Snapshots of modern mathematics
from Oberwolfach

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Closed geodesics on surfaces and Riemannian manifolds

Marco Radeschi[1]

Geodesics are special paths in surfaces and so-called *Riemannian manifolds* which connect close points in the shortest way. *Closed geodesics* are geodesics which go back to where they started. In this snapshot we talk about these special paths, and the efforts to find closed geodesics.

1 Geodesics

Imagine you are a tiny ant walking on a curved surface in space, for example on the surface of a donut. While walking, you would be able to tell, at each point of your journey, if your path is turning to the right, or turning to the left, or not turning at all. If the path you took was not turning at all, you would say that you took a "straight path". However, looking from far away, such paths would look all but straight. On the surface of the donut (mathematically called a *torus*), some "straight path" might look spiraling, while some other straight path might look oscillating up and down, see Figure 1.

Nevertheless, these paths still retain some remarkable properties of the usual straight paths we know: namely, they are the shortest paths between any two points along the path, provided these points are sufficiently close to each other. These "straight lines" are called *geodesics*. It turns out that, just as one might expect, wherever you are on the surface and for every possible direction, there is a unique way to move "straight" in that direction. In other words, there exists

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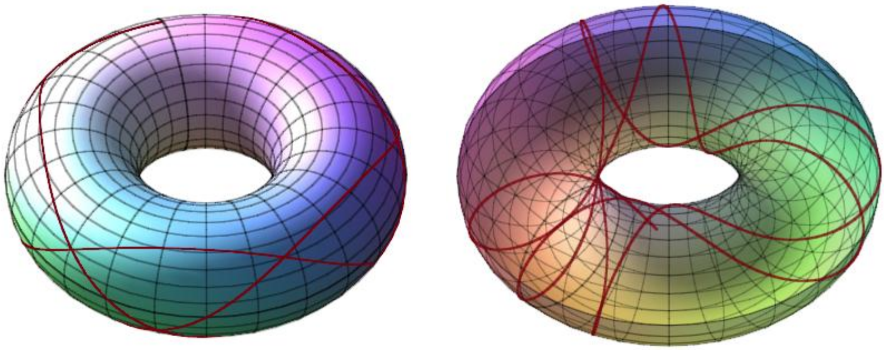


Figure 1: Two different geodesics on the torus.

a unique geodesic starting from that point in that direction. Moreover, this holds not just for surfaces, but for a whole class of spaces generalizing surfaces in higher dimension, called *Riemannian manifolds*. As a matter of fact, every result we will cite here holds for Riemannian manifolds in general!

When trying to understand certain properties of surfaces, it is very important to understand the behavior of geodesics. In general, this can be very complicated. The most special case is when a geodesic closes up, coming back to its initial point and pointing in the same direction as initially. Quite unimaginatively, these geodesics are called *closed geodesics*. Examples of this type are the equator of a round sphere, or the "meridians" of the torus. These geodesics have the special property of being *critical points for the length*: whenever $\{g_t : t \in (-1,1)\}$ is a family of closed paths that are "nearby" to a closed geodesic g_0 and $\ell(t)$ denotes the length of each path g_t, the following happens:

$$\frac{d\ell}{dt}(0) = 0.$$

Our goal here is to explore what is known about closed geodesics, in general and in some special cases.

2 Looking for closed geodesics

Here we ask ourselves the question: given a surface, can we find closed geodesics on that surface at all? And if so, how many distinct closed geodesics can we find? Some surfaces do not have closed geodesics: in the plane, for example, every geodesic is a straight line and so it can not close up. Notice that, unlike a sphere or a torus, the plane continues to infinity. We say that any surface that is contained in a bounded region of the space (and does not have any edges where you could fall off) is called *compact*.[2] For example, the sphere and the torus are compact, while the plane is not.

2.1 Topology and the existence of closed geodesics

If you think of a surface as a sheet of rubber in space, you can imagine deforming it, by squeezing and stretching some parts of it (without cutting or gluing). Whenever you apply such a deformation, some properties of the original surface will be modified, while others will not change. One distinguishes these two types of properties by saying that a certain property is *topological* if it does not depend on how the surface gets stretched or squeezed, while we say that a property is *geometric* if it depends on how the surface is sitting in space. For example, the *diameter* of a surface, defined as the maximal distance that two points on it can have (it says how "large" the surface is), is clearly a geometric concept, because it can be changed by squeezing and stretching the surface. On the other hand, properties like being compact, or having "holes" (like in the case of a torus) will not change after deformations of the surface and therefore are topological.

Since geodesics are (by definition) lines that locally minimize distance and the concept of distance depends on the geometry, geodesics are geometric objects, and in fact deforming the surface will change completely the shape of geodesics. If you try to keep the starting point and the initial direction of a closed geodesic but slightly deform the surface, it might happen that after the deformation the geodesic is not even closed any more! However, as we shall see in a moment, it has been proved that the mere *presence* of closed geodesics can be detected by topological properties! To put it in other words: there are certain *topological* properties of surfaces (and Riemannian manifolds in general) that tell us that – no matter how the surface is sitting in the surrounding space, and how deformed it looks – *somewhere* in the surface we will find a closed geodesic!

To illustrate how this is possible, consider the torus and a path on it which moves in some way around the "hollow" that is hidden by the surface (where in a real donut the jam filling might be). If you think of the path as a piece of string laid on the surface of the torus, you can imagine moving this path and trying to shorten it as much as possible. However, this string cannot be contracted to a single point on the torus because of the hole, and the shortest

[2] Strictly speaking, mathematicians usually call a surface also compact when it has an edge or *boundary*, provided the boundary is viewed as being part of the surface. But we do not want to consider this case in this snapshot.

possible path that you obtain will indeed be a closed geodesic. In short, the presence of the hole in the torus imposes the existence of at least one closed geodesic.

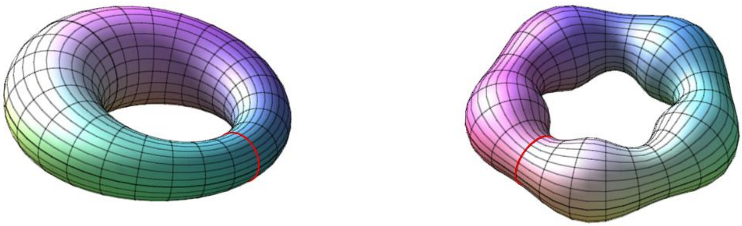


Figure 2: The hole of the torus imposes the existence of a closed geodesic (in red), independently of how the torus sits in space.

There are higher-dimensional analogues of the hole of the torus. For example, the hollow space of a sphere can be seen as a 2–dimensional hole". These 2–dimensional holes are again topological objects, and they can be used to produce closed geodesics as well, using a procedure that is somewhat similar to the one described above (but more complicated). For the interested reader, the concept of these "holes" is made precise using *homology theory* [3], and the procedure to obtain closed geodesics uses *Morse theory* [4].

Also, it turns our that *every* compact surface admits a hole of some kind, and therefore every compact surface has at least one closed geodesic.

2.2 On the number of closed geodesics

As we said, every compact surface (or any compact Riemannian manifold in general) admits at least one closed geodesic. However, how many closed geodesics are there? This question was answered, with remarkable generality, by two important theorems, by Gromoll and Meyer on the one hand, and Sullivan and Vigué-Poirrier on the other.

Theorem 1 (Gromoll–Meyer [2], Sullivan–Vigué-Poirrier [6]) *Any Riemannian manifold whose rational cohomology ring is generated by at least two elements contains infinitely many distinct closed geodesics.*

The condition that the "rational cohomology ring is generated by at least two elements" simply means that the manifold is "complicated enough", but it is a very mild condition. In the case of surfaces, for example, this condition is satisfied by every surface with the exception of the sphere and the so-called *real projective plane*. The real projective plane can be visualized as an upper hemisphere of a sphere, where each point on the equator is glued to its antipodal point (which also lies on the equator). [3] As one might imagine, the sphere and the real projective plane are closely related to each other. In fact, if one could prove that any 2–dimensional sphere has infinitely many closed geodesics, the same would be true for the projective plane as well.

Figure 3: A model of the real projective plane at MFO, Oberwolfach.

Whether it is possible to find infinitely many closed geodesics in every (deformed) sphere is still an open problem. However, something is known for this case as well.

Theorem 2 *Every* 2*–dimensional sphere contains at least three closed geodesics.*

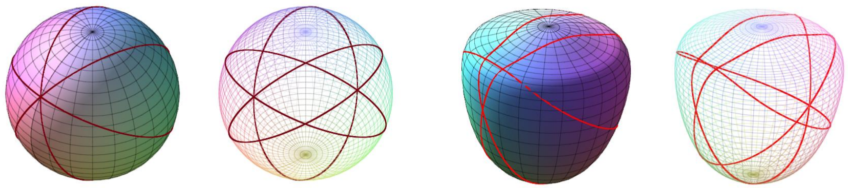
3 Riemannian manifolds with all geodesics closed

In the previous section, we saw that almost every surface is known to *always* admit infinitely many closed geodesics (independently of how it sits in the surrounding space!), with the possible exception of the sphere and the real projective plane; similarly, every compact Riemannian manifold has infinitely many closed geodesics, with the possible exception of "very simple manifolds", like (possibly higher-dimensional) spheres.

It is then very curious that, if we look at the extreme situation of Riemannian manifolds with *all* geodesics closed, the only manifolds that can admit this property are exactly these very simple manifolds, and not the others. One example? Take the round sphere: there, the geodesics are precisely the "great circles", and they all close up. As a matter of fact, the (possibly deformed) sphere and the (possibly deformed) real projective plane are the only two

[3] A more complicated model is given by the so-called Boy surface, shown in Figure 3.

surfaces on which all geodesics can be closed. Moreover, there are infinitely many different ways for the sphere to sit in space such that all its geodesics are closed!



(a) The round sphere. (b) A deformed sphere.

Figure 4: Different spheres with all geodesics closed. The red lines are geodesics.

Even when all geodesics are closed, however, their behaviour is not intuitively clear, and these closed gleodesics could in principle be very complicated. However, the following theorem shows a case where one still has some nice control.

Theorem 3 (Grove–Gromoll [1]) *For any metric on the* 2*–dimensional sphere with all geodesics closed, the geodesics have all the same length.*

That means that if all the geodesics on a sphere are closed then they all have the same length. For the round sphere, this is nothing new: it just means that all great circles have the same length. But the theorem states that no matter how the sphere is deformed, all of its geodesics will have the same length, provided that all the geodesics are closed!

This result has recently been extended in [5] for spheres of all dimensions, except for dimension 3.

Image credits

Fig. 3 "The Boy surface at the MFO, Oberwolfach". Author: Florian-TFW. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, https://commons.wikimedia.org/wiki/File:Boyflaeche.JPG, visited on February 6, 2016.

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