the i'th factor of $(M^N)^k$ by taking $\varphi(f)(m) = f|_{N_i}(m)$ for $m \in N_i$ and for $m \notin N_i$ letting $\varphi(f)(m)$ be the same constant as $f(C_i)$ where C_i is the component separating m from N_i.

In [CJ02], it was discovered that for the Cacti operad, a passage to the latter part of 1 can be done via spectra to obtain so-called umkehr maps, homologically reversing one of the arrows in the correspondences. We follow this idea, and as in [CJ02] work with something stronger than stated in 1, namely that the action is realized by taking homotopy groups of a stable map between spectra. where an extended and punctured version of the Cleavage Operad is introduced as technical assistance for this purpose. This paper will only indicate how the action works pointwise in $\mathcal{Cleav}_{S^n}$, as well as the 2-ary term of the operad – providing reason to the specifics of the definitions of $\mathcal{Cleav}_{S^n}$.

It turns out that working with the punctured cleavage operad corresponds to adjoining a unit to the algebra of $\mathcal{Cleav}_{S^n}$. We therefore stress that the algebra described in

Theorem A is a non-untial algebra, and that the coloured operad $\mathcal{C}leav_{S^n}$ does not have 0-ary terms.

Section 5 and beyond are used to show the following:

Theorem B The coloured operad $\mathcal{C}leav_{S^n}$ is a coloured E_{n+1}-operad. Taking a semidirect product of $\mathcal{C}leav_{S^n}$ by $\mathrm{SO}(n+1)$ provides a coloured operad $\mathcal{C}leav_{S^n} \rtimes \mathrm{SO}(n+1)$ whose homologous actions provides $(n+1)$-Batalin-Vilkovisky algebras.

The main technicality in proving the above theorem is the first statement of a coloured E_{n+1}-operad. We apply combinatorial methods of [Ber97] to show this theorem in section 5. The final part of the statement follows in section 6 by the construction of semidirects products as given in [SW03]. Briefly, since $\mathcal{C}leav_{S^n}$ and its action on M^{S^n} is well-behaved with respect to an action of $\mathrm{SO}(n+1)$, we can apply a coloured construction of a semidirect product.

Acknowledgement

2 Operadic and Categorical Concepts

In this section we shall introduce language from the world of operads, used troughout. For a general overview, we refer to the category theoretic [Lei04, 2.1], where they are called *multicategories*.

Definition 2.1 A *coloured operad* $\mathcal{C}$ consist of

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects* or *colours*.

- For each $k \in \mathbb{N}$ and $a, a_1, \ldots, a_k \in \mathrm{Ob}(\mathcal{C})$, a class of k-ary morphisms denoted $\mathcal{C}(a; a_1, \ldots, a_k)$.

- To $i \in \{1, \ldots, k\}$, $f \in \mathcal{C}(a; a_1, \ldots, a_k)$ k-ary and $g \in \mathcal{C}(a_i; b_1, \ldots, b_n)$ n-ary an operadiccomposition $f \circ_i g \in \mathcal{C}(a; a_1, \ldots, a_{i-1}, b_1, \ldots, b_n, a_{i+1}, \ldots, a_k)$ of arity $k + n - 1$.

- Units $\mathbb{1}_a \in \mathcal{C}(a; a)$ for any object a.

- An action of Σ_k, $\sigma.: \mathcal{C}(a; a_1, \ldots, a_k) \to \mathcal{C}(a; a_{\sigma(1)}, \ldots, a_{\sigma(k)})$ for all $k \in \mathbb{N}$ and $a_i \in \mathrm{Ob}(\mathcal{C})$.

These are subject to the following conditions, where we to a $H \subseteq \{1, \ldots, m\}$, denote by Σ_H the permutation group along the elements of H. As by convention let $\Sigma_{|H|}$ denote the permutation group on the first $|H|$ natural numbers. The unique monotone map $H \to \{1, \ldots, |H|\}$ defines an isomorphism $\rho_H \colon \Sigma_H \to \Sigma_{|H|}$:

- Associativity: For $f \in \mathcal{C}\left(a; a_1, \ldots, a_k\right), g \in \mathcal{C}\left(a_i; b_1, \ldots, b_m\right), h \in \mathcal{C}\left(b_j; c_1, \ldots, c_l\right)$ the identity
$$f \circ_i (g \circ_j h) = (f \circ_i) \circ_{j+i-1} h$$
holds for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$. For further $s \in \mathcal{C}(a_r; d_1, \ldots, d_u)$ where $i < r$ we require that

$$(f \circ_i g) \circ_r s = (f \circ_r s) \circ_{i+u-1} g.$$

- Σ_k-equivariance: For $\sigma \in \Sigma_{k+m}$ and $f \in \mathcal{C}(a; a_1, \ldots, a_k), g \in \mathcal{C}(a_i; b_1, \ldots, b_m)$ the identity
$$\sigma.(f \circ_i g) = \sigma|_I.f \circ_{\sigma|_I(i)} \sigma|_J.g$$
holds where $I := \{1, \ldots, i, i+m+1, \ldots, k+m\}$ is the set of integers from 1 to $k+m$ excluding the set $J := \{i+1, \ldots, i+m\}$. For $H \subseteq \{1, \ldots, l\}$ the permutation $\sigma|_H \in \Sigma_{|H|}$ is given as $\sigma|_H := \rho_H\left(\widetilde{\sigma|_H}\right)$ where in turn $\widetilde{\sigma|_H} \in \Sigma|_H$ is defined from σ by requiring that to $l, p \in H$ we have $\widetilde{\sigma|_H}(l) < \widetilde{\sigma|_H}(p)$ whenever $\sigma(l) < \sigma(p)$ so $\sigma|_H$ permutes the ordered symbols of H in the same way that σ permutes $\{1, \ldots, |H|\}$.

- Unit-identity: For $f \in \mathcal{C}(a; a_1, \ldots, a_k)$ we have

$$f \circ_i \mathbb{1}_{a_i} = f \text{ and } \mathbb{1}_a \circ_1 f = f$$

for all $i \in \{1, \ldots, k\}$.

Indeed, a classical *operad* is simply a coloured operad with a single object. We shall refer to such gadgets as *monochrome operads*. On the other hand, a category $\mathcal{C}$ is the same as a coloured operad with $\mathcal{C}(a, a_1, \ldots, a_k) = \varnothing$ for $k > 1$.

Familiar concepts like functors, hom-sets and adjoints are extended in the obvious ways to this multi-arity setting.

Definition 2.2 Let $(\mathcal{A}, \boxtimes)$ be a symmetric monoidal category. The *underlying coloured operad* $\mathcal{U}nd_{\mathcal{A}}$ is given by letting $\mathrm{Ob}(\mathcal{U}nd_{\mathcal{A}}) = \mathrm{Ob}(\mathcal{A})$ and

$$\mathcal{U}nd_{\mathcal{A}}(a; a_1, \ldots, a_n) := \mathrm{Hom}_{\mathcal{A}}(a_1 \boxtimes \cdots \boxtimes a_n, a).$$

The usual (monochrome) endomorphism operad $\mathcal{E}nd_A$ of an object $A \in \mathcal{A}$ is given by considering the full subcategory of $\mathcal{U}nd_{\mathcal{A}}$ generated by $\{A\} \subseteq \mathrm{Ob}(\mathcal{U}nd_{\mathcal{A}})$.

Definition 2.3 An *action* of a coloured operad $\mathcal{C}$ on $\mathcal{A}$ is a functor $\alpha \colon \mathcal{C} \to \mathcal{U}nd_{\mathcal{A}}$. A *monochrome action* of $\mathcal{C}$ is a functor $\alpha \colon \mathcal{C} \to \mathcal{E}nd_A$ for an object $A \in \mathcal{A}$.

In (string) topology, operads are sought after for their actions on topological entities. As stated in the introduction, we venture on the same basic safari, but seek monochrome actions of coloured operads. A point is that as long as we seek monochrome actions, the extra colours on the operad become somewhat opaque – and we get actions similar to that of monochrome operads. To have topological actions of course requires topology to enter the game:

To $\mathcal{O}$ a coloured operad, denote by $\mathcal{O}(-; k)$ the set of all k-ary morphisms of $\mathcal{O}$. Let $\mathcal{O}(A; k)$ be the restriction of $\mathcal{O}(-; k)$ with $A \in \mathrm{Ob}(\mathcal{O})$ incoming.

Definition 2.4 Let $\mathcal{O}$ be a coloured operad. We say that $\mathcal{O}$ is a coloured topological operad if both $\mathrm{Ob}(\mathcal{O})$ and $\mathcal{O}(-; k)$ are topological spaces, along with the data of the following commutative diagram involving a pullback for $m, k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$:

$$\mathcal{O}(-; k + m - 1) \xleftarrow{\circ_i} \mathcal{O}(-; k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-; m) \longrightarrow \mathcal{O}(-; m)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow ev_{\mathrm{in}}$$

$$\mathcal{O}(-; k) \xrightarrow{\quad ev_i \quad} \mathrm{Ob}(\mathcal{O})$$

where ev_i evaluates at the i'th outgoing colour and ev_{in} evaluates at the incoming colour

The structure should naturally adhere to the associativity, unit and Σ_k-equivariance conditions as specified in 2.1.

Note that homology does not in general preserve inverse limits such as a push-out, so applying the homology functor to the diagram 2.4 will not yield another push-out diagram. And in effect not lead to a similar structure in graded modules. One can however define the homology of a coloured topological operad as the coloured operad defined partially by the induced diagram

$$\begin{array}{c}
\mathrm{H}_* \left(\mathcal{O}(-; k+m-1) \right) \\
\Big\uparrow {\scriptstyle \circ_i} \\
\mathrm{H}_* \left(\mathcal{O}(-; k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-; m) \right)
\end{array}$$

$$
\mathrm{H}_*(\mathcal{O}(-;k)) \otimes_{\mathrm{H}_*(\mathrm{Ob}(\mathcal{O}))} \mathrm{H}_*(\mathcal{O}(-;m)) \longrightarrow \mathrm{H}_*(\mathcal{O}(-;m))
$$

$$
\mathrm{H}_*\left(\mathcal{O}(-;k)\right) \xrightarrow{\ (\mathrm{ev}_i)_*\ } \mathrm{H}_*\left(\mathrm{Ob}(\mathcal{O})\right)
$$

with vertical map $(\mathrm{ev}_{\mathrm{in}})_*$.

which in turn can be taken to lead to a notion of partially defined coloured operads. Partial in the sense that the dotted arrow to the pullback-space is not always invertible. For the purpose of actions of operads, such a slightly more technical notion of partially defined operads would generally suffice.

However, the operads we shall define in this thesis will all have contractible colours, and we can instead of introducing partial operads use the following proposition to see that in our case, applying the homology functor to our operads will result in classical operads.

Proposition 2.5 Assume that $\mathcal{O}$ is a coloured topological operad with $\mathrm{Ob}(\mathcal{O}) \simeq *$, and with evalution maps ev_i fibrations for all $i \in \{1, \ldots, k\}$, or with $\mathrm{ev}_{\mathrm{in}}$ a fibration. Then applying homology to $\mathcal{O}$ defines $\mathrm{H}_*(\mathcal{O})$ as a classical monochrome operad in the category of graded modules.

Proof. Since $\mathrm{Ob}(\mathcal{O}) \simeq *$, and the evaluation maps are fibrations, the long exact sequence of homotopy groups along with the 5-lemma tells us that the pullback spaces $\mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m)$ and $\mathcal{O}(-;k) \times \mathcal{O}(-;m)$ are homotopy equivalent for all $k, m \in \mathbb{N}$ so we have that

$$
\mathrm{H}_*(\mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m)) \cong \mathrm{H}_*(\mathcal{O}(-;k) \times \mathcal{O}(-;m)).
$$

The Künneth formula now gives the map

$$
\circ_i \colon \mathrm{H}_*(\mathcal{O}(-;k)) \otimes \mathrm{H}_*(\mathcal{O}(-;m)) \to \mathrm{H}_*(\mathcal{O}(-;k+m-1))
$$

used to define classical operads. By definition of coloured topological operads, this map satisfies the needed associativity, unity and Σ_k-invariance conditions. $\qquad \square$

Definition 2.6 A *morphism* $F \colon \mathcal{O} \to \mathcal{P}$ of topological operads internal to $(\mathcal{A}, \boxtimes)$ is given by morphisms

$$
F_{\mathrm{Ob}} \colon \mathrm{Ob}(\mathcal{O}) \to \mathrm{Ob}(\mathcal{P}), \ F_k \colon \mathcal{O}(-;k) \to \mathcal{P}(-;k), \ \mathcal{O}^{k,m} \to \mathcal{P}^{k,m} \tag{1}
$$

for all $k, m \in \mathbb{N}$, such that these morphisms provide a natural transformation of the diagrams of the type in 2.4 defining the structure for $\mathcal{O}$ and $\mathcal{P}$.

A *weak equivalence* of topological operads is given by a zig-zag of morphisms, where all continous maps of (1) are weak homotopy equivalences.

In particular, a weak equivalence $\mathcal{P} \simeq \mathcal{O}$ of topological coloured operads induces an isomorphism $H_*(\mathcal{P}) \cong H_*(\mathcal{O})$.

Example 2.7 Let $\mathcal{P}$ be a monochrome operad, and X a topological space. We form the trivial X-coloured operad over $\mathcal{P}$, $\mathcal{P} \times X$, by setting

- $\mathrm{Ob}(\mathcal{P} \times X) := X$

- $\mathcal{P} \times X(-;k) := \mathcal{P}(k) \times X$

evaluation maps $\mathcal{P}(k) \times X \rightarrow X$ are given by the projection map, $\circ_i$-composition, pointwise in X, the same as $\circ_i$-composition in $\mathcal{P}$.

Definition 2.8 We say that a coloured operad $\mathcal{O}$ is a *coloured E_n-operad* if there is a weak equivalences of operads between $\mathcal{O}$ and $\mathcal{P} \times \mathrm{Ob}(\mathcal{O})$, where $\mathcal{P}$ is a monochrome E_n-operad.

In 5.5, we use methods of [Ber97] to give a combinatorial way of detecting a coloured E_n-operad. We then use this to show that the operad we define in the next section is a coloured E_{n+1}-operad.

3 The Cleavage Operad

3.1 Definition of the Cleavage Operad

The operadic structure we shall define will be induced from the operadic structure of trees. The trees we consider will all, without further specification, be:

- Binary, in the sense that all vertices are univalent or trivalent.

- Rooted, in the sense that there is a distinguished univalent vertex called the root.

- Labelled, in the sense that for a k-ary tree, the k remaining univalent vertices are numbered from $1, \ldots, k$.

- Planar, specifying edges out of a trivalent vertex as left- right- or down-going.

Tree and *Tree*(k) denotes the set of isomorphism classes of trees, respectively k-ary trees. Grafting of trees defines *Tree* as a (monochrome) operad.

Let $\mathrm{Gr}_n(\mathbb{R}^{n+1})$ be the oriented Grassmanian of codimension 1 subvectorspaces of $\mathbb{R}^{n+1}$.

Definition 3.1 Let the space of *affine, oriented hyperplanes* be given as

$$\mathrm{Hyp}^{n+1} := \mathrm{Gr}_n(\mathbb{R}^{n+1}) \times \mathbb{R}$$

where a pair $(H, p) \in \mathrm{Hyp}^{n+1}$ defines an affine hyperplane $P \subset \mathbb{R}^{n+1}$ by translating H along p. We let the orientation of P be given as the orientation of H as a choice of normal-vector.

Definition 3.2 Let

$$\mathcal{T}\!ree_{\mathrm{Hyp}^{n+1}}(k) := \mathcal{T}\!ree(k) \times \left(\mathrm{Hyp}^{n+1}\right)^{k-1}$$

We call $(T, \underline{P}) \in \mathcal{T}\!ree_{\mathrm{Hyp}^{n+1}}(k)$ a $(n+1)$-decorated k-ary tree. Here $\underline{P} = (P_1, \ldots, P_{k-1})$ denotes the tuple of elements of Hyp^{n+1}. Specifying the trivalent vertices of T as $v_1, \ldots, v_{k-1}$, these are matched to the trivalent vertices of T – and we encourage the reader to pretend that P_i is dangling from v_i.

Denote by $\mathcal{T}\!ree_{\mathrm{Hyp}^{n+1}}$ the operad constituted from the pieces above.

Convention 3.3 Throughout this text, we denote by $N \subseteq \mathbb{R}^{n+1}$ an embedded, smooth manifold We assume further that N has a *recording area*, $\mathrm{Rec}(N) \subseteq \mathbb{R}^{n+1}$ where we have the requirement that N is the boundary of $\mathrm{Rec}(N)$; $N = \partial \mathrm{Rec}(N)$.

We shall allow for the recording area of N to be unspecified from the notation, as it will often be the obvious choice associated with it. However, as will become clear in the definition of $\mathcal{C}leav_N$, the choice of recording area is indeed a part of the data of the resulting operad, and a priori two different choices of recording area will result in two different operads.

Example 3.4 The main example in this paper is $N := S^n$ the unit-sphere inside $\mathbb{R}^{n+1}$. The associated recording area will always be $\mathrm{Rec}(S^n) := D^{n+1}$, the closed unit-disk inside $\mathbb{R}^{n+1}$.

For $P \in \mathrm{Hyp}^{n+1}$, $\mathbb{R}^{n+1} \backslash P$ consist of the two components $(\mathbb{R}^{n+1})_+^P$ and $(\mathbb{R}^{n+1})_-^P$, where $(\mathbb{R}^{n+1})_+^P$ is the space in the direction of the normal-vector of P. We say that P *bisects* $\mathbb{R}^{n+1}$ into these two open subsets of $\mathbb{R}^{n+1}$.

Let $(T, \underline{P}) \in \mathcal{T}\!ree_{Hyp^{n+1}}(k)$, and designate by V_T the set of vertices of T that are not the root. To our given manifold N and an open submanifold $U \subseteq N$, we associate for each internal vertex $v \in V_T$ a subspace $U_v \subseteq N$:

If v is the vertex attached through only a single edge to the root, we let $U_v = U$. Since T is binary, for $v \in V_T$ a trivalent vertex the left-going and right-going edge connect v to v_- and v_+, respectively. Let $P_v \in \mathrm{Hyp}^{n+1}$ be the decoration at v. We let $U_{v_-} = (U_v) \cap \left(\mathbb{R}_-^{n+1}\right)^{P_v}$ and $U_{v_+} = U_v \cap \left(\mathbb{R}_+^{n+1}\right)^{P_v}$. This determines U_v for all $v \in V_T$.

This timbering process is illustrated in figure 2 for the case $N = \mathbb{R}^2$, and three hyperplanes inside $\mathbb{R}^2$, it gives three different examples where trees are decorated by the hyperplanes in some way.

Definition 3.5 Let $U \subseteq N$ be an open submanifold. A tree $(T, \underline{P}) \in \mathcal{T}\!ree_{Hyp^{n+1}}$ is U-*cleaving* if we to each trivalent $v \in V_T$ decorated by P_v and associated with U_v in the recursive process above have that P_v intersects U_v non-trivially and transversally.

We let Timber_N be the set of subsets of N, called *timber*, where $U \in \mathrm{Timber}_N$ if there is an N-cleaving tree, T, with U associated to a leaf of T.

Hereby Timber_N consist of a subset of certain particular open submanifolds, since at each vertex of a N-cleaving tree, the submanifolds associated to the two vertices
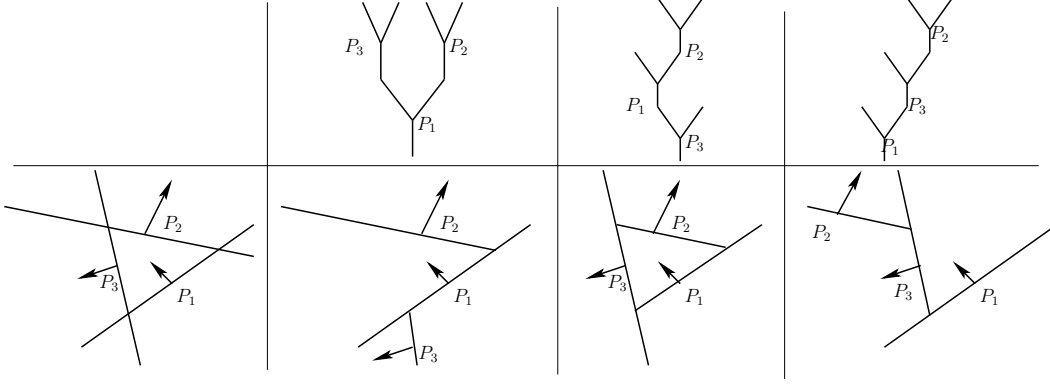
Figure 2: Three hyperplanes P_1, P_2, P_3 in $\mathbb{R}^2$ decorating three different trees. The four leafs of the individual trees will be decorated by the four subsets of $\mathbb{R}^2$ indicated by the associated picture below. Removing subtrees including and above a trivalent vertex, and the associated hyperplanes decorating the subtree in the picture below specifies the decoration of the other vertices.

above the vertex are again open submanifolds. Taking the closure inside N of these submanifolds will yield a codimension 0 submanifold potentially with boundary and corners.

Definition 3.6 We have specified a procedure to an N-cleaving tree $(T, \underline{P})$ that associates at each vertex v of T an open submanifold U_v. The exact same procedure can be extended to the recording area $\mathrm{Rec}(N)$, so that every vertex v of T has a subset $\mathrm{Rec}(U_v)$ associated to it, where the boundary of $\mathrm{Rec}(U_v)$ will be the space U_v. We call this the *associated recording area* of U_v. In case $\mathrm{Rec}(N)$ is a manifold, $\mathrm{Rec}(U_v)$ will in turn be a submanifold of $\mathrm{Rec}(N)$.

There is a natural topology on Timber_N – described by the space of hyperplanes giving rise to each timber, we assume this is given and wait until the next section with describing it explicitly to give the definition of the operad as fast as possible.

Definition 3.7 By the pre-N-cleavage operad, we shall understand the coloured operad $\widehat{\mathcal{C}leav}_N$, given by

- $\mathrm{Ob}(\widehat{\mathcal{C}leav}_N) = \mathrm{Timber}_N$

- $\widehat{\mathcal{C}leav}_N(U; k) := \left\{ (T, \underline{P}) \in \mathcal{T}ree_{Hyp^{n+1}}(k) \mid (T, \underline{P}) \text{ is } U\text{–cleaving} \right\}$

Granted the topology on Timber_N, we let

$$\widehat{\mathcal{C}leav}_N(-; k) = \coprod_{U \in \mathrm{Ob}(\widehat{\mathcal{C}leav}_N)} \widehat{\mathcal{C}leav}_N(U; k)$$

and endow this with a topology as a subset of $\mathcal{T}ree_{Hyp^n} \times \mathrm{Timber}_N$. The operadic composition

$$\circ_i \colon \widehat{Cleav}_N(U;k) \times_{\mathrm{Ob}(\widehat{Cleav}_N)} \widehat{Cleav}_N(-;m) \to \widehat{Cleav}_N(U;k+m-1)$$

is given by grafting indexing trees, and retaining all decorations of the result.

Definition 3.8 We let the N-cleavage operad, $Cleav_N$ be given by letting

$$\mathrm{Ob}(Cleav_N) = \{U \in \mathrm{Timber}_N \mid \complement U \simeq \coprod_{\text{finite}} *\}.$$

Here $\complement U$ denotes the complement of U as a subspace of N.

For the k-ary morphisms, we take the full suboperad of $\widehat{Cleav}_N$ on the objects $\mathrm{Ob}(Cleav_N)$ specified above, and apply a quotient: $Cleav_N(-;k) := \widehat{Cleav}_N(-;k)/\sim$, where $\sim$ is the equivalence relation given by letting $(T,\underline{P}) \sim (T',\underline{P}')$ if for all $i \in \{1,\ldots,k\}$ the i'th timber N_i associated to $(T,\underline{P})$ agrees with the i'th timber N_i' of $(T',\underline{P}')$. If $(T,\underline{P})$ and $(T',\underline{P}')$ are equivalent under $\sim$, we say they are *chop-equivalent*

Since the colours are left unchanged under $\sim$, taking operadic composition induced by $\widehat{Cleav}_N$ is well-defined.

We denote an element of $Cleav_N(-;k)$ by $[T,\underline{P}]$, where $(T,\underline{P})$ is a representative of the element.

Remark 3.9 A priori it would suffice to be given the set Timber_N of subsets beforehand, and from this define the operadic structure through these subsets, with k open substes whose closure cover U determining the k-ary information operations of $Cleav_N(U;k)$ – avoiding the introduction of trees and hyperplanes.

However, in our forthcoming computations we shall see that it is important that we have this very strict relationship between the trees decorated by hyperplanes and the associated timber. If one had given a more arbitrary space of subsets of N instead of Timber_N, the same combinatorial benefits would not be available for computations.

Remark 3.10 Note that for $N = S^n$, the complement inside S^n of $U \in \mathrm{Ob}(Cleav_{S^n})$ is by the generalized Schönflies theorem [Bro60] always given by a disjoint union of wedges of disks – the wedging occurring when hyperplanes intersect directly at S^n. We shall find this fact usual for obtaining homotopies of the complement of U.

Remark 3.11 That we for $Cleav_N$ have taken a subspace of objects; i.e. $\mathrm{Ob}(Cleav_N) \subset \mathrm{Timber}_N$ is necessary in order to obtain homological actions via umkehr maps, as we shall see in the next section.

For $N = S^1$ we have $\mathrm{Ob}(Cleav_{S^1}) = \mathrm{Timber}_{S^1}$; the complements $\complement U$ for $U \in \mathrm{Ob}(Cleav_{S^1})$ are always intervals.

Letting $N = S^n$ and $n > 1$, it follows by a simple Mayer-Vietoris argument of the H_0-groups along the closure of the timber $\overline{U}$ and $\complement U$ associated to $U \in \mathrm{Ob}(Cleav_{S^n})$, that we in taking the subspace $\mathrm{Ob}(Cleav_{S^n}) \subset \mathrm{Timber}_{S^n}$ are excluding the U that are disconnected[1]. Using the generalized Schöenflies theorem as stated in 3.10 we get that all $[T,\underline{P}]$ with in- and out-put connected are indeed in $Cleav_{S^n}$.

---

[1] such disconnected timber do exist; attempt for instance eating an apple conventionally to disconnect the peel in the last bite

If we take $N = S^1 \times S^1$, forming the subset $\mathrm{Ob}(\mathit{Cleav}_N) \subset \mathrm{Timber}_N$ removes all interesting operations from Cleav_N in the action defined in the next section. Basically since there will always be one complement of some timber that is either a cylinder or has genus.

The following two examples identify relations among trees in $\mathit{Cleav}_N(-;k)$. These two types of relations – it turns out – are essential in the forthcoming proofs.

To $P \in \mathrm{Hyp}^{n+1}$, let $-P \in \mathrm{Hyp}^{n+1}$ denote the hyperplane given by reversing orientation of P

Example 3.12 Assume that $(T, \underline{P})$ is an N-cleaving tree. Let v be an internal vertex of T, decorated by P. Let T' be the tree obtained from T by interchanging the branches above v, and let $\underline{P}'$ denote the set of hyperplanes with P interchanged with $-P$.
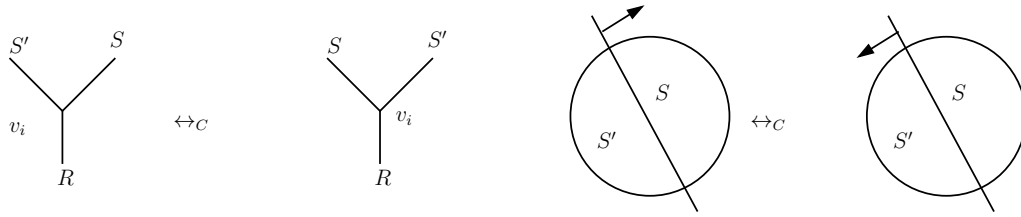
Figure 3: The $\leftrightarrow_C$-relation

Alternatively, the local picture 3 defines an equivalence relation $(T, \underline{P}) \leftrightarrow_C (T', \underline{P}')$. In $\mathit{Cleav}_N(-;k)$, we have that $[T, \underline{P}] = [T', \underline{P}']$

We say that two hyperplanes $P, Q \in \mathrm{Hyp}^{n+1}$ are *antipodally parallel* if $-Q$ can be obtained from P by translating P via its normal vector.

Example 3.13 The local picture between $(T, \underline{P})$ and $(T', \underline{P})$ N-cleaving in 4, describes when two N-cleaving trees are $\leftrightarrow_B$-related.

Figure 4: The $\leftrightarrow_B$-relation

Here the internal vertices v_j directly above v_i in T, decorated by P_i and P_j of $\underline{P}$ have swapped position – along with the branches specified in the picture – in T' compared to T. We let $(T, \underline{P}) \leftrightarrow_B (T', \underline{P})$ if P_i and P_j are antipodally parallel.

In this case we have $[T, \underline{P}] = [T', \underline{P}]$ in $\mathcal{C}leav_N(-; k)$.

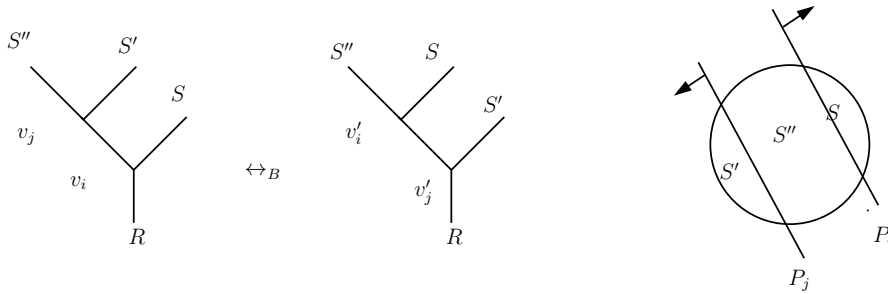We say that P and Q are parallel if either P, Q or $P, -Q$ are antipodally parallel.

Observation 3.14 Assume that we are given $(T, \underline{P}) \in \mathcal{C}leav_N(-; k)$, where all hyperplanes of $\underline{P}$ are pairwise parallel. Using the $\leftrightarrow_C$ and $\leftrightarrow_B$-relations of 3.12 and 3.13, we obtain that $[T, \underline{P}] = [L_k, \underline{P}']$, where L_k is a leftblown tree as in 3.15, and $\underline{P}'$ is obtained from $\underline{P}$ by reversing the orientations along some hyperplanes.

3.2 Topology on the Timber

Definition 3.15 We let the *arity k left-blown tree* be the tree $L_k \in \mathcal{T}ree(k)$, with the right-going edges all ending at leaves, let the only leaf on a left-going edge be labelled by k – and the leaf other leafs, at level $i + 1$, labelled by i.
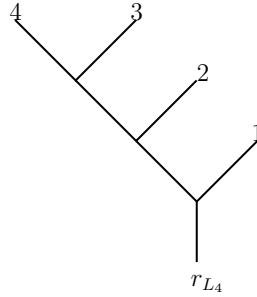

Figure 5: The left-blown tree L_4

Let L_1 denote the tree with $V_{L_1} = \varnothing$, and a single leaf and root.

Cultivating the cleaving tree appropriately, that is by reversing orientations of hyperplanes – swapping branches around as in 3.14 – and cutting away unnecessary branches, we can assume $U \in \mathrm{Timber}_N$ to be on the leaf labelled $k + 1$ of L_k for some $k \in \mathbb{N}$.

Construction 3.16 We specify a topology on $\mathrm{Timber}_{\mathbb{R}^{n+1}}$ as a subspace by seeing that there is an injection $\psi \colon \mathrm{Timber}_{\mathbb{R}^{n+1}} \to \coprod_{i \in \mathbb{N}} \left(\left(\mathrm{Hyp}^{n+1} \right)^i / \Sigma_i \right)$, where the permutation group Σ_i permutes the factors of the product.

To specify the injection ψ, note that for $U \in \mathrm{Timber}_{\mathbb{R}^{n+1}}$, the boundary of the closure in N of U, $\partial \overline{U}$ contain the information needed to reconstruct $(L_k, \underline{P})$ having U as the decoration on the top-leaf. Such hyperplanes are given by taking least affine subsets containing certain parts of $\partial \overline{U}$; either distinguished by different components of $\partial \overline{U}$ – and otherwise a corner of $\partial \overline{U}$ will be the distinguishing feature for $(L_k, \underline{P})$. The function ψ now maps $U \in \mathrm{Timber}_{\mathbb{R}^{n+1}}$ to the corresponding hyperplanes, $(P_1, \ldots, P_k) \in \left(\mathrm{Hyp}^{n+1} \right)^k$

13

decorating L_k and determined by $\partial \overline{U}$. This thus hits the component in the image of ψ indexed by k.

There is ambiguity in the above definition of ψ; any reordering of the hyperplanes $(P_1, \ldots, P_k)$ will give rise to the same top-level timber. We therefore quotient by Σ_i in the image of ψ.

Let $\mathrm{Timber}_N^{\varnothing} := \mathrm{Timber}_N \cup \{\varnothing\}$.

Construction 3.17 For $N \subset \mathbb{R}^{n+1}$, we have a surjection $\mu_N \colon \mathrm{Timber}_{\mathbb{R}^{n+1}} \to \mathrm{Timber}_N^{\varnothing}$, given by $\mu_N(U) = N \cap U$, and we specify a topology on $\mathrm{Timber}_N^{\varnothing}$ by letting μ_N be a quotient map.

We let $\mathrm{Timber}_N \subset \mathrm{Timber}_N^{\varnothing}$ be a subspace

Remark 3.18 Specifying μ_N as a quotient map means that certain elements $[L_k, \underline{P}] \in \boldsymbol{Cleav}_{\mathbb{R}^{n+1}}$ will give rise to the same $U \cap N$ at the top-leaf, under μ_N. In particular, if $[L_k, \underline{P}]$ giving rise to $U \cap N$ has some leaf decorated by $\varnothing$, we can instead consider $[L_{k-1}, \underline{\hat{P}}]$ as giving rise to $U \cap N$, where $\underline{\hat{P}}$ is given by removing the hyperplane from $\underline{P}$ decorating the vertex below said leaf, since the hyperplane in question will not be cleaving N.

Proposition 3.19 Let N be a compact submanifold of $\mathbb{R}^{n+1}$. Timber_N is contractible. Similarly, $\mathrm{Ob}(\boldsymbol{Cleav}_{S^n})$ is contractible for $S^n \subseteq \mathbb{R}^{n+1}$ the unit-sphere

Proof. Given a point $U \in \mathrm{Timber}_N$ will have $\mathcal{C}U$ consist of a disjoint union of sub-manifolds of N that has boundary at the points where U has been cleaved from N by hyperplanes $P_1, \ldots, P_k$. Each P_i has a normal-vector in the direction towards U, and one in the direction away from U. The topology on Timber_N, precisely determined by these hyperplanes makes it continuous in Timber_N to translate $P_1, \ldots, P_k$ in the direction away from U. Since N is compact, this translation will in finite time take each hyperplane past tangential hyperplanes of N. Hence each hyperplane eventually disappears from the cleaving data and by 3.18, eventually this translation provides an element of $\boldsymbol{Cleav}_N$ given by the 1-ary undecorated tree L_1 as an operation from N to N . This hence defines a homotopy $\Phi_t \colon \mathrm{Timber}_N \to \mathrm{Timber}_N$ with $\Phi_0(U) = U$ and $\Phi_1(U) = N$, and hence the desired null-homotopy onto $N \in \mathrm{Timber}_N$.

For the statement on $\mathrm{Ob}(\boldsymbol{Cleav}_{S^n})$, note that from the definition 3.8 and 3.10, that the submanifolds of $\mathcal{C}U$ will consist of a disjoint union of disks. The null-homotopy Φ_t above will in this case result in smaller and smaller disks as t increases, and so $\Phi_t(U)$ remains withing $\mathrm{Ob}(\boldsymbol{Cleav}_{S^n})$, and the null-homotopy is given as above.. $\qquad \square$

The following proposition tells us in conjunction with 3.19 that 2.5 applies to $\boldsymbol{Cleav}_N$.

Proposition 3.20 The evaluation map $\mathrm{ev}_{\mathrm{in}} \colon \boldsymbol{Cleav}_N(-; k) \to \mathrm{Ob}(\boldsymbol{Cleav}_N)$ is a fibration

Proof. To $[T, \underline{P}] \in \mathcal{C}leav_N(-; k)$ we shall first of all for each of the hyperplanes P_i of $\underline{P}$ prescribe the following transformation:

Under the relation 3.12, we have to make a choice of normal-vector ν_i of P_i, this defines an interval $J_i =]j_-, j_+[$ given by the maximal interval such that translating P_i along ν_i with $r \in J_i$ as a scalar the hyperplane still participates in a cleaving configuration as a decoration of T. Note that since $[T, \underline{P}]$ is cleaving we have $0 \in J_i$ and denote by $c(J_i)$ the center-point of the interval. Note that the other choice of normalvector $-\nu_i$ will give rise to the interval $-J_i$ which will leave the following invariant:

Fix $\varepsilon > 0$, if $j_{\min}^i := \min\{|j_-|, j_+\} < \varepsilon$, translate P_i by $\text{sgn}(c(J_i)) \cdot \min\{\varepsilon - j, c(J_i)\}$ where $\text{sgn}(c(J_i))$ is the sign of $c(J_i)$.

This translation can naturally be done to all the decorations of a decorated tree $[T, \underline{P}] \in \mathcal{C}leav_N(-; k)$ simultaneously. Call this transformation $\Gamma_\varepsilon(T, \underline{P})$, note that since we are moving all hyperplanes at once, dependent on how large ε is chosen, $\Gamma_\varepsilon([T, \underline{P}])$ does not a priori result in a cleaving tree.

We seek to find a lift in the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\;\varphi\;} & \mathcal{C}leav_N(-; k) \\
\downarrow & \nearrow^{\tilde{h}} & \downarrow{\text{ev}_{\text{in}}} \\
Y \times I & \xrightarrow{\;h\;} & \text{Ob}(\mathcal{C}leav_N)
\end{array}
$$

where we can assume that Y is compact, and therefore pick

$$
0 < \varepsilon < \inf_{y \in Y}(\min\{j_{\min}^i \mid P_i \text{ decorates } \varphi(y)\})
$$

where $j_{\min}^i$ is the minimal value where P_i can be translated in order to have it still participate in a cleavage as defined above.

The lift $\tilde{h}(y, t)$ is now given as $\Gamma_\varepsilon(\varphi(y))$ considered as a cleaving tree of the timber $h(y, t)$. Note that our choice of ε makes $\Gamma_\varepsilon(\varphi(y))$ result in an element of $\mathcal{C}leav_N(-; k)$, basically since along t, the timber $h(y, t)$ will change continuously and therefore by definition of Γ_ε will for a small neighborhood of $t \in I$ only give rise to a small change in how the configurations of hyperplanes change, guaranteeing their continual cleaving attributes. $\qquad\square$

4 Action of Cleavages

Let M be a compact manifold. We set $M^N := \{f\colon N \to M\}$ – i.e. the space of unbased, continous maps from N to M, endowed with the compact-open topology.

Above 3.5, we have specified a procedure to an N-cleaving tree $(T, \underline{P})$ associates at each vertex v of T an open submanifold U_v. As noted in 3.6, this procedure can be extended to the recording area $\text{Rec}(N)$, so that every vertex v of T has a subset $\text{Rec}(U)_v$ associated to it, where the boundary of $\text{Rec}(U)_v$ will be the space U_v. In case $\text{Rec}(N)$ is a manifold, $\text{Rec}(U)_v$ will in turn be a submanifold of $\text{Rec}(N)$.

Definition 4.1 Let $\mathrm{Rec}(U) \subseteq \mathrm{Rec}(N)$ denote the recording area of $U \in \mathrm{Timber}_N$ as given above. To a U-cleaving tree $(T, \underline{P})$, we associate the *blueprint* of $(T, \underline{P})$ to be the following subset of $\mathrm{Rec}(U)$:

$$\beta_{(T,\underline{P})} := \mathrm{Rec}(U) \backslash \bigcup_{i=1}^{k} \mathrm{Rec}(U)_i$$

where $\mathrm{Rec}(U)_i$ is the subset of $\mathrm{Rec}(U)$ associated to the i'th leaf of $(T, \underline{P})$. By definition of the recursive procedure above 3.5, $\beta_{(T,\underline{P})}$ will be contained in the collective union of all the hyperplanes of $\underline{P}$, loosely it will consist of all points of hyperplanes in $\underline{P}$ that have been involved in the recursive bisection process of $\mathrm{Rec}(U)$ described above 3.5.

In figure 1 of the introduction, the boundary of $\beta_{(T,\underline{P})}$ will be the collective boundary of the closure within S^n of the submanifolds in picture D.

Definition 4.2 We let $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$ denote the quotient of $\pi_0(\beta_{[T,\underline{P}]})$, where two pathcomponents are considered equivalent if the same hyperplane in $\underline{P}$ has given rise to these components of $\beta_{[T,\underline{P}]}$.

Example 4.3 If $N = S^n \subseteq \mathbb{R}^{n+1}$ as the unit-sphere, bounding the unit disk D^{n+1}, convexity of D^{n+1} entails that $\pi_0(\beta_{[T,\underline{P}]}) = \pi_0(\widehat{\beta_{[T,\underline{P}]}})$.

As an example of where the quotient matters, take N a standard-embedding of $S^1 \times S^1$ in $\mathbb{R}^3$, with recording area $D^1 \times S^1$, then cleaving $S^1 \times S^1$ with a single hyperplane into two annuli would have $\beta_{[T,\underline{P}]}$ consist of two components, whereas $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$ would still be trivial.

Let $\mathrm{Corr}(\mathcal{C})$ denote the correspondance category over $\mathcal{C}$ a co-complete category, as described in the introduction.

Construction 4.4 We construct a functor $\Phi_N \colon \boldsymbol{\mathcal{C}leav}_N \to \boldsymbol{\mathcal{E}nd}_{M^N}^{\mathrm{Corr(Top)}}$. That is, an action of $\boldsymbol{\mathcal{C}leav}_N$ on M^N as an object of the category of natural transformations of correspondances over Top.

Let $[T, \underline{P}] \in \boldsymbol{\mathcal{C}leav}_N(U; k)$. Let $\complement N_1, \ldots, \complement N_k$ denote the complement, inside of N, of the timbers associated to the leafs of T.

The action is given through the following pullback-diagram

$$\begin{array}{ccc} M_{[T,\underline{P}]}^N & \xrightarrow{\varphi^*} & \left(M^N\right)^k \\ \downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ M^{\pi_0(\widehat{\beta_{[T,\underline{P}]}})} & \xrightarrow{\varphi} & \prod_{i=1}^{k} M^{(\complement N_i)} \end{array} \tag{2}$$

where res is the restriction map onto each complement.

We define φ as the induced of a map $c \colon \coprod_{i=1}^{k} (\complement N_i) \to \pi_0(\widehat{\beta_{[T,\underline{P}]}})$. To define c, note that a component C of $\complement N_i$, will have $\partial C \backslash \partial \overline{U}$ be the result of some cuts of hyperplanes

16

decorating T. By definition of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$, and since C is a connected component, the cuts will all constitute the same element, $c(C) -$ of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$ making φ well-defined as the map constant map along the timber intersecting represenatives of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$ nontrivially.

By glueing the functions in the pullback space, we can identify $M^N_{[T,\underline{P}]}$ as the space of $f \in M^N$ such that f is constant along each subspace of the blueprint that is a representative of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$. We hence have a canonical inclusion $\iota_{[T,\underline{P}]}\colon M^N_{[T,\underline{P}]} \to M^N$, and in turn a correspondance

$$M^N \xleftarrow{\iota_{[T,\underline{P}]}} M^N_{[T,\underline{P}]} \xrightarrow{\varphi^*} \left(M^N\right)^k$$
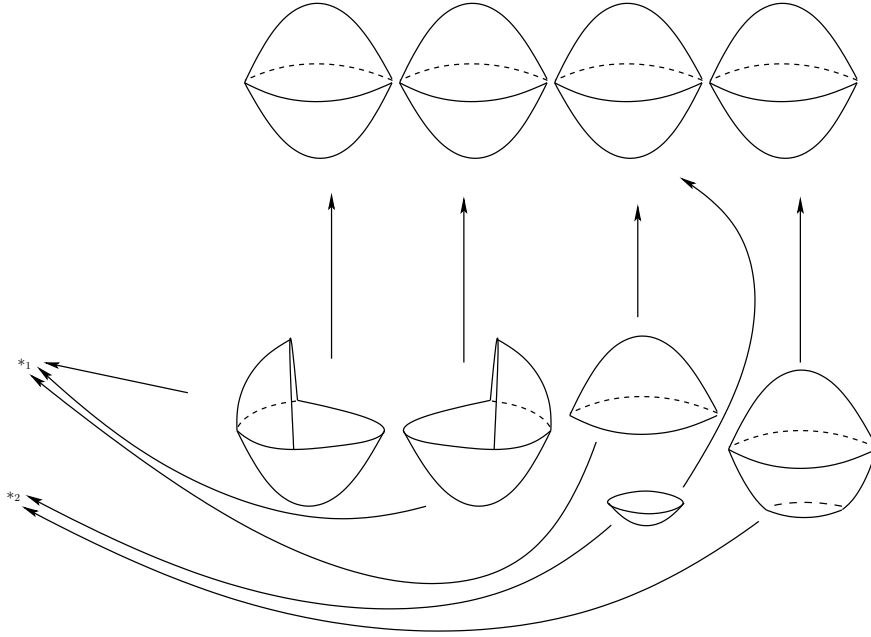
Figure 6: The 4-ary operation of the introduction has the complement of its timber $\complement N_1, \complement N_2, \complement N_3, \complement N_4$ drawn as the five disks on the bottom right corner. The upwards arrow are the inclusion maps, so that dualizing them provides the restriction map. The leftwards maps have as target two points, and these should be considered as collapsing the components of the blueprint $\beta_{[T,P_1,P_2,P_3]}$. These maps are given as the ones where the boundary of each disk is contained in a component of the blueprint.

Example 4.5 For the cleavage $[T, P_1, P_2, P_3] \in \mathcal{C}\!\mathit{leav}_{S^n}(S^n; 4)$ in picture D of figure 1 in the introduction, we can consider the morphisms defining diagram (2) through figure 6, using the mapping functor $M^{(-)}$ to dualize the morphisms indicated in the figure, we get the maps that define the pullback diagram (2).

Functoriality of the above pullback-construction in 4.4 gives us

Proposition 4.6 The construction of 4.4 defines an action of $\mathcal{Cleav}_N$ on M^N as an element of the symmetric monoidal category $(\mathrm{Corr}(\mathrm{Top}), \times)$.

Proposition 4.7 The map res is a fibration.

Proof. Follows directly since the inclusions $\complement N_i \to N$ are closed cofibarations, and res is the dualization under the mapping space functor M^-. □

Proposition 4.8 Consider the case of $N = S^n$. For any $[T, \underline{P}] \in \mathcal{Cleav}_{S^n}(-;k)$, with associated timber $N_1, \ldots, N_k$ the number

$$\left| \pi_0 \left(\coprod_{i=1}^{k} \complement N_i \right) \right| - \left| \pi_0 \left(\beta_{[T,\underline{P}]} \right) \right|$$

will be constantly $k - 1$ for all $[T, \underline{P}] \in \mathcal{Cleav}_{S^n}(-;k)$.

Proof. Note that $\beta_{[T,\underline{P}]}$ constitute the boundary of the disjoint union of wedges of disks $\coprod_{i=1}^{k} \complement N_i$, so the number $\left| \pi_0 \left(\coprod_{i=1}^{k} \complement N_i \right) \right|$ will be constant as long as $\left| \pi_0 \left(\beta_{[T,\underline{P}]} \right) \right|$ is constant.

Consider a path $\gamma \colon [0,1] \to \mathcal{Cleav}_{S^n}(-;k)$ with $\left| \pi_0(\beta_{\gamma(0)}) \right| = \left| \pi_0(\beta_{\gamma(1)}) \right| + 1$ and such that for a specific $t_0 \in [0,1]$, we have for all $t \leq t_0, \left| \pi_0(\beta_{\gamma(t)}) \right| = \left| \pi_0(\beta_{\gamma(0)}) \right|$ and for all $t > t_0, \left| \pi_0(\beta_{\gamma(t)}) \right| = \left| \pi_0(\beta_{\gamma(1)}) \right|$.

By definition of the cleaving proces, $\gamma(t_0)$ will have two hyperplanes P_l, P_r such that $P_l \cap P_r \cap \mathrm{Rec}(S^n)$ is a nontrivial subspace of $\partial \mathrm{Rec}(S^n) = S^n$, and for any $\varepsilon > 0$, the same intersection for the hyperplanes of $\gamma(t_0 + \varepsilon)$ will be trivial; meaning that for sufficiently small ε, such that the hyperplanes do not become parallel, $P_l \cap P_r \subseteq \mathbb{R}^{n+1}$ will be contained in $\mathbb{R}^{n+1} \backslash \mathrm{Rec}(N)$. This has the effect that there is precisely one $j \in \{1, \ldots, k\}$ such that the complement of the timber indexed by j, $\complement N_j$, for $\gamma(t_0)$ has a connected component containing $P_l \cap P_r \cap \mathrm{Rec}(N)$, whereas for $\gamma(t_0 + \varepsilon)$ this becomes disconnected with different boundary components of $\complement N_i$ being formed using intersections with P_l and P_r respectively. Hence for these basic types of paths, an increase in $\left| \pi_0(\beta_{[T,\underline{P}]}) \right|$ leads to an equal increase in $\left| \pi_0(\coprod_{i=1}^{k} \complement N_i) \right|$.

One can use these paths to parametrize a single cleaving hyperplane moving within $\mathcal{Cleav}_{S^n}(-;k)$, while the other $k - 2$ hyperplanes remain fixed. From such parametrizations, one puts together a general path moving all hyperplanes of $\mathcal{Cleav}_{S^n}(-;k)$ and the result follows.

To compute the constant, take a configuration of hyperplanes where all hyperplanes are parallel, so that $\beta_{[T,\underline{P}]}$ has $k - 1$ components. In this case the space of complements of the associated timber, $\coprod_{i=1}^{k} \complement N_i$ will have $(k - 2)$ spaces $\complement N_i$ that consist of two disjoint spaces, and the two extremal complements that consist of a single subdisks of S^n. Hence in total $2(k - 2) + 2 = 2(k - 1)$ components. In effect, the constant will be $2(k - 1) - (k - 1) = k - 1$.

□

Remark 4.9 The above 4.8 is only stated for the particular case $N := S^n$. However the only place in the proof where we use the nature of S^n is to identify that the number $\left| \pi_0 \left(\coprod_{i=1}^k \complement N_i \right) \right|$ will be constant as long as $\left| \pi_0 \left(\widehat{\beta_{[T,\underline{P}]}} \right) \right|$ is constant.

Again, we need to work with $\left| \pi_0 \left(\widehat{\beta_{[T,\underline{P}]}} \right) \right|$ as defined in 4.2 for more general N, however this also fits directly into the proof.

Since the hyperplanes of $\mathcal{C}leav_N(-;k)$ are required to cleave N transversally, one can indeed use Morse theory to obtain this initial statement of the proof for a general embedded manifold N. However, we shall not go into detail with this, as we shall only apply 4.8 in the case where $N = S^n$.

Of course in the more general case, this constant will no longer be $k-1$, and indeed the constant will potentially vary along components of $\mathcal{C}leav_N(-;k)$. With $\mathcal{C}leav_{S^n}(-;k)$ connected, basically since all hyperplanes will cleave S^n transversally, this variance along components is not part of the statement of 4.8.

Remark 4.10 Note that as we don't use it in the proof, 4.8 holds even if we drop the assumption in 3.8 that for all elements of $\mathcal{C}leav_{S^n}(-;k)$ the associated timber $N_1, \ldots, N_k$ should satisfy $\complement N_i \simeq \coprod_{\text{finite}} *$. In fact, this assumption is not needed for this paper, but will only be applied in [Bar11] to define homological actions from the correspondances of 4.4.

This is also the case for an extended version of 4.8 to $\mathcal{C}leav_N$ as indicated in 4.9

Remark 4.11 For the case $N = S^n$, as mentioned in the remarks above, the correspondance diagram (2) will by certain umkehr maps for the map φ^* eventually lead to a homological action in [Bar11]. What we in fact will show is something stronger, namely that there is a stable action map, residing in the category of spectra:

$$\mathcal{C}leav_{S^n}(-;k) \times \left(M^{S^n} \right)^k \to M^{S^n} \wedge S^{\dim(M) \cdot (k-1)} \tag{3}$$

Smashing the above map with the Eilenberg-Maclane spectrum, and taking homotopy groups yields the action mentioned below Theorem 1 of the introduction.

We shall in 4.12 below give an example of the action to illustrate the reasoning behind the definition of $\mathcal{C}leav_{S^n}$.

We shall first make some remarks on how the constructions of this section provides the foundation for how this map takes form.

First of all, note that since we in 3.8 have assumed that to $[T,\underline{P}] \in \mathcal{C}leav_N(-;k)$ the associated timber $N_1, \ldots, N_k$ will have $\complement N_i$ consist of a finite disjoint union of contractible spaces. Therefore, in the diagram (2), the space $\prod_{i=1}^k M^{\complement N_i}$ is a Poincaré duality space. That is, up to homotopy it is equivalent to a product of copies of M.

In the case $N = S^n$, taking such a specific $[T,\underline{P}]$ as a pointwise version of (3), we should be able to obtain a map $\left(M^{S^n} \right)^k \to M^{S^n} \wedge S^{\dim(M) \times (k-1)}$. Indeed, for fixed $[T,\underline{P}]$ there are methods available in the litterature to do this. For instance adapting the methods of [CK09, p.14/'Umkehr maps in String Topology] provides such a map. Here it is crucial to

note that the map φ is up to homotopy an embedding of codimension $\dim(M) \times (k-1)$ where the $(k-1)$-factor comes from 4.7, which provides the dimension-shift in (3).

Yet another method, formulated on the chain-level instead of spectra would be [FT09, Theorem A].

However, getting from these pointwise umkehr maps to a map such as (3) is a somewhat strenous hike. This is the focus of [Bar11]. Note that while the crucial 4.8 hints that it is possible to provide the maps for other N than the euclidean embedded unit-spheres S^n, the geometry involved makes us look only at the interesting case Cleav_{S^n}.

The complexity of the construction of the umkehr map rises with the arity of the involved maps. As the arity rises, the map φ of (2) will by 4.8 have constant codimension, whereas the actual dimension will be exposed to sudden jumps in the actual dimensions of the spaces φ is mapping between, as is indicated in the proof of 4.8. The coherence issues of higher arity lies in patching the instances of such jumps together. The following example illustrates the case of arity 2 operations where there are no such jumps:

Example 4.12 The 2-ary portion $\mathit{Cleav}_{S^n}(-;2)$ is a manifold, specified by a single cleaving hyperplane, it deformation retracts onto S^n, which is determined by the direction of the normal-vector of the cleaving hyperplane.

Consider the pull-back diagram

$$
\begin{array}{ccc}
M^{S^n}_{\mathit{Cleav}_{S^n}(-;2)} & \longrightarrow & \left(M^{S^n}\right)^2 \times \mathit{Cleav}_{S^n}(-;2) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\
M \times \mathit{Cleav}_{S^n}(-;2) & \longrightarrow & M^{\coprod_{i=1}^{2} \complement N_i}_{\mathit{Cleav}_{S^n}(-;2)}
\end{array}
\tag{4}
$$

Where $\complement N_1$ and $\complement N_2$ denotes the complement inside S^n of the timber associated to $[T,P] \in \mathit{Cleav}_{S^n}(-;2)$. Considering it as a set, $M^{\coprod_{i=1}^{2} \complement N_i}_{\mathit{Cleav}_{S^n}(-;2)}$ is given by the disjoint union $\coprod_{[T,P] \in \mathit{Cleav}_{S^n}(-;2)} M^{\complement N_i}$. The map res in the diagram is given by letting $\mathrm{res}(f_1, f_2, [T,P])$ be given as the the restriction of f_1 to $\complement N_1$ and f_2 to $\complement N_2$ along the component indexed by $[T,P]$. We topologize $M^{\coprod_{i=1}^{2} \complement N_i}_{\mathit{Cleav}_{S^n}(-;2)}$ by making res a quotient map.

Note that by 4.7, pointwise in $\mathit{Cleav}_{S^n}(-;2)$, res is a fibration. One sees that the lifts of this global map can be constructed to be continous in $\mathit{Cleav}_{S^n}(-;2)$ as well.

We hereby have homotopy-equivalences

$$
M^{\coprod_{i=1}^{2} \complement N_i}_{\mathit{Cleav}_{S^n}(-;2)} \simeq M^2 \times \mathit{Cleav}_{S^n}(-;2) \simeq M^2 \times S^n
$$

and

$$
M \times \mathit{Cleav}_{S^n}(-;2) \simeq M \times S^n.
$$

Stating that the lower portion of the diagram is an embedding of Poincaré duality spaces. For instance applying one of the methods mentioned in 4.11, this hereby provides the

first sign of an action of $\mathcal{C}leav_{S^n}$ on M^{S^n}, and taking homotopy groups of this map, we get a map in homology

$$\mathrm{H}_* \left(\mathcal{C}leav_{S^n}(-;2) \right) \otimes \mathrm{H}_*(M^{S^n})^{\otimes 2} \to \mathrm{H}_{*+\dim(M)}(M^{S^n})$$

5 Spherical Cleavages are E_{n+1}-operads

We now devote energy to prove that $\mathcal{C}leav_{S^n}$ is a coloured E_{n+1}-operad. A concept defined in 2.8

5.1 Combinatorics of Coloured E_n-operads

Definition 5.1 By a *full level (n,k)-graph*, we shall understand a graph G with k vertices $v_1,\ldots,v_k$ such that all pairs (v_i,v_j) are connected by precisely one edge e_{ij}. Let $\mathbf{n} := \{0,\ldots,n-1\}$. We let each of the $\binom{k}{2}$ edges of G be labelled by elements of $\mathbf{n}$.

We say that a full level (n,k)-graph G is oriented if there to each edge G e_{ij} is designated a direction; either from v_i to v_j or from v_j to v_i.

To $\sigma \in \Sigma_k$, there is a unique orientation of a full level (n,k)-graph, G. Namely by letting e_{ij} point from v_i to v_j if $\sigma(i) < \sigma(j)$, and point from v_j to v_i if $\sigma(j) < \sigma(i)$. We call σ the permutation associated to G.

Indeed, assuming that this orientation of G is oriented with no cycles and comes equipped with a sink and a source, one can reconstruct the permutation σ_G associated to G from the orientation of G: Let the index of the sink of G be mapped to k under σ_G, and successively remove the sink of an oriented full (n,i)-graph with sink, source and no cycles to obtain a full $(n,i-1)$-graph with induced orientation being guaranteed a sink by the pigeonhole principle. The index of this sink is mapped to i under σ_G. Continuing this process until only the source of G is left makes σ_G a well-defined permutation since G has no cycles.

Definition 5.2 Let $\mathcal{K}^n(k)$ denote the set of full level (n,k)-graphs, oriented via Σ_k as above. This gives a bijection $\mathbf{n}^{\binom{k}{2}} \times \Sigma_k \leftrightarrow \mathcal{K}^n(k)$.

We have that $\mathcal{K}^n$ is an operad: To $G_k \in \mathcal{K}^n(k)$ and $G_m \in \mathcal{K}^n(m)$, inserting the m vertices of G_m instead of i'th vertex of G^n, we obtain a full level $(n, m+k-1)$ graph $G_k \circ_i G_m$, by labelling and orienting the edges of $G_k \circ_i G_m$ in the following way:

Since we have replaced the i'th vertex of G_k with k vertices from $\mathcal{K}^n(m)$, G_k lives as a subgraph of $G_k \circ_i G_m$ in k different ways – one for each choice of replacement vertex for i in G_m. The graph G_m has all its vertices retained in $G_k \circ_i G_m$, so there is only one choice of subgraph for G_m.

We label and orient all edges of $G_k \circ_i G_m$ via the labellings and orientations of the possibilities of subgraphs G_k and G_m.
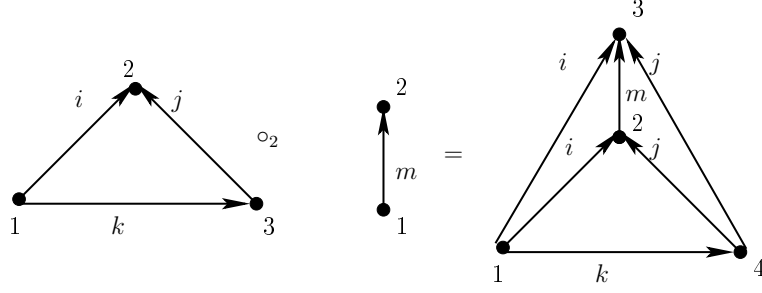
An example operadic composition is given in figure 7

Figure 7: An operadic composition of a 3-ary with a 2-ary operation in the full-graph operad. The labellings i, j, k, m are elements of $\mathbf{n}$. Note that in the 4-ary operation both graphs are contained in the final result, and we copy the labelling of the edges that are going to the vertex the operadic composition is happening at

Observation 5.3 Using 5.2, we see how $\mathcal{K}^n$ is an operad of posets. First of all, letting $\mathcal{K}^n(2) = \mathbf{n} \times \Sigma_2$ be given by setting $\Sigma_2 = \{\mathrm{id}, \tau\}$ and partially ordering through the arrows of the diagram

$$
\begin{array}{ccccccc}
(0, \mathrm{id}) & \longrightarrow & (1, \mathrm{id}) & \longrightarrow & \cdots & \longrightarrow & (n, \mathrm{id}) \\
& & & & & & \\
(0, \tau) & \longrightarrow & (1, \tau) & \longrightarrow & \cdots & \longrightarrow & (n, \tau)
\end{array}
\tag{5}
$$

Consider the maps $\gamma_{ij} \colon \mathcal{K}^n(k) \to \mathcal{K}^n(2)$, given by the subgraph, with one edge, of $G^n(k)$ spanned by v_i and v_j. Following [Ber97, 1.5], we let the partial ordering be given by the coarsest ordering such that γ_{ij} is order preserving for all $i < j \in \{0, \ldots, k\}$.

Definition 5.4 Given $\mathcal{O}$ a coloured operad, let $S(\mathcal{O})$ denote the coloured operad in posets, with objects the subsets of $\mathrm{Ob}(\mathcal{O})$, and k-ary morphisms subsets of $\mathcal{O}(-; k)$. The operad is an operad in posets through inclusions of subspaces.

That is, we let $\mathrm{Ob}(S(\mathcal{O})) = \{0, 1\}^{\mathrm{Ob}(\mathcal{O})}$. Let $S(\mathcal{O})(-; k) := \{0, 1\}^{\mathcal{O}(-; k)}$. These fit into the diagram

$$
\begin{array}{ccccc}
S(\mathcal{O})(-; k + m - 1) & \xleftarrow{\ \circ_i\ } & S(\mathcal{O})(-; m) \times_{\mathrm{Ob}(S(\mathcal{O}))} S(\mathcal{O})(-; k) & \longrightarrow & S(\mathcal{O})(-; k) \\
& & \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_i} \\
& & S(\mathcal{O})(-; m) & \xrightarrow{\ \mathrm{ev}_{\mathrm{in}}\ } & \mathrm{Ob}(S(\mathcal{O}))
\end{array}
$$

Where as usual ev_i evaluates at the i'th colour of $S(\mathcal{O})(-; k)$, given as a subset of $\mathrm{Ob}(\mathcal{O})$ and $\mathrm{ev}_{\mathrm{in}}$ evaluates the incoming colour of $S(\mathcal{O})(-; m)$.

Operadic composition is induced from composition in $\mathcal{O}$, pointwise. In the sense that $\circ_i \colon S(\mathcal{O})(-; m) \times_{\mathrm{Ob}(S(\mathcal{O}))} S(\mathcal{O})(-; k) \to S(\mathcal{O})(-; k + m - 1)$ is given by the subset of

$\mathcal{O}(-; k+m-1)$ obtained by to any point of $\mathcal{O}(-; m) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-; k)$ as an element of $S(\mathcal{O})(-; m) \times_{\mathrm{Ob}(S(\mathcal{O}))} S(\mathcal{O})(-; k)$ applying the $\circ_i$-operation from $\mathcal{O}(-; m) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-; k)$ to $\mathcal{O}(-; k+m-1)$, and taking the union over the specific subset of these compositions.

Recall from 2.7 that to $\mathcal{P}$ a monochrome operad, and X a space, the coloured topological operad $\mathcal{P} \times X$ is coloured over X and the k-ary morphisms are formed by taking the cartesian product of $\mathcal{P}(k)$ with X.

The Berger Cellularization Theorem, written in a monochrome fashion in [Ber97, Th. 1.16] hereby transfers to our coloured setting:

Theorem 5.5 Let $\mathcal{O}$ be a topological coloured operad. The operad $\mathcal{O}$ is a coloured E_n-operad, 2.7 if there is a functor $F_k \colon \mathcal{K}^n(k) \times S(\mathrm{Ob}(\mathcal{O})) \to S(\mathcal{O})(-; k)$ that is, both a functor with respect to the poset structure as well as a morphism of coloured operads, satisfying the following:

(A) Let $C_0 \in S(\mathrm{Ob}(\mathcal{O}))$. The *latching space of* $\alpha \in \mathcal{K}^n(k)$ is given by $L_{(\alpha, C_0)} := \bigcup_{\beta < \alpha} F_k(\beta, C_0)$. We require that the morphism

$$L_{(\alpha, C_0)} \hookrightarrow F_k(\alpha, C_0)$$

is a cofibration.

(B) For all $\alpha \in \mathcal{K}^n(k)$, we require a natural transformation of weak equivalences between the functor $F_k(\alpha, -)$ and the identity funtor on $S(\mathrm{Ob}(\mathcal{O}))$.

(C) $\mathrm{colim}_{(\alpha, C_0) \in \mathcal{K}^n(k) \times \mathrm{Ob}(S(\mathcal{O}))} F_k(\alpha, C_0) = \mathcal{O}(-; k)$, where the colimit is using the poset-structure on $\mathcal{K}^n(k)$ – given in 5.3 – and inclusions of subsets in $S(\mathrm{Ob}(\mathcal{O}))$. These inclusions should be compatible with the equation, in the sense that $\mathcal{O}(U; k)$ should be given by restricting the colimit to the $C_0 \in \mathrm{Ob}(S(\mathcal{O}))$ satisfying $C_0 \subseteq U$ in the indexing category.

Proof. First of all, note that $\mathcal{K}^n(k)$ as a finite poset is a Reedy Category in the sense of [Dug08, 13.1]; explicitly a degree function $\deg \colon \mathcal{K}^n(k) \to \mathbb{Z}$ can be given by letting $\deg(\alpha)$ be determined by the sum of the $\binom{k}{2}$ labels in $\{0, \dots, n-1\}$ of the edges of the graph α.

From the assumptions (A)-(C) we get the following homotopy equivalence:

$$
\begin{aligned}
\mathcal{O}(-; k) = \mathrm{colim}_{(\alpha, C_0) \in \mathcal{K}^n(k) \times S(\mathrm{Ob}(\mathcal{O}))} F_k(\alpha, C_0) &\cong \\
\mathrm{colim}_{C_0 \in S(\mathrm{Ob}(\mathcal{O}))} \mathrm{colim}_{\alpha \in \mathcal{K}^n(k)} F_k(\alpha, C_0) &\simeq \\
\mathrm{colim}_{C_0 \in S(\mathrm{Ob}(\mathcal{O}))} (\mathrm{hocolim}_{\alpha \in \mathcal{K}^n(k)} * \times C_0) &\cong \\
\mathrm{colim}_{C_0 \in S(\mathrm{Ob}(\mathcal{O}))} (|\mathcal{N}(\mathcal{K}^n)| \times C_0)(k) &\cong \\
(|\mathcal{N}(\mathcal{K}^n)| \times \mathrm{Ob}(\mathcal{O}))(k) &
\end{aligned}
$$

The first identification is given in assumption (C), which in turn splits out into a double colimit. To obtain the homotopy equivalence: Since $\mathcal{K}^n(k)$ is a Reedy Category, the assumption (A) allows us to apply [Dug08, 13.4] giving a homotopy equivalence $\mathrm{hocolim}_{\alpha \in \mathcal{K}^n(k)} F_k(\alpha, C_0) \to \mathrm{colim}_{\alpha \in \mathcal{K}^n(k)} F_k(\alpha, C_0)$ for fixed incoming colours $C_0 \in S(\mathrm{Ob}(\mathcal{O}))$. From (B) we get a homotopical identification of $F_k(\alpha, C_0)$ with C_0 and this computes the homotopy colimit, geometrically realizing the nerve of the full graph operad, along with a cartesian product of the colours $C_0 \in S(\mathrm{Ob}(\mathcal{O}))$ – independent of $\alpha \in \mathcal{K}^n(k)$. The final identification follows since the naturality of (B) supplies us with a cartesian product of the nerve along with an actual direct limit of all inclusions of $S(\mathrm{Ob}(\mathcal{O}))$ which can be identified with the final target of the inclusions, $\mathrm{Ob}(\mathcal{O})$.

One now utilizes F as a morphism between coloured operads to check that this gives an operadic weak equivalence $\mathcal{O} \simeq |\mathcal{N}(\mathcal{K}^n)| \times \mathrm{Ob}(\mathcal{O})$

$\square$

Definition 5.6 To $\mathcal{O}$ a coloured topological operad, we call $F \colon \mathcal{K}^n \times S(\mathrm{Ob}(\mathcal{O})) \to S(\mathcal{O})$ an E_n-*functor* if it satisifes the conditions of 5.5

Remark 5.7 In order to get an equivalence back to something known, let $\mathcal{D}isk_{n+1}$ denote the little disk operad. We hereby have that the coloured operad $\mathcal{D}isk_{n+1} \times \mathrm{Ob}(\mathcal{C}leav_{S^n})$ is a coloured E_{n+1}-operad in the sense of 2.8, coloured over the same objects as $\mathcal{C}leav_{S^n}$.

5.2 An E_{n+1}-functor for $\mathcal{C}leav_{S^n}$

Remark 5.8

Construction 5.9 We shall provide the combinatorial data, giving the link between the full graph operad, and the spherical cleavage operads.

For each $n \in \mathbb{N}$, let $I = [-1, 1] \hookrightarrow \mathbb{R}^{n+1}$ denote the interval as sitting inside the first coordinate axis of $\mathbb{R}^{n+1}$. Choosing $k - 1$ distinct points $x_1, \ldots, x_{k-1} \in I$ specifies a partition of I into k intervals $X_1 = [-1, x_1], X_2 = [x_1, x_2], \ldots, X_k = [x_{k-1}, 1]$. We endow the collection of these intervals with an ordering, determined by $\sigma \in \Sigma_k$, ordering them as $X_{\sigma(1)}, \ldots, X_{\sigma(k)}$.

Parametrize $S^n := \{\underline{s} = (s_1, \ldots, s_{n+1}) \in \mathbb{R}^{n+1} \mid \|\underline{s}\| = 1\}$ to consider the map $\eta \colon S^n \to I$ given by $\eta(s_1, \ldots, s_{n+1}) = s_1$. Any subinterval $X_i \subseteq I$ defines timber $\tilde{X}_i$ of S^n, by $\eta^{-1}(X_i)$.

Positioning hyperplanes $P_1, \ldots, P_{k-1}$ orthogonal to I – such that P_i contains the point $(x_i, 0, \ldots, 0)$, as decorations on a cleaving tree we can choose their normal-vector of P_i to point towards $(1, 0, \ldots, 0)$ and get colours that are labelled by 1 to k from left to right along the first coordinate axis.

Under the chop-equivalence of 3.8, we can always choose a representing cleaving tree with two leaves labelled i and $i+1$ directly above an internal vertex for any $i \in \{1, \ldots, k-1\}$. For this particular representative of a cleaving tree with the particular orientations of $P_1, \ldots, P_{k-1}$, applying the transposition between i and $i + 1$ corresponds exactly to inverting the orientation of the decoration at the vertex below the two leafs.

Since Σ_k is generated by these transpositions, we can permute the labelleing of the colours by $\sigma \in \Sigma_k$, and hereby obtain $[T_\sigma, \underline{P_\sigma}] \in \mathcal{Cleav}_{S^n}(Uk)$, with $U \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$ and outgoing colours decorated by $\sigma(1), \ldots, \sigma(k)$ from left to right along the first coordinate axis of $\mathbb{R}^{n+1}$.

Definition 5.10 For a given $\sigma \in \Sigma_k$, the collection of all $[T_\sigma, \underline{P_\sigma}]$ as given above specifies an element of the subsets of $\mathcal{Cleav}_{S^n}(-;k)$, where for $U \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$ this involves a restriction to the U-cleaving trees $(T_\sigma, \underline{P_\sigma})$.

Note that to $U \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$ choosing the hyperplanes $\underline{P_\sigma}$ to be in equidistant position from each other, and the closest hyperplanes that no longer cleave U, defines an embedding of $U \hookrightarrow \mathcal{Cleav}_{S^n}(U;k)$ for each $\sigma \in \Sigma_k$.

Observation 5.11 We define $J_U \subseteq [-1, 1]$ as sitting inside the first coordinate axis of $\mathbb{R}^{n+1}$ as the subspace where any U-cleaving $(T_\sigma, \underline{P_\sigma})$ have decorating hyperplanes contain points of J_U.

We shall for the sake of this section allow ourselves to assume that $J_U \subseteq [-1, 1]$ is a non-empty subinterval; formally, this can be done by redefining $\mathcal{Cleav}_{S^n}$ as a full suboperad of $\mathcal{Cleav}_{S^n}$ given by restricting $\mathrm{Ob}(\mathcal{Cleav}_{S^n})$ to the timber U for which J_U is U-cleaving.

Similar to 3.19, one can define a homotopy that pushes the hyperplanes defining U towards tangenthyperplanes of S^n to show that this restriction defines a deformation retraction of the objects, and hence makes the inclusion a weak equivalence of operads.

Remark 5.12 We find it enlightening to note that we can form a suboperad, the *caterpillar operad* [2] $\mathcal{Cater}_{S^n}$, of $\mathcal{Cleav}_{S^n}$ by taking the full suboperad under the condition that $[T, \underline{P}] \in \mathcal{Cater}_{S^n}(-;k)$ if $[T, \underline{P}]$ is of the form $[T_\sigma, \underline{P_\sigma}]$ as in 5.10 for some $\sigma \in \Sigma_k$.

The caterpillar operad will control the product structure on $\mathbb{H}_*(M^{S^n})$, however in order to obtain the higher bracket in the Gerstenhaber Algebra, we shall need more than just parallel hyperplanes – and engage all the ways hyperplanes can rotate in $\mathcal{Cleav}_{S^n}$, in contrast to the sole translational data of $\mathcal{Cater}_{S^n}$.

Said in a different way, there is an obvious operadic map from $\mathcal{Cater}_{S^n}$ to the little intervals operad, see e.g. [MS04, ch. 2], determined by how the hyperplanes of $\mathcal{Cater}_{S^n}$ partition the x-axis into intervals. The little intervals operad has the k-ary space given as the space of embeddings of k intervals inside $[0, 1]$. We can expand these little intervals linearly until they touch each other, and hereby similarly to $\mathcal{Cater}_{S^n}(-;k)$ partitioning $[0, 1]$ into k smaller intervals. This provides a map that is a weak equivalence of coloured operads from $\mathcal{Cater}_{S^n}$ to the little intervals operad, considered as a coloured operad with trivial colours. This hence provide the A_∞- or E_1-structure on the String Product associated to M^{S^n} for the action in spectra that is constructed in [Bar11]. The rest of this section is hence devoted to determining the rest of the E_{n+1}-structure of $\mathcal{Cleav}_{S^n}$.

In order to engage the combinatorics of this rotational data, i.e. define a E_{n+1}-functor, we shall prescribe explicit transformations of the hemispheres parametrizing the hyperplanes involved in the cleavages.

---

[2]collapse the components of the blueprint to a single point, to make a visual link to this name

Definition 5.13 We prescribe a function $\kappa\colon S^n \times \mathbb{R}^{n+1} \to \mathrm{Hyp}^{n+1}$ by letting $\kappa(s,t)$ be given as the oriented hyperplane that contains the point $s+t$ and has s as a normal vector. In comparison to the tangent plane at the unit-sphere at $s \in S^n \subseteq \mathbb{R}^{n+1}$, $\kappa(s,t)$ has been translated by t.

Definition 5.14 In the following, when referring to a sphere S^i, we shall generally consider it as sitting inside a string of inclusions

$$S^0 \overset{\iota^0}{\hookrightarrow} S^1 \overset{\iota^1}{\hookrightarrow} \cdots \overset{\iota^{n-1}}{\hookrightarrow} S^{n-1} \overset{\iota^{n-1}}{\hookrightarrow} S^n \tag{6}$$

where all are subsets of $\mathbb{R}^{n+1}$, where S^{i-1} is embedded equatorially into the first i coordinates of $S^i \subset \mathbb{R}^i$.

Let $S^i_+ = \{(x_0,\dots,x_n) \in S^n \mid x_i \geqslant 0, x_{i+1} = \cdots = x_{n-1} = 0\}$ and $S^i_- = \{(x_0,\dots,x_n \in \mathbb{R}^{n+1} \mid x_i \leqslant 0, x_{i+1} = \cdots = x_{n-1} = 0\}$. Restrictions of ι_i in (6), yields the partially ordered set of inclusions:

$$\tag{7}$$

Since these are all inclusions of closed lower-dimensional submanifolds, it is a partially ordered set of cofibrations.

Definition 5.15 Consider the partially ordered set I_k with objects $\alpha_j \subseteq \{1,\dots,k\}$, where j indicates that α_j is of cardinality j, and morphisms generated by the opposite arrows of *simple inclusions* $\alpha_{j-1} \to \alpha_j$.

To $f \in I_k$, where $f\colon \alpha_j \to \alpha_p$ let the domain be denoted by $D(f) := \alpha_j$ denote the domain of f, and the target $T(f) := \alpha_p$. A simle inclusion ι_l defines a *lost number* $j_{\iota_l} := T(\iota_l)\backslash D(\iota_l) \in \{1,\dots,k\}$.

An *i-string* of morphisms in I_k is given by a sequence $\underline{\iota} = (\iota_1,\dots,\iota_{k-1})$ of opposite arrows of simple inclusions such that $D(\iota_r) = T(\iota_{r+1})$ – and with $D(\iota_1) = \{1,\dots,k\}$ and $T(\iota_{k-1}) = \{i\}$.

Let $I_k|_i$ denote the set of *i*-strings of I_k.

Remark 5.16 While we shall mainly find it convenient to use the notation of 5.15, note that the data of $\underline{\iota} \in I_k|_i$ exactly corresponds to a permutation of $\{1,\dots,k\}$, where we to i assign the lost number of ι_i.

We shall thus allow ourselves to consider $\underline{\iota}$ as an element of Σ_k where we here use the notation $\underline{\iota}(i) \in \{1,\dots,k\}$ to indicate the value of i under the permutation.

Construction 5.17 We shall conglomerate the above constructions into a specific recursively defined function that provide the technical core in the definition of the E_n-functor.

For each $i \in \{1,\dots,k-1\}$, we want to define a function

$$\Theta_U : \Sigma_k \times I_{k-1}|_i \times \left(R_U \cap \left(S^n \times \mathbb{R}^{n+1} \right)^{k-1} \right) \to \mathit{Cleav}_{S^n}(U; k).$$

Where $R_U \subseteq (S^n \times \mathbb{R}^{n+1})^{k-1}$, amounts to a corestriction of each $(S^n \times \mathbb{R}^{n+1})$-factor that will be specified below.

Each ι_i will specify a hyperplane via its lost number, 5.15, as $P_{j_{\iota_i}}$ of $\underline{P}_\sigma$.

We shall produce a U-cleaving tree that is decorated by the hyperplanes

$$\kappa(s_1, r_1), \ldots, \kappa(s_{k-1}, r_{k-1}).$$

In order to specify the U-cleaving tree that these hyperplanes decorate, we utilize the ordering from left to right of the hyperplanes of $\underline{P}_\sigma$, specified in 5.9, as well as the i-string $\underline{\iota}$.

We build this tree recursively, and start by positioning $\kappa(s_1, r_1)$ as the decoration of a 2-ary tree T_2. We hereby restrict Θ_U by letting the first $(S^n \times \mathbb{R}^{n+1})$-factor of R_U be such that $(T_2, \kappa(s_1, r_1)$ is U-cleaving.

In the recursive step, assume that we have defined the first $l-1$ factors of R_U and that we are given an l-ary U-cleaving tree $(T_l, \kappa(s_1, r_1), \ldots, \kappa(s_{l-1}, r_{l-1}))$ such that taking $l-1$ hyperplanes of $\underline{P}_\sigma$, $P_{j_{\iota_1}}, \ldots, P_{j_{\iota_{l-1}}}$, and assigning $P_{j_{\iota_i}}$ to replace the decoaration $\kappa(s_i, r_i)$, also yields a U-cleaving tree.

The hyperplane $P_{j_{\iota_l}}$ cleaves timber associated to a specific leaf of the decorated tree $(T_l, P_{j_{\iota_1}}, \ldots, P_{j_{\iota_{l-1}}})$. We graft a 2-ary tree onto T_l at this leaf, to obtain the $(l+1)$-ary T_{l+1}. We let $\kappa(s_l, r_l)$ be the decoration at the new internal vertex of T_{l+1}, where we define the l'th factor of R_U by requiring that $(s_l, r_l) \in S^n \times \mathbb{R}^{n+1}$ makes the decorated $(T_{l+1}, \kappa(s_1, r_1), \ldots, \kappa(s_l, r_l))$ U-cleaving. The timber at the leafs of the decorated tree $(T_{l+1}, \kappa(s_1, r_1), \ldots, \kappa(s_l, r_l))$ are induced by the timber at the same leafs of $(T_{l+1}, P_{j_{\iota_1}}, \ldots, P_{j_{\iota_l}})$.

We make the following restriction on R_U the ensure that in each step of $\underline{\iota}$, the reparametrization is happening continously and the second a technical condition for the proof of 5.21:

($\star$) From R_U we exclude the tuppels $((s_1, r_1), \ldots, (s_{k-1}, r_{k-1}))$ where the hyperplane $\Theta_U(G, \underline{\iota}, (s_1, r_1), \ldots, (r_{k-1}, r_{k-1}))$ is not in the pathcomponent of $[T_{\sigma_G}, P_{\sigma_G}]$.

($\dagger$) We further intersect R_U by $\prod_{l=1}^{k-1} (S^n \times \text{Rec}(U_{\kappa(s_l, r_l)}))$, where the cleaved recording area $\text{Rec}(U_{\kappa(s_l, r_l)}) \subset \mathbb{R}^{n+1}$ associated to the vertex decorated by $\kappa(s_l, r_l)$ is given in 3.6.

Any edge e_{ij} of a graph $G \in \mathcal{K}^n(k)$ is uniquely determined as the edge attached to the vertices labelled by some ordered pair $i < j \in \{1, \ldots, k\}$. Let $\omega_G(i, j) \in \mathbb{Z}_2 =: \{'+', '-'\}$ be given by $\omega_G(i, j) =' +'$ if e_{ij} points from i to j and $\omega_G(i, j) =' -'$ if e_{ij} points from j to i.

Denote by $\lambda_G(i, j) \in \{0, \ldots, n-1\}$ the labelling of the edge e_{ij}.

Construction 5.18 We shall construct an E_{n+1}-functor for $\mathcal{C}\!\mathit{leav}_{S^n}$. That is, we are after a functor $D\colon \mathcal{K}^{n+1} \times S(\mathrm{Ob}(\mathcal{C}\!\mathit{leav}_{S^n})) \to S(\mathcal{C}\!\mathit{leav}_{S^n})$.

Let $G \in \mathcal{K}^{n+1}(k)$ be a graph with underlying permutation given by $\sigma_G \in \Sigma_k$

Given a i-string $\underline{\iota} \in I_{k-1}|_i$, for $l \in \{1, \ldots, k-1\}$, we consider the two lost numbers j_{ι_l} and $j_{\iota_{l-1}}$ in the sense of 5.15, to account for the case $l = 1$, we let $j_{\iota_0} := k$. Denote by $\iota_{l_{\max}} := \max\{j_{\iota_l}, j_{\iota_{l-1}}\}$ and $\iota_{l_{\min}} := \min\{j_{\iota_l}, j_{\iota_{l-1}}\}$.

Denote by $R_U(\iota_l)$ the restriction of the subspace $S^{\lambda_G(\iota_{l_{\min}}, \iota_{l_{\max}})}_{\omega_G(\iota_{l_{\min}}, \iota_{l_{\max}})} \times \mathbb{R}^{n+1}$ considered as the i'th input to Θ_U, i.e. specified by the space R_U of 5.17. Here, the hemispheres $S^{\lambda_G(\iota_{l_{\min}}, \iota_{l_{\max}})}_{\omega_G(\iota_{l_{\min}}, \iota_{l_{\max}})}$ are given in the diagram (7).

With this in mind, we define the k'th operadic constituent of D as follows:

$$D_k(G, A_0) =$$

$$\bigcup_{a_0 \in A_0} \bigcup_{i \in \{1, \ldots, k-1\}} \bigcup_{\underline{\iota} \in I_{k-1}|_i} \Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))) \tag{8}$$

5.3 Proof that Cleavages are E_{n+1}

To state the first lemma, note that maps $G \to G' \in \mathcal{K}^n(k)$ given by raising the index of an edge of G from i to l where $i < l$, will induce injective maps $D(G, A_0) \hookrightarrow D(G', \underline{A})$ given by a restriction of the inclusion $S^i \hookrightarrow S^l$ as one of the coordinates of $R_U(\iota_j)$ under $D_k(G, A_0) \to D_k(G', A_0)$. This describes how D is a functor of posets, and we use the the following three lemmas to check the conditions (A)-(C) of 5.5 to prove that $\mathcal{C}\!\mathit{leav}_{S^n}$ is a coloured E_{n+1}-operad.

Lemma 5.19 As in (A) of 5.5, consider the latching space $L_{(G, A_0)} = \bigcup_{G' < G} D_k(G', A_0)$. The induced map $L_{(G, A_0)} \hookrightarrow D_k(G', A_0)$ is a cofibration

Proof. The inclusions of submanifolds in (7) are of codimension strictly larger than 0, so these are automatically cofibrations. The maps out of $D_k(G', A_0)$ are built out of these maps by restricting to $a_0 \in A_0$-cleaving trees, along with pushouts and factors of cartesian products. Since whether a decoration of a a_0-cleaving tree is cleaving or not is an open condition (that is, if it holds for the hyperplane it holds for a small neighborhood of the hyperplane), the associated restriction of inclusions induced from (7) will again be an inclusion of submanifolds that are of codimension greater than 0. Each $D_k(G', A_0) \hookrightarrow D_k(G, A_0)$ will hence result in a cofibration, that can be obtained as a lower-dimensional skeleton of a CW-structure on $D_k(G, A_0)$. Since the latching space is given by a finite union of these lower-dimensional spaces, the map from the latching space is again a cofibration. $\qquad\square$

Lemma 5.20 $\mathrm{colim}_{(G, A_0) \in (\mathcal{K}^{n+1} \times \mathrm{Ob}(\mathcal{C}\!\mathit{leav}_{S^n}))(k)} D_k(G, A_0) = (\mathcal{C}\!\mathit{leav}_{S^n})(-; k)$

Proof. Any $[T, \underline{P}] \in \mathcal{C}\!\mathit{leav}_{S^n}(-; k)$ can be obtained as an element of (8) for some choice of hemispheres determined by $G \in \mathcal{K}^n(k)$. Note namely that the definition of Θ_U is a

function that exactly mimics the cleaving procedure above 3.5, and therefore will $[T, \underline{P}]$ as obtained by this cleaving procedure be obtained since the hemispheres involved in the image of $D_k(G, A_0)$ cover S^n, and since we in $(\star)$ of 5.17 are taking of path-components of $[T_\sigma, P_\sigma]$ for all $\sigma \in \Sigma_k$

$\square$

We say that for $(T, \underline{P})$ an S^n-cleaving tree that the hyperplane of the decoration P_j *dominates* another hyperplane P_l of the decoration of T if P_j and P_l intersect within D^{n+1} and there are points of P_l that lie on $\beta_{[T,\underline{P}]}$ and on one of the subspaces of $\mathbb{R}^{n+1}$ that has been bisected by P_j, but none on the other side.

Lemma 5.21 Given $A_0 \in S(\mathrm{Ob}(\textit{Cleav}_{S^n}))$ and $G \in \mathcal{K}^{n+1}(k)$, we have a homotopy equivalence of $D_k(G, A_0)$ with the subspace A_0; considered as a subspace of $\textit{Cleav}_{S^n}(-;k)$ by 5.10.

Another usage of terminology would be that we supply a deformation retraction onto A_0, that is not a strong deformation retraction, in that we supply a homotopy $F\colon \textit{Cleav}_{S^n}(-;k) \times [0,1] \to \textit{Cleav}_{S^n}(-;k)$, where we only guarantee $F(a_0, t) = a_0$ for $a_0 \in A_0 \subset \textit{Cleav}_{S^n}(-;k)$ included as above.

Proof. We break the proof into some steps

Step 1: Assume that the labelling of the edges of $G \in \mathcal{K}^{n+1}(k)$ satisfy $\lambda_G(i,j) = 0$ for all $i < j \in \{1, \ldots, k\}$, that is all edges of G are labelled by 0. In this case, we have that $D_k(G, A_0)$ will pointwise in A_0 be the space of all hyperplanes sets of $k-1$ hyperplanes $P_1, \ldots, P_{k-1}$ orthogonal to the first coordinate axis of $\mathbb{R}^{n+1}$ that cleaves $a_0 \in A_0$, where the parallel hyperplanes are ordered as P_{σ_G} of 5.9, where $\sigma_G \in \Sigma_k$ is the permutation associated to G.

The space $D_k(G, A_0)$ is homotoped onto A_0 by considering the interval $J_{a_0} \subseteq [-1, 1]$ of the first coordinate axis of $\mathbb{R}^{n+1}$ as given in 5.11. Homotoping $P_1, \ldots, P_k$ to be equidistant within J_{a_0} for each $a_0 \in A_0$ yields a deformation retraction onto A_0 in the sense of 5.10, since the topology of $\mathrm{Ob}(\textit{Cleav}_{S^n})$ as determined by hyperplanes forming the timber lets the endpoints of J_{a_0} – as points in $\mathbb{R}$ vary continuously as functions of $\mathrm{Ob}(\textit{Cleav}_{S^n})$.

Step 2: For a general $G \in \mathcal{K}^n(k)$ and a fixed i-string $\underline{\iota}$, and $a_0 \in A_0$, we see that the space $\Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1})))$ is weakly equivalent to a product of hemispheres, considered as a subspace of $\textit{Cleav}_{S^n}(-;k)$.

We homotope Θ in $k-1$ steps according to its recursive definition of 5.17. For the first step, the points $(s_1, r_1) \in R_{a_0}(\iota_1)$ defines a hyperplane by $\kappa(s_1, r_1)$ that cleaves a_0. We define a function $\mu\colon \mathrm{Ob}(\textit{Cleav}_{S^n}) \times R_{a_0}(\iota_1) \times [0,1] \to \mathrm{Ob}(\textit{Cleav}_{S^n})$, such that for any $s' \in S^n$, $\kappa(s', r_1)$ will cleave $\mu(a_0, (s_1, r_1), 1) \in \mathrm{Ob}(\textit{Cleav}_{S^n})$.

In order to do define this function, note that a_0 as an element of $\mathrm{Ob}(\textit{Cleav}_{S^n})$ is defined by choosing some hyperplanes $H_1, \ldots, H_r$ that cleave S^n. For any one of these r hyperplanes, H_i, there is a well-defined normal-vector ν_i that points away from the hyperplane $\kappa(s_1, r_1)$. Understood in the sense that translating H_i in the direction of ν_i will leave $\kappa(s_1, r_1)$ cleaving. Similar to the proof of 3.19, we can therefore define the function μ by translating the hyperplanes H_i simultaneously in the direction of ν_i, we

29

do this until the hyperplanes are close enough to tangent-hyperplanes of S^n, formally requiring that the minimal distance between $H_i \cap S^n$ and $H_j \cap S^n$ is larger than some given $\varepsilon > 0$ for all $i, j \in \{1, \ldots, r\}$. That any $s' \in S^n$ hereby will have $\kappa(s', r_1)$ cleave $\mu(a_0, (r_1, r_1), 1)$ can be seen by 3.11 since the complement of this subspace of S^n will consist of disjoint disks, and by (†) of 5.17, $r_1 \in \mathrm{Rec}(a_0) \subseteq \mathrm{Rec}(a_1)$ so $\kappa(s', r_1)$ will always intersect a_1 non-trivially.

This hence defines $a_1 := \mu_1(a_0, (s_1, r_1), 1) \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$. In a very similar fashion, we wish to further deform the $k-1$ hyperplanes involved in the recursively defined Θ_{a_0}. To this end, assume we are given the hyperplanes $H_1, \ldots, H_{r+l}$ where for all $i, j < r + l$ where $i \neq j$ $H_i \cap S^n$ and $H_j \cap S^n$ are of distance at least ε to $H_i \cap S^n$. We can take these $r + l$ as data for the l'th step of a deformation of $\Theta_{a_0}(R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))$, where the first r hyperplanes defines timber as defined under μ above, and the remaining l are hyperplanes that cleave this timber. By assumption in the definition of Θ_{a_0}, we are given a hyperplane $\kappa(s_{l+1}, r_{l+1})$ that cleave some $a_l \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$ as defined through a portion of the hyperplanes $H_1, \ldots, H_{l+r}$.

Only choosing $j := l+r$ will potentially satisfy $H_j \cap H_i \cap S^n \neq \varnothing$, and we therefore wish to push H_{r+l} in the direction of the normal-vector ν_{r+l}, away from $\kappa(s_{l+1}, r_{l+1})$. Blindly pushing $\kappa(s_{l+1}, r_{l+1})$ in the direction of ν_{r_l} will deform a_l to incorporate more hyperplanes than the ones used to defining a_l, and we need to ensure continuity with respect to these hyperplanes. Let therefore again ν_i denote the normal-vector of H_i in the direction away from $\kappa(s_{l+1}, r_{l+1})$. Let similarly $\mathrm{dist}(H_i) \in \mathbb{R}_+$ denote the distance by the hyperplane H_i must be translated along ν_i to become a hyperplane, let $A(U) \in \mathbb{R}_+$ denote the area of an $n-1$-dimensional subspace inside S^n. Assume that $H_{r+l} \cap S^n$ and $H_i \cap S^n$ are within ε distance of each other; potentially intersecting, for $i \in K_{H_{r+l}} \subseteq \{1, \ldots, l + r - 1\}$. We push the hyperplanes indexed by $K_{H_{r+l}}$ and H_{r+l} simultaneously to obtain hyperplanes $H_{r+l}(t)$ and $H_i(t)$ where $i \in K_{H_{r+l}(t)}$ at time t, hereby forming the timber $a_l(t)$. Let as usual $\partial \overline{a_l(t)}$ denote the boundary of the closure of a_l within S^n. We push H_i in the direction of ν_i with with speed $\mathrm{dist}(H_i(t)) \cdot (1 - \frac{A(\partial \overline{a_l(t)} \cap H_i(t))}{A(S^n \cap H_i(t))})$ while H_{r+l} is pushed with speed $\mathrm{dist}(H_{r+l}(t)) \cdot \max_{i \in K_{H_{r+l}}}(\frac{A(\partial \overline{a_l(t)} \cap H_i(t))}{A(S^n \cap H_i(t))})$, and we stop once $H_{r+l}(t)$ is at distance at least $\varepsilon \cdot \min_{i \in \{1, \ldots, r+l-1\}} \mathrm{dist}(H_i(t))$ from $H_i(t)$ for all $i \in \{1, \ldots, r_{r+l-1}\}$.

These formula ensure that H_{r+l} is only pushed past the subspace $H_i \cap S^n$ if there is a significant portion of this subspace forming part of $\partial \overline{a_l}$. Hereby, we hence obtain the timber a_l that $\kappa(s', r_{l+1})$ cleaves for all $s' \in S^n$, since the complement $\complement a_l$ will consist of a disjoint union of subdisks of S^n.

For $l = k - 1$, this hence gives a subspace of $\mathcal{Cleav}_{S^n}(-; k)$ parametrized by the space $\prod_{l=1}^{k-1}(S^{\lambda_{\sigma_G}(\iota_{l_{\min}}, \iota_{l_{\max}})}_{\omega_{\sigma_G}(\iota_{l_{\min}}, \iota_{l_{\max}})} \times \mathrm{Rec}(a_l))$, where we again use (†) of 5.17, to identify the parametrizing $\mathbb{R}^{n+1}$-factor. Since $\mathrm{Rec}(U)$ can be seen to be contractible, in fact convex, for any $U \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$, the subspace to our given i-string is weakly equivalent to a product of hemispheres.

Step 3: Given $\underline{\iota}$ a i-string, and $\underline{\lambda}$ a l-string, we show that the associated spaces of $\underline{\iota}$ and $\underline{\lambda}$,

$$\Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))) \text{ and } \Theta_{a_0}(\sigma_G, \underline{\lambda}, (R_{a_0}(\lambda_1), \ldots, R_{a_0}(\lambda_{k-1})))$$

respectively, are glued together along contractible subsets, where we consider these as subsets of $\mathcal{C}leav_{S^n}(-;k)$. Since step 2 tells us that the space associated to $\underline{\iota}$ and the one associated to $\underline{\lambda}$ are contractible, we have that the glued spaces are contractible. This follows combining the Mayer-Vietoris sequence, Hurewicz map and the Van Kampen theorem so that if A and B are two contractible spaces and $A \cap B$ is contractible then $A \cup B$ is again a contractible space. As we shall see, all spaces associated to elements of $\bigcup_{i=1}^{k-1} I_i$ will be glued together along the common basepoint determined by the hyperplanes $\underline{P}_{\sigma_G}$ orthogonal to the first coordinate axis of $\mathbb{R}^{n+1}$ as determined by the orientation of G. In effect, the glueing of all these spaces will result in a contractible space.

In order to see how the spaces are glued together, we see in 5.17 that the spaces associated to $\underline{\iota}$ and $\underline{\lambda}$ are given by choosing new normal-vectors for the hyperplanes $P_1, \ldots, P_k$ of $\underline{P}_{\sigma_G}$ and decorating them on different trees.

Considering $\underline{\iota}$ and $\underline{\lambda}$ as permutations in Σ_k as given in 5.16, it a neccesary but not sufficient condition that points in the spaces associated to $\underline{\iota}$ and $\underline{\lambda}$ determined by the tuppels $((s_1, r_1), \ldots, (s_{k-1}, r_{k-1})$ and $((s'_1, r'_1), \ldots, (s'_{k-1}, r'_{k-1}))$ satisfy

$$\left(s_{\underline{\iota}^{-1}(i)}, r_{\underline{\iota}^{-1}(i)}\right) = \left(s'_{\underline{\lambda}^{-1}(i)}, r'_{\underline{\lambda}^{-1}(i)}\right) \tag{9}$$

for all $i \in \{1, \ldots, k-1\}$, since the associated timber of the cleaving trees have to agree.

The condition (9) is not sufficient since the way the hyperplanes dominate each other might be different according to the two trees that the hyperplanes are decorating. Note first of all that if (9) is satisfied, and $s_{\underline{\iota}^{-1}(i)} = s_{\underline{\lambda}^{-1}(i)}$ is the point $S^0_{\omega(\iota(i))}$ all the hyperplanes are parallel and parrallel to the first coordinate axis, where there is no dominance amongst hyperplanes, identifying the points.

We have an ordering on the hyperplanes of the spaces associated to $\underline{\iota}$ and $\underline{\lambda}$, given by $P_{\underline{\iota}(1)}, \ldots, P_{\underline{\iota}(k-1)}$ and $P_{\underline{\lambda}(1)}, \ldots, P_{\underline{\lambda}(k-1)}$. Assume we are given $[T, P_{\underline{\iota}}]$ and $[T', P_{\underline{\lambda}}]$ satisfying (9) such that they agree as elements of $D_k(G, A_0)$. We want to show that there is a unique path homotoping them onto trees decorated by hyperplanes orthogonal to the first coordinate axis, in which case step 1 applies to provide the deformation retraction.

For the hyperplane $P_{\underline{\iota}(1)}$, no other hyperplanes $P_{\underline{\iota}(2)}, \ldots, P_{\underline{\iota}(k-1)}$ will dominate $P_{\underline{\iota}(1)}$.

We can use step 2 to assume that for the 2-ary tree decorated by only P_{ι_1}, the reparametization of the hyperplane along the geodesic path along the hemisphere $S^{\lambda_G(\iota_{1_{\min}}, \iota_{1_{\max}})}_{\omega_G(\iota_{1_{\min}}, \iota_{1_{\max}})}$ will always be contained in $\mathcal{C}leav_{S^n}(-;2)$.

However, we need to consider these paths for our trees decorated by multiple hyperplanes. The condition $(\star)$ of 5.17 tells us that the geodesic path along the hemisphere $S^{\lambda_G(\iota_{1_{\min}}, \iota_{1_{\max}})}_{\omega_G(\iota_{1_{\min}}, \iota_{1_{\max}})}$ parametrizing $P_{\underline{\iota}(1)}$ will be contained in the space associated to $\underline{\iota}$. Since the elements $[T, P_{\underline{\iota}}]$ and $[T', P_{\underline{\lambda}}]$ agree, the hyperplane $P_{\underline{\lambda}^{-1}(\underline{\iota}(1))}$ as a decoration of $[T', \underline{P}_{\underline{\lambda}}]$ will be bound to dominate the same hyperplanes as $P_{\underline{\iota}(1)}$ as an element of $[T, \underline{P}_{\underline{\iota}}]$, and this will remain true along the reparametization along the geodesic, until $P_{\underline{\iota}(1)}$ no longer dominates any other hyperplanes, since otherwise either $[T, \underline{P}_{\underline{\iota}}]$ or $[T, \underline{P}_{\underline{\lambda}}]$ would be reparametrized to non-cleaving trees. To ensure that no new dominance occurs after the potential step where $P_{\underline{\iota}(1)}$ no longer dominates any other hyperplanes, one translates

along the first coordinate of $\mathbb{R}^{n+1}$ in $S_{\omega_G(\iota_{1\min},\iota_{1\max})}^{\lambda_G(\iota_{1\min},\iota_{1\max})} \times \mathbb{R}^{n+1}$ at the same speed in the opposite direction as $P_{\underline{\iota}(1)}$ is moving away from the hyperplanes it used to dominate.

This eventually brings the hyperplanes $P_{\underline{\iota}(1)}$ parallel with the first coordinate axis of $\mathbb{R}^{n+1}$, and one iterates this construction for the remaining $P_{\underline{\iota}(2)}, \ldots, P_{\underline{\iota}(k-1)}$ in that order, to – along a geodesic in the parametrizing hemispheres – bring them all parallel to the first coordinate axis. Finally, having used step 2, one applies the inverse homotopy of μ in step 2, to make these hyperplanes cleave a_0.

As noted previously, step 1 now applies to finish the proof. $\qquad\square$

Theorem 5.22 $\mathcal{Cleav}_{S^n}$ is a coloured E_{n+1}-operad.

Proof. The lemmas 5.19, 5.20, 5.21 check (A)-(C) in 5.5 the functor D of 5.18 should satisfy in order for $\mathcal{Cleav}_{S^n}$ to be E_{n+1}. Note that 5.21 indeed gives the full naturality as stated in (B), since we give an explicit homotopy equivalence onto $\mathrm{Ob}(\mathcal{Cleav}_{S^n}) \subset \mathcal{Cleav}_{S^n}(-;k)$ that respects morphisms of $S(\mathrm{Ob}(\mathcal{Cleav}_{S^n}))$. $\qquad\square$

Corollary 5.23 There is an equivalence of operads

$$\mathrm{H}_*(\mathcal{Cleav}_{S^n}) \cong \mathrm{H}_*(\mathcal{Disk}_{n+1})$$

Proof. By 3.19 the colours of $\mathcal{Cleav}_{S^n}$ are contractible, and by further 3.20, we can apply 2.5, so $\mathrm{H}_*(\mathcal{Cleav}_{S^n})$ is a monochrome operad and 5.22 together with 5.7 gives the corollary. $\qquad\square$

6 Semidirect Products of the Cleavage Operads

Given a topological space, X, we let $\mathrm{Homeo}(X)$ denote the group of self-homeomorphisms of X.

Similar to having a monochrome G-operad in the category of G-spaces, for coloured topological operads we give the following definition

Definition 6.1 A topological group G *acts* on a coloured topological operad, $\mathcal{O}$ if we are given

- $.: G \to \mathrm{Homeo}(\mathrm{Ob}(\mathcal{O}))$ a continuous map. Given $g \in G$ and $U \in \mathrm{Ob}(\mathcal{O})$, under adjunction, we denote the corresponding acted upon element as $g.U$.

- $\alpha_i \colon G \to \mathrm{Homeo}(\mathcal{O}(-;m))$ continous maps for all $i \in \{1, \ldots, m\}$ and $m \in \mathbb{N}$.

and these respect the topological structure of operads; i.e. letting $g \in G$ the following

diagram should be commutative for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$:

$$\mathcal{O}(A; k + m - 1) \tag{10}$$

In the classical setting, when $\mathcal{O}$ is a monochrome operad, this recovers the notion of a G-operad. We in this spirit call a coloured operad satisfying the above a *coloured G-operad*.

In [SW03, 2.1], semidirect products of monochrome operads are introduced, and we can expand the notion to the coloured setting by only expanding a little on the operadic evaluation maps:

Definition 6.2 For a coloured topological operad $\mathcal{O}$ with an action of a group G, we can form the *semidirect product* of $\mathcal{O}$ by G, as for the monochrome setting denoted $\mathcal{O} \rtimes G$, by letting

- $\mathrm{Ob}(\mathcal{O} \rtimes G) = \mathrm{Ob}(\mathcal{O})$

- $\mathcal{O} \rtimes G = \mathcal{O}(-; k) \times G^k$

Letting ev_i and $\mathrm{ev}_{\mathrm{in}}$ denote the operadic evaluation maps of $\mathcal{O}$. The operadic evaluation maps ev_i^G and $\mathrm{ev}_{\mathrm{in}}^G$ for the semidirect product are given by $\mathrm{ev}_i^G(\omega, \rho_1, \ldots, \rho_k) = \rho_i . \mathrm{ev}_i(\omega)$ and $\mathrm{ev}_{\mathrm{in}}^G = \mathrm{ev}_{\mathrm{in}}$. As for the monochrome case, the operadic composition $\circ_i \colon (\mathcal{O} \rtimes G)(-; k) \times_{\mathrm{Ob}(\mathcal{O})} (\mathcal{O} \rtimes G)(-; m) \to (\mathcal{O} \rtimes G)(-; k + m - 1)$ is given by twisting the composition of $\mathcal{O}$ through the action of G in the following sense:

$$(\omega; \rho_1, \ldots, \rho_k) \circ_i (\omega'; \eta_1, \ldots, \eta_m) = (\omega \circ_i \rho_i.\omega'; \rho_1, \ldots, \rho_{i-1}, \rho_i.\eta_1, \ldots, \rho_i.\eta_m, \rho_{i+1}, \ldots, \rho_k)$$

Observation 6.3 As a gadget constructed from $\mathbb{R}^{n+1}$, there is an induced action of $\mathrm{SO}(n+1)$ on $\mathcal{C}\!\mathit{leav}_{S^n}$, precisely:

To an affine oriented hyperplane $P \in \mathrm{Hyp}^{n+1}$ and $\rho \in \mathrm{SO}(n+1)$, letting ρ act on $\mathbb{R}^{n+1}$ by rotation leads P to the affine oriented hyperplane $\rho.P$. Hereby $\mathrm{SO}(n+1)$ acts on the space Hyp^{n+1}.

There is also an action of $\mathrm{SO}(n+1)$ on $\mathrm{Ob}(\mathbf{Cleav}_{S^n})$ rotating timber $U \in \mathrm{Ob}(\mathbf{Cleav}_{S^n})$ obtained by cleaving S^n along some hyperplanes to the timber $\rho.U$ obtained by cleaving S^n along the hyperplanes rotated along ρ.

Let $[T, \underline{P}] \in \mathbf{Cleav}_{S^n}(U; k)$, for some $U \in \mathrm{Ob}(\mathbf{Cleav}_{S^n})$. Let $\rho.[T, \underline{P}] = [T, \rho.\underline{P}]$, understood in the sense that ρ rotates the decorations of T simultaneously. Having $[T, \underline{P}]$ an element of $\mathbf{Cleav}_{S^n}(U; k)$ this should be interpreted in the sense that $\rho.[T, \underline{P}]$ is an element of $\mathbf{Cleav}_{S^n}(\rho.U; k)$, ensuring that the decorated tree cleaving $\rho.U$ and in turn making this action of $\mathrm{SO}(n + 1)$ on $\mathbf{Cleav}_{S^n}$ satisfy 6.1.

The action of 6.3 defines the semidirect product of $\mathbf{Cleav}_{S^n}$ by $\mathrm{SO}(n+1)$, in the sense of 6.2.

Observation 6.4 We can extend the action of $\mathbf{Cleav}_{S^n}$ along correspondances as given in 4.4 to an action of $\mathbf{Cleav}_{S^n} \rtimes \mathrm{SO}(n + 1)$ on M^{S^n} by to $([T, \underline{P}], \rho_1, \dots, \rho_k)$ an element of $(\mathbf{Cleav}_{S^n} \rtimes \mathrm{SO}(n + 1))(-; k)$ considering the following pullback-diagram:

$$
\begin{array}{ccc}
M^{S^n}_{[T,\underline{P}]} & \xrightarrow{\varphi^*_{\underline{\rho}}} & \left(M^{S^n}\right)^k \\
\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{\underline{\rho}}} \\
M^{\pi_0(\widehat{\beta_{[T,\underline{P}]}})} & \xrightarrow{\varphi} & \prod_{i=1}^k M^{(\complement N_i)}
\end{array}
$$

Where all spaces are as in 4.4, as is the map φ. However, the *twisted restriction map* $\mathrm{res}_{\underline{\rho}}$ is given by at the i'th factor of $\left(M^N\right)^k$ be defined as $\mathrm{res}_{\complement N_i} . \rho_i^{-1}$, where the map $\mathrm{res}_X \colon M^{S^n} \to M^X$ denotes the restriction map to the space $X \subseteq S^n$. The element $\rho_i \in \mathrm{SO}(n + 1)$ is considered as a diffeomorphism of S^n, and $\mathrm{res}_X . \rho_i^{-1}$ denotes the precomposition of ρ_i^{-1} of the domain of $f \in M^{S^n}$ prior to applying the restriction map. This preapplication of ρ_i^{-1} allows us to consider $\mathrm{res}_{\complement N_i} . \rho_i^{-1}$ as a map that takes points of $\rho_i(\complement N_i) \subseteq S^n$ and brings these to $\complement N_i$ where a restriction map is subsequently applied.

Note that whereas this allows us to let φ be given as the same morphism as in 4.4, which in turn allows us to again identify the pullback space $M^{S^n}_{[T,\underline{P}]}$ as maps from S^n to M that are constant along the blueprint of $[T, \underline{P}]$. However, the fact that we are applying a twisted restriction map means that the associated map $\varphi^*_{\underline{\rho}}$ in the pullback will be different from 4.4. Concretely, $\varphi^*_{\underline{\rho}}$ is given by having the i'th image maps from S^n to M that are constant along $\rho_i(\complement N_i)$.

Note that this description of $\varphi^*_{\underline{\rho}}$ makes this action of correspondanes respect the operadic composition, in the sense that for $([T, \underline{P}], \rho_1, \dots, \rho_k)$ and $([T', \underline{P}'], \eta_1, \dots, \eta_m) \circ_i$-composable as elements of $\mathbf{Cleav}_{S^n} \rtimes \mathrm{SO}(n + 1)$, the change of colours by $\rho_i \colon S^n \to S^n$ in the operadic composition of the semidirect product makes it necessary for commutativity of operadic associatativity diagrams to map $M^{S^n}_{[T,\underline{P}] \circ_i \rho_i . [T',\underline{P}']}$ to the $i + j - 1$'th factor of $\left(M^{S^n}\right)^{k+m-1}$ that are constant along the subspace $\eta_j(\rho_i(\complement N_{i+j-1})) \subseteq S^n$, where $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, k\}$.

In [Bar11], we shall see that this makes the effect of $\mathrm{SO}(n+1)$ non-trivial in the action of $\mathbf{Cleav}_{S^n} \rtimes \mathrm{SO}(n + 1)$ compared to $\mathbf{Cleav}_{S^n}$

Proposition 6.5 Actions of $H_*(\mathcal{Cleav}_{S^n} \rtimes SO(n+1))$ are BV_{n+1}-algebras, when $H_*(-)$ denotes homology with coefficients in a field.

Proof. Since $\mathcal{Cleav}_{S^n}$ is a coloured E_{n+1}-operad; 5.22, with contractible colours as; 3.19, we have that $H_*(\mathcal{Cleav}_{S^n})$ is in particular a quadratic operad, with 3-ary operations relations determined by the Gerstenhaber relations.

Hereby, the statement follows directly by applying [SW03, 4.4], to see that operadic actions of $H_*(\mathcal{Cleav}_{S^n} \rtimes SO(n+1))$ agrees with operadic actions of $H_*(\mathcal{Disk}_{n+1} \rtimes SO(n+1))$.

$\square$

It is natural to conjecture that there is a weak equivalence of operads $\mathcal{Cleav}_{S^n} \rtimes SO(n+1) \simeq \mathcal{Disk}_{n+1} \rtimes SO(n+1)$. However the string of equivalences of 5.5 does not end up with an equivalence in 5.22 with intermediate terms, involving the nerve of the full graph operad, that does not see the action of $SO(n+1)$ on $\mathbb{R}^{n+1}$. One option for proving this could be to attempt at making $SO(n+1)$-equivariant versions of the nerve of the full graph operad. We shall however not go into these considerations in this paper.

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