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Abstract Bivariant Cuntz Semigroups

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ABSTRACT BIVARIANT CUNTZ SEMIGROUPS

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ABSTRACT. We show that abstract Cuntz semigroups form a closed symmetric monoidal category. Thus, given Cuntz semigroups S and T, there is another Cuntz semigroup $[\![S,T]\!]$ playing the role of morphisms from S to T. Applied to C^*-algebras A and B, the semigroup $[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!]$ should be considered as the target in analogues of the UCT for bivariant theories of Cuntz semigroups.

Abstract bivariant Cuntz semigroups are computable in a number of interesting cases. We explore its behaviour under the tensor product with the Cuntz semigroup of strongly self-absorbing C^*-algebras and the Jacelon-Razak algebra. We also show that order-zero maps between C^*-algebras naturally define elements in the respective bivariant Cuntz semigroup.

1. Introduction

The Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A is an invariant that plays an important role in the structure theory of C^*-algebras and the related Elliott classification program. It is defined analogously to the Murray-von Neumann semigroup, $V(A)$, by using equivalence classes of positive elements instead of projections; see [Cun78]. In general, however, the semigroup $\mathrm{Cu}(A)$ contains much more information than $V(A)$, and it is therefore also more difficult to compute.

The Cuntz semigroup has been successfully used in the classification program, both in the simple and nonsimple setting. For example, Toms constructed two simple AH-algebras that have the same Elliott invariants, but which are nevertheless not isomorphic, a fact that is detected by the Cuntz semigroup; see [Tom08]. On the other hand, Robert classified (not necessarily simple) inductive limits of one-dimensional NCCW-complexes with trivial K_1-group using the Cuntz semigroup; see [Rob12].

The connection of the Cuntz semigroup with the Elliott invariant has been explored in a number of instances; see for instance [PT07], [BPT08] and [Tik11]. In fact, for the class of simple, unital, nuclear C^*-algebras that are $\mathcal{Z}$-stable (that is, that tensorially absorb the Jiang-Su algebra $\mathcal{Z}$), the Elliott invariant and the Cuntz semigroup together with the K_1-group determine one another functorially; see [ADPS14]. When dropping the assumption of $\mathcal{Z}$-stability, it is not known whether the Elliott invariant together with the Cuntz semigroup provides a complete invariant for classification of simple, unital, nuclear C^*-algebras.

It is therefore very interesting to study the structural properties of the Cuntz semigroup of a C^*-algebra. This study was initiated by Coward, Elliott and Ivanescu in [CEI08], who introduced a category Cu and showed that the assignment $A \mapsto \mathrm{Cu}(A)$ is a sequentially continuous functor from C^*-algebras to Cu. The objects of Cu are called *abstract Cuntz semigroups* or Cu*-semigroups*. Working in this category allows one to provide elegant algebraic proofs for structural properties of C^*-algebras.

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A systematic study of the category Cu was undertaken in [APT14]. One of the main results obtained is that Cu has a natural structure as symmetric monoidal category (see Paragraph 2.14 for more details). This means, in particular, that Cu admits tensor products and that there is a bifunctor

$$\otimes\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$$

which is (up to natural isomorphisms) associative, symmetric, and has a unit object, namely the semigroup $\overline{\mathbb{N}}=\{0,1,2,\ldots,\infty\}$. The basic properties of this construction were studied in [APT14], relating in particular $\mathrm{Cu}(A\otimes B)$ with $\mathrm{Cu}(A)\otimes\mathrm{Cu}(B)$ for certain classes of C^*-algebras.

Following the line of thought above, it is very natural to ask whether Cu is also a *closed* category. This problem was left open in [APT14, Chapter 9]. Given Cu-semigroups S and T, the question is if there exists a Cu-semigroup $[\![S,T]\!]$ that plays the role of morphisms from S to T. In category theory, this is expressed by requiring that the functor $[\![T,\_]\!]$ is adjoint to the functor $\_\otimes T$, which means that for any other Cu-semigroup P we have a natural bijection

$$\mathrm{Cu}\big(S,[\![T,P]\!]\big)\cong\mathrm{Cu}\big(S\otimes T,P\big),$$

where $\mathrm{Cu}(\_,\_)$ denotes the set of morphisms in the category Cu. The morphisms in Cu, also called Cu-*morphisms*, are order-preserving monoid maps that preserve suprema of increasing sequences and that preserve the so-called *way-below relation*; see Definition 2.4.

One of the main objectives of this paper is to construct the Cu-semigroup $[\![S,T]\!]$ and to study its basic properties. We call $[\![S,T]\!]$ an *abstract bivariant Cuntz semigroup* or a *bivariant* Cu-*semigroup*. The construction defines a bifunctor

$$[\![\_,\_]\!]\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu},$$

referred to as the *internal-hom* bifunctor; see, for example, [Kel05].

An important motivation for our construction is to find an analogue of the universal coefficient theorem (UCT) for Cuntz semigroups. Recall that a separable C^*-algebra A is said to satisfy the UCT if for every separable C^*-algebra B there is a short exact sequence

$$0\to\bigoplus_{i=0,1}\mathrm{Ext}\big(K_i(A),K_i(B)\big)\to KK_0(A,B)\to\bigoplus_{i=0,1}\mathrm{Hom}\big(K_i(A),K_i(B)\big)\to 0.$$

We refer to [Bla98, Chapter 23] for details.

The near goal is then to replace $KK_0(A,B)$ by a suitable bivariant version of the Cuntz semigroup (for example, along the lines of [BTZ16]), and the Hom-functor in the category of abelian groups by the internal-hom functor $[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!]$ alluded to above. In this direction, the construction developed in [BTZ16] uses certain equivalence classes of completely positive contractive (abbreviated c.p.c.) order-zero maps between C^*-algebras, denoted here as $\mathrm{cpc}_\perp(A,B)$. In Section 8, we show that every c.p.c. order-zero map $\varphi\colon A\to B$ defines an element $\mathrm{Cu}(\varphi)$ in the abstract bivariant Cuntz semigroup $[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!]$.

The construction of bivariant Cu-semigroups resorts to the use of a more general class of maps than just Cu-morphisms. A *generalized* Cu-*morphism* is defined as an order-preserving monoid map that preserves suprema of increasing sequences (but not necessarily the way-below relation); see Definition 2.4. We denote the set of such maps by $\mathrm{Cu}[S,T]$. Since every Cu-morphism is also a generalized Cu-morphism, we have an inclusion $\mathrm{Cu}(S,T)\subseteq\mathrm{Cu}[S,T]$.

When equipped with poinwise order and addition, $\mathrm{Cu}[S,T]$ has a natural structure as a partially ordered monoid, but it is in general not a Cu-semigroup. Similarly, $\mathrm{Cu}(S,T)$ is usually not a Cu-semigroup. The solution is to consider *paths* in

$\mathrm{Cu}[S,T]$, that is, rationally indexed maps $\mathbb{Q}\cap(0,1)\to\mathrm{Cu}[S,T]$ that are 'rapidly increasing' in a certain sense. Equipped with a suitable equivalence relation, these paths define the desired Cu-semigroup $[\![S,T]\!]$.

This procedure can be carried out in a much more general setting. In Section 4 we introduce a category $\mathcal{Q}$ of partially ordered semigroups that, roughly speaking, is a weakening of the category Cu, in that the way-below relation is replaced by a possibly different binary relation (called *auxiliary relation*). We show that Cu is a full subcategory of $\mathcal{Q}$; see Proposition 4.4. The path construction we have delineated above yields a covariant functor

$$\tau\colon\mathcal{Q}\to\mathrm{Cu},$$

that turns out to be right adjoint to the natural inclusion functor; see Theorem 4.12. We refer to this functor as the τ*-construction*. This result has numerous advantages. Besides making clearer arguments of the results in the present paper available than going via a direct argument, it allows to transport many categorical properties from $\mathcal{Q}$ to Cu. Some of these constructions will be explored in [APT17]. In our setting, the functor applied to the semigroup of generalized Cu-morphisms $\mathrm{Cu}[S,T]$ yields the internal-hom of S and T. In other words, for Cu-semigroups S and T, we define

$$[\![S,T]\!]:=\tau(\mathrm{Cu}[S,T]);$$

see Definition 5.3.

We illustrate our results by computing a number of examples, that include the (Cuntz semigroups of the) Jiang-Su algebra $\mathcal{Z}$, the Jacelon-Razak algebra $\mathcal{W}$, UHF-algebras of infinite type, and purely infinite simple C^*-algebras. Interestingly, $[\![\mathrm{Cu}(\mathcal{W}),\mathrm{Cu}(\mathcal{W})]\!]$ is isomorphic to the Cuntz semigroup of a II_1-factor.

The fact that Cu is a closed category automatically adds additional features well known to category theory. For example, one obtains a *composition product* given in the form of a Cu-morphism:

$$\circ\colon[\![T,P]\!]\otimes[\![S,T]\!]\to[\![S,P]\!].$$

In the case where $S=T=P$, the above composition product equips $[\![S,S]\!]$ with the structure of a (not necessarily commutative) Cu-semiring.

Although the said features can be derived from general principles, in our setting they become concrete, and this is very useful in applications. In this direction, and bearing in mind that $[\![S,T]\!]$ is a semigroup built out of paths of morphisms from S to T, the composition product can be realized as the composition of paths. Another important example is the *evaluation map* which, for Cu-semigroups S and T is a Cu-morphism $\mathrm{e}_{S,T}\colon[\![S,T]\!]\otimes S\to T$ such that $\mathrm{e}_{S,T}(x\otimes s)$ can be interpreted as the evaluation of $x\in[\![S,T]\!]$ at $s\in S$. We therefore also write $x(s):=\mathrm{e}_{S,T}(x\otimes s)$. The evaluation map can be used to concretize the adjunction between the internal-hom bifunctor and the tensor product.

Likewise, the tensor product of generalized Cu-morphisms induces an *external tensor product*

$$\boxtimes\colon[\![S_1,T_1]\!]\otimes[\![S_2,T_2]\!]\to[\![S_1\otimes S_2,T_1\otimes T_2]\!],$$

which is associative and, like in KK-Theory, compatible with the composition product. This means that, for elements $x_k\in[\![S_k,T_k]\!]$ and $y_k\in[\![T_k,P_k]\!]$ (for $k=1,2$), we have

$$(y_2\boxtimes y_1)\circ(x_2\boxtimes x_1)=(y_2\circ x_2)\boxtimes(y_1\circ x_1).$$

We also deepen our study of Cu-semirings and their semimodules. Recall that, as shown in [APT14, Chapter 7], the Cuntz semigroup of a strongly self-absorbing C^*-algebra has a natural product giving it the structure of a Cu-semiring.

A particular instance of the composition product setting arises when we study the semiring $[\![S,S]\!]$. Any Cu-semigroup S becomes an $[\![S,S]\!]$-semibimodule. This can also be extended to the more general situation where T has a left R-action (for a Cu-semiring R with compact unit) and S is an arbitrary Cu-semigroup to obtain that $[\![S,T]\!]$ has a left R-action as well.

For any Cu-semiring R, the internal-hom construction makes it possible to define a left regular representation-like map $\pi_R\colon R\to[\![R,R]\!]$, which is always multiplicative (and unital in case the unit of R is a compact element).

Recall that a Cu-semiring R is *solid* in case the multiplication defines an isomorphism between $R\otimes R$ and R (see [APT14, Chapter 7]). This terminology was inspired by that of solid rings, and reflects the situation of strongly self-absorbing C^*-algebras, in the sense that the Cuntz semigroup of any strongly self-absorbing C^*-algebra satisfying the UCT is solid.

Under mild assumptions (namely, the so-called axioms (O5) and (O6)), all solid Cu-semirings were classified in [APT14, Theorem 8.3.3]. We show that a Cu-semiring R satisfying (O5) and (O6) is solid if and only if the evaluation map $\mathrm{e}_{R,R}\colon [\![R,R]\!]\otimes R\to R$ is an isomorphism. The question of whether this holds without assuming (O5) or (O6) remains open (Question 7.26).

We finally specialise to C^*-algebras and show that a c.p.c. order-zero map $\varphi\colon A\to B$ between C^*-algebras A and B naturally defines an element $\mathrm{Cu}(\varphi)$ in the bivariant Cu-semigroup $[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!]$; see Theorem 8.3 and Definition 8.4. We then analyse the induced map

$$\mathrm{cpc}_\perp(A,B)\to[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!],$$

and show it is surjective in a number of cases; namely for a UHF-algebra of infinite type, the Jiang-Su algebra, or the Jacelon-Razak algebra $\mathcal{W}$; see Example 8.9.

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2. Preliminaries

Throughout, $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space. Given a C^*-algebra A, we let A_+ denote the positive elements in A.

2.1. **The category Cu of abstract Cuntz semigroups.** In this subsection, we recall the definition of the category Cu of abstract Cuntz semigroups as introduced by Coward, Elliott and Ivanescu in [CEI08].

2.1. Let us first recall the basic theory of the category PoM of positively ordered monoids. We refer to [APT14, Appendix B.2] for details.

A *positively ordered monoid* is a commutative semigroup M, written additively with zero element 0, together with a partial order $\leq$ such that $a \leq b$ implies that $a + c \leq b + c$ for all $a, b, c \in M$, and such that $0 \leq a$ for all $a \in M$. We let PoM denote the category whose objects are positively ordered monoids, and whose morphisms are maps preserving addition, order and the zero element.

Let M, N and P be positively ordered monoids. We denote the set of PoM-morphisms from M to N by $\mathrm{PoM}(M, N)$. A map $\varphi\colon M \times N \to P$ is called a PoM-*bimorphism* if it is a PoM-morphism in each variable, that is, for each $m \in M$ the map $N \to P$ given by $n \mapsto \varphi(m, n)$ is a PoM-morphism, and analogously in the first variable. We denote the collection of such maps by $\mathrm{BiPoM}(M \times N, P)$. We equip both $\mathrm{PoM}(M, N)$ and $\mathrm{BiPoM}(M \times N, P)$ with pointwise order and addition, which gives them a natural structure as positively ordered monoids.

Given positively ordered monoids M and N, there exists a positively ordered monoid $M \otimes_{\mathrm{PoM}} N$ and a PoM-bimorphism $\omega\colon M \times N \to M \otimes_{\mathrm{PoM}} N$ with the following universal property: For every positively ordered monoid P, the assignment that maps a PoM-morphism $\alpha\colon M \otimes_{\mathrm{PoM}} N \to P$ to the PoM-bimorphism $\alpha \circ \omega\colon M \times N \to P$ is a bijection between the following (bi)morphism sets

$$\mathrm{PoM}\big(M \otimes_{\mathrm{PoM}} N, P\big) \cong \mathrm{BiPoM}\big(M \times N, P\big),$$

which moreover respects the structure of the (bi)morphism sets as positively ordered monoids. We call $M \otimes_{\mathrm{PoM}} N$ together with ω the *tensor product* of M and N (in the category PoM).

Recall that a set Λ with a binary relation $\prec$ is called *upward directed* if for all $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ such that $\lambda_1, \lambda_2 \prec \lambda$.

Following [GHK$^+$03, Definition I-1.11, p.57], we define auxiliary relations on partially ordered sets and monoids:

Definition 2.2. Let X be a partially ordered set. An *auxiliary relation* on X is a binary relation $\prec$ on X satisfying the following conditions for all $x, x', y, y' \in X$:

(1) If $x \prec y$ then $x \leq y$.
(2) If $x' \leq x \prec y \leq y'$ then $x' \prec y'$.

If X is also a monoid, then an auxiliary relation $\prec$ on X is said to be *additive* if it satisfies the following conditions for all $x, x_1, x_2, y_1, y_2 \in X$:

(3) We have $0 \prec x$.
(4) If $x_1 \prec y_1$ and $x_2 \prec y_2$, then $x_1 + x_2 \prec y_1 + y_2$.

An important example of an auxiliary relation is the so called *way-below relation*, which has its origins in domain theory (see [GHK$^+$03]). We recall below its sequential version, which is the one used to define abstract Cuntz semigroups.

Definition 2.3. Let X be a partially ordered set, and let $x, y \in X$. We say that x is *way-below* y, or that x is *compactly contained in* y, in symbols $x \ll y$, if whenever $(z_n)_n$ is an increasing sequence in X for which the supremum exists and which satisfies $y \leq \sup_n z_n$, then there exists $k \in \mathbb{N}$ with $x \leq z_k$. We say that x is *compact* if $x \ll x$. We let X_c denote the set of compact elements in X.

The following definition is due to Coward, Elliott and Ivanescu in [CEI08]. See also [APT14, Definition 3.1.2].

Definition 2.4. A Cu-*semigroup*, also called *abstract Cuntz semigroup*, is a positively ordered semigroup S that satisfies the following axioms (O1)-(O4):

(O1) Every increasing sequence $(a_n)_n$ in S has a supremum $\sup_n a_n$ in S.

(O2) For every element $a \in S$ there exists a sequence $(a_n)_n$ in S with $a_n \ll a_{n+1}$ for all $n \in \mathbb{N}$, and such that $a = \sup_n a_n$.
(O3) If $a' \ll a$ and $b' \ll b$ for $a', b', a, b \in S$, then $a' + b' \ll a + b$.
(O4) If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in S, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

Given Cu-semigroups S and T, a Cu-*morphism* from S to T is a map $f\colon S \to T$ that preserves addition, order, the zero element, the way-below relation and suprema of increasing sequences. A *generalized* Cu-*morphism* is a Cu-morphism that is not required to preserve the way-below relation. We denote the set of Cu-morphisms by $\mathrm{Cu}(S,T)$; and we denote the set of generalized Cu-morphisms by $\mathrm{Cu}[S,T]$.

We let Cu be the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.

Remark 2.5. Let S be a Cu-semigroup. Note that $0 \ll a$ for all $a \in S$. Thus, (O3) ensures that $\ll$ is an additive auxiliary relation on S.

2.6. Let A be a C^*-algebra, and let $a, b \in (A \otimes \mathcal{K})_+$. We say that a is *Cuntz subequivalent* to b, denoted $a \precsim b$, if there is a sequence $(x_n)_n$ in $A \otimes \mathcal{K}$ such that $a = \lim_n x_n b x_n^*$. We say that a and b are *Cuntz equivalent*, written $a \sim b$, provided $a \precsim b$ and $b \precsim a$. The set of equivalence classes

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/{\sim}$$

is called the (completed) Cuntz semigroup of A. One defines an addition on $\mathrm{Cu}(A)$ by setting $[a] + [b] := [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$ for $a, b \in (A \otimes \mathcal{K})_+$. (One uses that there is an isomorphism $M_2(\mathcal{K}) \cong \mathcal{K}$, and that the definition does not depend on the choice of isomorphism.) The class of $0 \in (A \otimes \mathcal{K})_+$ is a zero element for $\mathrm{Cu}(A)$, giving it the structure of a commutative monoid. One defines an order on $\mathrm{Cu}(A)$ by setting $[a] \leq [b]$ whenever $a \precsim b$. This gives $\mathrm{Cu}(A)$ the structure of a positively ordered monoid.

Theorem 2.7 ([CEI08]). *For every C^*-algebra A, the positively ordered monoid $\mathrm{Cu}(A)$ is a Cu-semigroup. Furthermore, if B is another C^*-algebra, then a *-homomorphism $\varphi\colon A \to B$ induces a Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ by*

$$\mathrm{Cu}(\varphi)([a]) := [\varphi(a)],$$

for $a \in (A \otimes \mathcal{K})_+$. This defines a functor from the category of C^*-algebras with *-homomorphisms to the category Cu.

Remark 2.8. Let A be a C^*-algebra. In order to show that (O2) holds for $\mathrm{Cu}(A)$ one proves that, for every $a \in (A \otimes K)_+$ and $\varepsilon > 0$ we have $[(a-\varepsilon)_+] \ll [a]$, and that moreover $[a] = \sup_{\varepsilon>0}[(a-\varepsilon)_+]$. One can then derive from this that the sequence $([(a-1/n)_+])_n$ satisfies the required properties in (O2).

This suggests the possibility of formally strengthening (O2) for every Cu-semigroup S in the following way: Given $a \in S$, there exists a $(0,1)$-indexed chain of elements $(a_\lambda)_{\lambda\in(0,1)}$ with the property that $a = \sup_\lambda a_\lambda$, and $a_{\lambda'} \ll a_\lambda$ whenever $\lambda' < \lambda$. Next, we show that this property holds for all Cu-semigroups.

Lemma 2.9. *Let S be a set equipped with a transitive binary relation $\prec$ that satisfies the following condition:*

(*) For each $a \in S$ there exists a sequence $(a_n)_n$ in S such that $a_n \prec a_{n+1} \prec a$ for all n; and such that whenever $a' \in S$ satisfies $a' \prec a$ then there exists n_0 with $a' \prec a_{n_0}$.

Then, for every $a \in S$, there exists a chain $(a_\lambda)_{\lambda\in(0,1)\cap\mathbb{Q}}$ such that $a_{\lambda'} \prec a_\lambda$ whenever $\lambda', \lambda \in (0,1)\cap\mathbb{Q}$ satisfy $\lambda' < \lambda$, and such that for every $a' \in S$ with $a' \prec a$ there exists $\mu \in (0,1)\cap\mathbb{Q}$ with $a' \prec a_\mu$.

Proof. Note that condition (*) implies the following: Whenever $b_1, b_2, b \in S$ satisfy $b_1, b_2 \prec b$, then there exists $b_3 \in S$ with $b_1, b_2 \prec b_3 \prec b$. This property, which we will refer to as the interpolation property, will be used throughout.

Given $a \in S$, first use (*) to fix an increasing sequence $0 \prec a_1 \prec a_2 \prec \cdots \prec a$ which is cofinal in $a^{\prec} := \{b \mid b \prec a\}$. (This means that, if $a' \in S$ satisfies $a' \prec a$, then there is $k \in \mathbb{N}$ with $a' \prec a_k$.)

Use the interpolation property to find $a_1^{(1)}$ such that $a_1 \prec a_1^{(1)} \prec a$ and consider the chain $0 \prec a_1^{(1)} \prec a$. Now use the interpolation property to refine the above chain as

$$\begin{array}{ccccccccc} 0 & & \prec & & a_1^{(1)} & & \prec & & a \\ \| & & & & \| & & & & \| \\ 0 & \prec & a_1^{(2)} & \prec & a_2^{(2)} & \prec & a_3^{(2)} & \prec & a\,, \end{array}$$

in such a way that moreover $a_2 \prec a_3^{(2)}$. We now proceed inductively, and thus suppose we have constructed a chain $0 \prec a_1^{(n)} \prec \cdots \prec a_{2^n-1}^{(n)} \prec a$ with $a_n \prec a_{2^n-1}^{(n)}$. Use the interpolation property to construct a new chain

$$0 \prec a_1^{(n+1)} \prec \cdots \prec a_{2^{n+1}-1}^{(n+1)} \prec a$$

such that

$$0 \prec a_1^{(n+1)} \prec a_1^{(n)}, \quad a_i^{(n)} \prec a_{2i+1}^{(n+1)} \prec a_{i+1}^{(n)}, \quad a_{2i}^{(n+1)} = a_i^{(n)}, \quad a_{2^n-1}^{(n)} \prec a_{2^{n+1}-1}^{(n+1)} \prec a,$$

and such that moreover $a_{n+1} \prec a_{2^{n+1}-1}^{(n+1)}$. This latter condition will ensure that the set of elements thus constructed is cofinal in $a^{\prec}$.

The index set $I := \{(n,i) \mid 1 \leq n, 1 \leq i \leq 2^n - 1\}$ can be totally ordered by setting $(n,i) \leq (m,j)$ provided $i2^{-n} \leq j2^{-m}$. It now follows from the construction above that $a_i^{(n)} \prec a_j^{(m)}$ whenever $(n,i) \leq (m,j)$.

The set I is order-isomorphic to the dyadic rationals in $(0,1)$. In fact, I is a countably infinite, totally ordered, dense set with no minimal nor maximal element. (Here, dense means that whenever $x < y$ in I there exists $z \in I$ such that $x < z < y$.) By a classical result of G. Cantor (see, for example, [Roi90, Theorem 27]), there is only one such set, up to order-isomorphism. We can therefore choose an order-preserving bijection $\psi\colon I \to (0,1)\cap\mathbb{Q}$ and, setting $a_\lambda = a_i^{(n)}$ whenever $\psi((n,i)) = \lambda$, the desired conclusion follows. □

Proposition 2.10. *Let S be a* Cu*-semigroup, and let $a \in S$. Then, there exists a family $(a_\lambda)_{\lambda\in(0,1]}$ in S with $a_1 = a$; such that $a_{\lambda'} \ll a_\lambda$ whenever $\lambda', \lambda \in (0,1]$ satisfy $\lambda' < \lambda$; and such that $a_\lambda = \sup_{\lambda'<\lambda} a_{\lambda'}$ for every $\lambda \in (0,1]$.*

Proof. Consider S equipped with the transitive relation $\ll$. Then (O2) ensures that condition (*) in Lemma 2.9 is fulfilled with $\ll$ in place of $\prec$. Hence, given $a \in S$ we can apply Lemma 2.9 to choose a $\ll$-increasing chain $(\bar{a}_\lambda)_{\lambda\in(0,1)\cap\mathbb{Q}}$ with $a = \sup_\lambda \bar{a}_\lambda$. For each $\lambda \in (0,1]$, define $a_\lambda := \sup\{\bar{a}_{\lambda'} : \lambda' < \lambda\}$. It is now easy to see that the chain $(a_\lambda)_{\lambda\in(0,1]}$ satisfies the conclusion. □

Let S be a Cu-semigroup, and let a be an element in S. We say that a is *soft* if for every $a' \in S$ with $a' \ll a$ we have $a' <_s a$, that is, there exists $k \in \mathbb{N}$ with $(k+1)a' \leq ka$; see [APT14, Definition 5.3.1]. We denote the set of soft elements in S by S_{soft}. Further, we set $S_{\mathrm{soft}}^\times := S_{\mathrm{soft}} \setminus \{0\}$.

The following result will be used later (in Section 7).

Lemma 2.11. *Let S and T be* Cu*-semigroups, let $\varphi\colon S \to T$ be a generalized* Cu*-morphism, and let $a \in S$ be a soft element. Then $\varphi(a)$ is soft.*

Proof. To verify that $\varphi(a)$ is soft, let $x \in T$ satisfy $x \ll \varphi(a)$. Using that φ preserves suprema of increasing sequences, we can choose $a' \in S$ with $a' \ll a$ and $x \leq \varphi(a')$. (Indeed, applying (O2) in S, chose a $\ll$-increasing sequence $(a_n)_n$ in S with supremum a. Then $\varphi(a) = \sup_n \varphi(a_n)$, whence there is n with $x \leq \varphi(a_n)$.) Since a is soft, we can chose $k \in \mathbb{N}$ such that $(k+1)a' \leq ka$. Then

$$(k+1)x \leq (k+1)\varphi(a') = \varphi((k+1)a') \leq \varphi(ka) = k\varphi(a),$$

which shows that $x <_{\mathrm{s}} \varphi(a)$, as desired. □

2.12. Given a C^*-algebra A, it is known that $\mathrm{Cu}(A)$ satisfies two additional axioms besides (O1)–(O4). The first one is usually referred to the *axiom of almost algebraic order* or axiom (O5), first considered in [RW10, Lemma 7.1] (and later also in [ORT11, Corollary 4.16] and [Rob13, 2.1]). The version we use here is a strengthening of the original formulation, introduced in [APT14, Definition 4.1]:

We say that a Cu-semigroup S satisfies (O5) if, for every $a', a, b', b, c \in S$ that satisfy $a + b \leq c, a' \ll a, b' \ll b$, there is $x \in S$ with $a' + x \leq c \leq a + x$ and $b' \leq x$.

The second axiom is known as the axiom of *almost Riesz decomposition* or axiom (O6), and was introduced in [Rob13]:

We say that a Cu-semigroup S satisfies (O6) if, for every $a', a, b, c \in S$ satisfying $a' \ll a \leq b + c$, there exist $b', c' \in S$ such that $a' \leq b' + c'$, $b' \leq a, b$ and $c' \leq a, c$.

2.2. **Closed, monoidal categories.**

In this subsection, we recall the basic notions from the theory of closed, monoidal categories. For details we refer to [Kel05] and [Mac71]. See also [APT14, Appendix A].

2.13. Recall that a monoidal category $\mathcal{V}$ consists of: a category $\mathcal{V}_0$ (which we assume is locally small), a bifunctor $\otimes\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$ (covariant in each variable) and a unit object I in $\mathcal{V}_0$ such that, whenever X, Y, Z are objects in $\mathcal{V}_0$, there are natural isomorphisms

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \quad \text{and} \quad X \otimes I \cong X, \quad \text{and} \quad I \otimes X \cong X,$$

that are subject to certain coherence axioms. An object or morphism in $\mathcal{V}$ means an object or morphism in $\mathcal{V}_0$, respectively. In concrete examples, such as the categories PoM and Cu, we will use the same notation for a monoidal category and its underlying category.

A monoidal category $\mathcal{V}$ is called *symmetric* provided that for each pair of objects X and Y there is a natural isomorphism $X \otimes Y \cong Y \otimes X$.

In many concrete examples of monoidal categories, the tensor product of two objects X and Y is the object $X \otimes Y$ (unique up to natural isomorphism) that linearizes bilinear maps from $X \times Y$. This is formalized by considering a functorial association of bimorphisms $\mathrm{Bimor}(X \times Y, Z)$ (covariant in Z, and contravariant in X and Y) such that $X \otimes Y$ represents the functor $\mathrm{Bimor}(X \times Y, \_)$, that is, for each Z there is a natural bijection

$$\mathrm{Bimor}\big(X \times Y, Z\big) \cong \mathrm{Mor}\big(X \otimes Y, Z\big).$$

One instance of this is the monoidal structure in the category Cu of abstract Cuntz semigroups, as introduced in [APT14]. We recall details in Subsection 2.3.

Another example is the category PoM of positively ordered monoids. There is a natural notion of bimorphisms in PoM, and the tensor product in PoM has the corresponding universal property of linearizing such bimorphisms; see Paragraph 2.1.

2.14. A monoidal category $\mathcal{V}$ is said to be *closed* provided that for each object Y, the functor $-\otimes Y\colon \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint, that we will denote by $[\![Y, -]\!]$. Thus, in a closed monoidal category, for all objects X, Y, Z, there is a natural bijection

$$\mathcal{V}_0\big(X \otimes Y, Z\big) \cong \mathcal{V}_0\big(X, [\![Y, Z]\!]\big),$$

where $\mathcal{V}_0(\_,\_)$ denotes the morphisms between two objects X and Y.

Let $\mathcal{V}$ be a monoidal category with unit object I. An enriched category $\mathcal{C}$ over $\mathcal{V}$ consists of: a collection of objects in $\mathcal{C}$; an object $\mathcal{C}(X,Y)$ in $\mathcal{V}$, for each pair of objects X and Y in $\mathcal{C}$ (playing the role of the morphisms in $\mathcal{C}$ from X to Y); a $\mathcal{V}$-morphism $j_X\colon I\to\mathcal{C}(X,X)$, called the identity on X, for each object X in $\mathcal{C}$ (playing the role of the identity morphism on X); and for each triple X, Y and Z of objects in $\mathcal{C}$, a $\mathcal{V}$-morphism

$$\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y)\to\mathcal{C}(X,Z),$$

that plays the role of a composition law and is subject to certain coherence axioms; see [Kel05, Section 1.2] for details.

It follows from general category theory that every closed symmetric monoidal category $\mathcal{V}$ can be enriched over itself. Let us recall some details. Given two objects X and Y in $\mathcal{V}$, the object $[\![X,Y]\!]$ in $\mathcal{V}$ plays the role of the morphisms from X to Y. Given an object X, the identity on X (for the enrichment) is defined as the $\mathcal{V}$-morphism $j_X\colon I\to[\![X,X]\!]$ that corresponds to the 'usual' identity morphism $\mathrm{id}_X\in\mathcal{V}_0\big(X,X\big)$ under the following natural bijections

$$\mathcal{V}_0\big(I,[\![X,X]\!]\big)\cong\mathcal{V}_0\big(I\otimes X,X\big)\cong\mathcal{V}_0\big(X,X\big).$$

It is easiest to construct the composition map by using the evaluation maps. Given objects X and Y, the evaluation (or counit) map is defined as the $\mathcal{V}$-morphism $\mathrm{e}_X^Y\colon[\![X,Y]\!]\otimes X\to Y$ that corresponds to the identity morphism in $\mathcal{V}_0([\![X,Y]\!],[\![X,Y]\!])$ under the natural bijection

$$\mathcal{V}_0\big([\![X,Y]\!]\otimes X,Y\big)\cong\mathcal{V}_0\big([\![X,Y]\!],[\![X,Y]\!]\big).$$

Then, given objects X, Y and Z, the composition

$$[\![Y,Z]\!]\otimes[\![X,Y]\!]\to[\![X,Z]\!]$$

is defined as the $\mathcal{V}$-morphism that corresponds to the composition

$$[\![Y,Z]\!]\otimes[\![X,Y]\!]\otimes X\xrightarrow{\mathrm{id}_{[\![Y,Z]\!]}\otimes\mathrm{e}_X^Y}[\![Y,Z]\!]\otimes Z\xrightarrow{\mathrm{e}_Y^Z}Z$$

under the natural bijection

$$\mathcal{V}_0\big([\![Y,Z]\!]\otimes[\![X,Y]\!],[\![X,Z]\!]\big)\cong\mathcal{V}_0\big([\![Y,Z]\!]\otimes[\![X,Y]\!]\otimes X,Z\big).$$

The natural question of whether the monoidal category Cu is closed was left open in [APT14, Problem 2]. One of the objectives of this paper is to show that this is indeed the case by applying the τ-construction that will be developed in Section 3 to a suitable semigroup of morphisms in the category Cu; see Definition 5.3 and Theorem 5.11.

2.3. **Tensor products in** Cu**.** In this subsection we recall the construction of tensor products of Cu-semigroups as introduced in [APT14]. We first recall the notion of Cu-bimorphisms.

Definition 2.15 ([APT14, Definition 6.3.1])**.** Let S,T and P be Cu-semigroups, and let $\varphi\colon S\times T\to P$ be a PoM-bimorphism. We say that φ is a Cu-*bimorphism* if it satisfies the following conditions:

(1) We have that $\sup_k\varphi(a_k,b_k)=\varphi(\sup_k a_k,\sup_k b_k)$, for every increasing sequences $(a_k)_k$ in S and $(b_k)_k$ in T.
(2) If $a',a\in S$ and $b',b\in T$ satisfy $a'\ll a$ and $b'\ll b$, then $\varphi(a',b')\ll\varphi(a,b)$.

We denote the set of Cu-bimorphisms by $\mathrm{BiCu}(S\times T,R)$.

Given Cu-semigroups S, T and P, we equip $\mathrm{BiCu}(S\times T, R)$ with pointwise order and addition, giving it the structure of a positively ordered monoid. Similarly, we consider the set of Cu-morphisms between two Cu-semigroups as a positively ordered monoid with the pointwise order and addition.

Theorem 2.16 ([APT14, Theorem 6.3.3])**.** *Let S and T be* Cu*-semigroups. Then there exists a* Cu*-semigroup $S\otimes T$ and a* Cu*-bimorphism $\omega\colon S\times T\to S\otimes T$ such that for every* Cu*-semigroup P the following universal properties hold:*

1. *For every* Cu*-bimorphism $\varphi\colon S\times T\to P$ there exists a (unique)* Cu*-morphism $\tilde{\varphi}\colon S\otimes T\to P$ such that $\varphi=\tilde{\varphi}\circ\omega$.*
2. *If $\alpha_1,\alpha_2\colon S\otimes T\to P$ are* Cu*-morphisms, then $\alpha_1\leq\alpha_2$ if and only if $\alpha_1\circ\omega\leq\alpha_2\circ\omega$.*

Thus, for every Cu*-semigroup P, the assignment that sends a* Cu*-morphism $\alpha\colon S\otimes T\to P$ to the* Cu*-bimorphism $\alpha\circ\omega\colon S\times T\to P$ defines a natural bijection between the following (bi)morphism sets*

$$\mathrm{Cu}\big(S\otimes T, P\big)\cong\mathrm{BiCu}\big(S\times T, P\big),$$

Moreover, this bijection respects the structure of the (bi)morphism sets as positively ordered monoids.

2.17. Let S and T be Cu-semigroups, and consider the universal Cu-bimorphism $\omega\colon S\times T\to S\otimes T$ from Theorem 2.16. Given $s\in S$ and $t\in T$, we set $s\otimes t:=\omega(s,t)$. We call $s\otimes t$ a *simple tensor*.

The tensor product in Cu is functorial in each variable: If $\varphi_1\colon S_1\to T_1$ and $\varphi_2\colon S_2\to T_2$ are Cu-morphisms, then there is a unique Cu-morphism $\varphi_1\otimes\varphi_2\colon S_1\otimes S_2\to T_1\otimes T_2$ with the property that $(\varphi_1\otimes\varphi_2)(a_1\otimes a_2)=\varphi_1(a_1)\otimes\varphi_2(a_2)$ for every $a_1\in S_1$ and $a_2\in S_2$.

Thus, the tensor product in Cu defines a bifunctor $\otimes\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$. The Cu-semigroup $\overline{\mathbb{N}}=\{0,1,2,\ldots,\infty\}$ (with usual addition and order) is a unit object, that is, for every Cu-semigroup S there are canonical isomorphisms $S\otimes\overline{\mathbb{N}}\cong S$ and $\overline{\mathbb{N}}\otimes S\cong S$. Further, for every Cu-semigroups S, T and P, there are natural isomorphisms

$$S\otimes(T\otimes P)\cong(S\otimes T)\otimes P\quad\text{ and }\quad S\otimes T\cong T\otimes S.$$

It follows that Cu is a symmetric, monoidal category; see also [APT14, 6.3.7].

3. The Path Construction

In this section we introduce a functorial construction from a category of monoids with a transitive relation to the category Cu. This construction, when restricted to the category $\mathcal{Q}$ introduced in Section 4 (a category that contains Cu) is a coreflection for the natural inclusion from Cu.

Definition 3.1. A *$\mathcal{P}$-semigroup* is a pair $(S,\prec)$, where S is a commutative monoid and where $\prec$ is a transitive relation on S, such that:

1. We have $0\prec a$ for all $a\in S$.
2. If $a_1,a_2,b_1,b_2\in S$ satisfy $a_1\prec b_1$ and $a_2\prec b_2$, then $a_1+a_2\prec b_1+b_2$.

We often supress the reference to the relation and denote a $\mathcal{P}$-semigroup $(S,\prec)$ simply by S.

A *$\mathcal{P}$-morphism* is a monoid morphism that preserves the relation. Given $\mathcal{P}$-semigroups $(S,\prec)$ and $(T,\prec)$, we denote the collection of all $\mathcal{P}$-morphisms by $\mathcal{P}((S,\prec),(T,\prec))$, or simply by $\mathcal{P}(S,T)$. We let $\mathcal{P}$ be the category whose objects are $\mathcal{P}$-semigroups and whose morphisms are $\mathcal{P}$-morphisms.

Remark 3.2. Conditions (1) and (2) of Definition 3.1 are the same as the conditions from Definition 2.2 for an auxiliary relation to be additive.

Definition 3.3. Let $I = (I, \prec)$ be a set with an upward directed transitive relation $\prec$. Let $S = (S, \prec)$ be a $\mathcal{P}$-semigroup. An *I-path* (or simply a *path*) in S is a map $f\colon I \to S$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda', \lambda \in I$ satisfy $\lambda' \prec \lambda$. We set

$$P(I, S) := \big\{ f\colon I \to S \text{ such that } f \text{ is a path in } S \big\}.$$

Given two paths f and g, we define their sum $f + g$ by setting $(f + g)(\lambda) := f(\lambda) + g(\lambda)$ for all $\lambda \in I$. Let $0 \in P(I, S)$ denote the path that satisfies $0(\lambda) = 0$ for all $\lambda \in I$.

We define a binary relation $\precsim$ on $P(I, S)$ by setting $f \precsim g$ for two paths f and g if and only if for every $\lambda \in I$ there exists $\mu \in I$ such that $f(\lambda) \prec g(\mu)$. Finally we antisymmetrize the relation $\precsim$ by setting $f \sim g$ if and only if $f \precsim g$ and $g \precsim f$.

Given $s \in S$ and $f \in P(I, S)$, we write $s \prec f$ if $s \prec f(\lambda)$ for all $\lambda \in I$; and we write $f \prec s$ provided $f(\lambda) \prec s$ for all $\lambda \in I$.

The proof of the following result is straightforward and therefore omitted.

Lemma 3.4. *Let I be a set with an upward directed transitive relation, and let S be a $\mathcal{P}$-semigroup. Then the addition and the zero element defined in Definition 3.3 give $P(I, S)$ the structure of a commutative monoid. Moreover, the relation $\precsim$ on $P(I, S)$ is transitive, reflexive and satisfies:*

(1) For every $f \in P(I, S)$ we have $0 \precsim f$.

(2) If $f_1, f_2, g_1, g_2 \in P(I, S)$ satisfy $f_1 \precsim g_1$ and $f_2 \precsim g_2$, then $f_1 + f_2 \precsim g_1 + g_2$.

Further, $\sim$ is an equivalence relation on $P(I, S)$.

Definition 3.5. Let I be a set with an upward directed transitive relation, and let S be a $\mathcal{P}$-semigroup. Let $\sim$ be the equivalence relation on $P(I, S)$ from Definition 3.3. We define

$$\tau_I(S) := P(I, S)/_{\sim}.$$

Given a path f in S, its equivalence class in $\tau_I(S)$ is denoted by $[f]$.

We define $0 \in \tau_I(S)$ as the equivalence class of the zero-path. We define $+$ and $\leq$ on $\tau_I(S)$ by setting $[f] + [g] := [f + g]$, and by setting $[f] \leq [g]$ provided $f \precsim g$.

The following results follows immediately from Lemma 3.4.

Proposition 3.6. *Let I be a set with an upward directed transitive relation, and let S be a $\mathcal{P}$-semigroup. Then the addition, the zero element, and the order defined in Definition 3.5 give $\tau_I(S)$ the structure of a positively ordered monoid.*

Remarks 3.7. (1) We call the construction of $\tau_I(S)$ the *τ-construction* or *path construction*. We call I the *path type*.

(2) Given a $\mathcal{P}$-semigroup S, the path construction $\tau_I(S)$ depends heavily on the choice of I. For instance, using the most simple case $I = (\{0\}, \leq)$, we obtain

$$\tau_{\{0\}}(S) \simeq \big\{ a \in S : a \prec a \big\}.$$

For $I = (\mathbb{N}, <)$, one can show that $\tau_I(S)$ is the (sequential) round ideal completion of S as considered for instance in [APT14, Proposition 3.1.6].

We will not pursue this general constructions further. Rather, motivated by the results in Lemma 2.9 and Proposition 2.10, we will focus on the concrete case where the path type is taken to be $\big(\mathbb{Q} \cap (0, 1), <\big)$.

Notation 3.8. We set $I_{\mathbb{Q}} := \big(\mathbb{Q} \cap (0, 1), <\big)$. Given a $\mathcal{P}$-semigroup S, we denote $P(I_{\mathbb{Q}}, S)$ and $\tau_{I_{\mathbb{Q}}}(S)$ by $P(S)$ and $\tau(S)$, respectively. If we want to stress the auxiliary relation on S, we also write $P(S, \prec)$ and $\tau(S, \prec)$.

Thinking of $I_{\mathbb{Q}}$ as an ordered index set, we will often denote a path in S as an indexed family $(a_\lambda)_{\lambda \in I_{\mathbb{Q}}}$.

Given a $\mathcal{P}$-semigroup S, we show in Theorem 3.15 that $\tau(S)$ is a Cu-semigroup when equipped with the order and addition in Definition 3.5. We split the proof into several lemmas.

Recall from Definition 3.3 that, given paths f and g in S, and given $\lambda \in I_{\mathbb{Q}}$, we write $f(\lambda) \prec g$ (respectively, $f \prec g(\lambda)$) if $f(\lambda) \prec g(\mu)$ (respectively, $f(\mu) \prec g(\lambda)$) for every $\mu \in I_{\mathbb{Q}}$.

Lemma 3.9. *Let $S = (S, \prec)$ be a $\mathcal{P}$-semigroup, let f be a path in S, and let $\lambda', \lambda \in I_{\mathbb{Q}}$ satisfy $\lambda' < \lambda$. Then there exists a path h in S such that $f(\lambda') \prec h \prec f(\lambda)$.*

Proof. Define $h\colon I_{\mathbb{Q}} \to S$ by
$$h(\gamma) := f\big(\gamma\lambda + (1-\gamma)\lambda'\big),$$
for $\gamma \in I_{\mathbb{Q}}$. Then h is a path satisfying $f(\lambda') \prec h \prec f(\lambda)$, as desired. □

Lemma 3.10. *Let $S = (S, \prec)$ be a $\mathcal{P}$-semigroup. Given a sequence $(f_n)_{n\geq 1}$ of paths in S, and given a sequence $(a_n)_{n\geq 1}$ in S such that*
$$0 \prec f_1 \prec a_1 \prec f_2 \prec a_2 \prec f_3 \prec a_3 \cdots,$$
there exists a path h in S such that $h(\frac{n}{n+1}) = a_n$ for all $n \geq 1$.

Proof. Define $h\colon I_{\mathbb{Q}} \to S$ as follows:
$$h(\lambda) := \begin{cases} f_n(\lambda), & \text{if } \lambda \in (\frac{n-1}{n}, \frac{n}{n+1}) \\ a_n, & \text{if } \lambda = \frac{n}{n+1} \end{cases}.$$
It is easy to see that h is a path and that $h(\frac{n}{n+1}) = a_n$, as desired. □

Lemma 3.11. *Let S be a $\mathcal{P}$-semigroup, and let $([f_n])_{n\geq 1}$ be an increasing sequence in $\tau(S)$. Then there exists a strictly increasing sequence $(\lambda_m)_{m\geq 1}$ in $I_{\mathbb{Q}}$ and a path f in S such that the following conditions hold:*

1. *(1) We have $\sup_m \lambda_m = 1$.*
2. *(2) We have $f_n(\lambda_m) \prec f_l(\lambda_l)$, whenever $n, m < l$.*
3. *(3) We have $f(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \geq 1$.*

Moreover, if f is a path in S for which there exists a strictly increasing sequence $(\lambda_m)_{m\geq 1}$ in $I_{\mathbb{Q}}$ satisfying conditions (1), (2) and (3) above, then $[f] = \sup_n [f_n]$ in $\tau(S)$. In particular, $\tau(S)$ satisfies (O1).

Proof. The proof is divided in two parts.

We inductively find $\lambda_m \in I_{\mathbb{Q}}$ and $h_m \in P(S)$ for $m \geq 1$ such that:

(a) $\lambda_{m-1} < \lambda_m$ and $\frac{m}{m+1} \leq \lambda_m$, for all m; and
(b) $f_n(\lambda_{m-1}) \prec h_m \prec f_m(\lambda_m)$, for all $n < m$.

Set $\lambda_1 := \frac{1}{2}$, and define the path h_1 by $h_1(\lambda) = f_1(\frac{\lambda}{2})$. Note that $0 \prec h_1 \prec f_1(\lambda_1)$.

Assume we have chosen λ_n and h_n for all $n < m$. For each $k = 1, \ldots, m-1$, using that $f_k \precsim f_m$, we choose $\lambda_{m,k} \in I_{\mathbb{Q}}$ such that $f_k(\lambda_{m-1}) \prec f_m(\lambda_{m,k})$. Let λ'_m be the maximum of $\lambda_{m,1}, \ldots, \lambda_{m,m-1}, \frac{m}{m+1}$. Choose $\lambda_m \in I_{\mathbb{Q}}$ with $\lambda'_m < \lambda_m$. Using Lemma 3.9, we choose a path h_m with $f_m(\lambda'_m) \prec h_m \prec f_m(\lambda_m)$.

Note that in particular we have the following relations:
$$0 \prec h_1 \prec f_1(\lambda_1) \prec h_2 \prec f_2(\lambda_2) \prec h_3 \prec f_3(\lambda_3) \prec h_4 \cdots$$
Applying Lemma 3.10, we choose $f \in P(S)$ with $f(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \geq 1$. Then it is easy to check that the sequence $(\lambda_m)_m$ and the path f satisfy conditions (1), (2) and (3).

For the second part of the proof, let $(\lambda_m)_{m\geq 1}$ be a strictly increasing sequence in $I_{\mathbb{Q}}$, and let $f \in P(S)$ satisfy (1), (2) and (3). We show that $[f] = \sup_n [f_n]$ in $\tau(S)$.

We first show that $[f_n] \leq [f]$ for each $n \geq 1$. Fix $n \geq 1$. To verify that $f_n \precsim f$, let λ be an element in $I_{\mathbb{Q}}$. Use (1) to choose m with $n < m$ and $\lambda < \lambda_m$. Using that f_n is a path at the first step, using condition (2) at the second step, and using (3) at the last step, we obtain that

$$f_n(\lambda) \prec f_n(\lambda_m) \prec f_{m+1}(\lambda_{m+1}) = f(\tfrac{m+1}{m+2}).$$

Hence $f_n \precsim f$, as desired.

Conversely, let $g \in P(S)$ satisfy $f_n \precsim g$ for all $n \geq 1$. To show that $f \precsim g$, take $\lambda \in I_{\mathbb{Q}}$. Choose m such that $\lambda < \frac{m}{m+1}$. Since $f_m \precsim g$, there exists $\mu \in I_{\mathbb{Q}}$ such that $f_m(\lambda_m) \prec g(\mu)$. Using this at the last step, using that f is a path at the first step, and using condition (3) at the second step, we get

$$f(\lambda) \prec f(\tfrac{m}{m+1}) = f_m(\lambda_m) \prec g(\mu).$$

This shows that $f \precsim g$, as desired. □

Definition 3.12. Let S be a $\mathcal{P}$-semigroup, let $f \in P(S)$, and let $\varepsilon \in I_{\mathbb{Q}}$. We define $f_\varepsilon \colon I_{\mathbb{Q}} \to S$ by

$$f_\varepsilon(\lambda) := \begin{cases} f(\lambda - \varepsilon), & \text{if } \lambda > \varepsilon \\ 0, & \text{otherwise} \end{cases}.$$

We will refer to f_ε as the *ε-cut down* of f.

Remark 3.13. It is easy to see that f_ε is a path in S. If t is a real number, we write t_+ for $\max\{0, t\}$. Then, under the convention that $f(0) = 0$, we have $f_\varepsilon(\lambda) = f((\lambda - \varepsilon)_+)$ for all $\lambda \in I_{\mathbb{Q}}$.

Lemma 3.14. *Let S be a $\mathcal{P}$-semigroup, and let $f \in P(S)$. Then $[f_\varepsilon] \ll [f_{\varepsilon'}]$ in $\tau(S)$, for every $\varepsilon', \varepsilon \in I_{\mathbb{Q}}$ with $\varepsilon' < \varepsilon$. Moreover, we have $[f] = \sup_{\varepsilon \in I_{\mathbb{Q}}} [f_\varepsilon]$ in $\tau(S)$. In particular, $\tau(S)$ satisfies (O2).*

Proof. It is routine to check that $[f] = \sup_\varepsilon [f_\varepsilon]$. Given $\varepsilon', \varepsilon \in I_{\mathbb{Q}}$ with $\varepsilon' < \varepsilon$, note that $f_\varepsilon = (f_{\varepsilon'})_{\varepsilon - \varepsilon'}$. Thus it is enough to show that $[f_\varepsilon] \ll [f]$ for every $\varepsilon > 0$.

Fix $\varepsilon > 0$. To show that $[f_\varepsilon] \ll [f]$, let $([g_n])_n$ be an increasing sequence in $\tau(S)$ with $[f] \leq \sup_n [g_n]$. By Lemma 3.11, there exists a path $h \in P(S)$ and an increasing sequence $(\lambda_n)_n$ in $I_{\mathbb{Q}}$ such that $[h] = \sup_n [g_n]$, and such that $h(\frac{m}{m+1}) = g_m(\lambda_m)$ for all $m \geq 1$.

Choose $m_0 \geq 1$ with $\frac{1}{m_0} < \varepsilon$. Since $f \precsim h$, there exists $\mu \in I_{\mathbb{Q}}$ satisfying $f(1 - \frac{1}{m_0}) \prec h(\mu)$. Choose $m_1 \geq 1$ such that $\mu < \frac{m_1}{m_1+1}$. Let us show that $f_\varepsilon \precsim g_{m_1}$. For every $\lambda \in I_{\mathbb{Q}}$, we have $\lambda - \varepsilon < 1 - \frac{1}{m_0}$. Therefore, using that f and h are paths at the second and fourth step, respectively, and using that $f(1 - \frac{1}{m_0}) \prec h(\mu)$ at the third step, we obtain that

$$f_\varepsilon(\lambda) = f\big((\lambda - \varepsilon)_+\big) \prec f(1 - \tfrac{1}{m_0}) \prec h(\mu) \prec h(\tfrac{m_1}{m_1+1}) = g_{m_1}(\lambda_{m_1}),$$

for every $\lambda \in I_{\mathbb{Q}}$. This proves that $[f_\varepsilon] \leq [g_{m_1}]$, as desired. □

Theorem 3.15. *Let S be a $\mathcal{P}$-semigroup. Then $\tau(S)$ is a* Cu*-semigroup.*

Proof. By Proposition 3.6, Lemma 3.11 and Lemma 3.14, $\tau(S)$ is a positively ordered monoid that satisfies axioms (O1) and (O2). It remains to show that $\tau(S)$ satisfies (O3) and (O4).

To verify (O3), let $[f'], [f], [g'], [g] \in \tau(S)$ satisfy $[f'] \ll [f]$ and $[g'] \ll [g]$. Using that $[f] = \sup_\varepsilon [f_\varepsilon]$, we can choose $\varepsilon_1 \in I_{\mathbb{Q}}$ such that $[f'] \leq [f_{\varepsilon_1}]$. Similarly we choose ε_2 for $[g]$. Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. We then have $[f'] \leq [f_\varepsilon]$ and $[g'] \leq [g_\varepsilon]$. Using that $f_\varepsilon + g_\varepsilon = (f+g)_\varepsilon$ at the third step, and using Lemma 3.14 at the fourth step, we deduce that

$$[f'] + [g'] \leq [f_\varepsilon] + [g_\varepsilon] = [f_\varepsilon + g_\varepsilon] = [(f+g)_\varepsilon] \ll [f+g] = [f] + [g],$$

which implies that $[f'] + [g'] \ll [f] + [g]$, as desired.

To prove (O4), let $([f_n])_n$ and $([g_n])_n$ be two increasing sequences in $\tau(S)$. It is clear that $\sup_n([f_n] + [g_n]) \leq \sup_n[f_n] + \sup_n[g_n]$. Let us prove the converse inequality.

By Lemma 3.11, there exist $f, g \in P(S)$ and increasing sequences $(\lambda_m)_m$ and $(\mu_m)_m$ in $I_{\mathbb{Q}}$ such that $[f] = \sup_n[f_n]$ and $[g] = \sup_n[g_n]$, and such that $f(\frac{m}{m+1}) = f_m(\lambda_m)$ and $g(\frac{m}{m+1}) = g_m(\mu_m)$ for all $m \in \mathbb{N}$. Given $\lambda \in I_{\mathbb{Q}}$, choose $m \in \mathbb{N}$ with $\lambda < \frac{m}{m+1}$. Choose $\tilde{\lambda} \in I_{\mathbb{Q}}$ such that $\lambda_m, \mu_m < \tilde{\lambda}$. We deduce that

$$f(\lambda) + g(\lambda) \prec (f+g)(\tfrac{m}{m+1}) = f_m(\lambda_m) + g_m(\mu_m) \prec f_m(\tilde{\lambda}) + g_m(\tilde{\lambda}).$$

It follows that $[f] + [g] \leq \sup_n([f_n] + [g_n])$, as desired. This verifies (O4) and completes the proof. $\square$

The following result provides a useful criterion for compact containment in $\tau(S)$.

Lemma 3.16. *Let S be a $\mathcal{P}$-semigroup, and let f', f be elements in $P(S)$. Then $[f'] \ll [f]$ in $\tau(S)$ if and only if there exists $\mu \in I_{\mathbb{Q}}$ such that $f' \prec f(\mu)$.*

Proof. Assume that $[f'] \ll [f]$. Since $[f] = \sup_\varepsilon[f_\varepsilon]$, there exists $\delta \in I_{\mathbb{Q}}$ such that $[f'] \leq [f_\delta]$. Let us show that $\mu = 1-\delta$ has the desired properties, that is, $f' \prec f(\mu)$. Given $\lambda \in I_{\mathbb{Q}}$, there is $\mu' \in I_{\mathbb{Q}}$ with $f'(\lambda) \prec f_\delta(\mu')$. Using that $(\mu'-\delta)_+ < 1-\delta = \mu$ at the last step, we deduce that

$$f'(\lambda) \prec f_\delta(\mu') = f\big((\mu'-\delta)_+\big) \prec f(\mu),$$

as desired.

Conversely, suppose that there exists $\mu \in I_{\mathbb{Q}}$ with $f' \prec f(\mu)$. Then, for every μ' with $\mu < \mu' < 1$ we have $[f'] \leq [f_{1-\mu'}] \ll [f]$, as desired. $\square$

Lemma 3.17. *Let S and T be $\mathcal{P}$-semigroups, and let $\alpha\colon S \to T$ be a $\mathcal{P}$-morphism. Then, for every $f \in P(S)$, the map $\alpha \circ f\colon I_{\mathbb{Q}} \to T$ belongs to $P(T)$. Moreover, the induced map $\tau(\alpha)\colon \tau(S) \to \tau(T)$ given by*

$$\tau(\alpha)([f]) := [\alpha \circ f],$$

for $f \in P(S)$, is a well-defined Cu*-morphism.*

Proof. Given $f \in P(S)$, it is easy to see that $\alpha\circ f$ belongs to $P(T)$. Moreover, given $f, g \in P(S)$ with $f \precsim g$ we have $\alpha \circ f \precsim \alpha \circ g$. This shows that $\tau(\alpha)$ is well-defined and order-preserving. It is also easy to see that $\tau(\alpha)$ preserves addition and the the zero element.

To show that $\tau(\alpha)$ preserves the way-below relation, let $f', f \in P(S)$ satisfy $[f'] \ll [f]$ in $\tau(S)$. By Lemma 3.16, there is $\mu \in I_{\mathbb{Q}}$ with $f' \prec f(\mu)$. Since α is a $\mathcal{P}$-morphism, we obtain that $\alpha \circ f' \prec (\alpha \circ f)(\mu)$. A second usage of Lemma 3.16 implies that $[\alpha \circ f'] \ll [\alpha \circ f]$, as desired.

To show that $\tau(\alpha)$ preserves suprema of increasing sequences, let $([f_n])_n$ be such a sequence in $\tau(S)$. By Lemma 3.11, there exist $f \in P(S)$ and a strictly increasing sequence $(\lambda_m)_m$ in $I_{\mathbb{Q}}$ such that the following conditions are satisfied:

(1) We have $\sup_m \lambda_m = 1$.
(2) We have $f_n(\lambda_m) \prec f_l(\lambda_l)$, whenever $n, m < l$.
(3) We have $f(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \geq 1$.

Further, for every $f \in P(S)$ satisfying these conditions, we have $[f] = \sup_n[f_n]$.

To show that $[\alpha\circ f] = \sup_n[\alpha\circ f_n]$, we verify that the path $\alpha\circ f$ and the sequence $(\lambda_m)_m$ satisfy the analogs of the above conditions with respect to the sequence $(\alpha \circ f_n)_n$. Condition (1) is unchanged. To verify the analog of condition (2), let

$n, m < l$. Since α is a $\mathcal{P}$-morphism, we have $(\alpha \circ f_n)(\lambda_m) \prec (\alpha \circ f_l)(\lambda_l)$, as desired. The analog of (3) holds, since

$$(\alpha \circ f)(\tfrac{n}{n+1}) = (\alpha \circ f_n)(\lambda_n),$$

for every $n \geq 1$. Thus, the path $\alpha \circ f$ satisfies conditions (1), (2) and (3) for the sequence $(\alpha \circ f_n)_n$, which implies that $[\alpha \circ f] = \sup_n [\alpha \circ f_n]$. Using this in the third step, we deduce that

$$\tau(\alpha)(\sup_n [f_n]) = \tau(\alpha)([f]) = [\alpha \circ f] = \sup_n [\alpha \circ f_n] = \sup_n \tau(\alpha)([f_n]),$$

as desired. Altogether, we have that $\tau(\alpha)$ is a Cu-morphism. □

Proposition 3.18. *The τ-construction defines a covariant functor $\tau\colon \mathcal{P} \to \mathrm{Cu}$ by sending a $\mathcal{P}$-semigroup S to the* Cu*-semigroup $\tau(S)$ (see Theorem 3.15), and by sending a $\mathcal{P}$-morphism $\alpha\colon S \to T$ to the* Cu*-morphism $\tau(\alpha)\colon \tau(S) \to \tau(T)$ (see Lemma 3.17).*

Proof. It follow easily from the construction that $\tau(\mathrm{id}_S) = \mathrm{id}_{\tau(S)}$ for every $\mathcal{P}$-semigroup S. It is also straightforward to check that $\tau(\alpha \circ \beta) = \tau(\alpha) \circ \tau(\beta)$ for every pair of composable $\mathcal{P}$-morphisms α and β. This shows that the τ-construction defines a covariant functor, as claimed. □

Although the τ-construction is a useful tool to derive Cu-semigroups from such simple objects as $\mathcal{P}$-semigroups, the next example shows that without additional care the τ-construction may just produce a trivial object.

Example 3.19. Consider $\mathbb{N} = \{0, 1, 2 \ldots\}$ with the usual structure as a monoid. We define $\prec$ on $\mathbb{N}$ by setting $k \prec l$ if $k < l$ or $k = l = 0$. It is easy to check that $(\mathbb{N}, \prec)$ is a $\mathcal{P}$-semigroup, and that the only path in $P(\mathbb{N}, \prec)$ is the constant path with value 0. It follows that $\tau(\mathbb{N}, \prec) \cong \{0\}$.

4. The category $\mathcal{Q}$

The category $\mathcal{P}$ introduced in the previous section, though useful in certain situations to construct Cu-semigroups from semigroups with very little structure, is too general to provide a nice categorical relation from $\mathcal{P}$ to Cu. In this section we introduce a subcategory of $\mathcal{P}$, which we denote by $\mathcal{Q}$, where Cu can be embedded as a full subcategory, and in such a way that the restriction of the τ-construction from Section 3 defines a coreflection $\mathcal{Q} \to \mathrm{Cu}$; see Theorem 4.12. In Subsection 4.2, we introduce the notion of internal-hom in the category $\mathcal{Q}$, which will be needed for the corresponding construction in Cu.

4.1. The coreflection $\tau\colon \mathcal{Q} \to \mathrm{Cu}$.

Recall the definition of an additive auxiliary relation from Definition 2.2.

Definition 4.1. A *$\mathcal{Q}$-semigroup* is a positively ordered monoid S together with an additive, auxiliary relation $\prec$ on S such that the following conditions are satisfied:

(O1) Every increasing sequence $(a_n)_n$ in S has a supremum $\sup_n a_n$ in S.

(O4) If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in S, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

We often drop the reference to the auxiliary relation and simply call S a $\mathcal{Q}$-semigroup.

Given $\mathcal{Q}$-semigroups S and T, a *$\mathcal{Q}$-morphism* from S to T is a map $S \to T$ that preserves addition, order, the zero element, the auxiliary relation and suprema of increasing sequences. We denote the set of $\mathcal{Q}$-morphisms by $\mathcal{Q}(S, T)$. A *generalized $\mathcal{Q}$-morphism* is a map that preserves addition, order, the zero element and suprema of increasing sequences. We denote the set of generalized $\mathcal{Q}$-morphisms by $\mathcal{Q}[S, T]$.

We let $\mathcal{Q}$ be the category whose objects are $\mathcal{Q}$-semigroups and whose morphisms are $\mathcal{Q}$-morphisms.

Remarks 4.2. (1) Axioms (O1) and (O4) in Definition 4.1 are the same as in Definition 2.4. Note that a generalized $\mathcal{Q}$-morphism is a $\mathcal{Q}$-morphism if and only if it preserves the auxiliary relation. Moreover, generalized $\mathcal{Q}$-morphisms are precisely the Scott continuous $\mathcal{P}$-morphisms. (See for example [GHK+03, Proposition II-2.1, p.157].)

(2) Let S, T be $\mathcal{Q}$-semigroups. The sets $\mathcal{Q}[S,T]$ and $\mathcal{Q}(S,T)$ of (generalized) $\mathcal{Q}$-morphism are positively ordered monoids, when equipped with the pointwise addition and order. It is easy to see that $\mathcal{Q}[S,T]$ satisfies (O1) and (O4). In Definition 4.17 we will define an auxiliary relation on $\mathcal{Q}[S,T]$ giving it the structure of a $\mathcal{Q}$-semigroup; see Proposition 4.18.

4.3. We define a functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$ as follows: Given a Cu-semigroup S, the (sequential) way-below relation $\ll$ is an additive auxiliary relation on S. It follows that $(S,\ll)$ is a $\mathcal{Q}$-semigroup, and we let ι map S to $(S,\ll)$.

Further, given Cu-semigroups S and T, a map $\varphi\colon S \to T$ is a Cu-morphism if and only if $\varphi\colon (S,\ll) \to (T,\ll)$ is a $\mathcal{Q}$-morphism. We let ι map a Cu-morphism to itself, considered as a $\mathcal{Q}$-morphism. It is easy to see that this defines a functor from Cu to $\mathcal{Q}$.

From our considerations, we clearly have:

Proposition 4.4. *The functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$ from Paragraph 4.3 embeds* Cu *as a full subcategory of $\mathcal{Q}$.*

Every $\mathcal{Q}$-semigroup can be considered as a $\mathcal{P}$-semigroup by forgetting its partial order. Therefore, if S is a $\mathcal{Q}$-semigroup with auxiliary relation $\prec$, then a *path* in S is a map $f\colon I_{\mathbb{Q}} \to S$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda', \lambda \in I_{\mathbb{Q}}$ satisfy $\lambda' < \lambda$; see Definition 3.3 and Notation 3.8. Recall that $P(S)$ denotes the set of paths in S.

Definition 4.5. Let S be a $\mathcal{Q}$-semigroup, and let $f \in P(S)$. We define the *endpoint* of f, denoted by $f(1)$, as $f(1) := \sup_{\lambda\in I_{\mathbb{Q}}} f(\lambda)$.

Proposition 4.6. *Let S be a $\mathcal{Q}$-semigroup, and let $f, g \in P(S)$. Then:*

1. *We have $(f+g)(1) = f(1) + g(1)$ in S.*
2. *If $f \precsim g$, then $f(1) \leq g(1)$ in S.*
3. *If $[f] \ll [g]$ in $\tau(S)$, then $f(1) \prec g(1)$.*
4. *If $([f_n])_n$ is an increasing sequence in $\tau(S)$ and $[f] = \sup_n [f_n]$, then $f(1) = \sup_n f_n(1)$ in S.*

Proof. (1): This is a consequence of the fact that S satisfies (O4).

(2): Given $\lambda \in I_{\mathbb{Q}}$, using that $f \precsim g$, there is $\mu \in I_{\mathbb{Q}}$ with $f(\lambda) \prec g(\mu) \leq g(1)$. Taking the supremum over λ, we obtain that $f(1) \leq g(1)$.

(3): Assuming $[f] \ll [g]$, we use Lemma 3.16 to choose $\mu \in I_{\mathbb{Q}}$ with $f \prec g(\mu)$. Then $f(1) \leq g(\mu) \prec g(1)$.

(4): Let $([f_n])_n$ be an increasing sequence in $\tau(S)$, and let $[f] = \sup_n [f_n]$. By (2), the endpoint of a path only depends on its equivalence class with respect to the relation $\sim$ from Definition 3.3.

By Lemma 3.11, there are $f' \in P(S)$ and an increasing sequence $(\lambda_m)_m$ in $I_{\mathbb{Q}}$ such that $\sup_m \lambda_m = 1$ and $[f'] = \sup_n [f_n]$, and such that $f'(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \in \mathbb{N}$. Using that $f' \sim f$ at the first step, and using the above property of f' at the fourth step we obtain that

$$f(1) = f'(1) = \sup_{\lambda\in I_{\mathbb{Q}}} f'(\lambda) = \sup_n f'(\tfrac{n}{n+1}) = \sup_n f_n(\lambda_n) \leq \sup_n f_n(1).$$

For each n, we have $f_n \precsim f$ and therefore $f_n(1) \leq f(1)$ by (2). It follows that $\sup_n f_n(1) \leq f(1)$, and therefore $f(1) = \sup_n f_n(1)$, as desired. $\square$

By Proposition 4.6, the endpoint of a path only depends on the equivalence class in $\tau(S)$. Therefore, the following definition makes sense.

Definition 4.7. Let S be a $\mathcal{Q}$-semigroup. We define a map $\varphi_S\colon \tau(S) \to S$ by

$$\varphi_S([f]) := f(1),$$

for all $f \in P(S)$. We refer to φ_S as the *endpoint map.*

Proposition 4.8. *Let S be a $\mathcal{Q}$-semigroup. Then the endpoint map $\varphi_S\colon \tau(S) \to S$ is a well-defined $\mathcal{Q}$-morphism (when considering $\tau(S)$ as a $\mathcal{Q}$-morphism via the inclusion functor ι from Paragraph 4.3.)*

Moreover, the endpoint map is natural in the sense that $\alpha \circ \varphi_S = \varphi_T \circ \tau(\alpha)$ for every $\mathcal{Q}$-morphism $\alpha\colon S \to T$ between $\mathcal{Q}$-semigroups S and T. This means that the following diagram commutes:

$$\begin{tikzcd} \tau(S) \arrow[r, "\varphi_S"] \arrow[d, "\tau(\alpha)"'] & S \arrow[d, "\alpha"] \\ \tau(T) \arrow[r, "\varphi_T"'] & T \end{tikzcd}$$

Proof. It follows directly from Proposition 4.6 that φ_S is a well-defined $\mathcal{Q}$-morphism. To show the commutativity of the diagram, let $f \in P(S)$. Using that α preserves suprema of increasing sequences at the second step, we deduce that

$$\alpha\left(\varphi_S([f])\right) = \alpha\left(\sup_{\lambda\in I_{\mathbb{Q}}} f(\lambda)\right) = \sup_{\lambda\in I_{\mathbb{Q}}} \alpha(f(\lambda)) = \varphi_T([\alpha\circ f]) = \varphi_T(\tau(\alpha)([f])),$$

as desired. $\square$

Remark 4.9. The naturality of the endpoint map as formulated in Proposition 4.8 means precisely that the $\mathcal{Q}$-morphisms φ_S, for S ranging over the objects in $\mathcal{Q}$, form the components of a natural transformation from $\iota\circ\tau$ to the identity functor on $\mathcal{Q}$.

In general, the endpoint map is neither surjective nor injective; see Examples 4.13 and 4.14. We now show that φ_S is an order-isomorphism if (and only if) S is a Cu-semigroup.

Proposition 4.10. *Let S be a* Cu*-semigroup, considered as a $\mathcal{Q}$-semigroup $(S, \ll)$. Then the endpoint map $\varphi_S\colon \tau(S,\ll) \to S$ is an order-isomorphism.*

Proof. We first prove that φ_S is an order-embedding. Let $[f],[g] \in \tau(S,\ll)$ satisfy $\varphi_S([f]) \leq \varphi_S([g])$. Then, by definition, $\sup_\mu f(\mu) \leq \sup_\mu g(\mu)$. To show that $f \precsim g$, let $\lambda \in I_{\mathbb{Q}}$. Choose $\tilde{\lambda} \in I_{\mathbb{Q}}$ with $\lambda < \tilde{\lambda}$. We deduce that

$$f(\lambda) \ll f(\tilde{\lambda}) \leq \sup_\mu f(\mu) \leq \sup_\mu g(\mu).$$

Therefore, there exists $\mu \in I_{\mathbb{Q}}$ such that $f(\lambda) \leq g(\mu)$. Choose $\tilde{\mu} \in I_{\mathbb{Q}}$ with $\mu < \tilde{\mu}$. Then $f(\lambda) \ll g(\tilde{\mu})$. This implies that $f \precsim g$ and thus $[f] \leq [g]$, as desired.

To show that φ_S is surjective, let $s \in S$. Choose a $\ll$-increasing chain $(s_\lambda)_{\lambda\in(0,1)}$ as in Proposition 2.10. In particular, we have $s = \sup_\lambda s_\lambda$, and $s_{\lambda'} \ll s_\lambda$ whenever $\lambda', \lambda \in I_{\mathbb{Q}}$ satisfy $\lambda' < \lambda$. Thus, if we define $f\colon I_{\mathbb{Q}} \to S$ by $f(\lambda) := s_\lambda$, for $\lambda \in I_{\mathbb{Q}}$, then f belongs to $P(S,\ll)$. By construction, $\varphi_S([f]) = s$, as desired. $\square$

Given $\mathcal{Q}$-semigroups S and T, recall that we equip the set of $\mathcal{Q}$-morphisms $\mathcal{Q}(S,T)$ with pointwise order and addition; see Remarks 4.2.

Proposition 4.11. *Let T be a* Cu*-semigroup, let S be a $\mathcal{Q}$-semigroup, and let $\varphi_S\colon \tau(S) \to S$ be the endpoint map from Definition 4.7. Then:*

1. *For every $\mathcal{Q}$-morphism $\alpha\colon T \to S$ there exists a* Cu*-morphism $\bar{\alpha}\colon T \to \tau(S)$ such that $\varphi_S \circ \bar{\alpha} = \alpha$.*
2. *We have $\varphi_S \circ \beta \leq \varphi_S \circ \gamma$ if and only if $\beta \leq \gamma$, for any pair of* Cu*-morphisms $\beta, \gamma\colon T \to \tau(S)$.*

Statement (1) means that for every α one can find $\bar{\alpha}$ making the following diagram commute:

$$\begin{tikzcd} \tau(S) \arrow[r, "\varphi_S"] & S \\ & T. \arrow[u, "\alpha"'] \arrow[ul, dashed, "\bar{\alpha}"] \end{tikzcd}$$

Proof. To show (1), let α be given. Since T is a Cu-semigroup, it follows from Proposition 4.10 that $\varphi_T\colon \tau(T, \ll) \to T$ is an order-isomorphism. Set $\bar{\alpha} := \tau(\alpha) \circ \varphi_T^{-1}$, which is clearly a Cu-morphism. By Proposition 4.8, we have $\varphi_S \circ \tau(\alpha) = \alpha \circ \varphi_T$. It follows that $\varphi_S \circ \bar{\alpha} = \alpha$. The maps are shown in the following diagram:

$$\begin{tikzcd} \tau(S) \arrow[r, "\varphi_S"] & S \\ \tau(T, \ll) \arrow[u, "\tau(\alpha)"] \arrow[r, "\sim", "\varphi_T"'] & T. \arrow[u, "\alpha"'] \arrow[ul, dashed, "\bar{\alpha}"] \end{tikzcd}$$

To show (2), let $\beta, \gamma\colon T \to \tau(S)$ be Cu-morphisms. It is clear that $\beta \leq \gamma$ implies that $\varphi_S \circ \beta \leq \varphi_S \circ \gamma$. Thus let us assume that $\varphi_S \circ \beta \leq \varphi_S \circ \gamma$.

To show that $\beta \leq \gamma$, let $t \in T$. Using that T satisfies (O2), choose a $\ll$-increasing sequence $(t_n)_n$ in T with supremum t. Fix n, and choose paths $f_n, g_n, g \in P(S)$ with $\beta(t_n) = [f_n]$, and $\gamma(t_n) = [g_n]$, and $\gamma(t) = [g]$. Since γ preserves the way-below relation, we have $[g_n] \ll [g]$ in $\tau(S)$. By Lemma 3.16, we can choose $\mu \in I_{\mathbb{Q}}$ such that $g_n(\lambda) \prec g(\mu)$ for all $\lambda \in I_{\mathbb{Q}}$. Passing to the supremum over λ, we obtain that $g_n(1) \leq g(\mu)$. Using this at the last step, and using the assumption that $\varphi_S \circ \beta \leq \varphi_S \circ \gamma$ at the second step, we deduce that

$$f_n(\lambda) \leq f_n(1) = \varphi_S(\beta(t_n)) \leq \varphi_S(\gamma(t_n)) = g_n(1) \leq g(\mu),$$

for every $\lambda \in I_{\mathbb{Q}}$. By definition, we have that $f_n \precsim g$, and hence $\beta(t_n) \leq \gamma(t)$.

Using that β preserves suprema of increasing sequences at the second step, and using the above observation $\beta(t_n) \leq \gamma(t)$ for each n at the last step, we deduce that

$$\beta(t) = \beta\left(\sup_n t_n\right) = \sup_n \beta(t_n) \leq \gamma(t),$$

as desired. $\square$

Theorem 4.12. *The category* Cu *is a coreflective, full subcategory of $\mathcal{Q}$; the functor $\tau\colon \mathcal{Q} \to$* Cu *is a right adjoint to the inclusion functor $\iota\colon$* Cu *$\to \mathcal{Q}$ from Paragraph 4.3.*

More precisely, let S be a $\mathcal{Q}$-semigroup, let $\varphi_S\colon \tau(S) \to S$ be the endpoint map from Definition 4.7, and let T be a Cu*-semigroup. Then the assignment that sends a* Cu*-morphism $\beta\colon T \to \tau(S)$ to the $\mathcal{Q}$-morphism $\varphi_S \circ \beta$ defines a natural bijection between the following morphism sets:*

$$\mathrm{Cu}\big(T, \tau(S)\big) \cong \mathcal{Q}(T, S).$$

Moreover, this bijection respects the structure of the bimoprhism sets as positively ordered monoids.

Proof. Let us denote the assignment from the statement by $\Phi\colon \mathrm{Cu}(T,\tau(S)) \to \mathcal{Q}(T,S)$. Then Φ is well-defined since φ_S is a $\mathcal{Q}$-morphism by Proposition 4.8. Statement (1) in Proposition 4.11 means exactly that Φ is surjective. Further, statement (2) in Proposition 4.11 shows that Φ is an order-embedding. Thus, Φ is an order-isomorphism, and in particular bijective. This shows that τ is right adjoint to ι, as desired. $\square$

We now consider examples of $\mathcal{Q}$-semigroups and their associated endpoint maps. The first example is analogous to Example 3.19 and shows that the τ-construction applied to $\mathcal{Q}$-semigroups may be trivial.

In Example 4.14 we introduce two important Cu-semigroups that are obtained by using the τ-construction. We denote these Cu-semigroups by M_1 and M_∞ since they turn out to be the Cuntz semigroups of II_1- and II_∞-factors, respectively; see Proposition 4.16. These are also examples where the endpoint map is not injective.

Example 4.13. In general, the endpoint map is not surjective. Consider for example $\overline{\mathbb{N}} = \{0,1,2,\ldots,\infty\}$ with auxiliary relation $\prec$ given by $k \prec l$ if and only if $k < l$ or $k = l = 0$. As in Example 3.19, we obtain that the only path in $(\overline{\mathbb{N}},\prec)$ is the constant path with value 0. It follows that $\tau(\overline{\mathbb{N}},\prec) = \{0\}$, and thus the endpoint map of this example is clearly not surjective.

Example 4.14. Consider $\overline{\mathbb{P}} := [0,\infty]$, with its usual structure as a positively ordered monoid. We define two relations $\prec_1$ and $\prec_\infty$ on S as follows: given $a,b \in S$ we set $a \prec_1 b$ if and only if $a < \infty$ and $a \leq b$; and we set $a \prec_\infty b$ if and only if $a \leq b$. It is easy to check that $(\overline{\mathbb{P}},\prec_1)$ and $(\overline{\mathbb{P}},\prec_\infty)$ are $\mathcal{Q}$-semigroups. We set

$$M_1 := \tau\big(\overline{\mathbb{P}},\prec_1\big), \quad \text{and} \quad M_\infty := \tau\big(\overline{\mathbb{P}},\prec_\infty\big).$$

Let us compute the precise structure of M_1 and M_∞. For the most part, the argument is the same in both cases, and we use $\prec_*$ to stand for either $\prec_1$ ar $\prec_\infty$. Recall that a path in $P(\overline{\mathbb{P}},\prec_*)$ is a $\prec_*$-increasing map $f\colon I_{\mathbb{Q}} \to \overline{\mathbb{P}}$. Given a path f, we let $f(1)$ denote the endpoint, that is, $f(1) = \sup_{\lambda\in I_{\mathbb{Q}}} f(\lambda)$; see Definition 4.5.

Let f,g be elements in $P(\overline{\mathbb{P}},\prec_*)$. If $f \precsim g$, then $f(1) \leq g(1)$, by Proposition 4.6 (2). Conversely, if $f(1) < g(1)$, then it is easy to deduce that $f \precsim g$. However, if $f(1) = g(1)$ then we do not necessarily have $f \sim g$. For example, the paths f and g given by $f(\lambda) = \lambda$ and $g(\lambda) = 1$ have the same endpoint but we do not have $g \precsim f$.

It is clear though that the equivalence class of a path only depends on its definition in $(1-\varepsilon,1) \cap I_{\mathbb{Q}}$, for some $\varepsilon > 0$. Therefore, all eventually constant paths with the same endpoint are equivalent and they majorize any path with the same endpoint. Furthermore, it is clear that two paths with equal endpoint which are not eventually constant are in fact equivalent.

Thus, for every $a \in (0,\infty)$ there are exactly two equivalence classes of paths with endpoint a: the classes $[f_a']$ and $[f_a]$ with f_a' and f_a given by $f_a'(\lambda) = \lambda a$ and $f_a(\lambda) = a$, for $\lambda \in I_{\mathbb{Q}}$.

The endpoints 0 and ∞ are particular: The only path with endpoint 0 is the constant path f_0 with value 0. So far, there was no difference between the case of auxiliary relation $\prec_1$ or $\prec_\infty$.

The only difference appears now for paths with endpoint ∞. There is no $\prec_1$-increasing path that is (eventually) constant with value ∞. Therefore, all paths in $P(\overline{\mathbb{P}},\prec_1)$ with endpoint ∞ are equivalent to f_∞' given by $f_\infty'(\lambda) = \frac{1}{1-\lambda}$. On the other hand, $P(\overline{\mathbb{P}},\prec_\infty)$ also contains the constant path f_∞ with value ∞. We obtain that

$$M_1 = \big\{[f_0],[f_a'],[f_a],[f_\infty'] : a \in (0,\infty)\big\}, \quad \text{and} \quad M_\infty = M_1 \cup \big\{[f_\infty]\big\}.$$

Thus, the only difference between M_1 and M_∞ is that M_∞ contains an additional infinite element. It is easy to see that the natural map $M_1 \to M_\infty$ is an additive order-embedding. Therefore, it is enough to describe the order and addition in M_∞. We have $[f_0] \leq [f'_a] \leq [f_a] \leq [f'_\infty] \leq [f_\infty]$ for every $a \in (0,\infty)$. Further, it is easy to see that for every $a,b \in (0,\infty)$ we have $[f'_a] \leq [f_b]$ if and only if $a \leq b$; and we have $[f_a] \leq [f'_b]$ if and only if $a < b$. We have $[f'_\infty] < [f_\infty]$.

It is straightforward to check that the addition in M_∞ is given by

$$[f_a] + [f_b] = [f_{a+b}], \quad \text{and} \quad [f'_a] + [f_b] = [f'_a] + [f'_b] = [f'_{a+b}],$$

for $a \in [0,\infty]$ and $b \in [0,\infty)$. We have that $[f'_\infty] + [f_\infty] = [f_\infty]$.

Abusing notation, we use a' and a to denote the classes of f'_a and f_a in M_∞, that is, we set $a' := [f'_a]$ and $a := [f_a]$, for $a \in (0,\infty]$. Further, we use 0 to denote the classes of f_0.

The compact elements in M_1 are 0 and a for $a \in (0,\infty)$. The soft elements in M_1 are 0 and a' for $a \in (0,\infty]$. The additional element ∞ in M_∞ is both soft and compact.

The endpoint map $M_1 \to \overline{\mathbb{P}}$ is not injective since it sends both $[f_a]$ and $[f'_a]$ to a, for every $a \in (0,\infty)$. Analogously, the endpoint map $M_\infty \to \overline{\mathbb{P}}$ is not injective either.

Remark 4.15. The Cu-semigroups M_1 and M_∞ from Example 4.14 are **m**assive objects in the sense that they contain uncountably many compact elements and are therefore not countably-based. (A Cu-semigroup S is *countably-based* if there exists a countable subset $B \subseteq S$ such that every element of S is the supremum of an increasing sequence from B. Every such B must contain all compact elements of S; see [APT14, Remarks 3.1.3].)

By [APT14, Proposition 3.2.3], the Cuntz semigroup of a separable C^*-algebra is countably-based. This shows that M_1 and M_∞ can not be realized as Cuntz semigroups of separable C^*-algebras.

Proposition 4.16. *We have* $\mathrm{Cu}(M) \cong M_1$ *for every* II_1*-factor* M*; and we have* $\mathrm{Cu}(N) \cong M_\infty$ *for every* II_∞*-factor* N*.*

Proof. Let M be a II_1-factor M, let $\tau\colon M_+ \to [0,1]$ denote its unique tracial state, and let $\tilde{\tau}\colon (M \otimes \mathcal{K})_+ \to [0,\infty]$ denote the unique extension to a tracial weight on the stabilization. Given projections $p,q \in (M \otimes \mathcal{K})_+$ we have $p \precsim q$ if and only if $\tilde{\tau}(p) \leq \tilde{\tau}(q)$. Moreover, for every $t \in [0,\infty)$, there exists a projection $p \in (M \otimes \mathcal{K})_+$ with $\tilde{\tau}(p) = t$. It follows that the Murray-von Neumann semigroup $V(M)$ is isomorphic to $[0,\infty)$, with the usual structure as a positively ordered monoid.

Recall that an interval in a positively ordered monoid is a nonempty, upward directed, order-hereditary subset. An interval is called countably-generated if it contains a countable cofinal subset. By [ABP11, Theorem 6.4], the Cuntz semigroup of a σ-unital C^*-algebra A with real rank zero can be computed as $\mathrm{Cu}(A) \cong \Lambda_\sigma(V(A))$, the set of countably-generated intervals in $V(A)$, with the natural addition and order given by set inclusion. By [BP91, Proposition 1.3], every von Neumann algebra has real rank zero.

It is easy to see that the (countably-generated) intervals of $[0,\infty)$ are given as: $I_0 := \{0\}$; $I'_a := [0,a)$ and $I_a := [0,a]$, for $a \in (0,\infty)$; and $I'_\infty := [0,\infty)$. We obtain an order-isomorphism $\Lambda_\sigma([0,\infty)) \cong M_1$ by mapping I_a to $[f_a]$, for $a \in [0,\infty)$, and by mapping I'_a to $[f'_a]$, for $a \in (0,\infty]$. Together, we obtain order-isomorphisms:

$$\mathrm{Cu}(M) \cong \Lambda_\sigma(V(M)) \cong \Lambda_\sigma([0,\infty)) \cong M_1.$$

Let N be a II_∞-factor. The argument runs analogous to the II_1-case, with the difference that N contains infinite projections. One obtains that $V(N) \cong [0,\infty]$. It follows that

$$\mathrm{Cu}(N) \cong \Lambda_\sigma(V(N)) \cong \Lambda_\sigma([0,\infty]) \cong M_\infty,$$

as desired. $\square$

4.2. **The internal hom in $\mathcal{Q}$.**

Definition 4.17. Let S and T be $\mathcal{Q}$-semigroups. We define a binary relation $\prec$ on the set of generalized $\mathcal{Q}$-morphisms $\mathcal{Q}[S,T]$ by setting $\varphi \prec \psi$ if and only $\varphi(a') \prec \psi(a)$ for all $a', a \in S$ with $a' \prec a$.

Proposition 4.18. *Let S and T be $\mathcal{Q}$-semigroups. Then the relation $\prec$ on $\mathcal{Q}[S,T]$, as defined in Definition 4.17, is an auxiliary relation. Moreover, $(\mathcal{Q}[S,T], \prec)$ is a $\mathcal{Q}$-semigroup.*

Proof. Since addition and order in $\mathcal{Q}[S,T]$ are defined pointwise, it is easy to verify that $\mathcal{Q}[S,T]$ is a positively ordered monoid. Given an increasing sequence $(\varphi_n)_n$ in $\mathcal{Q}[S,T]$, let $\varphi\colon S \to T$ be the pointwise supremum, that is, $\varphi(s) := \sup_n \varphi_n(s)$, for $s \in S$. Then clearly φ is a generalized $\mathcal{Q}$-morphism and $\sup_n \varphi_n = \varphi$ in $\mathcal{Q}[S,T]$. Thus, $\mathcal{Q}[S,T]$ satisfies (O1). It is also clear that taking suprema is compatible with addition and hence $\mathcal{Q}[S,T]$ also satisfies (O4).

Next, note that $\prec$ is an auxiliary relation on $\mathcal{Q}[S,T]$. To show that $\prec$ is additive, let $\varphi', \varphi, \psi', \psi \in \mathcal{Q}[S,T]$ satisfy $\varphi' \prec \varphi$ and $\psi' \prec \psi$. Given $s', s \in S$ with $s' \prec s$, we use that the auxiliary relation in T is additive at the second step to deduce that

$$(\varphi' + \psi')(s') = \varphi'(s') + \psi'(s') \prec \varphi(s) + \psi(s) = (\varphi + \psi)(s).$$

Hence, $\varphi' + \psi' \prec \varphi + \psi$, as desired. Therefore, $(\mathcal{Q}[S,T], \prec)$ is a $\mathcal{Q}$-semigroup. $\square$

Next, we define bimorphisms in the category $\mathcal{Q}$ analogous to the definition of Cu-bimorphisms; see Definition 2.15. Recall the definition of PoM-bimorphisms from Paragraph 2.1.

Definition 4.19. Let S, T and P be $\mathcal{Q}$-semigroups, and let $\varphi\colon S \times T \to P$ be a PoM-bimorphism. We say that φ is a *$\mathcal{Q}$-bimorphism* if it satisfies the following conditions:

(1) We have that $\sup_k \varphi(a_k, b_k) = \varphi(\sup_k a_k, \sup_k b_k)$, for every increasing sequences $(a_k)_k$ in S and $(b_k)_k$ in T.
(2) If $a', a \in S$ and $b', b \in T$ satisfy $a' \prec a$ and $b' \prec b$, then $\varphi(a', b') \prec \varphi(a, b)$.

We denote the set of $\mathcal{Q}$-bimorphisms by $\mathrm{Bi}\mathcal{Q}(S \times T, P)$.

Given $\mathcal{Q}$-semigroups S, T and P, we equip $\mathrm{Bi}\mathcal{Q}(S \times T, P)$ with pointwise order and addition, giving it the structure of a positively ordered monoid. Similarly, we consider the set of $\mathcal{Q}$-morphisms between two $\mathcal{Q}$-semigroups as a positively ordered monoid with the pointwise order and addition.

The proof of the following result follows straightforward from the definition of $\mathcal{Q}$-bimorphisms and is therefore omitted.

Lemma 4.20. *Let S, T and P be $\mathcal{Q}$-semigroups, and let $\varphi\colon S \times T \to P$ be a $\mathcal{Q}$-bimorphism. For each $a \in S$, define $\varphi_a\colon T \to P$ by $\varphi_a(b) = \varphi(a, b)$. Then φ_s belongs to $\mathcal{Q}[T,P]$. Moreover, if $a', a \in S$ satisfy $a' \prec a$, then $\varphi_{a'} \prec \varphi_a$.*

Notation 4.21. Let S, T and P be $\mathcal{Q}$-semigroups, and let $\varphi\colon S \times T \to P$ be a $\mathcal{Q}$-bimorphism. Using Lemma 4.20 we may define a map $\tilde{\varphi}\colon S \to \mathcal{Q}[T,P]$ by $\tilde{\varphi}(a) = \varphi_a$, for $a \in S$, which belongs to $\mathcal{Q}(S, \mathcal{Q}[T,P])$.

Theorem 4.22. *Let S, T and P be $\mathcal{Q}$-semigroups. Then:*

(1) For every $\mathcal{Q}$-morphism $\alpha\colon S \to \mathcal{Q}[T,P]$ there exists a $\mathcal{Q}$-bimorphism $\varphi\colon S \times T \to P$ such that $\alpha = \tilde{\varphi}$.

(2) If $\varphi, \psi\colon S \times T \to P$ are $\mathcal{Q}$-bimorphisms, then $\varphi \leq \psi$ if and only if $\tilde{\varphi} \leq \tilde{\psi}$.

Thus, the assignment Φ that sends a $\mathcal{Q}$-bimorphism $\varphi\colon S \times T \to P$ to the $\mathcal{Q}$-morphism $\tilde{\varphi}\colon S \to \mathcal{Q}[T, P]$ defines a natural bijection between the following (bi)morphism sets:

$$\mathrm{Bi}\mathcal{Q}\big(S \times T, P\big) \cong \mathcal{Q}\big(S, \mathcal{Q}[T, P]\big).$$

Moreover, this bijection respects the structure of the (bi)moprhism sets as positively ordered monoids.

Proof. To verify (1), let $\alpha\colon S \to \mathcal{Q}[T, P]$ be a $\mathcal{Q}$-morphism. Define $\varphi\colon S \times T \to P$ by $\varphi(s,t) = \alpha(s)(t)$. It is straightforward to check that φ is a $\mathcal{Q}$-bimorphism satisfying $\alpha = \tilde{\varphi}$, as desired.

Statement (2) is also easily verified. It follows that Φ is an order-isomorphism, and hence a bijection. It is also clear that Φ is additive and preserves the zero element. $\square$

Lemma 4.23. *Let S_1, S_2 and T be $\mathcal{Q}$-semigroups, and let $\alpha\colon S_1 \to S_2$ be a (generalized) $\mathcal{Q}$-morphism. Then the map $\alpha^*\colon \mathcal{Q}[S_2, T] \to \mathcal{Q}[S_1, T]$ given by $\alpha^*(f) := f \circ \alpha$, for $f \in \mathcal{Q}[S_2, T]$, is a (generalized) $\mathcal{Q}$-morphism.*

Analogously, given $\mathcal{Q}$-semigroups S, T_1 and T_2, and given a (generalized) $\mathcal{Q}$-morphism $\beta\colon T_1 \to T_2$, the map $\beta_*\colon \mathcal{Q}[S, T_1] \to \mathcal{Q}[S, T_2]$ defined by $\beta_*(f) := \beta \circ f$, for $f \in \mathcal{Q}[S, T_1]$, is a (generalized) $\mathcal{Q}$-morphism.

Proof. It is straightforward to check that α^* and β_* are generalized $\mathcal{Q}$-morphisms. Assume that α is a $\mathcal{Q}$-morphism. To show that α^* preserves the auxiliary relation, let $f_1, f_2 \in \mathcal{Q}[S_2, T]$ satisfy $f_1 \prec f_2$. To show that $\alpha^*(f_1) \prec \alpha^*(f_2)$, let $a', a \in S$ satisfy $a' \prec a$. Since α preserves the auxiliary relation, we have $\alpha(a') \prec \alpha(a)$. Using that $f_1 \prec f_2$ at the second step, we deduce that

$$\alpha^*(f_1)(a') = f_1(\alpha(a')) \prec f_2(\alpha(a)) = \alpha^*(f_2)(a),$$

as desired. Analogously, one shows that β_* preserves the auxiliary relation whenever β does. $\square$

4.24. Let T be a $\mathcal{Q}$-semigroup. We let $\mathcal{Q}[\_, T]\colon \mathcal{Q} \to \mathcal{Q}$ be the contravariant functor that sends a $\mathcal{Q}$-semigroup S to the $\mathcal{Q}$-semigroup $\mathcal{Q}[S, T]$ (see Proposition 4.18), and that sends a $\mathcal{Q}$-morphism $\alpha\colon S_1 \to S_2$ to the $\mathcal{Q}$-morphism $\alpha^*\colon \mathcal{Q}[S_2, T] \to \mathcal{Q}[S_1, T]$ as in Lemma 4.23.

Analogously, we obtain a covariant functor $\mathcal{Q}[S, \_]\colon \mathcal{Q} \to \mathcal{Q}$ for every $\mathcal{Q}$-semigroup S. Thus, we obtain a bifunctor

$$\mathcal{Q}[\_, \_]\colon \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}.$$

Remark 4.25. Given $\mathcal{Q}$-semigroups S and T, one can construct a $\mathcal{Q}$-semigroup $S \otimes_{\mathcal{Q}} T$ together with a $\mathcal{Q}$-bimorphism $\omega\colon S \times T \to S \otimes T$ that have the same universal properties as the tensor product in Cu. One can then show that $\mathcal{Q}$ is a closed, symmetric monoidal category, with $\mathcal{Q}[\_, \_]$ as internal-hom bifunctor. We omit the details since for our purpose it is not necessary to show that $\mathcal{Q}$ is a monoidal category.

5. Abstract bivariant Cuntz semigroups

In this section, we use the τ-construction developed in Sections 3 and 4 to prove that Cu is a *closed* symmetric monoidal category. Thus, given Cu-semigroups S and T, we construct a Cu-semigroup $[\![S, T]\!]$, playing the role of the set of morphisms from S to T, such that this 'internal-hom' is adjoint to the tensor product. This means that, given another Cu-semigroup P, there is a natural bijection between the morphism sets

$$\mathrm{Cu}\big(S, [\![T, P]\!]\big) \cong \mathrm{Cu}\big(S \otimes T, P\big),$$

as shown in Theorem 5.10. We will call $[\![S,T]\!]$ the *bivariant* Cu-*semigroup*, or the *abstract bivariant Cuntz semigroup* of S and T. The Cu-morphisms $S \to T$ actually correspond to compact elements in $[\![S,T]\!]$.

In Subsection 5.2, we compute the first examples of bivariant Cu-semigroups.

We also study the situation for algebraic Cu-semigroups; see Subsection 5.3. (Recall that a Cu-semigroup S is called algebraic if every element is the supremum of an increasing sequence of compact elements.) Although, in general, it is neither necessary nor sufficient that S and T are algebraic in order that $[\![S,T]\!]$ is algebraic, the full subcategory of Cu consisting of algebraic Cu-semigroups is also a closed, symmetric, monoidal category.

In Subsection 5.4, we analyse bivariant Cu-semigroups of ideals and quotients. More concretely, given an ideal J of T, we show that $[\![S,J]\!]$ is naturally identified with an ideal in $[\![S,T]\!]$. Similarly, given an ideal J of S, we can identify $[\![S/J,T]\!]$ with an ideal in $[\![S,T]\!]$. We provide examples to the effect that the ideal structure of the abstract bivariant Cu-semigroup $[\![S,T]\!]$ does not only depend on the ideal structure of S and T.

5.1. **Construction of abstract bivariant Cuntz semigroups.** Recall that a *generalized* Cu-*morphism* between Cu-semigroups S and T is a map that preserves order, addition, the zero element, and suprema of increasing sequences, and thus it is not necessarily a Cu-morphism; see Definition 2.4. In the language of domain theory, a generalized Cu-morphism is a monoid morphism that is sequentially Scott continuous (see [GHK$^+$03, Section II.2, p.157ff]). We denote the set of generalized Cu-morphisms $S \to T$ by $\mathrm{Cu}[S,T]$, and we equip it with the pointwise order and addition, giving it a natural structure as a positively ordered monoid.

Recall from Paragraph 4.3 that there is functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$ that embeds Cu is a full subcategory of $\mathcal{Q}$. This is given by considering a Cu-semigroup S as a $\mathcal{Q}$-semigroup for the auxiliary relation $\ll$.

In Definition 4.17 we introduced an auxiliary relation on the set of generalized $\mathcal{Q}$-morphisms, giving itself the structure of a $\mathcal{Q}$-semigroup; see Proposition 4.18. Let us transfer this definition to the setting of Cu-semigroups.

Definition 5.1. Let S and T be Cu-semigroups. We define a binary relation $\prec$ on the set of generalized Cu-morphisms $\mathrm{Cu}[S,T]$ by setting $\varphi \prec \psi$ if and only $\varphi(a') \ll \psi(a)$ for all $a', a \in S$ with $a' \ll a$.

Remarks 5.2. (1) The auxiliary relation $\prec$ on the set of generalized Cu-morphisms was already considered in [APT14, 6.2.6].

(2) Every Cu-morphism is also a generalized Cu-morphism, and we therefore consider $\mathrm{Cu}(S,T)$ as a subset of $\mathrm{Cu}[S,T]$. For $\varphi \in \mathrm{Cu}[S,T]$, we have $\varphi \prec \varphi$ if and only if φ is a Cu-morphism.

It follows from Proposition 4.18 that $\prec$ is an auxiliary relation on $\mathrm{Cu}[S,T]$ and that $(\mathrm{Cu}[S,T], \prec)$ is a $\mathcal{Q}$-semigroup. We may therefore apply the τ-construction.

Definition 5.3. Let S and T be Cu-semigroups. We define the *internal hom* from S to T as the Cu-semigroup

$$[\![S,T]\!] := \tau\big(\mathrm{Cu}[S,T], \prec\big).$$

We call $[\![S,T]\!]$ the *bivariant* Cu-*semigroup*, or the *abstract bivariant Cuntz semigroup* of S and T.

Remark 5.4. Recall that a path in $\mathrm{Cu}[S,T]$ is a map $f\colon I_{\mathbb{Q}} \to \mathrm{Cu}[S,T]$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda', \lambda \in I_{\mathbb{Q}}$ satisfy $\lambda' < \lambda$. We often denote $f(\lambda)$ by f_λ and we denote the path by $\mathbf{f} = (f_\lambda)_\lambda$.

By definition then, the elements of $[\![S,T]\!]$ are equivalence classes of paths in the $\mathcal{Q}$-semigroup $(\mathrm{Cu}[S,T],\prec)$.

5.5. Let we show that the internal-hom in Cu is functorial in both variables: contravariant in the first and covariant in the second variable.

Let T be a Cu-semigroup. Considering T as a $\mathcal{Q}$-semigroup, we have a contravariant functor $\mathcal{Q}[\_,T]\colon\mathcal{Q}\to\mathcal{Q}$ as in Paragraph 4.24. Precomposing with the inclusion $\iota\colon\mathrm{Cu}\to\mathcal{Q}$ from Paragraph 4.3 and postcomposing with the functor $\tau\colon\mathcal{Q}\to\mathrm{Cu}$, we obtain a functor $[\![\_,T]\!]\colon\mathrm{Cu}\to\mathrm{Cu}$.

Given Cu-semigroups S_1 and S_2, and a Cu-morphism $\alpha\colon S_1\to S_2$, we use α^* to denote the induced Cu-morphism $[\![S_2,T]\!]\to[\![S_1,T]\!]$. Thus, if we consider α as a $\mathcal{Q}$-morphism and if we let $\alpha^*_{\mathcal{Q}}\colon\mathcal{Q}[S_2,T]\to\mathcal{Q}[S_1,T]$ denote the induced $\mathcal{Q}$-morphism from Lemma 4.23, then α^* is given as $\alpha^*:=\tau(\alpha^*_{\mathcal{Q}})$.

Analogously, given a Cu-semigroup S, we define the functor $[\![S,\_]\!]\colon\mathrm{Cu}\to\mathrm{Cu}$ as the composition of the functors ι, the functor $\mathcal{Q}[S,\_]$ from Paragraph 4.24 and τ.

Given Cu-semigroups T_1 and T_2, and a Cu-morphism $\beta\colon T_1\to T_2$, we use β_* to denote the induced Cu-morphism $[\![S,T_1]\!]\to[\![S,T_2]\!]$. If we consider β as a $\mathcal{Q}$-morphism and if we let $\beta_*^{\mathcal{Q}}\colon\mathcal{Q}[S,T_1]\to\mathcal{Q}[S,T_2]$ denote the induced $\mathcal{Q}$-morphism from Lemma 4.23, then β_* is given as $\beta_*:=\tau(\beta_*^{\mathcal{Q}})$.

Thus, the internal-hom in the category Cu is a bifunctor

$$[\![\_,\_]\!]\colon\mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}.$$

In Subsection 6.3 below we will generalize these basic functoriality properties by describing a composition product $[\![T,P]\!]\otimes[\![S,T]\!]\to[\![S,P]\!]$.

Next, we transfer the concept of the endpoint map from Definition 4.7 to the setting of bivariant Cu-semigroups. To simplify notation, we write $\sigma_{S,T}$ for $\varphi_{\mathrm{Cu}[S,T]}$, the endpoint map associated to the $\mathcal{Q}$-semigroup $\mathrm{Cu}[S,T]$. The next definition makes this precise.

Definition 5.6. Let S and T be Cu-semigroups. We let $\sigma_{S,T}\colon[\![S,T]\!]\to\mathrm{Cu}[S,T]$ be defined by

$$\sigma_{S,T}([\mathbf{f}])(a)=\sup_{\lambda\in I_{\mathbb{Q}}}f_\lambda(a),$$

for a path $\mathbf{f}=(f_\lambda)_\lambda$ in $\mathrm{Cu}[S,T]$ and $a\in S$. We refer to $\sigma_{S,T}$ as the *endpoint map.*

Lemma 5.7. *Let S, T and P be* Cu*-semigroups, and let $\alpha\colon S\to[\![T,P]\!]$ be a* Cu*-morphism. Let $\sigma_{T,P}\colon[\![T,P]\!]\to\mathrm{Cu}[T,P]$ be the endpoint map from Definition 5.6. Define $\bar{\alpha}\colon S\times T\to P$ by*

$$\bar{\alpha}(a,b)=\sigma_{T,P}(\alpha(a))(b),$$

for $a\in S$ and $b\in T$. Then $\bar{\alpha}$ is a Cu*-bimorphism.*

Proof. We write σ for $\sigma_{T,P}$. To show that $\bar{\alpha}$ is a generalized Cu-morphism in the first variable, let $b\in T$. Since α and σ are both additive and order preserving, we conclude that $\bar{\alpha}(\_,b)=\sigma(\alpha(\_))(b)$ is additive and order preserving as well. To show that $\bar{\alpha}(\_,b)$ preserves suprema of increasing sequences, let $(a_n)_n$ be an increasing sequence in S. Set $a:=\sup_n a_n$. Since both α and σ preserve suprema of increasing sequences, we obtain that

$$\sigma(\alpha(a))=\sup_n\sigma(\alpha(a_n)),$$

in $\mathrm{Cu}[T,P]$. Since the supremum of an increasing sequence in $\mathrm{Cu}[T,P]$ is the pointwise supremum, we get that $\bar{\alpha}(a,b)=\sup_n\bar{\alpha}(a_n,b)$, as desired.

For each $a\in S$, we have $\bar{\alpha}(a,\_)=\sigma(\alpha(a))$, which is an element in $\mathrm{Cu}[T,P]$. Therefore, $\bar{\alpha}$ is a generalized Cu-morphism in the second variable.

Lastly, to show that $\bar{\alpha}$ preserves the joint way-below relation, let $a', a \in S$ and $b', b \in T$ satisfy $a' \ll a$ and $b' \ll b$. Since α is a Cu-morphism we have $\alpha(a') \ll \alpha(a)$. Using that σ is a $\mathcal{Q}$-morphism, it follows that $\sigma(\alpha(a')) \prec \sigma(\alpha(a))$. Therefore, applying the definition of the auxiliary relation $\prec$ at the second step, we obtain that

$$\bar{\alpha}(a', b') = \sigma(\alpha(a'))(b') \ll \sigma(\alpha(a))(b) = \bar{\alpha}(a, b),$$

as desired. □

We omit the straightforward proof of the following result.

Lemma 5.8. *Let S, T and P be* Cu*-semigroups, and let $\varphi\colon S \times T \to P$ be a map. Then φ is a* Cu*-bimorphism if and only if φ, considered as a map between $\mathcal{Q}$-semigroups, is a $\mathcal{Q}$-bimorphism. Thus, we have a canonical bijection*

$$\mathrm{Bi}\mathcal{Q}(S \times T, P) \cong \mathrm{BiCu}(S \times T, P).$$

Moreover, this bijection respects the structure of the bimoprhism sets as positively ordered monoids.

Lemma 5.9. *Let S, T and P be* Cu*-semigroups. Then the assignment that sends a* Cu*-morphism $\alpha\colon S \to [\![T, P]\!]$ to the* Cu*-bimorphism $\tilde{\alpha}\colon S \times T \to P$ given in Lemma 5.7 defines a natural bijection:*

$$\mathrm{Cu}\big(S, [\![T, P]\!]\big) \cong \mathrm{BiCu}\big(S \times T, P\big).$$

Moreover, this bijection respects the structure of the moprhism sets as positively ordered monoids.

Proof. By definition, we have $\mathrm{Cu}\big(S, [\![T, P]\!]\big) = \mathrm{Cu}\big(S, \tau(\mathcal{Q}[T, P])\big)$. Further, we have natural bijections, respecting the structure as positively ordered monoids, using Theorem 4.12 at the first step, using Theorem 4.22 at the second step, and using Lemma 5.8 at the last step:

$$\mathrm{Cu}\big(S, \tau(\mathcal{Q}[T, P])\big) \cong \mathcal{Q}\big(S, \mathcal{Q}[T, P]\big) \cong \mathrm{Bi}\mathcal{Q}\big(S \times T, P\big) \cong \mathrm{BiCu}\big(S \times T, P\big).$$

It is straightforward to check that the composition of these bijections identifies a Cu-morphism α with the Cu-bimorphism $\tilde{\alpha}$ as defined in Lemma 5.7. □

Theorem 5.10. *Let S, T and P be* Cu*-semigroups. Then there are natural bijections*

$$\mathrm{Cu}\big(S, [\![T, P]\!]\big) \cong \mathrm{BiCu}\big(S \times T, P\big) \cong \mathrm{Cu}\big(S \otimes T, P\big).$$

Moreover, these bijection respects the structure of the (bi)moprhism sets as positively ordered monoids.

The first bijection is given by assigning to a Cu*-morphism $\alpha\colon S \to [\![T, P]\!]$ the* Cu*-bimorphism $\tilde{\alpha}\colon S \times T \to P$ as in Lemma 5.7, that is, $\tilde{\alpha}(a, b) = \sigma_{T,P}(\alpha(a))(b)$, for $(a, b) \in S \times T$. The second bijection is given by assigning to a* Cu*-morphism $\beta\colon S \otimes T \to P$ the* Cu*-bimorphism $S \times T \to P$, $(a, b) \mapsto \beta(a \otimes b)$, for $(a, b) \in S \times T$.*

Proof. The first bijection is obtained from Lemma 5.9. The second bijection follows from Theorem 2.16. It is also shown in these results that the bijections respect the structure of the (bi)morphism sets as positively ordered monoids. □

Let T be a Cu-semigroup. We consider the functor $\_ \otimes T\colon \mathrm{Cu} \to \mathrm{Cu}$ given by tensoring with T. It follows from Theorem 5.10 that the functor $[\![T, \_]\!]$ is a right adjoint of $\_ \otimes T$. By definition, this shows that the monoidal category Cu is closed, and we obtain the following result:

Theorem 5.11. *The category* Cu *of abstract Cuntz semigroups is a closed, symmetric, monoidal category.*

Every closed symmetric monoidal category is enriched over itself, as noted in Paragraph 2.14. Let us make this precise for the category Cu. (See Section 6 for further details.) Given Cu-semigroups S and T, the Cu-semigroup $[\![S,T]\!]$ plays the role of morphisms from S to T. First, we show that Cu-morphisms $S \to T$ correspond to compact elements in $[\![S,T]\!]$.

Proposition 5.12. *Let S and T be* Cu*-semigroups. Then there is a natural bijection between* Cu*-morphisms $S \to T$ and compact elements in $[\![S,T]\!]$:*

$$[\![S,T]\!]_c \cong \mathrm{Cu}(S,T).$$

A Cu*-morphism $\varphi\colon S \to T$ is associated with the class in $[\![S,T]\!]$ of the constant path with value φ. Conversely, given a compact element in $[\![S,T]\!]$ represented by a path $(\varphi_\lambda)_\lambda$, then for λ close enough to 1 the map φ_λ is a* Cu*-morphism and independent of λ.*

Proof. It is straightforward to verify that the described associations are well-defined and inverses of each other. Alternatively, note that for every Cu-semigroup P, there is a natural identification of P_c with $\mathrm{Cu}(\overline{\mathbb{N}},P)$, by associating to a Cu-morphism $\varphi\colon \overline{\mathbb{N}} \to P$ the compact element $\varphi(1)$. Using this fact at the first step, using Theorem 5.10 at the second step, and using the isomorphism $\overline{\mathbb{N}} \otimes S \cong S$ at the third step, we obtain that

$$[\![S,T]\!]_c \cong \mathrm{Cu}\big(\overline{\mathbb{N}}, [\![S,T]\!]\big) \cong \mathrm{Cu}\big(\overline{\mathbb{N}} \otimes S, T\big) \cong \mathrm{Cu}(S,T),$$

as desired. □

In particular, the identity Cu-morphism $\mathrm{id}_S\colon S \to S$ naturally corresponds to a compact element in $[\![S,S]\!]$, also denoted by id_S. Further, id_S also naturally corresponds to a Cu-morphism $j_S\colon \overline{\mathbb{N}} \to [\![S,S]\!]$, which is the identity of S for the enrichment of Cu over itself.

Given Cu-semigroups S and T, recall that the *counit map*, or *evaluation map* is the Cu-morphism $\mathrm{e}_T^S\colon [\![S,T]\!] \otimes S \to T$ that corresponds to $\mathrm{id}_{[\![S,T]\!]}$ under the identification $\mathrm{Cu}\big([\![S,T]\!], [\![S,T]\!]\big) \cong \mathrm{Cu}\big([\![S,T]\!] \otimes S, T\big)$.

Given Cu-semigroups S, T and P, consider the following Cu-morphism:

$$([\![T,P]\!] \otimes [\![S,T]\!]) \otimes S \xrightarrow{\cong} [\![T,P]\!] \otimes ([\![S,T]\!] \otimes S) \xrightarrow{\mathrm{id} \otimes \mathrm{e}_T^S} [\![T,P]\!] \otimes T \xrightarrow{\mathrm{e}_P^T} P.$$

Under the identification $\mathrm{Cu}\big([\![T,P]\!] \otimes [\![S,T]\!], [\![S,P]\!]\big) \cong \mathrm{Cu}\big([\![T,P]\!] \otimes [\![S,T]\!] \otimes S, P\big)$, the above Cu-morphism corresponds to a Cu-morphism

$$\circ\colon [\![T,P]\!] \otimes [\![S,T]\!] \to [\![S,P]\!],$$

that we will call the *composition product*. The composition product implements the composition of morphisms when viewing the category Cu as enriched over itself. (See Section 6 for further details.) We obtain:

Theorem 5.13. *The category* Cu *of abstract Cuntz semigroups is enriched over itself.*

5.2. **First examples.** In this subsection, we compute several examples of bivariant Cu-semigroups $[\![S,T]\!]$. We mostly consider the case that S and T are the Cuntz semigroups of the Jacelon-Razak algebra $\mathcal{W}$, of the Jiang-Su algebra $\mathcal{Z}$, of a UHF-algebra of infinite type, or of the Cuntz algebra $\mathcal{O}_2$.

Recall that $\overline{\mathbb{P}}$ denotes the semigroup $[0,\infty]$ with the usual order and addition. It is known that $\overline{\mathbb{P}} \cong \mathrm{Cu}(\mathcal{W})$, the Cuntz semigroup of the Jacelon-Razak algebra $\mathcal{W}$ introduced in [Jac13] (see [Rob13]). The product of real numbers extends to a natural product on $[0,\infty]$ giving $\overline{\mathbb{P}}$ the structure of a solid Cu-semiring; see [APT14, Definition 7.1.5, Example 7.1.7], and also Subsection 7.3. The Cu-semiring $\overline{\mathbb{P}}$ and its Cu-semimodules were studied in [APT14, Section 7.5].

Let M_1 be defined as in Example 4.14. By Proposition 4.16, M_1 is the Cuntz semigroup of a II_1-factor M.

Proposition 5.14. *There is a natural isomorphism $[\![\overline{\mathbb{P}},\overline{\mathbb{P}}]\!] \cong M_1$.*

Proof. We show that the $\mathcal{Q}$-semigroup $(\mathrm{Cu}[\overline{\mathbb{P}},\overline{\mathbb{P}}],\prec)$ is isomorphic to $(\overline{\mathbb{P}},\prec_1)$, where $\prec_1$ is the auxiliary relation defined in Example 4.14. Applying the τ-construction, and using the arguments in Example 4.14 at the last step, we then obtain

$$[\![\overline{\mathbb{P}},\overline{\mathbb{P}}]\!] = \tau\left(\mathrm{Cu}[\overline{\mathbb{P}},\overline{\mathbb{P}}],\prec\right) \cong \tau\left(\overline{\mathbb{P}},\prec_1\right) = M_1.$$

Since $\overline{\mathbb{P}}$ is a solid Cu-semiring, every generalized Cu-morphism $\overline{\mathbb{P}} \to \overline{\mathbb{P}}$ is automatically $\overline{\mathbb{P}}$-linear; see Proposition 7.24, and [APT14, Proposition 7.1.6]. Thus, given a generalized Cu-morphism $\varphi\colon \overline{\mathbb{P}} \to \overline{\mathbb{P}}$, we have $\varphi(x) = \varphi(1)x$ for all $x \in \overline{\mathbb{P}}$. This allows us to identify $\mathrm{Cu}[\overline{\mathbb{P}},\overline{\mathbb{P}}]$ with $\overline{\mathbb{P}}$ by $\varphi \mapsto \varphi(1)$ and this is easily seen to be an additive order-isomorphism. To conclude the argument, we need to show that under this identification, the auxiliary relation $\prec$ on $\mathrm{Cu}[\overline{\mathbb{P}},\overline{\mathbb{P}}]$ corresponds to the auxiliary relation $\prec_1$ on $\overline{\mathbb{P}}$ as defined in Example 4.14.

Let $\varphi,\psi \in \mathrm{Cu}[\overline{\mathbb{P}},\overline{\mathbb{P}}]$. Clearly $\varphi \prec \psi$ implies $\varphi(1) \leq \psi(1)$. Moreover, if $\varphi(1) = \psi(1) = \infty$, then $\varphi \not\prec \psi$, since $\varphi(1) = \infty \not\ll \infty = \psi(2)$ while $1 \ll 2$. Thus, $\varphi \prec \psi$ implies $\varphi(1) \prec_1 \psi(1)$. Conversely, assume that $\varphi(1) \prec_1 \psi(1)$. By definition, $\varphi(1)$ is finite, and $\varphi(1) \leq \psi(1)$. To show that $\varphi \prec \psi$, let $s,t \in \overline{\mathbb{P}}$ satisfy $s \ll t$. Using that $\varphi(1)$ is finite at the second step, we deduce that

$$\varphi(s) = \varphi(1)s \ll \varphi(1)t \leq \psi(1)t = \psi(t),$$

as desired. $\square$

We let Z be the disjoint union $\mathbb{N} \sqcup (0,\infty]$, with elements in $\mathbb{N}$ being compact, and with elements in $(0,\infty)$ being soft. It is known that Z is isomorphic to the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$ introduced in [JS99] (see [PT07] and also [BT07]). We next recall some details.

To distinguish elements in both parts, we write a' (with a prime symbol) for the soft element of value a. For example, the compact one, denoted 1, corresponds the class of the unit in $\mathcal{Z}$; and the soft one, denoted $1'$, corresponds to the class of a positive element x in $\mathcal{Z}$ that has spectrum $[0,1]$ and with $\lim_{n\to\infty}\tau(x^{1/n}) = 1$, for the unique trace τ on $\mathcal{Z}$.

Order and addition are the usual inside the components $\mathbb{N}$ and $(0,\infty]$ of Z. Given $a \in \mathbb{N}$ and $b' \in (0,\infty]$, we have $a + b' = (a+b)'$ (the soft part is absorbing), and we have $a \leq b'$ if and only $a' < b'$, and we have $a \geq b'$ if and only if $a' \geq b'$.

We have a natural commutative product in Z, extending the natural products in the components $\mathbb{N}$ and $(0,\infty]$, and such that $0a = 0$ for every $a \in Z$, and such that $ab' = (ab)'$ for $a \in \mathbb{N}_{>0}$ and $b' \in (0,\infty)$. Note that 1 (the compact one) is a unit for this semiring, but $1'$ is not. Indeed, we have $1'1 = 1'$.

This gives Z the structure of a solid Cu-semiring. The Cu-semiring Z and its Cu-semimodules were studied in [APT14, Section 7.3].

Given a supernatural number q satisfying $q = q^2 \neq 1$, we let $\mathbb{N}[\frac{1}{q}]$ denote the set of nonnegative rational numbers whose denominators divide q, with usual addition. Let $R_q = \mathbb{N}[\frac{1}{q}] \sqcup (0,\infty]$, with elements in $\mathbb{N}[\frac{1}{q}]$ being compact, and with elements in $(0,\infty]$ being soft. The interplay of addition and order with the two components of R_q is analogous to the situation for Z. If we now let M_q denote the UHF-algebra of type q, then it is known that $\mathrm{Cu}(M_q) \cong R_q$.

Analogous to the case for Z, we can define a commutative multiplication on R_q, giving it the structure of a solid Cu-semiring. The Cu-semiring R_q and its Cu-semimodules were studied in [APT14, Section 7.4].

We exclude zero as a supernatural number. However, 1 is supernatural number that agrees with its square. It is consistent to let R_1 denote the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$. Thus, we set $R_1 := Z$, which simplifies the statement of Proposition 5.15 below.

Given supernatural numbers p and q satisfying $p = p^2$ and $q = q^2$, we have $R_p \otimes R_q \cong R_{pq}$. In particular, $Z \otimes R_p = R_1 \otimes R_p \cong R_p$. Moreover, if we let $Q = \mathbb{Q}^+ \sqcup (0, \infty]$, then Q is isomorphic to the Cuntz semigroup of the universal UHF-algebra (whose K_0-group is isomorphic to the rational numbers). We have $Q \otimes R_p \cong Q$.

Proposition 5.15. *Let p and q be supernatural numbers with $p = p^2$ and $q = q^2$. If p divides q, then $[\![R_p, R_q]\!] \cong R_q$. If p does not divide q, then* $\mathrm{Cu}(R_p, R_q) = \{0\}$ *and $[\![R_p, R_q]\!] \cong \overline{\mathbb{P}}$.*

Proof. First, assume that p divides q. Then $R_p \cong R_p \otimes R_p$ and $R_q \cong R_q \otimes R_p$. Let $\varphi\colon R_p \to R_q$ be a generalized Cu-morphism. It follows from [APT14, Proposition 7.1.6] that φ is R_p-linear. Thus, φ is determined by the image of the unit. Moreover, for every $a \in R_q$, there is a generalized Cu-morphism $\varphi\colon R_p \to R_q$ with $\varphi(1) = a$, given by $\varphi(t) = ta$ for $t \in R_p$. Thus, there is a bijection $\mathrm{Cu}[R_p, R_q] \cong R_q$ given by identifying φ with $\varphi(1)$. It is straightforward to check that under this identification, the relation $\prec$ on $\mathrm{Cu}[R_p, R_q]$ corresponds precisely to the way-below relation on R_q. It follows that

$$[\![R_p, R_q]\!] = \tau\big(\mathrm{Cu}[R_p, R_q], \prec\big) \cong \tau\big(R_q, \ll\big) \cong R_q,$$

as desired.

Assume now that p does not divide q. Let r be a prime number dividing p but not q. Every element of R_p is divisible by arbitrary powers of r.

On the other hand, we claim that only the soft elements of R_q are divisible by arbitrary powers of r. Indeed, every element of R_q is either compact or nonzero and soft. Moreover, the sum of a nonzero soft element with any other element in R_q is soft. Thus, if $x \in R_q$ is compact, and if $x = ky$ for some $k \in \mathbb{N}$ and $y \in R_q$, then y is necessarily compact. Thus, if a compact element of R_q is divisible in R_q then it is also divisible in the monoid of compact elements of R_q, which we identify with $\mathbb{N}[\frac{1}{q}]$. However, since r does not divide q, the only element in $\mathbb{N}[\frac{1}{q}]$ that is divisible by arbitrary powers of r is the zero element, which is soft.

It follows that every generalized Cu-morphism $R_p \to R_q$ has its image contained in the soft part of R_q.

In particular, if $\varphi\colon R_p \to R_q$ is a Cu-morphism, then every compact element of R_p is sent to zero by φ. Using that R_p is simple, it follows that φ is the zero map. Thus, $\mathrm{Cu}(R_p, R_q) = \{0\}$, as desired.

We identify $\overline{\mathbb{P}}$ with the soft part of R_p, and similarly for R_q. Let $\varphi\colon R_p \to R_q$ be a generalized Cu-morphism. We have seen that $\varphi(1)$ belongs to $\overline{\mathbb{P}} = (R_q)_{\mathrm{soft}}$. Moreover, for every $a \in \overline{\mathbb{P}}$ there is a generalized Cu-morphism $\varphi\colon R_p \to \overline{\mathbb{P}} \subseteq R_q$ with $\varphi(1) = a$, given by $\varphi(t) = ta$ for $t \in R_p$. Thus, there is a bijection $\mathrm{Cu}[R_p, R_q] \cong \overline{\mathbb{P}}$ given by identifying φ with $\varphi(1)$. It is straightforward to check that under this identification, the relation $\prec$ on $\mathrm{Cu}[R_p, R_q]$ corresponds to the way-below relation on $\overline{\mathbb{P}}$. As above, it follows that

$$[\![R_p, R_q]\!] = \tau\big(\mathrm{Cu}[R_p, R_q], \prec\big) \cong \tau\big(\overline{\mathbb{P}}, \ll\big) \cong \overline{\mathbb{P}},$$

as desired. $\square$

Example 5.16. Recall that Z and Q are isomorphic to the Cuntz semigroups of the Jiang-Su algebra and the universal UHF-algebra, respectively. By Proposition 5.15, there are natural isomorphisms

$$[\![Z,Z]\!] \cong Z \quad \text{and} \quad [\![Q,Q]\!] \cong Q.$$

More generally, there are natural isomorphisms

$$[\![Z,R_q]\!] \cong R_q \quad \text{and} \quad [\![R_q,Q]\!] \cong Q,$$

for every supernatural number q with $q = q^2$,

Example 5.17. Let q be a supernatural number with $q = q^2$. Then there are natural isomorphisms

$$[\![R_q,\overline{\mathbb{P}}]\!] \cong \overline{\mathbb{P}} \quad \text{and} \quad [\![\overline{\mathbb{P}},R_q]\!] \cong M_1,$$

which can be proved similarly as Proposition 5.15 and Proposition 5.14. In particular, we have $[\![Z,\overline{\mathbb{P}}]\!] \cong \overline{\mathbb{P}}$ and $[\![\overline{\mathbb{P}},Z]\!] \cong M_1$.

Given $k \in \mathbb{N}$, we set $E_k := \{0,1,\ldots,k,\infty\}$, equipped with the natural order and addition as a subset of $\overline{\mathbb{N}}$, with the convention that $a+b=\infty$ whenever $a+b>k$ in $\overline{\mathbb{N}}$. With the obvious multiplication, E_k is a solid Cu-semiring (see, e.g. [APT14, Example 8.1.2]). Note that $E_0 = \{0,\infty\}$ is the Cuntz semigroup of the Cuntz algebra $\mathcal{O}_2$ (or of any other simple, purely infinite C^*-algebra). The Cu-semiring $E_0 = \{0,\infty\}$ and its Cu-semimodules were studied in [APT14, Section 7.2].

Proposition 5.18. *Let k,l be natural numbers. Let $\lceil \frac{l+1}{k+1} \rceil$ denote the smallest natural number larger than or equal to $\frac{l+1}{k+1}$. Then $[\![E_k,E_l]\!]$ is isomorphic to the sub-*Cu*-semigroup $\{0,\lceil \frac{l+1}{k+1} \rceil,\ldots,l,\infty\}$ of E_l.*

Proof. Let $\varphi\colon E_k \to E_l$ be a generalized Cu-morphism. Then φ is determined by the image of 1, which can be zero or any element $a \in E_l$ such that $(k+1)a = \infty$. Thus, for every $a \geq \frac{l+1}{k+1}$ there is a unique generalized Cu-morphism $E_k \to E_l$ given by $x \mapsto ax$. Moreover, each such a map preserves the way-below relation and is therefore a Cu-morphism. The desired result follows. ☐

Example 5.19. It follows from Proposition 5.18 that there is a natural isomorphism

$$[\![\{0,\infty\},\{0,\infty\}]\!] \cong \{0,\infty\},$$

and more generally $[\![E_k,E_k]\!] \cong E_k$ for every $k \in \mathbb{N}$.

Recall the definition of axiom (O5) (almost algebraic order) from Paragraph 2.12. Note that E_k satisfies (O5) for every $k \in \mathbb{N}$. On the other hand, $[\![E_k,E_l]\!]$ does not satisfy (O5) whenever $l > k > 1$. For example, in $[\![E_2,E_3]\!] = \{0,2,3,\infty\}$ we have $2 \ll 2 \leq 3$, yet there is no element x with $2+x=3$. This shows that (O5) does not pass to abstract bivariant Cuntz semigroups.

Probably, (O6) (almost Riesz decomposition) does not pass to bivariant Cu-semigroups either, but we are currently not aware of any example.

5.3. **Algebraic** Cu**-semigroups.**

5.20. Recall that a Cu-semigroup S is *algebraic* if the subset S_c of compact elements is sup-dense in S, that is, if every element in S is the supremum of an increasing sequence of compact elements. We always endow S_c with the addition and partial order inherited from S, giving it the structure of a positively ordered monoid.

A natural source of algebraic semigroups comes from C^*-algebras of real rank zero. More concretely, the Cuntz semigroup of a C^*-algebra with real rank zero is always algebraic. Conversely, if a C^*-algebra A has stable rank one and $\mathrm{Cu}(A)$ is algebraic, then A has real rank zero; see [CEI08]. Another natural source comes from positively ordered monoids. Specifically, given any positively ordered monoid M, then its Cu-completion $\gamma(M,\leq)$ as constructed in [APT14, Proposition 3.1.6] (see also [APT14, 5.5.3]) is always an algebraic Cu-semigroup.

By [APT14, Proposition 5.5.5], there is an equivalence of the following categories: the category PoM of positively ordered monoids, and the full subcategory of Cu, consisting of algebraic Cu-semigroups. This means in particular that there is a natural bijection $\mathrm{Cu}(S,T) \cong \mathrm{PoM}(S_c, T_c)$, whenever S and T are algebraic Cu-semigroups.

It is natural to ask whether $[\![S,T]\!]$ is an algebraic Cu-semigroup whenever S and T are. In general, this is not the case. Consider for examples the Cuntz semigroups R_{2^∞} and R_{3^∞} of the UHF-algebras of type 2^∞ and 3^∞, respectively. Note that R_{2^∞} and R_{3^∞} are algebraic. However, by Proposition 5.15 we have $[\![R_{2^\infty}, R_{3^\infty}]\!] \cong \overline{\mathbb{P}}$, which is not algebraic.

Conversely, if $[\![S,T]\!]$ is algebraic, then it need not follow that S or T is algebraic. Indeed, while $\overline{\mathbb{P}}$ is not algebraic, we showed in Proposition 5.14 that $[\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!]$ is isomorphic to the algebraic Cu-semigroup M_1 from Example 4.14.

Thus, the natural problem to determine when an abstract bivariant Cuntz semigroup is algebraic has probably no simple answer.

Let $\mathrm{Cu}_{\mathrm{alg}}$ denote the full subcategory of Cu whose objects are algebraic Cu-semigroups. The tensor product of two algebraic Cu-semigroups is again algebraic, as shown in [APT14, Corollary 6.4.8]. Thus, $\mathrm{Cu}_{\mathrm{alg}}$ has a natural symmetric, monoidal structure.

Theorem 5.21. *The category* $\mathrm{Cu}_{\mathrm{alg}}$ *is closed. Given algebraic* Cu*-semigroups* S *and* T*, the internal-hom of* S *and* T *in* $\mathrm{Cu}_{\mathrm{alg}}$ *is the algebraic* Cu*-semigroup* $[\![S,T]\!]_{\mathrm{alg}}$ *given as the* Cu*-completion of the positively ordered monoid* $\mathrm{Cu}(S,T)$*:*

$$[\![S,T]\!]_{\mathrm{alg}} = \gamma\big(\mathrm{Cu}(S,T)\big).$$

Proof. Let S,T and R be algebraic Cu-semigroup. Then $R\otimes S$ is also algebraic, with $(R\otimes S)_c \cong R_c \otimes_{\mathrm{PoM}} S_c$, the latter tensor product being the one in the category of partially ordered monoids; see [APT14, Proposition 6.4.7 and Corollary 6.4.8]. We deduce that there are natural bijections

$$\begin{aligned}\mathrm{Cu}\big(R\otimes S, T\big) &\cong \mathrm{PoM}\big(R_c\otimes S_c, T_c\big) \cong \mathrm{PoM}\big(R_c, \mathrm{PoM}(S_c,T_c)\big)\\ &\cong \mathrm{PoM}\big(R_c, \mathrm{Cu}(S,T)\big) \cong \mathrm{Cu}\big(R, \gamma(\mathrm{Cu}(S,T))\big).\end{aligned}$$

Thus, the functor $\_ \otimes S\colon \mathrm{Cu}_{\mathrm{alg}} \to \mathrm{Cu}_{\mathrm{alg}}$ has a right adjoint given by the functor $\gamma(\mathrm{Cu}(S,\_))\colon \mathrm{Cu}_{\mathrm{alg}} \to \mathrm{Cu}_{\mathrm{alg}}$, as desired. □

5.4. **Bivariant Cuntz semigroups of ideals and quotients.** Let T be a Cu-semigroup, and let $S\subseteq T$ be a submonoid. We call S a *sub*-Cu*-semigroup* of T if S is a Cu-semigroup for the partial order inherited from T and such that the inclusion $S\to T$ is a Cu-morphism. It is easy to see that S is a sub-Cu-semigroup of T if and only if S is closed under passing to suprema of increasing sequences and if the way-below relation in S and T agree.

Lemma 5.22. *Let* S *and* T *be* Cu*-semigroups, and let* $T'\subseteq T$ *be a sub*-Cu*-semigroup. Then the inclusion* $\iota\colon T'\to T$ *induces an order-embedding* $\iota_*\colon [\![S,T']\!] \to [\![S,T]\!]$.

Proof. Let $\mathbf{f} = (f_\lambda)_\lambda$ be a path in $\mathrm{Cu}[S,T']$. Then $\tilde{\mathbf{f}} = (\iota\circ f_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S,T]$. It follows from Paragraph 5.5 that $\iota_*([\mathbf{f}]) = [\tilde{\mathbf{f}}]$.

To show that ι_* is an order-embedding, let $x,y\in [\![S,T']\!]$ with $\iota_*(x)\leq\iota_*(y)$. Choose paths $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{Cu}[S,T']$ representing x and y, respectively. We have $(\iota\circ f_\lambda)_\lambda \precsim (\iota\circ g_\lambda)_\lambda$. Thus, for every $\lambda\in I_{\mathbb{Q}}$, there is $\mu\in I_{\mathbb{Q}}$ such that $\iota\circ f_\lambda \prec \iota\circ g_\mu$. Using that $T'\subseteq T$ is a sub-Cu-semigroup, for such λ and μ we deduce that $f_\lambda\prec g_\mu$. (We use that for $a',a\in T'$ we have $a'\ll a$ in T' if and only if $\iota(a')\ll\iota(a)$ in T.) It follows that $\mathbf{f}\precsim\mathbf{g}$, and hence $x\leq y$, as desired. □

Recall that $J \subseteq S$ is called an *ideal* (of S) if J is a submonoid of S that is closed under passing to suprema of increasing sequences and that is downward-hereditary (if $a, b \in S$ satisfy $a \leq b$ and $b \in J$, then $a \in J$). We write $J \lhd S$ to mean that J is an ideal of S. Note that every ideal is in particular a sub-Cu-semigroup. (See [APT14, Section 5.1] for an account on ideals and quotients.)

Proposition 5.23. *Let S and T be* Cu*-semigroups, and let J be an ideal of T. Let $\iota\colon J \to T$ denote the inclusion map. Then the induced* Cu*-morphism $\iota_*\colon [\![S,J]\!] \to [\![S,T]\!]$ is an order-embedding that identifies $[\![S,J]\!]$ with an ideal of $[\![S,T]\!]$. Moreover, $x \in [\![S,T]\!]$ belongs to $[\![S,J]\!]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing x, each f_λ takes image in J.*

Proof. By Lemma 5.22, ι_* is an order-embedding. It follows that ι_* identifies $[\![S,J]\!]$ with a submonoid of $[\![S,T]\!]$ that is closed under passing to suprema of increasing sequences.

Let $x \in [\![S,T]\!]$ be represented by a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[S,T]$. If each f_λ takes values in J, then we can consider $\mathbf{f}$ as a path in $\mathrm{Cu}[S,J]$ whose class is an element $x' \in [\![S,J]\!]$ satisfying $\iota_*(x') = x$. Conversely, assume that x belongs to $[\![S,J]\!]$. Then there is a path $\mathbf{g} = (g_\mu)_\mu$ in $\mathrm{Cu}[S,J]$ with $\iota_*([\mathbf{g}]) = x$. Let $\lambda \in I_{\mathbb{Q}}$. Since $\mathbf{f} \precsim \mathbf{g}$, we can choose $\mu \in I_{\mathbb{Q}}$ with $f_\lambda \prec g_\mu$. Since g_μ takes values in J, and since J is downward-hereditary, it follows that f_λ takes values in J, as desired.

A similar argument shows that $[\![S,J]\!]$ is downward-hereditary in $[\![S,T]\!]$. $\square$

5.24. Given S, let us study whether the functor $[\![S,\_]\!]\colon \mathrm{Cu} \to \mathrm{Cu}$ is exact. More precisely, let $J \lhd T$ be an ideal, with inclusion map $\iota\colon J \to T$ and with quotient map $\pi\colon T \to T/J$. This induces the following Cu-morphisms:

$$[\![S,J]\!] \xrightarrow{\iota_*} [\![S,T]\!] \xrightarrow{\pi_*} [\![S,T/J]\!].$$

By Proposition 5.23, ι_* identifies $[\![S,J]\!]$ with an ideal in $[\![S,T]\!]$. Since $\pi \circ \iota$ is the zero map, so is $\pi_* \circ \iota_*$. Thus, π_* vanishes on the ideal $[\![S,J]\!] \lhd [\![S,T]\!]$. It follows that π_* induces a Cu-morphism

$$\widehat{\pi_*}\colon [\![S,T]\!]/[\![S,J]\!] \to [\![S,T/J]\!].$$

Problem 5.25. Study the order-theoretic properties of the Cu-morphism $\widehat{\pi_*}$ from Paragraph 5.24. In particular, when is $\widehat{\pi_*}$ an order-embedding, when is it surjective?

We are currently not aware of any example for S and $J \lhd T$ such that the map $\widehat{\pi_*}\colon [\![S,T]\!]/[\![S,J]\!] \to [\![S,T/J]\!]$ is not an isomorphism.

Remark 5.26. There are several possible concepts of projectivity for objects in Cu. One could say that a Cu-semigroup S is projective, if for every Cu-semigroup T and every ideal $J \lhd T$, with induced quotient map $\pi\colon T \to T/J$, and for every Cu-morphism $\varphi\colon S \to T/J$, there exists a Cu-morphism $\tilde{\varphi}\colon S \to T$ (called a lift) such that $\varphi = \pi \circ \tilde{\varphi}$. However, this concept seems very restrictive, since not even $\overline{\mathbb{N}}$ is projective in this sense.

A more promising approach is to say that S is projective, if for every T and every ideal $J \lhd T$, with induced quotient map $\pi\colon T \to T/J$, and for every $x \in [\![S,T/J]\!]$, there exists $\tilde{x} \in [\![S,T]\!]$ such that $x = \pi \circ \tilde{x}$. This means precisely that the map $\widehat{\pi_*}\colon [\![S,T]\!]/[\![S,J]\!] \to [\![S,T/J]\!]$ from Paragraph 5.24 is surjective for every $J \lhd T$.

In this sense, $\overline{\mathbb{N}}$ is projective. More generally, one can show that the product $\prod_I \overline{\mathbb{N}}$ is projective. The theory of products and, more generally, limits in Cu is developed in [APT17]. It follows that there are many projective objects in Cu, meaning that for every Cu-semigroup T there is a projective Cu-semigroup P and a surjective Cu-morphism $P \to T$. This could be the starting point to develop a theory of projective resolutions and derived functors, in particular an ext-functor in Cu. We will not pursue this line of thought here.

5.27. Let J be an ideal of T, and assume that J has a largest element z_J. (Every simple and every countably-based Cu-semigroup has a largest element. For Cuntz semigroups of C^*-algebras this is connected to the question whether the C^*-algebra contains a full element. See [APT14, Section 5.1] for more details.) Then, every element $a \in T/J$ has a largest preimage in T, given as $x + z_J$ for any choice of preimage x. Let $\omega\colon T/J \to T$ be the map that sends an element in T/J to its largest preimage in T. It is straightforward to check that ω is a generalized Cu-morphism. (However, in general, it does not preserve the way-below relation.) It is a natural question, closely related to Problem 5.25, to determine whether ω is the endpoint of a path in $\mathrm{Cu}[T/J, T]$.

The following result and its proof are analogous to Proposition 5.23.

Proposition 5.28. *Let S and T be* Cu*-semigroups, let $J \lhd S$, and let $\pi\colon S \to S/J$ denote the quotient map. Then the induced* Cu*-morphism $\pi^*\colon [\![S/J, T]\!] \to [\![S, T]\!]$ is an order-embedding that identifies $[\![S/J, T]\!]$ with an ideal in $[\![S, T]\!]$. Moreover, $x \in [\![S, T]\!]$ belongs to $[\![S/J, T]\!]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing x, each f_λ vanishes on J.*

5.29. By Propositions 5.23 and 5.28, ideals in S and T naturally induce ideals in $[\![S, T]\!]$. More precisely, if $J \lhd S$ and $K \lhd T$, we can identify $[\![S/J, K]\!]$ with an ideal in $[\![S, T]\!]$. Let $\mathrm{Lat}(P)$ denote the ideal lattice of a Cu-semigroup P. We therefore obtain a map

$$\mathrm{Lat}(S)^{\mathrm{op}} \times \mathrm{Lat}(T) \to \mathrm{Lat}([\![S, T]\!]).$$

However, this map need not be injective. For example, consider $S = Z$ and $T = \overline{\mathbb{N}} \oplus Z$ with the ideal $J = 0 \oplus Z$. Note that every generalized Cu-morphism $Z \to \overline{\mathbb{N}} \oplus Z$ necessarily takes values in the ideal $0 \oplus Z$. It follows that in this case $[\![S, J]\!] = [\![S, T]\!]$.

The following example shows that the above map is also not surjective in general. In fact, the example shows that there exists a simple Cu-semigroup S such that $[\![S, S]\!]$ is not simple.

Example 5.30. Let $S := [0, 1] \cup \{\infty\}$, considered with order and addition as a subset of $\overline{\mathbb{P}}$, with the convention that $a + b = \infty$ whenever $a + b > 1$ in $\overline{\mathbb{P}}$. It is easy to check that S is a simple Cu-semigroup.

Given $t \in \{0\} \cup [1, \infty]$, let $\varphi_t\colon S \to S$ be the map given by $\varphi_t(a) := ta$, where ta is given by the usual multiplication in $\overline{\mathbb{P}}$ applying the above convention that an element is ∞ as soon as it is larger than 1. Then φ_t is a generalized Cu-morphism. One can show that every generalized Cu-morphism $S \to S$ is of this form. We deduce that $\mathrm{Cu}[S, S]$ is isomorphic to $\{0\} \cup [1, \infty]$, identifying $\prec$ with $\leq$. It follows that

$$[\![S, S]\!] = \tau\big(\mathrm{Cu}[S, S], \prec\big) \cong \tau\big(\{0\} \cup [1, \infty], \leq\big) \cong \{0\} \sqcup [1, \infty] \sqcup (1, \infty],$$

which is a disjoint union of compact elements corresponding to $\{0\} \cup [1, \infty]$ and nonzero soft elements corresponding to $(1, \infty]$. (Similar to the decomposition of Z and R_q.)

In particular, $[\![S, S]\!]$ contains a compact infinite element ∞, and a noncompact infinite element ∞'. The set $J := \{x : x \leq \infty'\}$ is an ideal in $[\![S, S]\!]$. We have $\infty \notin J$, which shows that $[\![S, S]\!]$ is not simple.

Problem 5.31. Characterize when $[\![S, T]\!]$ is simple. In particular, given simple Cu-semigroups S and T, give necessary and sufficient criteria for $[\![S, T]\!]$ to be simple.

6. Concretization of categorical constructions for Cu

In this section, we give concrete pictures of general constructions in closed, symmetric, monoidal categories for the category Cu. This will be used in the next section. In Subsection 6.1 we study unit and counit maps, which are natural Cu-morphisms

$$\mathrm{d}_{S,T}\colon S \to [\![T, S\otimes T]\!] \quad \text{and} \quad \mathrm{e}_{S,T}\colon [\![S,T]\!]\otimes S \to T,$$

respectively. In the particular case that $T=\overline{\mathbb{N}}$, and after applying the isomorphism $S\otimes\overline{\mathbb{N}}\cong S$, the unit map $\mathrm{d}_{S,\overline{\mathbb{N}}}$ takes the form $S\to[\![\overline{\mathbb{N}}, S]\!]$ and is denoted by i_S. We will see that i_S is a natural isomorphism between S and $[\![\overline{\mathbb{N}}, S]\!]$.

In Subsection 6.2 we generalize the tensor product of Cu-morphisms (as defined in Paragraph 2.17) by introducing the *external tensor product map*

$$\boxtimes\colon [\![S_1,T_1]\!]\otimes[\![S_2,T_2]\!]\to[\![S_1\otimes S_2, T_1\otimes T_2]\!].$$

In Subsection 6.3, we study the composition product, which is the generalization of the composition of morphisms in a category to a notion of composition between internal-hom objects in a closed category; see the comments after Proposition 5.12. We then show that the external tensor product and the composition product are compatible; see Proposition 6.22.

In Subsection 6.4, we show how the unit and counit maps, the tensor and the composition product can be used to give concrete formulas for the correspondence

$$\mathrm{Cu}\big(S,[\![T,P]\!]\big)\cong\mathrm{Cu}\big(S\otimes T,P\big),$$

proved in Theorem 5.10; see Proposition 6.23.

We also study functorial properties of the endpoint map, and of the unit and counit maps. Finally, similar to KK-theory for C^*-algebras, we have a general form of the product that simultaneously generalizes the composition product and the external tensor product; see Paragraph 6.30.

6.1. **Unit and counit.**

Definition 6.1. Given Cu-semigroups S and T, we define the *unit map* as the Cu-morphism $\mathrm{d}_{S,T}\colon S\to[\![T,S\otimes T]\!]$ that under the identification

$$\mathrm{Cu}\big(S,[\![T,S\otimes T]\!]\big)\cong\mathrm{Cu}\big(S\otimes T,S\otimes T\big)$$

corresponds to the identity map on $S\otimes T$.

Proposition 6.2. *Let S and T be* Cu*-semigroups, and let $s\in S$. Let $(s_\lambda)_{\lambda\in I_{\mathbb{Q}}}$ be a path in $(S,\ll)$ with endpoint s. Then for each $\lambda\in I_{\mathbb{Q}}$, the map $s_\lambda\otimes\_\colon T\to S\otimes T$, sending $t\in T$ to $s_\lambda\otimes t$, is a generalized* Cu*-morphism. Moreover, $(s_\lambda\otimes\_)_{\lambda\in I_{\mathbb{Q}}}$ is a path in $(\mathrm{Cu}[T,S\otimes T],\prec)$, and we have $\mathrm{d}_S^T(s)=[(s_\lambda\otimes\_)_\lambda]$.*

Proof. The map $\omega\colon S\times T\to S\otimes T$, given by $\omega(a,b)=a\otimes b$, is a Cu-bimorphism. This implies that $a\otimes\_\colon T\to S\otimes T$ is a generalized Cu-morphism for each $a\in S$. Moreover, using that ω preserves the joint way-below relation, we obtain that $a'\otimes\_\prec a\otimes\_$ for $a',a\in S$ satisfying $a'\ll a$. In particular, if $(a_\lambda)_{\lambda\in I_{\mathbb{Q}}}$ is a path in S, then $(a_\lambda\otimes\_)_{\lambda\in I_{\mathbb{Q}}}$ is a path in $(\mathrm{Cu}[T,S\otimes T],\prec)$. We define $\alpha\colon S\to[\![T,S\otimes T]\!]$ by sending $a\in S$ to $[(a_\lambda\otimes\_)_\lambda]$ for some choice of path $(a_\lambda)_\lambda$ in S with endpoint a. It is straightforward to check that α is a well-defined Cu-morphism.

Let us show that $\alpha=\mathrm{d}_{S,T}$. Consider the bijections

$$\mathrm{Cu}\big(S,[\![T,S\otimes T]\!]\big)\cong\mathrm{BiCu}\big(S\times T,S\otimes T\big)\cong\mathrm{Cu}\big(S\otimes T,S\otimes T\big)$$

from Theorem 5.10. Under the first bijection, α corresponds to the Cu-bimorphism $\bar\alpha$ given by

$$\bar\alpha(a,b)=\sigma_{T,S\otimes T}(\alpha(a))(b),$$

for $a \in S$ and $b \in T$, where $\sigma_{T,S\otimes T}$ is the endpoint map. We compute

$$\bar{\alpha}(a,b) = \sigma_{T,S\otimes T}(\alpha(a))(b) = \sup_{\lambda\in I_{\mathbb{Q}}} (a_\lambda \otimes \_)(b) = \sup_{\lambda\in I_{\mathbb{Q}}} (a_\lambda \otimes b) = a \otimes b,$$

for every path $(a_\lambda)_\lambda$ with endpoint $a \in S$, and every $b \in T$. It follows that $\bar{\alpha}$ corresponds to $\mathrm{id}_{S\otimes T}$ under the second bijection. By definition of $\mathrm{d}_{S,T}$, this show that $\alpha = \mathrm{d}_{S,T}$, as desired. ☐

Notation 6.3. Given Cu-semigroups S, T and P, and a Cu-bimorphism $\alpha\colon S \times T \to P$, we shall often use the notation $\bar{\alpha}\colon S \to [\![T, P]\!]$ to refer to the Cu-morphism that corresponds to α under the identification in Theorem 5.10.

Corollary 6.4. *Let S and T be Cu-semigroups. Then the composition*

$$\sigma_{T,S\otimes T} \circ \mathrm{d}_{S,T}\colon S \xrightarrow{\mathrm{d}_{S,T}} [\![T, S\otimes T]\!] \xrightarrow{\sigma_{T,S\otimes T}} \mathrm{Cu}[T, S \otimes T].$$

satisfies $(\sigma_{T,S\otimes T} \circ \mathrm{d}_{S,T}(s) = s \otimes \_$, for every $s \in S$. In particular

$$(\sigma_{T,S\otimes T} \circ \mathrm{d}_{S,T}(s)(t) = s \otimes t,$$

for $s \in S$ and $t \in T$.

Proof. Let $s \in S$ and $t \in T$. Choose a path $(s_\lambda)_\lambda$ in S with endpoint s. Then $\mathrm{d}_{S,T}(s) = [(s_\lambda \otimes \_)_\lambda]$ by Proposition 6.2. The supremum of the maps $s_\lambda \otimes \_$ in $\mathrm{Cu}[S, T \otimes S]$ is the map $s \otimes \_$. Thus, $(\sigma T, S \otimes T \circ \mathrm{d}_{S,T}(s) = s \otimes \_$, as desired. ☐

Definition 6.5. Given Cu-semigroups S and T, recall that the *counit map* (also called the *evaluation map*) is defined as the Cu-morphism $\mathrm{e}\, S, T\colon [\![S, T]\!] \otimes S \to T$ that under the identification

$$\mathrm{Cu}\big([\![S, T]\!] \otimes S, T\big) \cong \mathrm{Cu}\big([\![S, T]\!], [\![S, T]\!]\big)$$

corresponds to the identity map on $[\![S, T]\!]$. (See Paragraph 2.14 and the comments after Proposition 5.12.) Given $x \in [\![S, T]\!]$ and $s \in S$, we denote $\mathrm{e}_{S,T}(x \otimes s)$ by $x(s)$.

Proposition 6.6. *Let S and T be Cu-semigroups, let $x \in [\![S, T]\!]$, and let $s \in S$. Then $\mathrm{e}_{S,T}(x \otimes s) = \sigma_{S,T}(x)(s)$. Thus, if $\mathbf{f} = (f_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S, T]$, then*

$$[\mathbf{f}](s) = \mathrm{e}_{S,T}([\mathbf{f}] \otimes s) = \sup_{\lambda<1} f_\lambda(s).$$

Proof. Consider the bijections

$$\mathrm{Cu}\big([\![S, T]\!], [\![S, T]\!]\big) \cong \mathrm{BiCu}\big([\![S, T]\!] \times S, T\big) \cong \mathrm{Cu}\big([\![S, T]\!] \otimes S, T\big)$$

from Theorem 5.10. To simplify notation, we denote the identity map on $[\![S, T]\!]$ by id. Under the first bijection, id corresponds to the Cu-bimorphism $\bar{\mathrm{id}}$ satisfying

$$\bar{\mathrm{id}}(y, b) = \sigma_{S,T}(\mathrm{id}(y))(b),$$

for all $y \in [\![S, T]\!]$ and $b \in S$. We obtain that

$$\mathrm{e}_{S,T}(x \otimes s) = \bar{\mathrm{id}}(x, s) = \sigma_{S,T}(\mathrm{id}(x))(s) = \sigma_{S,T}(x)(s),$$

as desired. ☐

Remark 6.7. Let $\varphi\colon S \to T$ be a Cu-morphism, and let $s \in S$. Considering φ as an element of $[\![S, T]\!]$, the notation $\varphi(s)$ for $\mathrm{e}_{S,T}(\varphi \otimes s)$ is consistent with the usual notation of $\varphi(s)$ for the evaluation of φ at s.

Lemma 6.8. *Let S be a Cu-semigroup. Let $\mathrm{ev}_1\colon \mathrm{Cu}[\overline{\mathbb{N}}, S] \to S$ be given by $\mathrm{ev}_1(f) = f(1)$ for $f \in \mathrm{Cu}[\overline{\mathbb{N}}, S]$. Then ev_1 is an isomorphism of $\mathcal{Q}$-semigroups. That is, ev_1 is an additive order-isomorphism and we have $f \prec g$ if and only if $\mathrm{ev}_1(f) \ll \mathrm{ev}_1(g)$, for $f, g \in \mathrm{Cu}[\overline{\mathbb{N}}, S]$.*

It follows that $(\mathrm{Cu}[\overline{\mathbb{N}}, S], \prec)$ is a Cu-semigroup (naturally isomorphic to S via ev_1). Moreover, the endpoint map $\sigma_{\overline{\mathbb{N}},S}\colon [\![\overline{\mathbb{N}}, S]\!] \to \mathrm{Cu}[\overline{\mathbb{N}}, S]$ from Definition 5.6 is *an isomorphism. We obtain a commutative diagram of* Cu*-semigroups and isomorphisms:*

$$\begin{tikzcd} [\![\overline{\mathbb{N}}, S]\!] = \tau\big(\mathrm{Cu}[\overline{\mathbb{N}}, S]\big) \arrow[r, "\sigma_{\overline{\mathbb{N}},S}", "\cong"'] \arrow[d, "\tau(\mathrm{ev}_1)"', "\cong"] & \mathrm{Cu}[\overline{\mathbb{N}}, S] \arrow[d, "\cong"', "\mathrm{ev}_1"] \\ \tau(S) \arrow[r, "\cong", "\varphi_S"'] & S \end{tikzcd}$$

Proof. It is straightforward to prove that ev_1 is an isomorphism of $\mathcal{Q}$-semigroups. By Proposition 4.10, the endpoint map of a Cu-semigroup is an isomorphism. Thus, the endpoint maps φ_S and $\varphi_{\mathrm{Cu}[\overline{\mathbb{N}},S]}$ are isomorphisms. By definition, $\sigma_{\overline{\mathbb{N}},S} = \varphi_{\mathrm{Cu}[\overline{\mathbb{N}},S]}$. Since ev_1 is an isomorphism, so is $\tau(\mathrm{ev}_1)$. By Proposition 4.8 the endpoint map is natural, which implies that the diagram is commutative. □

Definition 6.9. Given a Cu-semigroup S, we let $\mathrm{i}_S\colon S \to [\![\overline{\mathbb{N}}, S]\!]$ be the Cu-morphism that under the identification

$$\mathrm{Cu}\big(S, [\![\overline{\mathbb{N}}, S]\!]\big) \cong \mathrm{Cu}\big(S \otimes \overline{\mathbb{N}}, S\big)$$

corresponds to the natural isomorphism $\mathrm{r}_S\colon S \otimes \overline{\mathbb{N}} \to S$.

We leave the proof of the following result to the reader.

Proposition 6.10. *Let S be a* Cu*-semigroup. Then the map $\mathrm{i}_S\colon S \to [\![\overline{\mathbb{N}}, S]\!]$ from Definition 6.9 is an isomorphism. The inverse of i_S is $\mathrm{ev}_1 \circ \sigma_{\overline{\mathbb{N}},S}$, where ev_1 is evaluation at 1 as in Lemma 6.8, and where $\sigma_{\overline{\mathbb{N}},S}\colon [\![\overline{\mathbb{N}}, S]\!] \to \mathrm{Cu}[\overline{\mathbb{N}}, S]$ denotes the endpoint map from Definition 5.6. Moreover, we have $\mathrm{i}_S = (\mathrm{r}_S)_* \circ \mathrm{d}_{S,\overline{\mathbb{N}}}$.*

6.2. **External tensor product.**

6.11. Let S_k and T_k be Cu-semigroups, and let $\varphi_k\colon S_k \to T_k$ be (generalized) Cu-morphisms, for $k = 1, 2$. Recall from Paragraph 2.17 that the map $\varphi_1 \times \varphi_2\colon S_1 \times S_2 \to T_1 \otimes T_2$, defined by

$$(\varphi_1 \times \varphi_2)(s_1, s_2) := f_1(s_1) \otimes f_2(s_2),$$

for $s_1 \in S_1$ and $s_2 \in S_2$, is (generalized) Cu-bimorphism. We denote the induced (generalized) Cu-morphism by $\varphi_1 \otimes \varphi_2\colon S_1 \otimes S_2 \to T_1 \otimes T_2$ and we call the map $\varphi_1 \otimes \varphi_2$, as customary, the *tensor product* of φ_1 and φ_2.

Next, we generalize this construction and define an external tensor product between elements of internal-homs.

Definition 6.12. Given Cu-semigroups S_1, S_2, T_1 and T_2, we define the *external tensor product map* $\boxtimes\colon [\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$ as the Cu-morphism that under the identification

$$\begin{aligned} &\mathrm{Cu}\big([\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!], [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]\big) \\ &\quad \cong \mathrm{Cu}\big([\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \otimes S_1 \otimes S_2, T_1 \otimes T_2\big), \end{aligned}$$

corresponds to the composition

$$(\mathrm{e}_{S_1,T_1} \otimes \mathrm{e}_{S_2,T_2}) \circ (\mathrm{id}_{[\![S_1,T_1]\!]} \otimes \sigma \otimes \mathrm{id}_{S_2}),$$

where $\sigma\colon [\![S_2, T_2]\!] \otimes S_1 \to S_1 \otimes [\![S_2, T_2]\!]$ denotes the flip isomorphism.

Given $x_1 \in [\![S_1, T_1]\!]$ and $x_2 \in [\![S_2, T_2]\!]$, we denote the image of $x_1 \otimes x_2$ under this map by $x_1 \boxtimes x_2$, and we call it the *external tensor product* of x_1 and x_2.

Remark 6.13. Let $\varphi_1\colon S_1 \to T_1$ and $\varphi_2\colon S_2 \to T_2$ be Cu-morphisms. Using Proposition 5.12, we identify φ_1 with a compact element in $[\![S_1, T_1]\!]$, and similarly for φ_2. It is easy to see that the element $\varphi_1 \boxtimes \varphi_2$ from Definition 6.12 agrees with the compact element in $[\![S_1\otimes S_2, T_1\otimes T_2]\!]$ that is identified with the tensor product map $\varphi_1\otimes\varphi_2\colon S_1\otimes S_2\to T_1\otimes T_2$ from Paragraph 6.11.

Notice that there is a certain ambiguity with the notation $\varphi_1\otimes\varphi_2$, in that it may refer to a Cu-morphism (identified with a compact element in $[\![S_1\otimes S_2, T_1\otimes T_2]\!]$), and also to an element in $[\![S_1,T_1]\!]\otimes[\![S_2,T_2]\!]$. However, the precise meaning will be clear from the context.

Proposition 6.14. *Let S_1, S_2, T_1 and T_2 be* Cu*-semigroups, and let $\mathbf{f}=(f_\lambda)_\lambda$ and $\mathbf{g}=(g_\lambda)_\lambda$ be paths in* $\mathrm{Cu}[S_1,T_1]$ *and* $\mathrm{Cu}[S_2,T_2]$*, respectively. For each λ, consider the generalized* Cu*-morphism $f_\lambda\otimes g_\lambda\colon S_1\otimes S_2\to T_1\otimes T_2$. Then $(f_\lambda\otimes g_\lambda)_\lambda$ is a path in* $\mathrm{Cu}[S_1\otimes S_2, T_1\otimes T_2]$ *and we have*

$$[\mathbf{f}]\boxtimes[\mathbf{g}] = [(f_\lambda\otimes g_\lambda)_\lambda].$$

Proof. To show that $(f_\lambda\otimes g_\lambda)_\lambda$ is a path, let $\lambda',\lambda\in I_{\mathbb{Q}}$ satisfy $\lambda'<\lambda$. To show that $f_{\lambda'}\otimes g_{\lambda'}\prec f_\lambda\otimes g_\lambda$, let $t',t\in S_1\otimes S_2$ satisfy $t'\ll t$. By properties of the tensor product in Cu, we can choose $n\in\mathbb{N}$, elements $a_k',a_k\in S_1$ and $b_k',b_k\in S_2$ satisfying $a_k'\ll a_k$ and $b_k'\ll b_k$ for $k=1,\ldots,n$, and such that

$$t'\leq\sum_{k=1}^n a_k'\otimes b_k', \quad\text{and}\quad \sum_{k=1}^n a_k\otimes b_k\leq t.$$

We have $f_{\lambda'}\prec f_\lambda$ and $g_{\lambda'}\prec g_\lambda$, and therefore $f_{\lambda'}(a_k')\ll f_\lambda(a_k)$ and $g_{\lambda'}(b_k')\ll g_\lambda(b_k)$ for $k=1,\ldots,n$. Using this at the third step we deduce that

$$\begin{aligned}
(f_{\lambda'}\otimes g_{\lambda'})(t') &\leq (f_{\lambda'}\otimes g_{\lambda'})\left(\sum_{k=1}^n a_k'\otimes b_k'\right)\\
&= \sum_{k=1}^n f_{\lambda'}(a_k')\otimes g_{\lambda'}(b_k')\\
&\ll \sum_{k=1}^n f_{\lambda}(a_k)\otimes g_{\lambda}(b_k)\\
&= (f_\lambda\otimes g_\lambda)\left(\sum_{k=1}^n a_k\otimes b_k\right)\leq (f_\lambda\otimes g_\lambda)(t),
\end{aligned}$$

as desired.

Thus, given paths $\mathbf{p}=(p_\lambda)_\lambda$ and $\mathbf{q}=(q_\lambda)_\lambda$ in $\mathrm{Cu}[S_1,T_1]$ and $\mathrm{Cu}[S_2,T_2]$, respectively, then $(p_\lambda\otimes q_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S_1\otimes S_2,T_1\otimes T_2]$. Moreover, it is tedious but straightforward to check that the map $[\![S_1,T_1]\!]\times[\![S_2,T_2]\!]\to[\![S_1\otimes S_2,T_1\otimes T_2]\!]$ that sends a pair $([\mathbf{p}]\otimes[\mathbf{q}])$ to $[(p_\lambda\boxtimes q_\lambda)_\lambda]$ is a well-defined Cu-bimorphism. We let $\alpha\colon[\![S_1,T_1]\!]\otimes[\![S_2,T_2]\!]\to[\![S_1\otimes S_2,T_1\otimes T_2]\!]$ be the induced Cu-morphism.

To show that $[\mathbf{f}]\boxtimes[\mathbf{g}]=[(f_\lambda\otimes g_\lambda)_\lambda]$, we will prove that the external tensor product $\boxtimes$ and the map α correspond to the same Cu-morphism under the bijection

$$\begin{aligned}
&\mathrm{Cu}\big([\![S_1,T_1]\!]\otimes[\![S_2,T_2]\!], [\![S_1\otimes S_2,T_1\otimes T_2]\!]\big)\\
&\quad\cong \mathrm{Cu}\big([\![S_1,T_1]\!]\otimes[\![S_2,T_2]\!]\otimes S_1\otimes S_2, T_1\otimes T_2\big)
\end{aligned}$$

from Theorem 5.10.

Let $\mathbf{p}=(p_\lambda)_\lambda$ and $\mathbf{q}=(q_\lambda)_\lambda$ be paths in $\mathrm{Cu}[S_1,T_1]$ and $\mathrm{Cu}[S_2,T_2]$, respectively, and let s_i be elements in S_i, for $i=1,2$. By definition of $\boxtimes$ (see Definition 6.12 and Notation 6.3), we have

$$\bar{\boxtimes}([\mathbf{p}]\otimes[\mathbf{q}]\otimes s_1\otimes s_2) = [\mathbf{p}](s_1)\otimes[\mathbf{q}](s_2) = p_1(s_1)\otimes q_1(s_2).$$

Using Theorem 5.10 at the first step, we obtain that

$$\begin{aligned}\bar{\alpha}([\mathbf{p}]\otimes[\mathbf{q}]\otimes s_1\otimes s_2) &= \sigma_{S_1\otimes S_2,T_1\otimes T_2}(\alpha([\mathbf{p}]\otimes[\mathbf{q}]))(s_1\otimes s_2)\\ &= \sigma_{S_1\otimes S_2,T_1\otimes T_2}([(p_\lambda\otimes q_\lambda)_\lambda])(s_1\otimes s_2)\\ &= (p_1\otimes q_1)(s_1\otimes s_2) = p_1(s_1)\otimes q_1(s_2).\end{aligned}$$

It follows that $\boxtimes = \alpha$ and therefore

$$[\mathbf{f}]\boxtimes[\mathbf{g}] = \boxtimes([\mathbf{f}]\otimes[\mathbf{g}]) = \alpha([\mathbf{f}]\otimes[\mathbf{g}]) = [(f_\lambda\otimes g_\lambda)_\lambda],$$

as desired. □

The following result shows that the external tensor product is associative.

Proposition 6.15. *Let S_1, S_2, T_1, T_2, P_1 and P_2 be* Cu*-semigroups, let $x \in [\![S_1, S_2]\!]$, $y \in [\![T_1, T_2]\!]$, and let $z \in [\![P_1, P_2]\!]$. For $k = 1, 2$, we identify $(S_k \otimes T_k) \otimes P_k$ with $S_k \otimes (T_k \otimes P_k)$ using the natural isomorphism from the monoidal structure of* Cu *(see Paragraph 2.17). Then*

$$(x \boxtimes y) \boxtimes z = x \boxtimes (y \boxtimes z).$$

Proof. Given $f \in \mathrm{Cu}[S_1, S_2]$, $g \in \mathrm{Cu}[T_1, T_2]$ and $h \in \mathrm{Cu}[P_1, P_2]$, it is straightforward to check that

$$(f \otimes g) \otimes h = f \otimes (g \otimes h),$$

as generalized Cu-morphisms $S_1 \otimes T_1 \otimes P_1 \to S_2 \otimes T_2 \otimes P_2$. The result follows by applying Proposition 6.14. □

Problem 6.16. Study the order-theoretic properties of the external tensor product map $\boxtimes\colon [\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$. In particular, when is this map an order-embedding, when is it surjective?

6.3. **Composition product.**

Definition 6.17. Given Cu-semigroups S, T and P, we define the *composition product*

$$\circ\colon [\![T, P]\!] \otimes [\![S, T]\!] \to [\![S, P]\!]$$

as the Cu-morphism that under the identification

$$\mathrm{Cu}\big([\![T, P]\!] \otimes [\![S, T]\!], [\![S, P]\!]\big) \cong \mathrm{Cu}\big([\![T, P]\!] \otimes [\![S, T]\!] \otimes S, P\big)$$

corresponds to the composition $\mathrm{e}_{T,P} \circ (\mathrm{id}_{[\![T,P]\!]} \otimes \mathrm{e}_{S,T})$. Given $x \in [\![S, T]\!]$ and $y \in [\![T, P]\!]$, we denote the image of $y \otimes x$ under the composition product by $y \circ x$.

Given $x \in [\![S, T]\!]$, we let $x^*\colon [\![T, P]\!] \to [\![S, P]\!]$ be given by $x^*(y) := y \circ x$ for $y \in [\![T, P]\!]$. Analogously, given $y \in [\![T, P]\!]$, we let $y_*\colon [\![S, T]\!] \to [\![S, P]\!]$ be given by $y_*(x) := y \circ x$ for $x \in [\![S, T]\!]$.

Proposition 6.18. *Let S, T and P be* Cu*-semigroups, and let $\mathbf{f} = (f_\lambda)_\lambda$ and $\mathbf{g} = (g_\lambda)_\lambda$ be paths in* $\mathrm{Cu}[S, T]$ *and* $\mathrm{Cu}[T, P]$*, respectively. For each λ, consider the generalized* Cu*-morphism $g_\lambda \circ f_\lambda\colon S \to P$. Then $(g_\lambda \circ f_\lambda)_\lambda$ is a path in* $\mathrm{Cu}[S, P]$ *and*

$$[\mathbf{g}] \circ [\mathbf{f}] = [(g_\lambda \circ f_\lambda)_\lambda].$$

Proof. It is easy to check that $(g_\lambda \circ f_\lambda)_\lambda$ is a path. Moreover, it is tedious but straightforward to check that the map $[\![T, P]\!] \times [\![S, T]\!] \to [\![S, P]\!]$ that sends a pair $([\mathbf{p}], [\mathbf{q}])$ to $[(q_\lambda \circ p_\lambda)_\lambda]$ is a well-defined Cu-bimorphism. We let $\alpha\colon [\![T, P]\!] \otimes [\![S, T]\!] \to [\![S, P]\!]$ be the induced Cu-morphism.

To show that $[\mathbf{g}] \circ [\mathbf{f}] = [(g_\lambda \otimes f_\lambda)_\lambda]$, we will prove that the composition product $\circ$ and the map α correspond to the same Cu-morphism under the bijection

$$\mathrm{Cu}\big([\![T, P]\!] \otimes [\![S, T]\!], [\![S, P]\!]\big) \cong \mathrm{Cu}\big([\![T, P]\!] \otimes [\![S, T]\!] \otimes S, P\big)$$

from Theorem 5.10.

Let $\mathbf{p} = (p_\lambda)_\lambda$ and $\mathbf{q} = (q_\lambda)_\lambda$ be paths in $\mathrm{Cu}[S,T]$ and $\mathrm{Cu}[T,P]$, respectively, and let $s \in S$. Set $p_1 := \sup_{\lambda<1} p_\lambda$ and $q_1 := \sup_{\lambda<1} q_\lambda$. By definition, we have

$$\begin{aligned}\bar{\circ}([\mathbf{q}] \otimes [\mathbf{p}] \otimes s) &= \mathrm{e}_{T,P} \circ (\mathrm{id}_{[\![T,P]\!]} \otimes \mathrm{e}_{S,T})([\mathbf{q}] \otimes [\mathbf{p}] \otimes s) \\ &= \mathrm{e}_{T,P}([\mathbf{q}] \otimes \mathrm{e}_{S,T}([\mathbf{p}] \otimes s)) \\ &= \mathrm{e}_{T,P}([\mathbf{q}] \otimes p_1(s)) \\ &= q_1(p_1(s)).\end{aligned}$$

On the other hand, using Theorem 5.10 at the first step, we obtain that

$$\begin{aligned}\bar{\alpha}([\mathbf{q}] \otimes [\mathbf{p}] \otimes s) &= \sigma_{S,P}(\alpha([\mathbf{q}] \otimes [\mathbf{p}]))(s) \\ &= \sigma_{S,P}([(q_\lambda \circ p_\lambda)_\lambda])(s) \\ &= (q_1 \circ p_1)(s) \\ &= q_1(p_1(s)).\end{aligned}$$

It follows that $\circ = \alpha$ and therefore

$$[\mathbf{g}] \circ [\mathbf{f}] = \circ([\mathbf{g}] \otimes [\mathbf{f}]) = \alpha([\mathbf{g}] \otimes [\mathbf{f}]) = [(g_\lambda \otimes f_\lambda)_\lambda],$$

as desired. $\square$

Note that, in Proposition 6.18, the composition product of two Cu-morphisms, viewed as compact elements in the internal-hom set, is the usual composition of morphisms as maps.

The following result shows that the composition product is associative and that the identity element $\mathrm{id}_S \in \mathrm{Cu}(S,S) \subseteq [\![S,S]\!]$ acts as a unit for the composition product (as expected). It follows that $[\![S,S]\!]$ and $[\![T,T]\!]$ are (not necessarily commutative) Cu-semirings and that $[\![S,T]\!]$ has a natural left $[\![S,S]\!]$- and right $[\![T,T]\!]$-semimodule structure; see Propositions 7.1 and 7.7 in the next section.

Proposition 6.19. *Let S, T, P and Q be* Cu*-semigroups, let $x \in [\![S,T]\!]$, $y \in [\![T,P]\!]$, and let $z \in [\![P,Q]\!]$. Then*

$$(z \circ y) \circ x = z \circ (y \circ x).$$

Further, for the identity Cu*-morphisms $\mathrm{id}_S \in \mathrm{Cu}(S,S)$ and $\mathrm{id}_T \in \mathrm{Cu}(T,T)$, we have*

$$\mathrm{id}_T \circ x = x = x \circ \mathrm{id}_S .$$

Proof. Given $f \in \mathrm{Cu}[S,T]$, $g \in \mathrm{Cu}[T,P]$ and $h \in \mathrm{Cu}[P,Q]$, it is straightforward to check that

$$(h \circ g) \circ f = h \circ (g \circ f),$$

in $\mathrm{Cu}[S,Q]$. The result follows by applying Proposition 6.18. The statement about the composition with id_S and id_T follows also directly from Proposition 6.18. $\square$

Next we show that the composition product is compatible with the evaluation map in the expected way. It will follow later that the evaluation map $\mathrm{e}_S^S\colon [\![S,S]\!] \otimes S \to S$ defines a natural left $[\![S,S]\!]$-semimodule structure on S; see Proposition 7.3.

Lemma 6.20. *Let S, T and P be* Cu*-semigroups, let $x \in [\![S,T]\!]$, and let $y \in [\![T,P]\!]$. Then*

$$\sigma_{S,P}(y \circ x) = \sigma_{T,P}(y) \circ \sigma_{S,T}(x).$$

Proof. Let $\mathbf{f} = (f_\lambda)_\lambda$ be a path in $\mathrm{Cu}[S,T]$ representing x, and let $\mathbf{g} = (g_\lambda)_\lambda$ be a path in $\mathrm{Cu}[T,P]$ representing y. Let $s \in S$. By Proposition 6.18, we have $y \circ x = [(g_\lambda \circ f_\lambda)_\lambda]$. Using this at the first step, we obtain that

$$\sigma_{S,P}(y \circ x)(s) = \sup_{\lambda \in I_\mathbb{Q}} (g_\lambda \circ f_\lambda)(s) = \sup_{\mu \in I_\mathbb{Q}} g_\mu \left(\sup_{\lambda \in I_\mathbb{Q}} f_\lambda(s) \right) = \sigma_{T,P}(y) \left(\sigma_{S,T}(x)(s) \right),$$

as desired. □

By combining Lemma 6.20 with Proposition 6.6, we obtain:

Proposition 6.21. *Let S, T and P be* Cu*-semigroups, let $x \in [\![S,T]\!]$, let $y \in [\![T,P]\!]$, and let $s \in S$. Then*

$$(y \circ x)(s) = y(x(s)).$$

Moreover, for the identity Cu*-morphism $\mathrm{id}_S \in \mathrm{Cu}(S,S)$, we have $\mathrm{id}_S(s) = s$.*

The following result shows that the external tensor product and the composition product commute.

Proposition 6.22. *Let S_1, S_2, T_1 and T_2 be* Cu*-semigroups. Given $x_k \in [\![S_k, T_k]\!]$ and $y_k \in [\![T_k, P_k]\!]$ for $k = 1, 2$, we have*

$$(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1).$$

Proof. Let $\mathbf{f^{(k)}} = (f_\lambda^{(k)})_\lambda$ be a path in $\mathrm{Cu}[S_k, T_k]$ representing x_k, for $k = 1, 2$, and let $\mathbf{g^{(k)}} = (g_\lambda^{(k)})_\lambda$ be a path in $\mathrm{Cu}[T_k, P_k]$ representing y_k, for $k = 1, 2$. Given λ, it is straightforward to check that

$$(g_\lambda^{(2)} \otimes g_\lambda^{(1)}) \circ (f_\lambda^{(2)} \otimes f_\lambda^{(1)}) = (g_\lambda^{(2)} \circ f_\lambda^{(2)}) \otimes (g_\lambda^{(1)} \circ f_\lambda^{(1)}).$$

Using this at the second step, and using Proposition 6.14 and Proposition 6.18 at the first and last step, we obtain that

$$\begin{aligned} (y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) &= \left[\left((g_\lambda^{(2)} \otimes g_\lambda^{(1)}) \circ (f_\lambda^{(2)} \otimes f_\lambda^{(1)}) \right)_\lambda \right] \\ &= \left[\left((g_\lambda^{(2)} \circ f_\lambda^{(2)}) \otimes (g_\lambda^{(1)} \circ f_\lambda^{(1)}) \right)_\lambda \right] \\ &= (y_2 \circ x_2) \boxtimes (y_1 \circ x_1), \end{aligned}$$

as desired. □

6.4. Implementation of the adjunction using unit and counit.

Proposition 6.23. *Let S, T and P be* Cu*-semigroups. Then the bijection*

$$\mathrm{Cu}\big(S, [\![T,P]\!]\big) \cong \mathrm{Cu}\big(S \otimes T, P\big)$$

from Theorem 5.10 identifies a Cu*-morphism $f \colon S \to [\![T,P]\!]$ with*

$$\mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T) \colon S \otimes T \xrightarrow{f \otimes \mathrm{id}_T} [\![T,P]\!] \otimes T \xrightarrow{\mathrm{e}_{T,P}} P.$$

Conversely, a Cu*-morphism $g \colon S \otimes T \to P$ is identified with*

$$g_* \circ \mathrm{d}_{S,T} \colon S \xrightarrow{\mathrm{d}_{S,T}} [\![T, S \otimes T]\!] \xrightarrow{g_*} [\![T,P]\!].$$

In particular, we have

$$f = \left(\mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T)\right)_* \circ \mathrm{d}_{S,T}, \quad \textit{and} \quad g = \mathrm{e}_{T,P} \circ ((g_* \circ \mathrm{d}_{S,T}) \otimes \mathrm{id}_T).$$

Proof. Let $f \colon S \to [\![T,P]\!]$ be a Cu-morphism. Under the natural bijection from Theorem 5.10, f corresponds to the Cu-morphism $\bar{f} \colon S \otimes T \to P$ with

$$\bar{f}(s \otimes t) = \sigma_P^T(f(s))(t),$$

for a simple tensor $s\otimes t\in S\otimes T$. On the other hand, we have

$$(\mathrm{e}_{T,P}\circ(f\otimes\mathrm{id}_T))(s\otimes t)=\mathrm{e}_{T,P}(f(s)\otimes t)=\sigma_{T,P}(f(s))(t),$$

for a simple tensor $s\otimes t\in S\otimes T$. Thus $\bar f$ and $\mathrm{e}_{T,P}\circ(f\otimes\mathrm{id}_T)$ agree on simple tensors, and consequently $\bar f=\mathrm{e}_{T,P}\circ(f\otimes\mathrm{id}_T)$, as desired.

Let $g\colon S\otimes T\to P$ be a Cu-morphism. Set $\alpha:=g_*\circ\mathrm{d}_{S,T}$. Under the natural bijection from Theorem 5.10, α corresponds to the Cu-morphism $\bar\alpha\colon S\otimes T\to P$ with

$$\bar\alpha(s\otimes t)=\sigma_{T,P}(\alpha(s))(t),$$

for a simple tensor $s\otimes t\in S\otimes T$. It is straightforward to verify that $\sigma_{T,P}\circ g_*=g_*\circ\sigma_{T,S\otimes T}$. (See Proposition 6.25.) Using this at the third step, and using Corollary 6.4 at the fourth step, we deduce that

$$\begin{aligned}\bar\alpha(s\otimes t)=\sigma_{T,P}(\alpha(s))(t)&=(\sigma_{T,P}\circ g_*\circ\mathrm{d}_{S,T})(s)(t)\\&=(g_*\circ\sigma_{T,S\otimes P}\circ\mathrm{d}_{S,T})(s)(t)=g(s\otimes t),\end{aligned}$$

for every simple tensor $s\otimes t\in S\otimes T$. Thus, $\bar\alpha=g$, as desired. $\square$

Applying the previous result to the identity morphisms, we obtain:

Corollary 6.24. *Let S and T be* Cu*-semigroups. Then*

$$\mathrm{id}_{[\![S,T]\!]}=(\mathrm{e}^T_S)_*\circ\mathrm{d}^S_{[\![S,T]\!]},\quad\textit{and}\quad\mathrm{id}_{S\otimes T}=\mathrm{e}^{S\otimes T}_T\circ(\mathrm{d}^T_S\otimes\mathrm{id}_T).$$

6.5. **Functorial properties.** Next, we study the functorial properties of the endpoint map, the unit map, and the counit map.

Proposition 6.25. *Let S,S',T and T' be* Cu*-semigroups, let $x\in[\![S,T]\!]$, and let $f\colon T\to T'$ and $g\colon S'\to S$ be* Cu*-morphisms. Then*

$$\sigma_{S,T'}\circ f_*=f_*\circ\sigma_{S,T},\quad\textit{and}\quad\sigma_{S,T}\circ g^*=g^*\circ\sigma_{S',T},$$

which means that the following diagrams commute:

$$\begin{tikzcd}[\![S,T]\!] \arrow[r, "f_*"] \arrow[d, "\sigma_{S,T}"'] & [\![S,T']\!] \arrow[d, "\sigma_{S,T'}"] \\ \mathrm{Cu}[S,T] \arrow[r, "f_*"] & \mathrm{Cu}[S,T']\end{tikzcd}\qquad\begin{tikzcd}[\![S',T]\!] \arrow[r, "g^*"] \arrow[d, "\sigma_{S',T}"'] & [\![S,T]\!] \arrow[d, "\sigma_{S,T}"] \\ \mathrm{Cu}[S',T] \arrow[r, "g^*"] & \mathrm{Cu}[S,T].\end{tikzcd}$$

Proof. This follows from the definition of the abstract bivariant Cu-semigroup (see Definition 5.3) and the naturality of the endpoint map (see Proposition 4.8). $\square$

The proofs of the following results are straightforward and left to the reader.

Proposition 6.26. *Let S_1,S_2 and T be* Cu*-semigroups, and let $f\colon S_1\to S_2$ be a* Cu*-morphism. Then*

$$\mathrm{d}_{S_2,T}\circ f=(f\otimes\mathrm{id}_T)_*\circ\mathrm{d}_{S_1,T},$$

which means that the left diagram below commutes.

Analogously, if S,T_1 and T_2 are Cu*-semigroups, and if $g\colon T_1\to T_2$ is a* Cu*-morphism, then*

$$g\circ\mathrm{e}_{S,T_1}=\mathrm{e}_{S,T_2}\circ(g_*\otimes\mathrm{id}_S),$$

which means that the right diagram below commutes.

$$\begin{tikzcd}S_1 \arrow[r, "\mathrm{d}_{S_1,T}"] \arrow[d, "f"'] & [\![T,S_1\otimes T]\!] \arrow[d, "(f\otimes\mathrm{id}_T)_*"] \\ S_2 \arrow[r, "\mathrm{d}_{S_2,T}"'] & [\![T,S_2\otimes T]\!]\end{tikzcd}\qquad\begin{tikzcd}[\![S,T_1]\!]\otimes S \arrow[r, "\mathrm{e}_{S,T_1}"] \arrow[d, "g_*\otimes\mathrm{id}_S"'] & T_1 \arrow[d, "g"] \\ [\![S,T_2]\!]\otimes S \arrow[r, "\mathrm{e}_{S,T_2}"'] & T_2.\end{tikzcd}$$

6.6. **General form of unit and product.** Given Cu-semigroups S and T, we consider the unit map $\mathrm{d}_{S,T}\colon S \to [\![T, S\otimes T]\!]$ from Definition 6.1. Next, we introduce a more general form of the unit map.

Definition 6.27. Let S, T and T' be Cu-semigroups. We define *the general left unit map* $S \otimes [\![T', T]\!] \to [\![T', S\otimes T]\!]$ as the Cu-morphism that under the identification

$$\mathrm{Cu}\big(S \otimes [\![T', T]\!], [\![T', S\otimes T]\!]\big) \cong \mathrm{Cu}\big(S \otimes [\![T', T]\!] \otimes T', S \otimes T\big)$$

corresponds to the map $\mathrm{id}_S \otimes \mathrm{e}_{T',T}$. Given $a \in S$ and $x \in [\![T', T]\!]$, we denote the image of $a \otimes x$ under this map by ${}_a x$.

Analogously, we define *the general right unit map* $[\![T', T]\!] \otimes S \to [\![T', T\otimes S]\!]$ as the Cu-morphism that under the identification

$$\mathrm{Cu}\big([\![T', T]\!] \otimes S, [\![T', T\otimes S]\!]\big) \cong \mathrm{Cu}\big([\![T', T]\!] \otimes S \otimes T', T \otimes S\big)$$

corresponds to the map $(\mathrm{e}_{T',T} \otimes \mathrm{id}_S) \circ (\mathrm{id}_{[\![T',T]\!]} \otimes \sigma)$, where σ denotes the flip isomorphism. Given $a \in S$ and $x \in [\![T', T]\!]$, we denote the image of $x \otimes a$ under this map by x_a.

We leave the proofs of the following results to the reader.

Proposition 6.28. *Let S, T and T' be* Cu*-semigroups, let a be an element in S, and let x be an element in $[\![T', T]\!]$. Let $\mathrm{i}_S\colon S \to [\![\overline{\mathbb{N}}, S]\!]$ be the isomorphism from Definition 6.9, and let $\mathrm{l}_{T'}\colon \overline{\mathbb{N}} \otimes T' \to T'$ and $\mathrm{r}_{T'}\colon T' \otimes \overline{\mathbb{N}} \to T'$ be the natural* Cu*-isomorphism. Then*

$${}_a x = (\mathrm{i}_S(a) \otimes x) \circ \mathrm{l}_{T'}^{-1} = \mathrm{d}_{S,T}(a) \circ x = (\mathrm{id}_S \otimes x) \circ \mathrm{d}_{S,T'}(a).$$

and analogously $x_a = (x \otimes \mathrm{i}_S(a)) \circ \mathrm{r}_{T'}^{-1}$. Further, for the unit map $\mathrm{d}_{S,T}\colon S \to [\![T, S\otimes T]\!]$, we have $\mathrm{d}_{S,T}(a) = {}_a(\mathrm{id}_T)$ for every $a \in S$.

Lemma 6.29. *Let S, T, T' and T'' be* Cu*-semigroups. Given $x \in [\![T, T']\!]$, $y \in [\![T', T'']\!]$, and $a \in S$, we have*

$${}_a(y \circ x) = (\mathrm{id}_S \otimes y) \circ {}_a x = {}_a y \circ x, \quad \textit{and} \quad (y \circ x)_a = (\mathrm{id}_S \otimes y) \circ x_a = y_a \circ x,$$

in $[\![T, S \otimes T'']\!]$ and $[\![T, T'' \otimes S]\!]$, respectively.

Given another Cu*-semigroup S' and $a' \in S'$, we have*

$${}_{a'}({}_a x) = {}_{a'\otimes a} x, \quad \textit{and} \quad (x_a)_{a'} = x_{a \otimes a'},$$

in $[\![T, S' \otimes S \otimes T']\!]$ and $[\![T, T' \otimes S \otimes S']\!]$, respectively.

6.30. As in KK-theory, one can define a 'general form of the product' which generalizes both the composition product and the external tensor product; see [Bla98, Section 18.9, p.180f].

Let P, S_1, S_2, T_1 and T_2 be Cu-semigroups. We let

$$\boxtimes_P\colon [\![S_1 \otimes P, T_1]\!] \otimes [\![S_2, P \otimes T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!],$$

be the Cu-morphism that under the identification

$$\begin{aligned} &\mathrm{Cu}\big([\![S_1 \otimes P, T_1]\!] \otimes [\![S_2, P \otimes T_2]\!], [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]\big) \\ &\quad \cong \mathrm{Cu}\big([\![S_1 \otimes P, T_1]\!] \otimes [\![S_2, P \otimes T_2]\!] \otimes S_1 \otimes S_2, T_1 \otimes T_2\big) \end{aligned}$$

corresponds to the composition

$$(\mathrm{e}_{S_1\otimes P, T_1} \otimes \mathrm{id}_{T_2}) \circ (\mathrm{id}_{[\![S_1\otimes P, T_1]\!]} \otimes \mathrm{e}_{S_2, P\otimes T_2}) \circ (\mathrm{id}_{[\![S_1\otimes P, T_1]\!]} \otimes \sigma_{[\![S_2, P\otimes T_2]\!], S_1} \otimes \mathrm{id}_{S_2}),$$

where $\sigma_{[\![S_2, P\otimes T_2]\!], S_1}$ denotes the flip isomorphism.

Given $x \in [\![S_1 \otimes P, T_1]\!]$ and $y \in [\![S_2, P \otimes T_2]\!]$, we have

$$x \boxtimes_P y = (x \boxtimes \mathrm{id}_{T_2}) \circ (\mathrm{id}_{S_1} \boxtimes y).$$

Specializing to the case $P = \overline{\mathbb{N}}$, we obtain the external tensor product, after applying the usual isomorphisms $S_1 \otimes \overline{\mathbb{N}} \cong S_1$ and $\overline{\mathbb{N}} \otimes T_2 \cong T_2$.

Specializing to the case $T_2 = S_1 = \overline{\mathbb{N}}$, we obtain the composition product, after applying the natural isomorphisms $\overline{\mathbb{N}} \otimes P \cong P \cong P \otimes \overline{\mathbb{N}}$.

Remark 6.31. The order of the product in KK-theory is reversed from the one used here for the category Cu, that is, given C^*-algebras A, B and D, the product in KK-theory is as a bilinear map

$$KK(A, D) \times KK(D, B) \to KK(A, B);$$

see [Bla98, Section 18.1, p166] and [JT91, Before Lemma 2.2.9, p.73].

We have mainly two reasons for our choice of ordering for the composition product in the category Cu: First, the composition product extends the usual composition of Cu-morphisms and our choice is compatible with the standard notation for composition of maps. Second, our ordering agrees with that of the composition law of internal-homs in closed categories; see [Kel05, Section 1.6, p.15].

7. Cu-semirings and Cu-semimodules

In Subsection 7.1, we first recall the definitions of Cu-semirings and of left and right Cu-semimodules. Given a Cu-semigroup S, the abstract bivariant Cuntz semigroup $[\![S, S]\!]$ has a natural Cu-semiring structure; see Proposition 7.1. In Example 7.9 we will see that $[\![S, S]\!]$ is noncommutative in general.

The evaluation map $\mathrm{e}_{S,S}\colon [\![S, S]\!] \otimes S \to S$ defines a natural left $[\![S, S]\!]$-semimodule structure on S; see Proposition 7.3. Given a Cu-semigroup R with compact unit and a Cu-semigroup T with left R-action, for every Cu-semigroup S the bivariant Cu-semigroup $[\![S, T]\!]$ has a natural left R-action; see Proposition 7.6. It follows that $[\![S, T]\!]$ has a natural left action by the Cu-semiring $[\![T, T]\!]$, and a compatible right $[\![S, S]\!]$-action; see Proposition 7.7.

In Subsection 7.2, we study the connection between a Cu-semiring R and the associated Cu-semiring $[\![R, R]\!]$. We show that there is a natural multiplicative Cu-morphism $\pi_R\colon R \to [\![R, R]\!]$ that is an order-embedding; see Definition 7.10 and Theorem 7.13 and Proposition 7.17. If the unit of R is compact, then π_R is unital. Thus, R naturally is a (unital) sub-Cu-semiring of $[\![R, R]\!]$.

In Subsection 7.3, we study the situation for solid Cu-semirings. We first show that a (not necessarily commutative) Cu-semiring R is solid whenever the map $\mu\colon R \otimes R \to R$ (induced by the multiplication in R) is injective; see Lemma 7.20. In Theorem 7.25, we relate the property of R being solid with other natural properties of Cu-semirings. In particular, a Cu-semiring R with compact unit is solid if and only if the evaluation map $\mathrm{e}_{R,R}\colon [\![R, R]\!] \otimes R \to R$ is an isomorphism; see Remark 7.27.

If R is a solid Cu-semiring with compact unit, and if T is an R-stable Cu-semigroup, then $[\![R \otimes S, T]\!] \cong [\![S, T]\!]$ for any S; see Proposition 7.30. In particular, we have $[\![R, T]\!] \cong T$ for every R-stable Cu-semigroup T; see Corollary 7.31.

7.1. **The Cu-semiring $[\![S, S]\!]$ and the Cu-semibimodule $[\![S, T]\!]$.** A (unital) *Cu-semiring* is a Cu-semigroup R together with a Cu-bimorphism $R \times R \to R$, denoted by $(r_1, r_2) \mapsto r_1 r_2$, and a distinguished element $1 \in R$, called the unit of R, such that $r_1(r_2 r_3) = (r_1 r_2) r_3$ and $r1 = r = 1r$ for all $r, r_1, r_2, r_3 \in R$. This concept was introduced and studied in [APT14, Chapter 7], where it is further assumed that the product in R be commutative. We will not make this assumption here.

We often let $\mu\colon R \otimes R \to R$ denote the Cu-morphism induced by the multiplication in a Cu-semiring R.

Proposition 7.1. *Let S be a* Cu*-semigroup. Then $[\![S,S]\!]$ is a* Cu*-semiring with product given by the composition product $\circ\colon [\![S,S]\!]\otimes[\![S,S]\!]\to[\![S,S]\!]$, and with unit element given by the identity map $\mathrm{id}_S\in[\![S,S]\!]$.*

Proof. It follows directly from Proposition 6.19 that the composition product on $[\![S,S]\!]$ is associative, and that id_S is a unit element for $[\![S,S]\!]$. ☐

Remark 7.2. Let S be a Cu-semigroup. The identity map $\mathrm{id}_S\colon S\to S$ is a Cu-morphism. Therefore, the unit of the Cu-semiring $[\![S,S]\!]$ is compact.

In Example 7.9, we will see that $[\![S,S]\!]$ is noncommutative in general.

Given a Cu-semiring R, a *left* Cu*-semimodule* over R is a Cu-semigroup S together with a Cu-bimorphim $R\times S\to S$, denoted by $(r,a)\mapsto ra$, such that for all $r_1,r_2\in R$ and $a\in S$, we have $(r_1r_2)a=r_1(r_2a)$ and $1a=a$. We also say that S has a *left action* on R if S is a left Cu-semimodule over R. Right Cu-semimodules are defined analogously. If R_1 and R_2 are Cu-semirings, we say that a Cu-semigroup S is a (R_1,R_2)-Cu-*semibimodule* if it has a left R_1-action and a right R_2-action that satisfy $r_1(ar_2)=(r_1a)r_2$ for all $r_1\in R_1$, $r_2\in R_2$ and $a\in S$.

We refer the reader to [APT14, Chapter 7] for a discussion on commutative Cu-semirings and their Cu-semimodules.

Proposition 7.3. *Let S be a* Cu*-semigroup. Then $\mathrm{e}_{S,S}\colon[\![S,S]\!]\otimes S\to S$ defines a left action of $[\![S,S]\!]$ on S.*

Proof. It follows directly from Proposition 6.21 that the action of $[\![S,S]\!]$ on S is associative and that id_S acts as a unit. ☐

Assume that R is a Cu-semiring with a compact unit. Next, we show that a left action of R on a Cu-semigroup T induces a left action of R on $[\![S,T]\!]$, for every Cu-semigroup S.

7.4. Let R be a Cu-semiring with multiplication given by $\mu\colon R\otimes R\to R$, let T be a Cu-semigroup with a left R-action $\alpha\colon R\otimes T\to T$, and let S be a Cu-semigroup. Consider the general left unit map $R\otimes[\![S,T]\!]\to[\![S,R\otimes T]\!]$ from Definition 6.27. Postcomposing with $\alpha_*\colon[\![S,R\otimes T]\!]\to[\![S,T]\!]$ we obtain a Cu-morphism that we denote by α_S:

$$\alpha_S\colon R\otimes[\![S,T]\!]\to[\![S,R\otimes T]\!]\xrightarrow{\alpha_*}[\![S,T]\!].$$

Let $r\in R$ and $x\in[\![S,T]\!]$. We denote $\alpha_S(r\otimes x)$ by rx. Applying Proposition 6.28 at the third step, we have

$$rx=\alpha_S(r\otimes x)=\alpha_*\circ({}_rx)=\alpha_*\circ\mathrm{d}_{R,T}(r)\circ x.$$

Lemma 7.5. *We retain the notation from Paragraph 7.4. Let $r\in R$, and let $\mathbf{f}=(f_\lambda)_\lambda$ be a path in* $\mathrm{Cu}[S,T]$*. Choose a path $(r_\lambda)_\lambda$ in R with endpoint r. For each λ, let $r_\lambda f_\lambda\colon S\to T$ be given by $s\mapsto r_\lambda f_\lambda(s)$. Then $(r_\lambda f_\lambda)_\lambda$ is a path in* $\mathrm{Cu}[S,T]$ *and*

$$r[\mathbf{f}]=[(r_\lambda f_\lambda)_\lambda].$$

Proof. Using the equation at the end of Paragraph 7.4 at the first step, using Proposition 6.2 at the second step, and using Proposition 6.18 at the third step, we deduce that

$$\begin{aligned} r[\mathbf{f}]=\alpha_*\circ\mathrm{d}_{R,T}(r)\circ[\mathbf{f}] &= \alpha_*\circ[(r_\lambda\otimes\_)_\lambda]\circ[(f_\lambda)_\lambda] \\ &= \alpha_*\circ[(r_\lambda\otimes f_\lambda(\_))_\lambda]=[(r_\lambda f_\lambda)_\lambda], \end{aligned}$$

as desired. ☐

Proposition 7.6. *Let R be a* Cu*-semiring with compact unit, let S be a* Cu*-semigroup, and let T be a* Cu*-semigroup with a left R-action $\alpha\colon R\otimes T\to T$. Then the map $\alpha_S\colon R\otimes[\![S,T]\!]\to[\![S,T]\!]$ from Paragraph 7.4 defines a left R-action on $[\![S,T]\!]$.*

Proof. Let $r, r' \in R$ and $x \in [\![S,T]\!]$. Choose a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[S,T]$ representing x. Choose paths $(r_\lambda)_\lambda$ and $(r'_\lambda)_\lambda$ in R with endpoints r and r', respectively. Then $(r_\lambda r'_\lambda)_\lambda$ is a path in R with endpoint rr'. Using Lemma 7.5 at the first, third and last step, we deduce that

$$(rr')x = [((r_\lambda r'_\lambda)f_\lambda)_\lambda] = [(r_\lambda(r'_\lambda f_\lambda))_\lambda] = r[(r'_\lambda f_\lambda)_\lambda] = r(r'x),$$

as desired.

Let 1 denote the unit element of R. For every $f \in \mathrm{Cu}[S,T]$, we have $1f = f$. Since 1 is compact, the constant function with value 1 is a path in R with endpoint 1. Using Lemma 7.5 at the first step, we deduce that

$$1x = [(1f_\lambda)_\lambda] = [(f_\lambda)_\lambda] = x,$$

as desired. □

Proposition 7.7. *Let S and T be* Cu*-semigroups. Then the composition product* $\circ\colon [\![T,T]\!] \otimes [\![S,T]\!] \to [\![S,T]\!]$ *defines a left action of the* Cu*-semiring* $[\![T,T]\!]$ *on* $[\![S,T]\!]$. *Analogously, we obtain a right action of* $[\![S,S]\!]$ *on* $[\![S,T]\!]$. *These actions are compatible and thus* $[\![S,T]\!]$ *is a* $([\![T,T]\!],[\![S,S]\!])$-Cu*-semibimodule.*

Proof. This follows directly from the associativity of the composition product; see Proposition 6.19. □

Remark 7.8. Let S and T be Cu-semigroups. By Proposition 7.3, the evaluation map $\mathrm{e}_{T,T}\colon [\![T,T]\!] \otimes T \to T$ from Definition 6.5 defines a left action of $[\![T,T]\!]$ on T. By Proposition 7.6, this induces a left action of $[\![T,T]\!]$ on $[\![S,T]\!]$. This action agrees with that from Proposition 7.7.

Example 7.9. Given $k \in \mathbb{N}$, we let $\overline{\mathbb{N}}^k$ denote the Cuntz semigroup of the C^*-algebra $\mathbb{C}^k$. We think of an element $\mathbf{v} \in \overline{\mathbb{N}}^k$ as a tuple $(v_1, \ldots, v_k)^T$ with k entries in $\overline{\mathbb{N}}$. We let $\mathbf{e^{(1)}}, \ldots, \mathbf{e^{(k)}}$ denote the 'standard basis vectors' of $\overline{\mathbb{N}}^k$, such that $\mathbf{v} = \sum_{i=1}^k v_i \mathbf{e^{(i)}}$.

Let $k, l \in \mathbb{N}$. Let us show that $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ can be identified with $M_{l,k}(\overline{\mathbb{N}})$, the $l \times k$-matrices with entries in $\overline{\mathbb{N}}$, with order and addition defined entrywise. Thus, as a Cu-semigroup, $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ is isomorphic to $\overline{\mathbb{N}}^{kl}$. However, the presentation as matrices allows to expatiate the composition product.

First, let $\varphi\colon \overline{\mathbb{N}}^k \to \overline{\mathbb{N}}^l$ be a generalized Cu-morphism. For each $j \in \{1, \ldots, k\}$, we consider the vector $\varphi(\mathbf{e^{(j)}})$ in $\overline{\mathbb{N}}^l$ and we let $x_{1,j}, \ldots, x_{l,j}$ denote its coefficients. This defines a matrix $\mathbf{x} = (x_{i,j})_{i,j}$ with $l \times k$ entries in $\overline{\mathbb{N}}$. It is then readily verfied that the coefficients of $\varphi(\mathbf{v})$ are obtained by multiplication of the matrix $\mathbf{x}$ with the vector of coefficients of $\mathbf{v}$. We identify φ with the associated matrix $\mathbf{x}$ in $M_{l,k}(\overline{\mathbb{N}})$.

Let $\varphi, \psi\colon \overline{\mathbb{N}}^k \to \overline{\mathbb{N}}^l$ be generalized Cu-morphisms with associated matrixes $\mathbf{x}$ and $\mathbf{y}$ in $M_{l,k}(\overline{\mathbb{N}})$. It is straightforward to check that $\varphi \prec \psi$ if and only if $x_{i,j}$ is finite and $x_{i,j} \leq y_{i,j}$ for each i, j. It follows that $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ can be identified with $M_{l,k}(\overline{\mathbb{N}})$, with addition and order defined entrywise.

Given $k, l, m \in \mathbb{N}$, consider the composition product

$$[\![\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^m]\!] \otimes [\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!] \to [\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^m]\!].$$

After identifying $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ with $M_{l,k}(\overline{\mathbb{N}})$, identifying $[\![\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^m]\!]$ with $M_{m,l}(\overline{\mathbb{N}})$, and identifying $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^m]\!]$ with $M_{m,k}(\overline{\mathbb{N}})$, the composition product is given as a map

$$M_{m,l}(\overline{\mathbb{N}}) \otimes M_{l,k}(\overline{\mathbb{N}}) \to M_{m,k}(\overline{\mathbb{N}}).$$

It is straightforward to check that this map is induced by matrix multiplication. In particular, the Cu-semiring $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^k]\!]$ can be identified with $M_{k,k}(\overline{\mathbb{N}})$. Thus, for $k \geq 2$, the Cu-semiring $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^k]\!]$ is not commutative.

The left action of the Cu-semiring $[\![\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^l]\!] = M_{l,l}(\overline{\mathbb{N}})$ on the Cu-semigroup $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!] = M_{l,k}(\overline{\mathbb{N}})$ is given by matrix multiplication, and similarly for the right action of $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^k]\!] = M_{k,k}(\overline{\mathbb{N}})$.

7.2. **The embedding of a Cu-semiring R in $[\![R, R]\!]$.** Given a Cu-semiring R, recall that $\mu\colon R \otimes R \to R$ denotes the Cu-morphism induced by multiplication in R.

Definition 7.10. Given a Cu-semiring R, we let $\pi_R\colon R \to [\![R, R]\!]$ be the Cu-morphism that corresponds to μ under the identification

$$\mathrm{Cu}\big(R, [\![R, R]\!]\big) \cong \mathrm{Cu}\big(R \otimes R, R\big).$$

Remark 7.11. The Cu-morphism π_R plays the role of the left regular representation. In a similar way, one might define an analogue of the right regular representation as the Cu-morphism $\bar{\pi}_R\colon R \to [\![R, R]\!]$ corresponding to $\mu\sigma$, where σ is the flip automorphism.

Lemma 7.12. *We have $\pi_R = \mu_* \circ \mathrm{d}_{R,R}$ and $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) = \mu$.*

Proof. The first equality follows from Proposition 6.23. By Proposition 6.26, we have $\mathrm{e}_{R,R} \circ (\mu_* \otimes \mathrm{id}_R) = \mu \circ \mathrm{e}_{R,R\otimes R}$. Further, we have $\mathrm{e}_{R,R\otimes R} \circ (\mathrm{d}_{R,R} \otimes \mathrm{id}_R) = \mathrm{id}_{R\otimes R}$ by Corollary 6.24 that . Using these equations at the second and third step, we deduce that

$$\begin{aligned}
\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) &= \mathrm{e}_{R,R} \circ ((\mu_* \circ \mathrm{d}_{R,R}) \otimes \mathrm{id}_R) \\
&= \mathrm{e}_{R,R} \circ (\mu_* \otimes \mathrm{id}_R) \circ (\mathrm{d}_{R,R} \otimes \mathrm{id}_R) \\
&= \mu \circ \mathrm{e}_{R,R\otimes R} \circ (\mathrm{d}_{R,R} \otimes \mathrm{id}_R) \\
&= \mu,
\end{aligned}$$

as desired. □

Theorem 7.13. *Let R be a* Cu*-semiring. Then the map $\pi_R\colon R \to [\![R, R]\!]$ from Definition 7.10 is multiplicative. If the unit element of R is compact, then π_R is unital.*

Proof. Let $M\colon [\![R, R]\!] \otimes [\![R, R]\!] \to [\![R, R]\!]$ denote the composition map. We need to show that $M \circ (\pi_R \otimes \pi_R) = \pi_R \circ \mu$.

Given r and $s \in R$, choose paths $\mathbf{r} = (r_\lambda)_\lambda$ and $\mathbf{s} = (s_\lambda)$ in $(R, \ll)$ with endpoints r and s, respectively. For each λ, let $f_\lambda\colon R \to R$ and $g_\lambda\colon R \to R$ be the generalized Cu-morphism given by left multiplication with r_λ and s_λ, respectively. By Proposition 6.2, we have $\mathrm{d}_{R,R}(r) = [(r_\lambda \otimes \_)_\lambda]$, where $r_\lambda \otimes \_ \colon R \to R \otimes R$ is the map sending $t \in R$ to $r_\lambda \otimes t$. We also have $\mu \circ (r_\lambda \otimes \_) = f_\lambda$. Since $\pi_R = \mu_* \circ \mathrm{d}_{R,R}$ by Lemma 7.12, it follows that $\pi_R(r) = [(f_\lambda)_\lambda]$. Likewise, we deduce $\pi_R(s) = [(g_\lambda)_\lambda]$. By Proposition 6.18, we obtain $M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)_\lambda]$.

As the product in R is associative, the composition $f_\lambda \circ g_\lambda$ is the generalized Cu-morphism h_λ defined by left multiplication with $r_\lambda s_\lambda$. Notice that $(r_\lambda s_\lambda)_\lambda$ is a path in $(R, \ll)$ with endpoint rs. Therefore, $\pi_R(rs) = [(h_\lambda)_\lambda]$. Altogether, this implies

$$M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)_\lambda] = [(h_\lambda)_\lambda] = \pi_R(\mu(r \otimes s)),$$

as desired.

To show the second statement, let us assume that the unit 1_R of R is compact. Then the constant function with value 1_R is a path in $(R, \ll)$ with endpoint 1_R.

Then it follows easily as in the first part of the proof that $\pi_R(1_R) = [(\mathrm{id}_R)_\lambda] = \mathrm{id}_R$. $\square$

Definition 7.14. Given a Cu-semiring R, we let $\varepsilon_R\colon [\![R,R]\!] \to R$ be the generalized Cu-morphism given by

$$\varepsilon_R([\mathbf{f}]) = \sup_\lambda f_\lambda(1),$$

for a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[R,R]$.

Remark 7.15. Let $\sigma_{R,R}\colon [\![R,R]\!] \to \mathrm{Cu}[R,R]$ denote the endpoint map as introduced in Definition 5.6, and let 1 denote the unit of R. Then $\varepsilon_R(x) = \sigma_{R,R}(x)(1)$ for every $x \in [\![R,R]\!]$.

Lemma 7.16. *We have $\varepsilon_R \circ \pi_R = \mathrm{id}_R$.*

Proof. Let r be an element in R. Proceeding as in Theorem 7.13 we know that, if $(r_\lambda)_\lambda$ is a path in $(R,\ll)$ with endpoint r and $f_\lambda\colon R \to R$ is given by left multiplication by r_λ, then $\pi_R(r) = [(f_\lambda)_\lambda]$. Given $r \in R$, we deduce that

$$\varepsilon_R(\pi_R(r)) = \varepsilon_R((f_\lambda)_\lambda) = \sup_\lambda f_\lambda(1) = \sup_\lambda (r_\lambda 1) = r,$$

as desired. $\square$

Proposition 7.17. *Let R be a* Cu*-semiring. Then $\pi_R\colon R \to [\![R,R]\!]$ is a multiplicative order-embedding. Thus, in a natural way, R is a sub-semiring of $[\![R,R]\!]$. If the unit of R is compact, then R is even a unital sub-semiring of $[\![R,R]\!]$. (One could call this a sub-*Cu*-semiring.)*

Proof. By Lemma 7.16, we have $\varepsilon_R \circ \pi_R = \mathrm{id}_R$, which implies that π_R is an order-embedding. By Theorem 7.13, π_R is a (unital) multiplicative Cu-morphism. $\square$

Recall that $\overline{\mathbb{P}} = [0,\infty]$ is isomorphic to the Cuntz semigroup of the Jacelon-Razak algebra. The usual multiplication of real numbers extends to $\overline{\mathbb{P}}$. This gives $\overline{\mathbb{P}}$ the structure of a commutative Cu-semiring.

Example 7.18. Recall that M_1 denotes the Cuntz semigroup of a II_1-factor; see Example 4.14 and Proposition 4.16. Note that M_1 is the disjoint union of compact elements $[0,\infty)$ and nonzero soft elements $(0,\infty]$. We identify $\overline{\mathbb{P}} = [0,\infty]$ with the sub-Cu-semigroup of soft elements in M_1. We define a Cu-morphism $\varrho\colon M_1 \to \overline{\mathbb{P}} \subseteq M_1$ by fixing all soft elements and by sending a compact to the soft element of the same value.

We define a product on M_1 as follows: We equip the compact part $[0,\infty)$ with the usual multiplication of real numbers, and similarly for the product in $(0,\infty]$. The product of any element with 0 is 0. Given a nonzero compact element a and a nonzero soft element b, their product is defined as the soft element $ab := \varrho(a)b$.

This gives M_1 the structure of a commutative Cu-semiring. Moreover, we may identify $\overline{\mathbb{P}}$ with the (nonunital) sub-Cu-semiring of soft elements in M_1. The map $\varrho\colon M_1 \to \overline{\mathbb{P}}$ is multiplicative. One can show that the map $\pi_{M_1}\colon M_1 \to [\![M_1,M_1]\!]$ is an isomorphism.

Example 7.19. We have $[\![\overline{\mathbb{P}},\overline{\mathbb{P}}]\!] \cong M_1$. The map $\pi_{\overline{\mathbb{P}}}\colon \overline{\mathbb{P}} \to [\![\overline{\mathbb{P}},\overline{\mathbb{P}}]\!]$ embeds $\overline{\mathbb{P}}$ as the sub-Cu-semiring of soft elements in M_1. In particular, $\pi_{\overline{\mathbb{P}}}$ is not unital.

Proof. We have $[\![\overline{\mathbb{P}},\overline{\mathbb{P}}]\!] \cong M_1$ by Proposition 5.14. By Proposition 7.17, $\pi_{\overline{\mathbb{P}}}$ is a multiplicative order-embedding. Note that every element of $\overline{\mathbb{P}}$ is soft. By Lemma 2.11, a generalized Cu-morphism maps soft elements to soft elements. Thus, the image of $\pi_{\overline{\mathbb{P}}}$ is contained in the soft elements of M_1. It easily follows that $\pi_{\overline{\mathbb{P}}}$ identifies $\overline{\mathbb{P}}$ with the soft elements in M_1. Since the unit of M_1 is compact, it also follows that $\pi_{\overline{\mathbb{P}}}$ is not unital. $\square$

7.3. **Solid** Cu**-semirings.** Throughout this subsection, R denotes a (not necessarily commutative) Cu-semiring, with multiplication given by the Cu-morphism $\mu\colon R\otimes R\to R$.

Recall from [APT14, Definition 7.1.5] that R is said to be *solid* if $\mu\colon R\otimes R\to R$ is an isomorphism. In [APT14], all Cu-semirings were required to be commutative, and thus a solid Cu-semiring was assumed to be commutative. Next, we show that this assumption is not necessary since a Cu-semiring is automatically commutative as soon as μ is injective.

Lemma 7.20. *Let R be a (not necessarily commutative)* Cu*-semiring such that $\mu\colon R\otimes R\to R$ is injective. Then R is commutative and μ is an isomorphism (and consequently R is solid.)*

Proof. To show that R is commutative, let $a,b\in R$. We have

$$\mu(1\otimes a)=a=\mu(a\otimes 1),$$

and therefore $1\otimes a=a\otimes 1$ in $R\otimes R$. Consider the shuffle Cu-morphism $\alpha\colon R\otimes R\otimes R\to R\otimes R\otimes R$ that satisfies $\alpha(x\otimes y\otimes z)=y\otimes x\otimes z$ for every $x,y,z\in R$. It follows that

$$1\otimes b\otimes a=\alpha(b\otimes 1\otimes a)=\alpha(b\otimes a\otimes 1)=a\otimes b\otimes 1$$

in $R\otimes R\otimes R$. By the associativity of the product in R, this implies that $ba=ab$, as desired.

Thus, if μ is injective, then R is commutative and $1\otimes a=a\otimes 1$ in $R\otimes R$, for every $a\in R$. Using [APT14, Proposition 7.1.6], this implies that R is solid. $\square$

7.21. Let R be a solid Cu-semiring, and let S be a Cu-semigroup. It was shown in [APT14, Corollary 7.1.8] that any two R-actions on S agree. (Since R is commutative, we need not distinguish between left and right R-actions.) Thus, S either has a (unique) R-action, or it does not admit any R-action. Thus, having an R-action is a property rather than an additional structure for S, which justifies the following definition.

Definition 7.22. Let R be a solid Cu-semiring, and let S be a Cu-semigroup. We say that S is *R-stable* if S has an R-action.

Remark 7.23. In [APT14], we said that S 'has R-multiplication' if it has an R-action. Given a solid ring R, it was shown [APT14, Theorem 7.1.12] that S is R-stable if and only if $S\cong R\otimes S$.

Recall that a C^*-algebra A is said to be *$\mathcal{Z}$-stable* if $A\cong\mathcal{Z}\otimes A$, and similarly one defines being UHF-stable and $\mathcal{O}_\infty$-stable. Thus, the terminology of being 'R-stable' for Cu-semigroups is analogous to the terminology used for C^*-algebras.

The following fact about solid Cu-semirings will be used in the sequel.

Proposition 7.24 ([APT14, Proposition 7.1.6])**.** *Let R be a solid* Cu*-semiring, and let S and T be R-stable* Cu*-semigroups. Then every generalized* Cu*-morphism $\varphi\colon S\to T$ is automatically R-linear, that is, we have $\varphi(ra)=r\varphi(a)$ for all $r\in R$ and $a\in S$.*

Theorem 7.25. *Given a* Cu*-semiring R, consider the following statements:*

(1) R is solid, that is, $\mu\colon R\otimes R\to R$ is an isomorphism.
(2) The map $\mathrm{e}_{R,R}\colon [\![R,R]\!]\otimes R\to R$ is an isomorphism.
(3) The map $\pi_R\otimes\mathrm{id}_R\colon R\otimes R\to[\![R,R]\!]\otimes R$ is an isomorphism.
(4) The map $\pi_R\colon R\to[\![R,R]\!]$ is an isomorphism.
(5) The map $\varepsilon_R\colon[\![R,R]\!]\to R$ is an isomorphism.

Then the following implications hold:

$$(1) \Leftarrow (2) \Rightarrow (3) \Leftarrow (4) \Leftrightarrow (5).$$

Further, if R satisfies (1) and (3), then it satisfies (2). The Cu*-semiring $\overline{\mathbb{P}}$ satisfies (1),(2) and (3), but not (4); see Example 7.19. The* Cu*-semiring M_1 from Example 4.14 satisfies (3) and (4) but neither (1) nor (2); see Example 7.18.*

Proof. By Lemma 7.16, we have $\varepsilon_R \circ \pi_R = \mathrm{id}_R$. It follows that ε_R is an isomorphism if and only if π_R is, which shows the equivalence of (4) and (5). We also deduce that

$$(\varepsilon_R \otimes \mathrm{id}_R) \circ (\pi_R \otimes \mathrm{id}_R) = \mathrm{id}_R \otimes \mathrm{id}_R \, .$$

Therefore, the map $\pi_R \otimes \mathrm{id}_R$ is always an order-embedding. It is obvious that (4) implies (3).

To show that (2) implies (1), assume that $\mathrm{e}_{R,R}$ is an isomorphism. Then the composition $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R)$ is an order-embedding. By Lemma 7.12, we have $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) = \mu$, which shows that μ is an order-embedding. By Lemma 7.20, this implies that R is solid.

Using again that $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) = \mu$, if any two of the three maps $\mathrm{e}_{R,R}$, $\pi_R \otimes \mathrm{id}_R$ and μ are isomorphisms, then so is the third. This shows that (2) implies (3), and that the combination of (1) and (3) implies (2). □

Question 7.26. Given a solid Cu-semiring R, is the evaluation map $\mathrm{e}_{R,R} \colon [\![R,R]\!] \otimes R \to R$ an isomorphism?

Remark 7.27. Let R be a solid Cu-semiring. The answer to Question 7.26 is 'yes' in the following cases:

1. If the unit of R is compact; see Remark 7.32 below.
2. If R satisfies (O5) and (O6). This follows from the classification of solid Cu-semirings with (O5) obtained in [APT14, Theorem 8.3.13] which shows that each such Cu-semiring is either isomorphic to $\overline{\mathbb{P}}$ or has a compact unit. In either case, Question 7.26 has a positive answer.

In particular, a Cu-semiring R with compact unit is solid if and only if the evaluation map $\mathrm{e}_{R,R} \colon [\![R,R]\!] \otimes R \to R$ is an isomorphism.

Theorem 7.28. *Let R be a solid* Cu*-semiring with compact unit, and let S and T be* Cu*-semigroups. Assume that T is R-stable. Then $[\![S,T]\!]$ is R-stable, and hence $[\![S,T]\!] \cong R \otimes [\![S,T]\!]$.*

Proof. Since the unit of R is compact, it follows from Proposition 7.6 that $[\![S,T]\!]$ has a left R-action. Since R is solid, this implies that $[\![S,T]\!]$ is R-stable. □

Lemma 7.29. *Let R be a solid* Cu*-semiring, let S and T be* Cu*-semigroups, and let $f, g \colon R \otimes S \to T$ be a generalized* Cu*-morphisms. Assume that T is R-stable. Then $f \leq g$ if and only if $f(1 \otimes a) \leq g(1 \otimes a)$ for all $a \in S$.*

If the unit of R is compact, then $f \prec g$ if and only if $f(1 \otimes a') \ll g(1 \otimes a)$ for all $a', a \in S$ with $a' \ll a$.

Proof. The forward implications are obvious. To show the converse of the first statement, assume that $f(1 \otimes a) \leq g(1 \otimes a)$ for all $a \in S$. To verify $f \leq g$, it is enough to show that $f(r \otimes a) \leq g(r \otimes a)$ for all $r \in R$ and $a \in S$. Note that $R \otimes S$ and T are R-stable. Since R is solid, every generalized Cu-morphism between R-stable Cu-semigroups is automatically R-linear; see Proposition 7.24. Thus, given $r \in R$ and $a \in S$, we obtain

$$f(r \otimes a) = f(r(1 \otimes a)) = rf(1 \otimes a) \leq rg(1 \otimes a) = g(r \otimes a),$$

as desired.

To show the converse of the second statement, assume that $f(1\otimes a') \ll g(1\otimes a)$ for all $a', a \in S$ with $a' \ll a$. To verify $f \prec g$, it is enough to show that $f(r'\otimes a') \ll g(r \otimes a)$ for all $r', r \in R$ and $a', a \in S$ with $r' \ll r$ and $a' \ll a$. Given such r', r, a' and a, we use at the second step that multiplication in R preserves the joint way-below relation, to deduce

$$f(r' \otimes a') = r'f(1 \otimes a') \ll rg(1 \otimes a) = g(r \otimes a),$$

as desired. ☐

Proposition 7.30. *Let R be a solid* Cu*-semiring with compact unit, and let S and T be* Cu*-semigroups. Assume that T is R-stable. Let $\alpha\colon S \to R\otimes S$ be the* Cu*-morphism given by $\alpha(a) = 1 \otimes a$, for $a \in S$. Then the induced map $\alpha^*\colon [\![R \otimes S, T]\!] \to [\![S, T]\!]$ is an isomorphism.*

Proof. Consider the map $\alpha^*_{\mathcal{Q}}\colon \mathrm{Cu}[R \otimes S, T] \to \mathrm{Cu}[S, T]$ given by sending a generalized Cu-morphism $f\colon R \otimes S \to T$ to the generalized Cu-morphism $\alpha^*_{\mathcal{Q}}(f)$ given by

$$\alpha^*_{\mathcal{Q}}(f)(a) = f(1 \otimes a),$$

for $a \in S$. It follows from Lemma 7.29 that $\alpha^*_{\mathcal{Q}}$ is an isomorphism of $\mathcal{Q}$-semigroups.

Since α^* is obtained by applying the functor τ to $\alpha^*_{\mathcal{Q}}$ (see Paragraph 5.5), it follows that α^* is an isomorphism, as desired. ☐

Corollary 7.31. *Let R be a solid* Cu*-semiring with compact unit, and let T be an R-stable* Cu*-semigroup. Then there is a natural isomorphism $[\![R, T]\!] \cong T$.*

Proof. Applying Proposition 7.30 for $S := \overline{\mathbb{N}}$, we obtain $[\![R, T]\!] \cong [\![\overline{\mathbb{N}}, T]\!]$. By Proposition 6.10, we have a natural isomorphism $[\![\overline{\mathbb{N}}, T]\!] \cong T$. ☐

Remark 7.32. Let R be a solid Cu-semiring with compact unit. Since R is R-stable itself, it follows from Corollary 7.31 that $[\![R, R]\!] \cong R$. It follows that the evaluation map $\mathrm{e}_{R,R}\colon [\![R, R]\!] \otimes R \to R$ is an isomorphism.

For the solid Cu-semiring $\overline{\mathbb{P}}$, we have seen in Proposition 5.14 that $[\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!] \cong M_1 \not\cong \overline{\mathbb{P}}$. This shows that Proposition 7.30 and Corollary 7.31 cannot be generalized to solid Cu-semirings without compact unit.

8. Applications to C^*-algebras

Given C^*-algebras A and B, recall that a map $\varphi\colon A \to B$ is called *completely positive contractive* (abbreviated c.p.c.) if it is linear, contractive and for each $n \in \mathbb{N}$ the amplification to $n\times n$-matrices $\varphi \otimes \mathrm{id}\colon A \otimes M_n \to B \otimes M_n$ is positive. Every c.p.c. map $\varphi\colon A \to B$ induces a contractive, positive map $\varphi\otimes\mathrm{id}\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$.

Two elements a and b in a C^*-algebra are called *orthogonal*, denoted $a \perp b$, if $ab = a^*b = ab^* = a^*b^* = 0$. If a and b are self-adjoint, then $a \perp b$ if and only if $ab = 0$. A c.p.c. map φ is said to have *order-zero* if for all $a, b \in A$ we have that $a \perp b$ implies $\varphi(a) \perp \varphi(b)$. We denote the set of c.p.c. order-zero maps by $\mathrm{cpc}_\perp(A, B)$.

The concept of c.p.c. order-zero maps was studied by Winter and Zacharias, [WZ09], who also gave a useful structure theorem for such maps. We present their result in a slightly different way.

Theorem 8.1 (Winter and Zacharias, [WZ09, Theorem 3.3])**.** *Let A and B be C^*-algebras, and let $\varphi\colon A \to B$ be a c.p.c. order-zero map. Set $C := C^*(\varphi(A))$, the sub-C^*-algebra of B generated by the image of φ. Then there exist a unital *-homomorphism $\pi_\varphi\colon \widetilde{A} \to M(C)$, from the minimal unitization of A to the multiplier algebra of C, such that*

$$\varphi(ab) = \varphi(a)\pi_\varphi(b) = \pi_\varphi(a)\varphi(b),$$

for all $a, b \in \widetilde{A}$.

In particular, the element $h := \varphi(1_{\widetilde{A}})$ *is contractive, positive, it commutes with the image of* π_φ*, and we have* $\varphi(a) = h\pi_\varphi(a) = \pi_\varphi(a)h$ *for all* $a \in A$.

This structure theorem has many interesting applications. For instance, it implies that c.p.c. order-zero maps induce generalized Cu-morphisms. Let us recall some details. Let $\varphi\colon A \to B$ be a c.p.c. order-zero map. Then the amplification $\varphi \otimes \mathrm{id}\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is a c.p.c. order-zero map as well; see [WZ09, Corollary 4.3]. Define $\mathrm{Cu}[\varphi]\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ by

$$\mathrm{Cu}[\varphi]([a]) := [(\varphi \otimes \mathrm{id})(a)],$$

for $a \in (A \otimes \mathcal{K})_+$. Then $\mathrm{Cu}[\varphi]$ is a generalized Cu-morphism; see [WZ09, Corollary 4.5] and [APT14, 2.2.7, 3.2.5]. We thus obtain a natural map

$$\mathrm{cpc}_\perp(A, B) \to \mathrm{Cu}[\mathrm{Cu}(A), \mathrm{Cu}(B)].$$

Below, we will show that this map factors through $[\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!]$.

8.2. The theorem of Winter and Zacharias also allows us to define functional calculus for order-zero maps: Let $\varphi\colon A \to B$ be a c.p.c. order-zero map. Choose C, π_φ and h as in Theorem 8.1. Given a continuous function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$, we define $f(\varphi)\colon A \to B$ by $f(\varphi)(a) := f(h)\pi_\varphi(a)$ for $a \in A$; see [WZ09, Corollary 4.2].

In particular, this allows us to define 'cut-downs' of c.p.c. order-zero maps: Given $\varepsilon > 0$, we may apply the function $(\_ - \varepsilon)_+$ to φ. To simplify notation, we set $\varphi_\varepsilon := (\varphi - \varepsilon)_+$. Thus, for $a \in A$ we have

$$\varphi_\varepsilon(a) = (h - \varepsilon)_+ \pi_\varphi(a).$$

Theorem 8.3. *Let* A *and* B *be* C^**-algebras, and let* $\varphi\colon A \to B$ *be a c.p.c. order-zero map. For each* $\varepsilon > 0$*, let* $f_\varepsilon\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ *be the generalized* Cu*-morphism induced by the c.p.c. order-zero map* $\varphi_\varepsilon\colon A \to B$*. Then* $\mathbf{f} = (f_{1-\lambda})_\lambda$ *is a path in* $\mathrm{Cu}[\mathrm{Cu}(A), \mathrm{Cu}(B)]$*. Moreover, the endpoint of* $\mathbf{f}$ *is* $\mathrm{Cu}[\varphi]$*, the generalized* Cu*-morphism induced by* φ.

Proof. We have already observed that every f_ε is a generalized Cu-morphism. To verify that $(f_{1-\lambda})_\lambda$ is a path, we need to show that $f_{\varepsilon'} \prec f_\varepsilon$ for $\varepsilon' > \varepsilon > 0$. Since $f_{\varepsilon+\delta} = (f_\varepsilon)_\delta$, it is enough to show the following:

Claim: We have $f_\varepsilon \prec f$. To show the claim, let $a, b \in (A \otimes \mathcal{K})_+$ such that $[a] \ll [b]$ in $\mathrm{Cu}(A)$. Recall that two positive elements x and y in a C^*-algebra satisfy $[x] \ll [y]$ if and only if there exists $\delta > 0$ with $[x] \leq [(y - \delta)_+]$. Thus, we can choose $\delta > 0$ such that $[a] \leq [(b - \delta)_+]$. Note that if x and y are commuting positive elements in a C^*-algebra, then $(x - \varepsilon)_+(y - \delta)_+ \leq (xy - \varepsilon\delta)_+$. Using this at the last step, we deduce that

$$\begin{aligned} \varphi_\varepsilon(a) \precsim \varphi_\varepsilon((b - \delta)_+) &= (h - \varepsilon)_+ \pi_\varphi((b - \delta)_+) \\ &= (h - \varepsilon)_+ (\pi_\varphi(b) - \delta)_+ \leq (h\pi_\varphi(b) - \varepsilon\delta)_+ = (\varphi(b) - \varepsilon\delta)_+, \end{aligned}$$

which implies that

$$f_\varepsilon([a]) = [\varphi_\varepsilon(a)] \ll [\varphi(b)] = f([b]),$$

as desired. This proves the claim and shows that $\mathbf{f}$ is a path.

Let f be the generalized Cu-morphism induced by φ. To show that the endpoint of $\mathbf{f}$ is f, let $a \in (A \otimes \mathcal{K})_+$. We have

$$\lim_{\lambda \to 1} \varphi_{1-\lambda}(a) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(a) = \lim_{\varepsilon \to 0} (h - \varepsilon)_+ \pi_\varphi(a) = h\pi_\varphi(a) = \varphi(a).$$

This implies that $\sup_{\lambda < 1} f_\lambda([a]) = f([a])$ in $\mathrm{Cu}(A)$, as desired. $\square$

Definition 8.4. Let A and B be C^*-algebras, and let $\varphi\colon A \to B$ be a c.p.c. order-zero map. We let $\mathrm{Cu}(\varphi)$ be the element in $[\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!]$ that is the class of the path $(\mathrm{Cu}[\varphi_{1-\lambda}])_\lambda$ as constructed in Theorem 8.3.

Remark 8.5. Let $\varphi\colon A \to B$ be a *-homomorphism. In the definition of the functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$ we denoted $\mathrm{Cu}(\varphi)$ as the Cu-morphism $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$ given by $\mathrm{Cu}(\varphi)([a]) = [(\varphi \otimes \mathrm{id})(a)]$ for $a \in (A \otimes \mathcal{K})_+$.

On the other hand, in Definition 8.4 we defined $\mathrm{Cu}(\varphi)$ as the class of the path $(\mathrm{Cu}[\varphi_{1-\lambda}])_\lambda$ as constructed in Theorem 8.3. Given $\varepsilon > 0$, it is easy to verify that $\varphi_\varepsilon = (1-\varepsilon)_+\varphi$. It follows that $\mathrm{Cu}[\varphi_\varepsilon] = \mathrm{Cu}[\varphi]$ for $\varepsilon \in [0,1)$. Thus, the path $(\mathrm{Cu}[\varphi_{1-\lambda}])_\lambda$ is constant with value $\mathrm{Cu}[\varphi]$.

We identify a Cu-morphism $f\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ with the compact element in $[\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!]$ given by the constant path with value f; see Proposition 5.12. It follows that the notation $\mathrm{Cu}(\varphi)$ for a *-homomorphism φ is unambiguous.

8.6. The functor $C^* \to \mathrm{Cu}$ defines a map

$$\mathrm{Cu}\colon \mathrm{Hom}(A,B) \to \mathrm{Cu}(\mathrm{Cu}(A), \mathrm{Cu}(B)).$$

By Definition 8.4 we obtain a well-defined map

$$\mathrm{cpc}_\perp(A,B) \to [\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!].$$

As noticed in Remark 8.5, these assignemnts are compatible, which means that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{cpc}_\perp(A,B) \arrow[r, "\mathrm{Cu}"] & {[\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!]} \\
\mathrm{Hom}(A,B) \arrow[u, hook] \arrow[r, "\mathrm{Cu}"] & \mathrm{Cu}(\mathrm{Cu}(A), \mathrm{Cu}(B)) \arrow[u, hook]
\end{tikzcd}$$

Problem 8.7. Study the properties of the map $\mathrm{cpc}_\perp(A,B) \to [\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!]$. In particular, when is this map surjective?

Example 8.8. Recall that $\mathcal{W}$ denotes the Jacelon-Razak algebra. We know that $\mathrm{Cu}(\mathcal{W}) \cong \overline{\mathbb{P}}$. By Proposition 5.14, we have $[\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!] \cong M_1$, and recall that $M_1 = [0,\infty) \sqcup (0,\infty]$. We claim that the map

$$\mathrm{cpc}_\perp(\mathcal{W},\mathcal{W}) \to [\![\mathrm{Cu}(\mathcal{W}), \mathrm{Cu}(\mathcal{W})]\!] \cong [\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!] \cong M_1$$

is surjective.

The idea is to choose a unital, simple, AF-algebra A with unique tracial state and a suitable element $x \in (A \otimes \mathcal{K})_+$ and consider the map $\mathcal{W} \to \mathcal{W} \otimes A$, given by $y \mapsto y \otimes x$, followed by a *-isomorphism $\mathcal{W} \otimes A \cong \mathcal{W}$.

Let A be a unital, simple AF-algebra with unique tracial state. We claim that $\mathcal{W} \otimes A \cong \mathcal{W}$. By construction, $\mathcal{W}$ is an inductive limit of the building blocks considered by Razak in [Raz02]. Since A is an AF-algebra, $\mathcal{W} \otimes A$ is an inductive limit of Razak building blocks as well. Since A is simple and has a unique tracial state, $\mathcal{W}$ and $\mathcal{W} \otimes A$ have the same invariant used for the classification [Raz02, Theorem 1.1], which gives the desired *-isomorphism $\mathcal{W} \otimes A \cong \mathcal{W}$.

Given $a \in M_1$, let us define a c.p.c. order-zero map $\mathcal{W} \to \mathcal{W}$ corresponding to a. We distinguish two cases:

Case 1: Assume that a is nonzero and soft. Let $\mathcal{U}$ denote the universal UHF-algebra. We have $\mathrm{Cu}(\mathcal{U}) \cong \mathbb{Q}_+ \sqcup (0,\infty]$. We consider a as a soft element in $\mathrm{Cu}(\mathcal{U})_{\mathrm{soft}} = [0,\infty]$. Choose $x_a \in (\mathcal{U} \otimes \mathcal{K})_+$ with Cuntz class a. (For example, let x_a be a positive element with spectrum $[0,1]$ - ensuring that its Cuntz class is soft - and such that for the unique normalized extended trace $\tau\colon (\mathcal{U} \otimes \mathcal{K})_+ \to [0,\infty]$ we have $\lim_{n\to\infty} \tau(x_a^{1/n}) = a$.)

Consider the map $\varphi_a\colon \mathcal{W}\to\mathcal{W}\otimes\mathcal{U}$ given by $\varphi_a(y)=y\otimes x_a$ for $y\in\mathcal{W}$. It is easy to see that φ_a is a c.p.c. order-zero map. Let $\psi\colon \mathcal{W}\otimes\mathcal{U}\to\mathcal{W}$ be an isomorphism. Then $\psi\circ\varphi_a$ is a c.p.c. order-zero map $\mathcal{W}\to\mathcal{W}$ with the desired properties.

Case 2: Assume that a is compact. We claim that there exists a unital, simple AF-algebra A with unique normalized trace $\tau\colon (A\otimes\mathcal{K})_+\to[0,\infty]$ and a projection $p_a\in(A\otimes\mathcal{K})_+$ with $\tau(p_a)=a$. Indeed, if a is rational, then we can take $A=\mathcal{U}$. If a is irrational, then we use that $\mathbb{Z}+a\mathbb{Z}$ is a dimension group for the order and addition inherited as a subgroup of $\mathbb{R}$. Moreover, $\mathbb{Z}+a\mathbb{Z}$ has a unique normalized state. It follows that there is a unique unital AF-algebra A such that $(K_0(A),K_0(A)_+,[1])$ is isomorphic to $(\mathbb{Z}+a\mathbb{Z},(\mathbb{Z}+a\mathbb{Z})\cap[0,\infty),1)$. By construction, there exists a projection $p_a\in A\otimes\mathcal{K}$ with $\tau(p_a)=a$.

Define $\varphi_a\colon\mathcal{W}\to\mathcal{W}\otimes A$ by $\varphi_a(y)=y\otimes p_a$ for $y\in\mathcal{W}$. Then φ_a is a *-homomorphism. Postcomposing with a *-isomorphism $\mathcal{W}\otimes R_\theta\cong\mathcal{W}$, we obtain a *-homomorphism $\mathcal{W}\to\mathcal{W}$ with the desired properties.

Example 8.9. With similar methods as in Example 8.8, one can show that the map $\mathrm{cpc}_\perp(A,B)\to[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!]$ is surjective whenever A and B are any of the following C^*-algebras: a UHF-algebra of infinite type, the Jiang-Su algebra, the Jacelon-Razak algebra $\mathcal{W}$.

Remark 8.10. In [BTZ16, Definition 2.27], Bosa, Tornetta and Zacharias introduced a bivariant Cuntz semigroup, denoted $WW(A,B)$, as suitable equivalence classes of c.p.c. order-zero maps $A\otimes\mathcal{K}\to B\otimes\mathcal{K}$. It would be interesting to study if the map from Problem 8.7 factors through $WW(A,B)$, that is, if the following diagram can be completed to be commutative:

$$\begin{tikzcd} \mathrm{cpc}_\perp(A,B) \arrow[r] \arrow[d] & {[\![\mathrm{Cu}(A),\mathrm{Cu}(B)]\!]} \\ WW(A,B) \arrow[ur, dashed] & \end{tikzcd}.$$

Observe that, in order for this to be satisfied, one needs to show that, given φ and ψ in $\mathrm{cpc}_\perp(A,B)$ such that $\varphi\precsim\psi$ in the sense of [BTZ16] then, for $\epsilon>0$, there is $\delta>0$ such that $\mathrm{Cu}[\varphi_{1-\epsilon}]\prec\mathrm{Cu}[\psi_{1-\delta}]$.

References

[ABP11] R. Antoine, J. Bosa, and F. Perera, Completions of monoids with applications to the Cuntz semigroup, *Internat. J. Math.* **22** (2011), 837–861. MR 2812090 (2012d:46146). Zbl 1239.46042.

[ADPS14] R. Antoine, M. Dadarlat, F. Perera, and L. Santiago, Recovering the Elliott invariant from the Cuntz semigroup, *Trans. Amer. Math. Soc.* **366** (2014), 2907–2922. MR 3180735. Zbl 06303118.

[APT14] R. Antoine, F. Perera, and H. Thiel, Tensor products and regularity properties of Cuntz semigroups, Mem. Amer. Math. Soc. (to appear), preprint (arXiv:1410.0483 [math.OA]), 2014.

[APT17] R. Antoine, F. Perera, and H. Thiel, Cuntz semigroups of ultraproduct C^*-algebras, in preparation, 2017.

[Bla98] B. Blackadar, *K-theory for operator algebras*, second ed., *Mathematical Sciences Research Institute Publications* **5**, Cambridge University Press, Cambridge, 1998. MR 1656031 (99g:46104). Zbl 0597.46072.

[BTZ16] J. Bosa, G. Tornetta, and J. Zacharias, A bivariant theory for the Cuntz semigroup, preprint (arXiv:1602.02043 [math.OA]), 2016.

[BP91] L. G. Brown and G. K. Pedersen, C^*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149. MR 1120918. Zbl 0776.46026.

[BPT08] N. P. Brown, F. Perera, and A. S. Toms, The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras, *J. Reine Angew. Math.* **621** (2008), 191–211. MR 2431254 (2010a:46125).

[BT07] N. P. Brown and A. S. Toms, Three applications of the Cuntz semigroup, *Int. Math. Res. Not. IMRN* (2007), Art. ID rnm068, 14. MR 2359541 (2009a:46104). Zbl 1134.46040.

[CEI08] K. T. Coward, G. A. Elliott, and C. Ivanescu, The Cuntz semigroup as an invariant for C^*-algebras, *J. Reine Angew. Math.* **623** (2008), 161–193. MR 2458043 (2010m:46101). Zbl 1161.46029.

[Cun78] J. Cuntz, Dimension functions on simple C^*-algebras, *Math. Ann.* **233** (1978), 145–153. MR 0467332 (57 #7191). Zbl 0354.46043.

[GHK+03] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous lattices and domains*, *Encyclopedia of Mathematics and its Applications* **93**, Cambridge University Press, Cambridge, 2003. MR 1975381 (2004h:06001). Zbl 1088.06001.

[Jac13] B. Jacelon, A simple, monotracial, stably projectionless C^*-algebra, *J. Lond. Math. Soc. (2)* **87** (2013), 365–383. MR 3046276. Zbl 1275.46047.

[JT91] K. K. Jensen and K. Thomsen, *Elements of KK-theory*, *Mathematics: Theory & Applications*, Birkhäuser Boston, Inc., Boston, MA, 1991. MR 1124848. Zbl 1155.19300.

[JS99] X. Jiang and H. Su, On a simple unital projectionless C^*-algebra, *Amer. J. Math.* **121** (1999), 359–413. MR 1680321 (2000a:46104). Zbl 0923.46069.

[Kel05] G. M. Kelly, Basic concepts of enriched category theory, *Repr. Theory Appl. Categ.* (2005), vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR 2177301. Zbl 1086.18001.

[Mac71] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 5. MR 0354798 (50 #7275). Zbl 0906.18001.

[ORT11] E. Ortega, M. Rørdam, and H. Thiel, The Cuntz semigroup and comparison of open projections, *J. Funct. Anal.* **260** (2011), 3474–3493. MR 2781968 (2012d:46148). Zbl 1222.46043.

[PT07] F. Perera and A. S. Toms, Recasting the Elliott conjecture, *Math. Ann.* **338** (2007), 669–702. MR 2317934 (2008g:46110). Zbl 1161.46035.

[Raz02] S. Razak, On the classification of simple stably projectionless C^*-algebras, *Canad. J. Math.* **54** (2002), 138–224. MR 1880962. Zbl 1038.46051.

[Rob12] L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, *Adv. Math.* **231** (2012), 2802–2836. MR 2970466. Zbl 1268.46041.

[Rob13] L. Robert, The cone of functionals on the Cuntz semigroup, *Math. Scand.* **113** (2013), 161–186. MR 3145179. Zbl 1286.46061.

[Roi90] J. Roitman, *Introduction to modern set theory*, *Pure and Applied Mathematics (New York)*, John Wiley & Sons, Inc., New York, 1990, A Wiley-Interscience Publication. MR 1028781. Zbl 0689.04001.

[RW10] M. Rørdam and W. Winter, The Jiang-Su algebra revisited, *J. Reine Angew. Math.* **642** (2010), 129–155. MR 2658184 (2011i:46074). Zbl 1209.46031.

[Tik11] A. Tikuisis, The Cuntz semigroup of continuous functions into certain simple C^*-algebras, *Internat. J. Math.* **22** (2011), 1051–1087. MR 2826555 (2012h:46102).

[Tom08] A. S. Toms, On the classification problem for nuclear C^*-algebras, *Ann. of Math. (2)* **167** (2008), 1029–1044. MR 2415391 (2009g:46119). Zbl 1181.46047.

[WZ09] W. Winter and J. Zacharias, Completely positive maps of order zero, *Münster J. Math.* **2** (2009), 311–324. MR 2545617 (2010g:46093). Zbl 1190.46042.

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