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ON UNIPOTENT RADICALS OF PSEUDO-REDUCTIVE GROUPS

MICHAEL BATE, BENJAMIN MARTIN, GERHARD RÖHRLE, AND DAVID I. STEWART

Abstract. We establish some results on the structure of the geometric unipotent radicals of pseudo-reductive k-groups. In particular, let k' be a purely inseparable field extension of k of degree p^e and let G denote the Weil restriction of scalars $\mathrm{R}_{k'/k}(G')$ of a reductive k'-group G'. We prove that the unipotent radical $\mathscr{R}_u(G_{\bar{k}})$ of the extension of scalars of G to the algebraic closure $\bar{k}$ of k has exponent e. Our main theorem is to give bounds on the nilpotency class of geometric unipotent radicals of standard pseudo-reductive groups, which are sharp in many cases.

1. Introduction

Let G be a smooth affine algebraic k-group over an arbitrary field k. Then G is said to be *pseudo-reductive* if G is connected and the largest k-defined connected smooth normal unipotent subgroup $\mathscr{R}_{u,k}(G)$ of G is trivial. J. Tits introduced pseudo-reductive groups to the literature some time ago in a series of courses at the Collège de France ([Tit92] and [Tit93]), but they have resurfaced rather dramatically in recent years thanks to the monograph [CGP10], many of whose results were used in B. Conrad's proof of the finiteness of the Tate–Shafarevich sets and Tamagawa numbers of arbitrary linear algebraic groups over global function fields [Con12, Thm. 1.3.3]. The main result of that monograph is [CGP10, Thm. 5.1.1] which says that unless one is in some special situation over a field of characteristic 2 or 3, then any pseudo-reductive group is *standard*. This means it arises after a process of modification of a Cartan subgroup of a certain Weil restriction of scalars of a given reductive group (we assume reductive groups are connected). More specifically, a standard pseudo-reductive group G can be expressed as a quotient group of the form

$$G = (\mathrm{R}_{k'/k}(G') \rtimes C)/\mathrm{R}_{k'/k}(T')$$

corresponding to a 4-tuple $(G', k'/k, T', C)$, where k' is a non-zero finite reduced k-algebra, G' is a k'-group with reductive fibres over $\operatorname{Spec} k'$, T' is a maximal k'-torus of G' and C is a commutative pseudo-reductive k-group occurring in a factorisation

$$\mathrm{R}_{k'/k}(T') \xrightarrow{\phi} C \xrightarrow{\psi} \mathrm{R}_{k'/k}(T'/Z_{G'})$$

of the natural map $\varpi : \mathrm{R}_{k'/k}(T') \to \mathrm{R}_{k'/k}(T'/Z_{G'})$. Here $Z_{G'}$ is the (scheme-theoretic) centre of G'[1] and C acts on $\mathrm{R}_{k'/k}(G')$ via ψ followed by the functor $\mathrm{R}_{k'/k}$ applied to the conjugation action of $T'/Z_{G'}$ on G'; we regard $\mathrm{R}_{k'/k}(T')$ as a central subgroup of $\mathrm{R}_{k'/k}(G') \rtimes C$ via the map $h \mapsto (i(h)^{-1}, \phi(h))$, where i is the natural inclusion of $\mathrm{R}_{k'/k}(T')$ in $\mathrm{R}_{k'/k}(G')$.

The structure of general connected linear algebraic groups over perfect fields k is well-understood: the geometric unipotent radical $\mathscr{R}_u(G_{\bar{k}}) = \mathscr{R}_{u,\bar{k}}(G_{\bar{k}})$ descends to a subgroup $\mathscr{R}_u(G)$ of G, and the quotient $G^{\mathrm{red}} := G/\mathscr{R}_u(G)$ is reductive. If one further insists k be separably closed one even has

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[1] Note that ϖ is *not* surjective when $Z_{G'}$ has non-étale fibre at a factor field k'_i of k' that is not separable over k.

that G^{red} is split, so G^{red} is the central product of its semisimple derived group $\mathscr{D}(G)$ and a central torus, with G^{red}/Z_G semisimple—in fact, the direct product of its simple factors.

Most of this theory goes wrong over imperfect fields k, hence in particular the need to consider pseudo-reductive groups, whose geometric unipotent radicals may not be defined over k. In order to understand the structure of a given smooth affine algebraic group G over k it is therefore instructive to extend scalars to the (perfect) algebraically closed field $\bar{k}$ and analyse the structure of $G_{\bar{k}}$, where, for example, one sees the full unipotent radical. We pursue this approach in this paper and discuss the structure of $G_{\bar{k}}$ where G is a standard pseudo-reductive group G arising from a 4-tuple $(G', k'/k, T', C)$. The reductive part $G_{\bar{k}}^{\mathrm{red}} = G_{\bar{k}}/\mathscr{R}_u(G_{\bar{k}})$ is not especially interesting in that the universal property of Weil restriction implies that $G_{\bar{k}}^{\mathrm{red}}$ has the same root system as G'. Further results [CGP10, Thm. 3.4.6, Cor. A.5.16] even furnish us, under some restrictive conditions, with a Levi subgroup for G: a smooth subgroup H such that $H_{\bar{k}}$ is a complement to the geometric unipotent radical $\mathscr{R}_u(G_{\bar{k}})$. However, the precise structure of $\mathscr{R}_u(G_{\bar{k}})$ is rather more mysterious. While one knows that there is a composition series of $\mathscr{R}_u(G_{\bar{k}})$ whose composition factors are related to the adjoint $G'_{\bar{k}}$-module $\mathfrak{g}' = \mathrm{Lie}(G'_{\bar{k}})$ (see Lemma 2.2), it is unclear what the structure of $\mathscr{R}_u(G_{\bar{k}})$ is *qua* group. Since $\mathscr{R}_u(G_{\bar{k}})$ is a p-group, one may consider some standard invariants, which measure the order of its elements and the extent to which it is non-abelian. One major purpose of this paper is to show that as soon as the root system associated to G is non-trivial and k'/k is finite inseparable of odd characteristic, then $\mathscr{R}_u(G_{\bar{k}})$ is highly non-abelian (in a way that depends on the characteristic of the field amongst other things).

Recall that the exponent of a finite purely inseparable field extension k'/k in characteristic p is the minimum integer e such that $t^{p^e} \in k$ for all $t \in k'$. If k'/k is any finite field extension then k' is purely inseparable over the separable closure k_1 of k in k' and we refer to k'/k_1 as the *inseparable part* of the extension k'/k; in this case we say that the *exponent* of k'/k is that of k'/k_1. Lastly if $k' = \prod_{i\in I} k_i$ is a non-zero finite reduced k-algebra, with factor fields k_i, its *exponent* is the maximal exponent of the k_i/k. If the reductive k'-group G' is non-commutative (resp., non-trivial) then we define $e_{G'}$ (resp., $\widetilde{e}_{G'}$) to be the maximum of the exponents of k_i/k, where we range over all $i \in I$ such that the fibre G_i of G' over $\operatorname{Spec} k_i$ is non-commutative (resp., non-trivial); if k' is a field then $e_{G'}$ (resp., $\widetilde{e}_{G'}$) is just the exponent of k'/k, whenever G' is non-commutative (resp. non-trivial).

Theorem 1.1. *Let G be a non-commutative standard pseudo-reductive group over a field k, arising from a 4-tuple $(G', k'/k, T', C)$ with k' a non-zero finite reduced k-algebra. Suppose p is odd and the centre $Z_{G'}$ is smooth. Then the unipotent radical $\mathscr{R}_u(G_{\bar{k}})$ has nilpotency class at least $p^{e_{G'}} - 1$.*

Moreover, if the inseparable parts of the factor fields k_i of k' are all primitive extensions of k, then $\mathscr{R}_u(G_{\bar{k}})$ has precisely this nilpotency class. In particular, if $e_{G'} = 1$, then the nilpotency class of $\mathscr{R}_u(G_{\bar{k}})$ is exactly $p - 1$.

Note that in particular, $Z_{G'}$ is smooth if p is very good for G' or if G' is semisimple and of adjoint type.

What the theorem indicates is that the precise structure of $\mathscr{R}_u(G_{\bar{k}})$, including its nilpotency class, appears to depend in a very particular way on the nature of the field extension k'/k used in Weil restriction, rather than on, for instance, the reductive group G'. Specifically, the structure appears to depend primarily on the lattice of subfields of k' containing k. (This is made a little more precise in Conjecture 1.3 below.)

One gets a little further towards understanding this arrangement when considering the orders of the elements of $\mathscr{R}_u(G_{\bar{k}})$ in the case that G is a Weil restriction of a reductive group. Recall that the exponent of a p-group R is the maximum e such that R has an element of order p^e. This definition can be extended to a smooth unipotent k-group scheme U by defining its exponent to be the minimum e such that the p^e-power map $U \to U$ factors through the trivial group scheme. This is equivalent to asking for the minimal e such that $x^{p^e} = 1$ for all $x \in U(A)$ and for all k-algebras A.

Theorem 1.2. *Let k' be a non-zero finite reduced k-algebra and let $G = \mathrm{R}_{k'/k}(G')$ for some non-trivial reductive k'-group G'. Then the unipotent radical $\mathscr{R}_u(G_{\bar{k}})$ of $G_{\bar{k}}$ has exponent $\widetilde{e}_{G'}$.*

Note that Theorem 1.2 fails in the context of a general standard pseudo-reductive group. For instance, given a finite purely inseparable field extension k'/k with exponent e, we can take G to be the standard group arising from the 4-tuple $(G', k'/k, T', C)$, where $G' = T' = 1$ and $C = \mathrm{R}_{k''/k}(\mathbb{G}_m)$, with k''/k a finite purely inseparable field extension of arbitrarily large exponent f: for then $G \cong C$, and the theorem itself tells us that $\mathscr{R}_u(G_{\bar{k}})$ has exponent f. The proof of the theorem we give uses an easy calculation in the case G' is isomorphic to the multiplicative group $\mathbb{G}_m$ together with the density of Cartan subgroups; one may also appeal to [CGP10, Prop. A.5.12].

If $p = 2$ and k'/k has exponent 1 in Theorem 1.2 then $\mathscr{R}_u(G_{\bar{k}})$ is abelian (see Corollary 4.2). To sharpen this point, note that here k'/k can have arbitrarily large degree, yet the nilpotency class of $\mathscr{R}_u(G_{\bar{k}})$ remains constant, equal to 1. This is evidence towards the following (see also Example 2.3).

Conjecture 1.3. *Let G be a non-commutative standard pseudo-reductive group arising from a 4-tuple $(G', k'/k, T', C)$ such that k'/k is a non-zero finite reduced k-algebra. Then the nilpotency class of $\mathscr{R}_u(G_{\bar{k}})$ is $p^{e_{G'}} - 1$.*

One striking aspect of the conjecture, and the evidence we have collected above, is that the structure of $\mathscr{R}_u(G_{\bar{k}})$ depends as little as possible on the reductive group G' one starts with. For example, while the nilpotency class of a Borel subgroup of G' can grow arbitrarily with the rank of a simple factor, nevertheless the root system of G' does not feature in the conclusion at all. Certainly we observe that the exponent of $\mathscr{R}_u(G_{\bar{k}})$ is independent of the root system, while the possible orders of arbitrary unipotent elements of G are most definitely not.

Finally, let us remark that the proof of Theorem 1.1 reduces to a bare-hands calculation with matrices arising from the Weil restriction $\mathrm{R}_{k'/k}(\mathrm{GL}_2)$ and may be of interest to anyone who would like to see some examples of pseudo-reductive groups in an explicit description by matrices.

2. Notation and Preliminaries

We follow the notation of [CGP10]. In particular, k is always a field of characteristic $p > 0$. All algebraic groups are assumed to be affine and of finite type over the ground ring, and all subgroups are closed. Reductive and pseudo-reductive groups are assumed to be smooth and connected.

Let us recall the definition of Weil restriction and some relevant features. We consider algebraic groups scheme-theoretically, so that an algebraic k-group G is a functor $\{k\text{-algebras} \to \text{groups}\}$ which is representable via a finitely-presented k-algebra $k[G]$. Let k' be a non-zero finite reduced k-algebra. Then for any smooth k'-group G' with connected fibres over $\operatorname{Spec} k'$, the Weil restriction $G = \mathrm{R}_{k'/k}(G')$ is a smooth connected k-group of dimension $[k' : k] \dim G'$, characterised by the property $G(A) = G'(k' \otimes_k A)$ functorially in k-algebras A. If H' is a subgroup of G' then $\mathrm{R}_{k'/k}(H')$ is a subgroup of $\mathrm{R}_{k'/k}(G')$. For a thorough treatment of this, one may see [CGP10, A.5]. Important for us is the fact that Weil restriction is right adjoint to base change: that is, we have a bijection

$$\mathrm{Hom}_k(M, \mathrm{R}_{k'/k}(G')) \cong \mathrm{Hom}_{k'}(M_{k'}, G')$$

natural in the k-group scheme M. We need two special cases. First, if $M = G = \mathrm{R}_{k'/k}(G')$ then the identity morphism $G \to G$ corresponds to a map $q_{G'} : G_{k'} \to G'$. When k' is a finite purely inseparable field extension of k then by [CGP10, Thm. 1.6.2], $q_{G'}$ is smooth and surjective, $\ker q_{G'}$ coincides with $\mathscr{R}_{u,k'}(G_{k'})$ and $\mathscr{R}_{u,k'}(G_{k'})$ is a descent of $\mathscr{R}_u(G_{\bar{k}})$ (so $\mathscr{R}_u(G_{\bar{k}})$ is defined over k'). In particular, $\dim \mathscr{R}_{u,k'}(G_{k'}) = ([k' : k] - 1) \dim G'$. Second, if H is a reductive k-group and $M = G' = H_{k'}$ then the identity morphism $H_{k'} \to H_{k'}$ corresponds to a map $i_H : H \to \mathrm{R}_{k'/k}(H_{k'})$. When k' is a finite purely inseparable field extension of k then by [CGP10, Cor. A.5.16], the composition $H_{k'} \xrightarrow{(i_H)_{k'}} \mathrm{R}_{k'/k}(H_{k'}) \xrightarrow{q_{H_{k'}}} H_{k'}$ is the identity, so we may regard H as a subgroup of $\mathrm{R}_{k'/k}(H_{k'})$ via i_H; in fact, H is a Levi subgroup of $\mathrm{R}_{k'/k}(H_{k'})$.

Let $k' = \prod_{i=1}^n k_i$ be a non-zero finite reduced k-algebra with factor fields k_i. Then any algebraic k'-group G' decomposes as a product $G' = \prod_{i=1}^n G_i$ where each G_i is an algebraic k_i-group (it is the fibre of G' over $\mathrm{Spec}\, k_i$). In such a situation, we let $\mathscr{R}_{u,k'}(G')$ denote the unipotent subgroup $\prod_{i=1}^n \mathscr{R}_{u,k_i}(G_i)$. We denote by $\mathfrak{g}'$ the Lie algebra of G'.

Suppose now that H is a smooth algebraic group over k and let $[g, h] = ghg^{-1}h^{-1}$ denote the commutator of the elements $g, h \in H(\bar{k})$. Let $\{\mathscr{D}_m(H)\}_{m \geq 0}$ be the lower central series of H; note that $\mathscr{D}_m(H)(\bar{k})$ is the m^{th} term in the lower central series for the abstract group $H(\bar{k})$. We say that H is *nilpotent* if there exists some integer m such that $\mathscr{D}_m(H) = 1$. The *nilpotency class* $\mathrm{cl}(H)$ of H is the smallest integer m such that $\mathscr{D}_m(H) = 1$. Since $[H_{k'}, K_{k'}] = ([H, K])_{k'}$ for any two smooth connected k-groups H and K (cf. [Bor91, I.2.4]), extending the base field does not change the nilpotency class of H.

In proving Theorem 1.1 and intermediate results, we sometimes want to reduce to the case that k is separably closed, guaranteeing that k_i/k is purely inseparable for k_i a factor field of k'. This also allows us to assume that the group G' has *split* reductive fibres. We denote by k_s the separable closure of k in its algebraic closure $\bar{k}$, and we set $k'_s = k' \otimes_k k_s$, a non-zero finite reduced k_s-algebra. Even when k' is a field, k'_s need not be a field, but in this case the exponents of the field extensions k_i/k_s are all equal to the exponent of k'/k, where $k'_s = \prod_{i \in I} k_i$.

Lemma 2.1. *Let G be a standard pseudo-reductive group arising from $(G', k'/k, T', C)$. Then*

(i) G_{k_s} is isomorphic to the standard pseudo-reductive group arising from $(G'_{k'_s}, k'_s/k_s, T'_{k'_s}, C_{k_s})$.

(ii) $\mathscr{R}_{u,k'}(G_{k'})_{k'_s} = \mathscr{R}_{u,k'_s}(G_{k'_s})$.

Proof. (i). We have $G = (\mathrm{R}_{k'/k}(G') \rtimes C)/\mathrm{R}_{k'/k}(T')$. In order to see that

$$G_{k_s} \cong (\mathrm{R}_{k'_s/k_s}(G'_{k'_s}) \rtimes C_{k_s})/\mathrm{R}_{k'_s/k_s}(T'_{k'_s})$$

it suffices to see that: (a) the sequences

$$1 \to \mathrm{R}_{k'/k}(T') \to \mathrm{R}_{k'/k}(G') \rtimes C \to G \to 1$$

and

$$1 \to \mathrm{R}_{k'/k}(G') \to \mathrm{R}_{k'/k}(G') \rtimes C \to C \to 1$$

remain exact after taking the base change to k_s, with the second remaining split; and (b) the formation of the Weil restriction $\mathrm{R}_{k'/k}(G')$ commutes with base change to k_s.

These facts are standard: for (a) this follows since algebras over a field k are flat; and (b) is [CGP10, A.5.2(1)].

(ii). Write $k'_s = \prod_{i\in I} k_i$. Since

$$\mathscr{R}_{u,k'}(G_{k'})_{\bar{k}} = \mathscr{R}_u(G_{\bar{k}}) = \mathscr{R}_u((G_{k_s})_{\bar{k}}) = \mathscr{R}_u((G_{k_i})_{\bar{k}}) = \mathscr{R}_{u,k_i}(G_{k_i})_{\bar{k}},$$

we have $\mathscr{R}_{u,k'}(G_{k'})_{k_i} = \mathscr{R}_{u,k_i}(G_{k_i})$ for each $i \in I$. The result now follows. □

The following result will help us to identify the nilpotency class of $\mathscr{R}_u(G)$ when G is a pseudo-reductive group.

Lemma 2.2. *Let $G = \mathrm{R}_{k'/k}(G')$ for k' a non-zero finite reduced k-algebra and G' a reductive k'-group. Then $\mathscr{R}_u(G_{\bar{k}})$ has a filtration whose successive quotients are $G'_{\bar{k}}$-equivariantly isomorphic to subquotients of the adjoint $G'_{\bar{k}}$-module.*

In case k' is a finite purely inseparable field extension of k, G' is absolutely simple and p is very good for G', then $\mathrm{Lie}(\mathscr{R}_u(G_{\bar{k}}))$ is an isotypic $G'_{\bar{k}}$-module containing precisely $[k' : k] - 1$ copies of the adjoint module.

Proof. By Lemma 2.1 we may assume k is separably closed. Clearly we can assume k' is a finite purely inseparable field extension of k. Set $q = [k' : k]$. Let $f' \colon G' \to \mathrm{GL}(\mathfrak{g}')$ denote the adjoint representation. By the discussion in [CGP10, Sec. A.7], we may identify $\mathrm{Lie}(\mathrm{R}_{k'/k}(G'))$ with $\mathrm{R}_{k'/k}(\mathfrak{g}')$, and the adjoint representation f of $\mathrm{R}_{k'/k}(G')$ on $\mathrm{R}_{k'/k}(\mathfrak{g}')$ is given by the composition $\mathrm{R}_{k'/k}(G') \to \mathrm{R}_{k'/k}(\mathrm{GL}(\mathfrak{g}')) \to \mathrm{GL}(\mathrm{R}_{k'/k}(\mathfrak{g}'))$, where the first map is $\mathrm{R}_{k'/k}(f')$ and the second is the natural inclusion.

Now G' is split over k', so by the standard structure theory of reductive groups, there is a reductive k-group H such that $G' = H_{k'}$. We obtain a representation of H via the composition $H \stackrel{i_H}{\to} \mathrm{R}_{k'/k}(H_{k'}) \stackrel{f}{\to} \mathrm{GL}(\mathrm{R}_{k'/k}(\mathfrak{h}_{k'}))$, where $\mathfrak{h} := \mathrm{Lie}\, H$. We claim that

$(*)$ the H-module $\mathrm{R}_{k'/k}(\mathfrak{h}_{k'})$ is the direct sum of q copies of the adjoint module $\mathfrak{h}$.

In fact, one can easily show that the analogous result holds if we replace the adjoint representation $H \to \mathrm{GL}(\mathfrak{h})$ in this construction with any representation $\rho \colon H \to \mathrm{GL}(V)$; we simply choose explicit bases for V over k and for k' over k (cf. the calculations at the start of Section 3). Note that we need not assume here that k is separably closed.

If G' is absolutely simple and p is very good for G' then $\mathfrak{g}'_{\bar{k}} = \mathfrak{h}_{\bar{k}}$ is irreducible as a $G_{k'}$-module. It follows from $(*)$ after extending scalars that $\mathfrak{g}_{\bar{k}} = \mathrm{Lie}(\mathrm{R}_{k'/k}(G')_{\bar{k}})$ is the sum of q copies of the adjoint module $\mathfrak{g}'_{\bar{k}}$. Since $G_{\bar{k}} \cong \mathscr{R}_u(G_{\bar{k}}) \rtimes G'_{\bar{k}}$, we obtain the second assertion of the lemma.

By [McN14, Thm. B] (or [Ste13, Prop. 3.3.5]), $\mathscr{R}_u(\mathrm{R}_{k'/k}(G')_{\bar{k}})$ has a filtration by $G'_{\bar{k}}$-modules for the group $G'_{\bar{k}}$; more precisely, there is a filtration of $\mathscr{R}_u(\mathrm{R}_{k'/k}(G')_{\bar{k}})$ by vector groups V_i such that each is $G'_{\bar{k}}$-equivariantly isomorphic to $\mathrm{Lie}(V_i)$. Since any such must consist of subquotients of $\mathfrak{g}'_{\bar{k}}$, the first assertion of the lemma follows from $(*)$. □

Example 2.3. The result $(*)$ from Lemma 2.2 does not extend to an arbitrary standard pseudo-reductive group. For example, if $p = 2$, $G' = \mathrm{SL}_2$, k'/k is a purely inseparable field extension of degree 2 and G is the standard pseudo-reductive group $\mathrm{R}_{k'/k}(G')/\mathrm{R}_{k'/k}(\mu_2)$, then G has dimension $\dim G' \cdot [k' : k] - (p-1) = 5$; see [CGP10, Ex. 1.3.2]. Since there is at least one copy of the k-space $\mathfrak{g}' = \mathrm{Lie}(G')$ corresponding to the canonical copy of G' in $G_{k'}$, we see that $\mathrm{Lie}(G_{\bar{k}})$ cannot be a sum of copies of $\mathfrak{g}'_{\bar{k}}$, just for dimensional reasons. (Similar examples can be given for any G' whose centre is not smooth.) Note in this example that $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G'))$ is abelian by Corollary 4.2, so $\mathscr{R}_{u,k'}(G)$ is abelian; this gives further evidence for Conjecture 1.3.

3. Proof of Theorem 1.1.

Up until Proposition 3.2, we allow p to be any prime, not necessarily odd. We will write elements of $\mathrm{GL}_2(k')$ in the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{ij} \in k'$ such that $a_{11}a_{22} - a_{12}a_{21} \in (k')^{\times}$.

Let k' have basis $\{1 = \alpha_1, \alpha_2, \dots, \alpha_q\}$ as a k-vector space. Set

$$A = \begin{pmatrix} a_{11}^1 + \alpha_2 a_{11}^2 + \cdots + \alpha_q a_{11}^q & a_{12}^1 + \alpha_2 a_{12}^2 + \cdots + \alpha_q a_{12}^q \\ & \\ a_{21}^1 + \alpha_2 a_{21}^2 + \cdots + \alpha_q a_{21}^q & a_{22}^1 + \alpha_2 a_{22}^2 + \cdots + \alpha_q a_{22}^q \end{pmatrix},$$

where $1 \le i, j \le 2$, $1 \le r \le q$ and each a_{ij}^r belongs to k—note that here we use r as an indexing superscript; it does not indicate a power. Note also that we really think of A as being a variable matrix, depending on the parameters α_i and a_{ij}^r, which we suppress when invoking A. Then $\mathrm{R}_{k'/k}(\mathrm{GL}_2)(k)$ is the collection of matrices A such that A is invertible, as the parameters a_{ij}^r run over all elements of k.

Let V be the natural module for GL_2, so that $V(k') \cong k' \oplus k'$ has k'-basis $\{b_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), b_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$. Then $\mathrm{R}_{k'/k}(\mathrm{GL}_2)$ acts on the $2q$-dimensional k-vector space $\mathrm{R}_{k'/k}(V)$. Now $\mathrm{R}_{k'/k}(V)(k)$ has k-basis $\{b_1, \alpha_2 b_1, \dots, \alpha_q b_1, b_2, \alpha_2 b_2, \dots, \alpha_q b_2\}$, so $\mathrm{R}_{k'/k}(\mathrm{GL}_2)$ naturally identifies with a smooth subgroup of GL_{2q} via its action on this k-basis. In the case when $k' = k(t)$ is a primitive, purely inseparable extension of k with exponent e (so that $t^{p^e} \in k$ and $q = p^e$), then we may write $\alpha_i = t^{i-1}$. Hence we have $\mathrm{R}_{k'/k}(\mathrm{GL}_2) \hookrightarrow \mathrm{GL}_{2q}$ via $A \mapsto \widehat{A}$, where

$$\widehat{A} := \left(\begin{array}{cccccc|cccccc}
a_{11}^1 & t^q a_{11}^q & t^q a_{11}^{q-1} & t^q a_{11}^{q-2} & \dots & t^q a_{11}^2 & a_{12}^1 & t^q a_{12}^q & t^q a_{12}^{q-1} & t^q a_{12}^{q-2} & \dots & t^q a_{12}^2 \\
a_{11}^2 & a_{11}^1 & t^q a_{11}^q & t^q a_{11}^{q-1} & \dots & t^q a_{11}^3 & a_{12}^2 & a_{12}^1 & t^q a_{12}^q & t^q a_{12}^{q-1} & \dots & t^q a_{12}^3 \\
a_{11}^3 & a_{11}^2 & a_{11}^1 & t^q a_{11}^q & \dots & t^q a_{11}^4 & a_{12}^3 & a_{12}^2 & a_{12}^1 & t^q a_{12}^q & \dots & t^q a_{12}^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{11}^q & a_{11}^{q-1} & a_{11}^{q-2} & a_{11}^{q-3} & \dots & a_{11}^1 & a_{12}^q & a_{12}^{q-1} & a_{12}^{q-2} & a_{12}^{q-3} & \dots & a_{12}^1 \\
\hline
a_{21}^1 & t^q a_{21}^q & t^q a_{21}^{q-1} & t^q a_{21}^{q-2} & \dots & t^q a_{21}^2 & a_{22}^1 & t^q a_{22}^q & t^q a_{22}^{q-1} & t^q a_{22}^{q-2} & \dots & t^q a_{22}^2 \\
a_{21}^2 & a_{21}^1 & t^q a_{21}^q & t^q a_{21}^{q-1} & \dots & t^q a_{21}^3 & a_{22}^2 & a_{22}^1 & t^q a_{22}^q & t^q a_{22}^{q-1} & \dots & t^q a_{22}^3 \\
a_{21}^3 & a_{21}^2 & a_{21}^1 & t^q a_{21}^q & \dots & t^q a_{21}^4 & a_{22}^3 & a_{22}^2 & a_{22}^1 & t^q a_{22}^q & \dots & t^q a_{22}^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{21}^q & a_{21}^{q-1} & a_{21}^{q-2} & a_{21}^{q-3} & \dots & a_{21}^1 & a_{22}^q & a_{22}^{q-1} & a_{22}^{q-2} & a_{22}^{q-3} & \dots & a_{22}^1
\end{array}\right).$$

Of course, $\mathrm{R}_{k'/k}(\mathrm{GL}_2)(k')$ is obtained from $\mathrm{R}_{k'/k}(\mathrm{GL}_2)(k)$ by allowing the coefficients a_{ij}^r to take values in k'. If we take H to be the k-group GL_2 then the map $i_H \colon H \to \mathrm{R}_{k'/k}(H_{k'})$ is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}I & a_{12}I \\ a_{21}I & a_{22}I \end{pmatrix}$$

for $a_{ij} \in k$, where I is the $q \times q$ identity matrix. Note that as an H-module over k, $(k')^2 \cong k^{2q}$ is a sum of q copies of the natural module for H (cf. Lemma 2.2).

Lemma 3.1. *Let G' be the k'-group GL_2 and let $G = \mathrm{R}_{k'/k}(G')$ for k' a purely inseparable field extension of k with $[k' : k] = q = p^e$. Then $\mathscr{R}_u(G_{\bar{k}})(\bar{k})$ is found as the subset of $G(\bar{k})$ by setting*

$$a_{ij}^1 = \delta_{ij} - (\alpha_2 a_{ij}^2 + \cdots + \alpha_q a_{ij}^q).$$

Proof. The canonical map $q_{G'} : G_{k'} \to G'$ satisfies $\ker q_{G'} = \mathscr{R}_{u,k'}(G_{k'})$ and this is a descent of the geometric unipotent radical. Following the concrete description of the map $q_{G'}$ in [CGP10, A.5.7], we see that on k'-points, $q_{G'}$ is realised by sending $\widehat{A} \in G_{k'}(k') \subseteq \mathrm{GL}_{2q}(k')$ back to the matrix A, where the coefficients a_{ij}^r are now taken in k'. The k'-unipotent radical of $G_{k'}$ is then the kernel of this map, thus is given by choosing the a_{ij}^r in such a way that the resulting matrix is the identity. This is achieved by setting a_{ij}^1 to be as proposed in the lemma. □

Again let $k' = k(t)$ be a purely inseparable extension with exponent e. Then we consider the element $X := \widehat{A} \in \mathscr{R}_{u,k'}(G_{k'})(k') \subseteq \mathrm{GL}_{2q}(k')$ arising from setting $a_{ij}^r = \delta_{i1}\delta_{ij}\delta_{2r}t^{-1} + \delta_{i2}\delta_{ij}\delta_{1r}$. Explicitly, we have

$$X = \left(\begin{array}{cccccc|cccccc} 0 & 0 & 0 & \dots & 0 & t^{q-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ t^{-1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & t^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t^{-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{array}\right),$$

where by inspection the top-left block matrix M is given by $M = t^{-1}M'$, with M' a companion matrix for the polynomial

$$s(x) = x^q - t^q.$$

In particular, as the minimal polynomial of X is $s(x)$ and M is unipotent, the matrix $M - I$ is nilpotent of rank $q - 1$. Thus, over the algebraic closure $\bar{k}$, it is similar to a full Jordan block.

Now let Q be the subgroup of $\mathscr{R}_{u,k'}(G_{k'})(k')$ consisting of the matrices of the form

$$Y = \begin{pmatrix} I & N \\ 0 & I \end{pmatrix},$$

where I and N are $q \times q$ matrices and I denotes the identity. As X is of the form

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix},$$

it follows that X normalises Q. Let $[X, Y] = XYX^{-1}Y^{-1}$ denote the commutator of the two matrices X and Y. Then one checks that

$$[X, Y] = \begin{pmatrix} I & (M - I)N \\ 0 & I \end{pmatrix}$$

so that the iterated commutator is given by

$$[\underbrace{X,[X,\ldots,[X}_{n},Y]]] = \begin{pmatrix} I & (M-I)^n N \\ 0 & I \end{pmatrix}. \tag{1}$$

Let $\nu\colon Q \to k^q$ be the function that maps $\begin{pmatrix} I & N \\ 0 & I \end{pmatrix}$ to the first column of N. Clearly ν is X-equivariant, and the image $U := \nu(Q)$ is a $(q-1)$-dimensional subspace of k^q. Explicitly, U consists of all vectors of the form

$$(-ta_{12}^2 - t^2 a_{12}^3 - \cdots - t^{q-1}a_{12}^q, a_{12}^2, a_{12}^3, \ldots, a_{12}^q)^{\mathrm{tr}},$$

with each $a_{12}^r \in k'$. Now since the nullity of $(M-I)^{q-2}$ is $q-2$, the kernel in V of $(M-I)^{q-2}$ cannot contain the whole of the $(q-1)$-dimensional space U. This shows that the right-hand side of (1) is non-zero for some $Y \in Q$ when $n = q-2$. We conclude that the nilpotency class of $\mathscr{R}_{u,k'}(G_{k'})$ is at least $q-1$. However, we actually have the following, essentially a verification of Theorem 1.1 in the rank 1 split case.

Proposition 3.2. *Let $k' = k(t)$ be a purely inseparable primitive field extension of k with exponent e, where k is a field of characteristic $p > 2$. Set $q = p^e$. Then the nilpotency classes of the groups $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{GL}_2)_{k'})$, $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{SL}_2)_{k'})$ and $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{PGL}_2)_{k'})$ coincide with the integer $q-1$.*

To prove this we need the following two lemmas, which we also use in the proof of Theorem 1.1.

Lemma 3.3. *Let k' be a non-zero finite reduced k-algebra and let G' be a semisimple k'-group such that $Z' := Z_{G'}$ is smooth. Let $G'_{\mathrm{ad}} = G'/Z'$ and let $G := \mathrm{R}_{k'/k}(G')$. Then there is a natural isomorphism*

$$\mathscr{R}_{u,k'}(G_{k'}) \cong \mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'}).$$

Proof. Since Z' is smooth, there is a smooth isogeny giving rise to the exact sequence

$$1 \to Z' \to G' \xrightarrow{\pi} G'_{\mathrm{ad}} \to 1.$$

By [CGP10, A.5.4(3)], Weil restriction preserves the exactness of this sequence, and so there is an exact sequence

$$1 \to \mathrm{R}_{k'/k}(Z') \to G \xrightarrow{\mathrm{R}_{k'/k}(\pi)} \mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \to 1.$$

This gives, after base change to k', an exact sequence

$$1 \to \mathrm{R}_{k'/k}(Z')_{k'} \to G_{k'} \xrightarrow{\mathrm{R}_{k'/k}(\pi)_{k'}} \mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'} \to 1.$$

Now Z' is a smooth finite group scheme, so $Z' \cong \mathrm{R}_{k'/k}(Z')_{k'}$ as algebraic groups. This implies that $\mathrm{R}_{k'/k}(\pi)_{k'}$ is a smooth isogeny. It follows that the map $\mathrm{R}_{k'/k}(\pi)_{\bar{k}}\colon G_{\bar{k}} \to \mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{\bar{k}}$ obtained by base change to $\bar{k}$ gives rise to an isomorphism from $\mathscr{R}_u(G_{\bar{k}})$ to $\mathscr{R}_u(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{\bar{k}})$. But $\mathscr{R}_u(G_{\bar{k}})$ and $\mathscr{R}_u(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{\bar{k}})$ are defined over k', so $\mathrm{R}_{k'/k}(\pi)_{k'}$ gives an isomorphism from $\mathscr{R}_{u,k'}(G_{k'})$ to $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'})$, as required. ∎

Lemma 3.4. *Keep the notation and hypotheses of Lemma 3.3. Further, let C be a commutative pseudo-reductive k-group occurring in a factorisation of the map $\mathrm{R}_{k'/k}(T') \to \mathrm{R}_{k'/k}(T'/Z')$ for T' a maximal k'-torus of G'. Then*

$$\mathrm{cl}(\mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G') \rtimes C)_{k'})) = \mathrm{cl}(\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'})) = \mathrm{cl}(\mathscr{R}_{u,k'}(((\mathrm{R}_{k'/k}(G') \rtimes C)/\mathrm{R}_{k'/k}(T'))_{k'})),$$

where in the final term we quotient by the usual central copy of $\mathrm{R}_{k'/k}(T')$ *occurring in the standard construction.*

Proof. Let $\phi : \mathrm{R}_{k'/k}(T') \to C$ be the map in the factorisation. Set $Z = \mathrm{R}_{k'/k}(Z')$. Thanks to the isomorphism of groups $Z' \cong Z_{k'}$ arising from the hypothesis on Z', we have by the arguments of Lemma 3.3 a smooth isogeny

$$1 \to Z' \times \phi_{k'}(Z') \to \mathrm{R}_{k'/k}(G')_{k'} \rtimes C_{k'} \to \mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'} \rtimes C_{k'}/\phi_{k'}(Z') \to 1,$$

or equivalently

$$(*) \qquad 1 \to Z' \times \phi_{k'}(Z') \to (\mathrm{R}_{k'/k}(G') \rtimes C)_{k'} \to (\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)_{k'} \to 1,$$

where we write D in place of $C/\phi(Z)$. The argument of Lemma 3.3 yields an isomorphism

$$\mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G') \rtimes C)_{k'}) \cong \mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)_{k'}).$$

The map $C \to \mathrm{R}_{k'/k}(T'/Z')$ gives rise to a map $\kappa : D \to \mathrm{R}_{k'/k}(G'_{\mathrm{ad}})$. We claim that the map $\tau : \mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D \to \mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \times D$ given on the level of $\bar{k}$-points by $(g, d) \mapsto (g\kappa(d), d)$ is an isomorphism. To see this, first recall $(g, d)(h, e) = (g\kappa(d)h\kappa(d)^{-1}, de)$. Then $(g, d)(h, e) \mapsto (g\kappa(d)h\kappa(d)^{-1}\kappa(de), de) = (g\kappa(d)h\kappa(e), de)$ which is the product of $(g\kappa(d), d)$ and $(h\kappa(e), e)$ in the direct product as required.

With this in mind, we get

$$\mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)_{k'}) \cong \mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \times D)_{k'}) \cong \mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'}) \times \mathscr{R}_{u,k'}(D_{k'});$$

but the commutativity of D implies that $\mathrm{cl}(\mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D))_{k'}) = \mathrm{cl}(\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}})_{k'}))$. This proves the first equality of the lemma.

To see the second equality, observe that $\mathrm{R}_{k'/k}(T'/Z') \cong \mathrm{R}_{k'/k}(T')/Z$ since Z' is smooth. We see that the map $(\mathrm{R}_{k'/k}(G') \rtimes C)_{k'} \to (\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)_{k'}$ from $(*)$ takes the copy of $\mathrm{R}_{k'/k}(T')_{k'}$ onto the copy of $\mathrm{R}_{k'/k}(T'/Z')_{k'}$, so the induced map of quotient groups yields an isogeny

$$1 \to N \to (\mathrm{R}_{k'/k}(G') \rtimes C)_{k'}/\mathrm{R}_{k'/k}(T')_{k'} \to (\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)_{k'}/\mathrm{R}_{k'/k}(T'/Z')_{k'} \to 1.$$

for some N. This isogeny is smooth because the canonical projections to the quotient groups are smooth. By the argument of Lemma 3.3 again, we get an isomorphism

$$\mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G') \rtimes C)_{k'}/\mathrm{R}_{k'/k}(T')_{k'}) \cong \mathscr{R}_{u,k'}((\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)_{k'}/\mathrm{R}_{k'/k}(T'/Z')_{k'}).$$

Now $\phi_{k'}$ induces a map Φ from $\mathrm{R}_{k'/k}(T'/Z')_{k'}$ onto a smooth subgroup of D, and it is easily checked that τ gives rise to an isomorphism

$$(\mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \rtimes D)/\mathrm{R}_{k'/k}(T'/Z') \cong \mathrm{R}_{k'/k}(G'_{\mathrm{ad}}) \times D/\Phi(\mathrm{R}_{k'/k}(T'/Z')).$$

The commutativity of D again gives the result. ∎

Now we give the

Proof of Proposition 3.2. Let G be one of the groups $\mathrm{R}_{k'/k}(\mathrm{GL}_2)$, $\mathrm{R}_{k'/k}(\mathrm{PGL}_2)$ or $\mathrm{R}_{k'/k}(\mathrm{SL}_2)$, as in the statement of the proposition. According to [CGP10, Thm. 4.1.1], one may take a standard presentation for G as in Section 1: so G arises from the standard construction applied to the 4-tuple $(G', k'/k, T', C)$ where G' has simple and simply connected fibres over $\mathrm{Spec}\, k'$. In our case, this simply means that G' is of type SL_2. Now $Z_{G'}$ is smooth since $p \neq 2$, so by Lemmas 3.3 and 3.4 the nilpotency classes of $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{GL}_2)_{k'})$, $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{PGL}_2)_{k'})$ and $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{SL}_2)_{k'})$ are all equal. Hence we can now assume $G = \mathrm{R}_{k'/k}((\mathrm{SL}_2)_{k'})$. Since we have shown in the discussion before Proposition 3.2 that $\mathrm{cl}(\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathrm{GL}_2)_{k'})) \geq q-1$, we will be done if we can show that $\mathrm{cl}(\mathscr{R}_{u,k'}(G_{k'})) \leq q-1$.

For this, note that as $R := \mathscr{R}_{u,k'}(G_{k'})$ is unipotent, it is nilpotent and hence each successive quotient $\mathscr{D}_i(R)/\mathscr{D}_{i+1}(R)$ of terms of the lower central series of R is non-trivial. The subgroups $\mathscr{D}_i(R)$ are $G_{k'}$-stable, so $G_{k'}$ acts on each quotient $\mathscr{D}_i(R)/\mathscr{D}_{i+1}(R)$. Since G' is defined over k and is absolutely simple, we conclude from $(*)$ in Lemma 2.2 that the Lie algebra of R is an isotypic G'-module with composition factors isomorphic to $\mathfrak{g}'$, where G' is identified with its canonical copy $G' \subseteq \mathrm{R}_{k'/k}(G')_{k'} = G_{k'}$. Now each successive quotient $\mathscr{D}_i(R)/\mathscr{D}_{i+1}(R)$ gives rise to a non-trivial G'-module $\mathrm{Lie}(\mathscr{D}_i(R)/\mathscr{D}_{i+1}(R))$. Using again the irreducibility of the adjoint module $\mathfrak{g}'$, each successive quotient $\mathscr{D}_i(R)/\mathscr{D}_{i+1}(R)$ must be at least 3-dimensional, as $\dim \mathfrak{g}' = 3$. But as $\dim R = q \dim G' - \dim G' = 3(q-1)$, there can only be at most $q-1$ non-trivial terms in the lower central series of R. This proves that the unipotent radical R of G has nilpotency class at most $q-1$, so by our earlier remarks, it has exactly this nilpotency class. □

Remark 3.5. The earlier matrix calculations of this section did not require any assumption on the characteristic of k and show that the nilpotency class of $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G')_{k'})$ is at least $q-1$ when $G' \cong \mathrm{GL}_2$. Furthermore, Theorem 1.2 (yet to be proved) will give the nilpotency class of $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G')_{k'})$ as exactly 1 when $q = 2$ for G' being any non-trivial reductive group, so Proposition 3.2 holds in those cases too.

One final lemma before the proof of the main theorem sets down some useful interactions between unipotent radicals arising from Weil restrictions across towers of field extensions and arising from Weil restrictions of subgroups.

Lemma 3.6. *(i) Let G be a reductive k-group, and let $k'/\tilde{k}/k$ be a tower of finite purely inseparable field extensions. Let R be the unipotent radical $\mathscr{R}_{u,\tilde{k}}(\mathrm{R}_{\tilde{k}/k}(G_{\tilde{k}})_{\tilde{k}})$. Then $R_{k'}$ naturally identifies with a subgroup of $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G_{k'})_{k'})$.*

(ii) Let H be a reductive k-subgroup of the reductive k-group G, and let k'/k be a finite purely inseparable field extension. Then we have an inclusion of unipotent radicals $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(H_{k'})_{k'}) \subseteq \mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G_{k'})_{k'})$.

Proof. (i). We may regard $G_{\tilde{k}}$ as as a Levi subgroup of $\mathrm{R}_{k'/\tilde{k}}(G_{k'})$ via $(i_G)_{\tilde{k}}$. Now applying transitivity of Weil restriction, *viz.* $\mathrm{R}_{k'/k}(G_{k'}) = \mathrm{R}_{\tilde{k}/k}(\mathrm{R}_{k'/\tilde{k}}(G_{k'}))$, we see that $\mathrm{R}_{\tilde{k}/k}(G_{\tilde{k}})$ is a subgroup of $\mathrm{R}_{k'/k}(G_{k'})$. In particular, $R = \mathscr{R}_{u,\tilde{k}}(\mathrm{R}_{\tilde{k}/k}(G_{\tilde{k}})_{\tilde{k}})$ is a subgroup of $\mathrm{R}_{k'/k}(G_{k'})_{\tilde{k}}$. Now $G_{\tilde{k}}$ is a Levi subgroup of $\mathrm{R}_{\tilde{k}/k}(G_{\tilde{k}})_{\tilde{k}}$, so R is a complement to $G_{\tilde{k}}$ by virtue of being the unipotent radical of $\mathrm{R}_{\tilde{k}/k}(G_{\tilde{k}})_{\tilde{k}}$. Base-changing the inclusion maps to k', we get a subgroup $R_{k'}$ of $\mathrm{R}_{k'/k}(G_{k'})_{k'}$ having trivial projection to the Levi subgroup $G_{k'} = (G_{\tilde{k}})_{k'}$ under $q_{G'}$. Thus $R_{k'}$ is contained in $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G_{k'})_{k'})$.

(ii). The canonical map $\mathrm{R}_{k'/k}(G_{k'})_{k'} \to G_{k'}$ restricts to the identity on the canonical copy of $G_{k'}$ in $\mathrm{R}_{k'/k}(G_{k'})_{k'}$. Hence it commutes with restriction to the canonical copy of $H_{k'}$ in $\mathrm{R}_{k'/k}(H_{k'})_{k'}$; so the composite map $H_{k'} \to \mathrm{R}_{k'/k}(H_{k'})_{k'} \to \mathrm{R}_{k'/k}(G_{k'})_{k'} \to G_{k'}$ is an isomorphism on to its image. In particular, the kernel $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(H_{k'})_{k'})$ of the map $\mathrm{R}_{k'/k}(H_{k'})_{k'} \to G_{k'}$ is contained in the kernel $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(G_{k'})_{k'})$ of $\mathrm{R}_{k'/k}(G_{k'})_{k'} \to G_{k'}$. □

Proof of Theorem 1.1. Recall that $\mathscr{R}_u(G_{\bar{k}})$ has a descent to $\mathscr{R}_{u,k'}(G_{k'})$. We begin with a series of reductions using material already in place. By [CGP10, Thm. 4.1.1], we may assume that G' has simple and simply connected fibres over Spec k'. Using Lemma 3.4 we may prove the result in the case $G = \mathrm{R}_{k'/k}(G')$.

By Lemma 2.1, we have $\mathscr{R}_{u,k'}(G_{k'})_{k'_s} = \mathscr{R}_{u,k'_s}(G_{k'_s})$, so we can assume k is separably closed; in particular, we can assume G is split. Clearly we can reduce to the case when k'/k is a finite purely inseparable extension of fields. Further, using Lemma 3.3 we may assume G' is a direct product of simply connected simple groups.

Since Weil restriction, base change, and $\mathscr{R}_{u,k'}(\,\cdot\,)$ all distribute over direct products, it suffices to treat the case that G' is simply connected and absolutely simple. By the standard structure theory of reductive groups, there is a split simply connected absolutely simple k-group M such that $M_{k'} = G'$.

First we prove that $\mathrm{cl}(\mathscr{R}_{u,k'}(G_{k'})) \geq p^e - 1$. There must be an intermediate field $k \subseteq \tilde{k} \subseteq k'$ such that $\tilde{k}/k$ is primitive and has exponent e. But then Lemma 3.6(i) (applied to the k-group M) furnishes a containment $\mathscr{R}_{u,\tilde{k}}(\mathrm{R}_{\tilde{k}/k}(M_{\tilde{k}})_{\tilde{k}})_{k'} \subseteq \mathscr{R}_{u,k'}(G_{k'})$ of unipotent radicals. Hence it is enough to show that $\mathrm{cl}(\mathscr{R}_{u,\tilde{k}}(\mathrm{R}_{\tilde{k}/k}(M_{\tilde{k}})_{\tilde{k}})) \geq p^e - 1$; so we can assume that $k' = \tilde{k}$ is a primitive extension of k with exponent p^e. Since M is simple, simply connected and split, it must contain a k-subgroup H that is isomorphic to SL_2. It follows that $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(H_{k'})_{k'}) \subseteq \mathscr{R}_{u,k'}(G_{k'})$, invoking Lemma 3.6(ii) (applied to the inclusion of H in M). But $\mathrm{cl}(\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(H)_{k'})) \geq p^e - 1$ by Proposition 3.2 since p is odd, so $\mathrm{cl}(\mathscr{R}_{u,k'}(G_{k'}))$ is at least $p^e - 1$, as required.

It remains to show that when k' is a product of extension fields of k for which the inseparable part of the extension is primitive, then the nilpotency class of $\mathscr{R}_u(G_{\bar{k}})$ is at most $p^e - 1$. Given our reductions, it suffices to treat the case where $k' = k(t)$ is a primitive purely inseparable extension with exponent e; i.e., $[k' : k] = p^e$. By Lemma 2.2, each term of the lower central series of $\mathscr{R}_u(G_{\bar{k}})$ is a non-trivial isotypic $G'_{\bar{k}}$-module consisting of copies of the adjoint module $\mathfrak{g}'$, that module being irreducible by our hypothesis on p. Hence each successive term is of dimension at least $\dim \mathfrak{g}'$. But now there are at most $\dim \mathscr{R}_u(G_{\bar{k}})/\dim \mathfrak{g}' = ([k' : k] - 1) \dim G'/\dim \mathfrak{g}' = p^e - 1$ non-zero terms. This proves that the unipotent radical $\mathscr{R}_{u,k'}(G_{\bar{k}})$ of $G_{\bar{k}}$ has nilpotency class at most $p^e - 1$, so it has exactly this nilpotency class. ☐

4. Proof of Theorem 1.2

We begin by proving Theorem 1.2 for Weil restrictions of the multiplicative group $\mathbb{G}_m$. One bound on the exponent of the unipotent radical of the Weil restriction is essentially [CGP10, Ex. 1.1.3]. The other is an easy matrix calculation.

Proposition 4.1. *Let k' be a non-zero finite reduced k-algebra with exponent e and set $G = \mathrm{R}_{k'/k}(\mathbb{G}_m)_{k'}$. Then the exponent of $\mathscr{R}_{u,k'}(G_{k'})$ is e.*

Proof. We can assume by Lemma 2.1 that $k = k_s$ and $k' = k'_s$. If $k' = \prod_{i \in I} k_i$, then since $\mathrm{R}_{k'/k}(\mathbb{G}_m)$ is the direct product of the $\mathrm{R}_{k_i/k}(\mathbb{G}_m)$, it clearly suffices to prove the result in the case k' is a field. Now, as $((k')^\times)^{p^e} \subseteq k^\times$, the p^e-power map takes G into the canonical copy of $\mathbb{G}_m$ contained as a subgroup (cf. [CGP10, Ex. 1.1.3]). After extension to k' we see then that the image of the p^e-power map factors through the natural quotient map $q_{G'} : G_{k'} \to \mathbb{G}_m$, the kernel of which is $\mathscr{R}_{u,k'}(G_{k'})$. This shows that the exponent of $\mathscr{R}_{u,k'}(G_{k'})$ is no more than e.

Let $t \in k'$ such that $\tilde{k} := k(t)$ is a purely inseparable extension of k with exponent e. As in §3, an explicit computation identifying $\mathrm{R}_{\tilde{k}/k}(\mathbb{G}_m)$ with a subgroup of GL_{p^e} shows that

$$X = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & t^{q-1} \\ t^{-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & t^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & t^{-1} & 0 \end{pmatrix}$$

is an element of order $q := p^e$ in $R := \mathscr{R}_{u,\tilde{k}}(\mathrm{R}_{\tilde{k}/k}(\mathbb{G}_m)_{\tilde{k}})$; but from Lemma 3.6(i) we see that $R_{k'}$ is a subgroup of $\mathscr{R}_{u,k'}(G_{k'})$. Thus the exponent of $\mathscr{R}_{u,k'}(G_{k'})$ is at least e, proving the proposition. □

Proof of Theorem 1.2. First we show that R has exponent at most e. We may employ Lemma 2.1 to assume $k = k_s$. It clearly suffices to treat the case when k' is a field. Let e be the exponent of k'/k and let $R = \mathscr{R}_{u,k'}(G_{k'})$. It is enough to show that $x^{p^e} = 1$ for all $x \in R(\bar{k})$. Let T' be a maximal torus of G'. By [CGP10, A.5.15], $C := \mathrm{R}_{k'/k}(T')$ is a Cartan subgroup of G, so $C_{\bar{k}}$ is a Cartan subgroup of $G_{\bar{k}}$. The union of the conjugates of $C_{\bar{k}}$ contains a dense open subset U of $G_{\bar{k}}$, by [Spr98, Thm. 6.4.5(iii)]. Now $(R \cap U)(\bar{k})$ is nonempty (it contains the identity), so it is dense in $R(\bar{k})$. Hence it is enough to show that $x^{p^e} = 1$ for all $x \in (R \cap U)(\bar{k})$. The group C is abelian, so the unipotent elements of $C(\bar{k})$ are precisely the elements of $\mathscr{R}_u(C_{\bar{k}})(\bar{k})$, so any element of $(R \cap U)(\bar{k})$ is $G(\bar{k})$-conjugate to an element of $\mathscr{R}_u(C_{\bar{k}})(\bar{k})$. Hence it is enough to show that $\mathscr{R}_u(C_{\bar{k}})$ has exponent at most e. Since $k = k_s$, T' is split, so C is isomorphic to a product of copies of $\mathrm{R}_{k'/k}(\mathbb{G}_m)$. Proposition 4.1 now implies that $\mathscr{R}_{u,k'}(C_{k'})$ has exponent e, and we deduce that R has exponent at most e.

In light of Lemma 3.6(ii), the fact that some element of $\mathscr{R}_{u,k'}(\mathrm{R}_{k'/k}(\mathbb{G}_m)_{k'})$ has order p^e implies the same of R, so R has exponent at least e. Thus the exponent of R is e, as required. □

We have the following corollary (cf. Conjecture 1.3).

Corollary 4.2. *Let $p = 2$, let k' be a non-zero finite reduced k-algebra with exponent* 1 *and let $G = \mathrm{R}_{k'/k}(G')$ for some k'-group G' whose fibres over* $\mathrm{Spec}\, k'$ *are reductive. Then $\mathscr{R}_u(G_{\bar{k}})$ is abelian.*

Proof. Our hypotheses imply that $\widetilde{e}_{G'} \leq 1$. If all the fibres of G' over the $\mathrm{Spec}\, k'$ are trivial then $\mathscr{R}_u(G_{\bar{k}})$ is trivial, hence abelian. Otherwise Theorem 1.2 applies, so $\mathscr{R}_u(G_{\bar{k}})$ has exponent 1, so every non-trivial element of $\mathscr{R}_u(G_{\bar{k}})(\bar{k})$ has order 2. By a well-known result in undergraduate group theory, this means $\mathscr{R}_u(G_{\bar{k}})$ is abelian. □

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