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Gradient Canyons, Concentration of Curvature, and Lipschitz Invariants

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GRADIENT CANYONS, CONCENTRATION OF CURVATURE, AND LIPSCHITZ INVARIANTS

LAURENŢIU PĂUNESCU AND MIHAI TIBĂR

Abstract. We find new bi-Lipschitz invariants of holomorphic functions of two variables by using the gradient canyons and by combining analytic and geometric viewpoints on the concentration of curvature.

1. Introduction

By the classical work of Pham and Teissier [PT] it is well-known that two plane curve germs are bi-Lipschitz equivalent for the outer metric if and only if they have the same embedded topology; this was recently complemented by Neumann and Pichon [NP1], see also [Fe]. Beyond the case of curves, the bi-Lipschitz equivalence is no more controlled by the topology only. A bi-Lipschitz classification of normal surfaces for the inner metric has been found by Birbrair, Neumann and Pichon [BNP]. More recently Neumann and Pichon [NP2] characterised the Lipschitz triviality (for the outer metric) of a family of normal surfaces by the equisingularity of the absolute polar curve of the surface.

In case of functions of two variables, the first bi-Lipschitz invariants of holomorphic functions, different from the topological invariants, were found by Henry and Parusiński [HP1, HP2]. More recently the "contact bi-Lipschitz equivalence" has been characterised by Birbrair, Fernandes and Grandjean [BFG], and all their invariants are still topological at this "contact" level.

We take another step toward the bi-Lipschitz classification of function germs by finding more other bi-Lipschitz invariants. Our proof calls three complementary viewpoints: the canyon method of Kuo, Koike and Păunescu [KKP1, KKP2] for studying the bumps of curvature on the Milnor fibre, the Henry and Parusiński [HP1, HP2] study of the by-Lipschitz equivalence of functions, and the Garcia Barroso and Teissier [GT] geometric viewpoint upon the integral curvature of the Milnor fibre.

The gradient canyons were defined by Kuo, Koike and Păunescu [KKP1, KKP2] as key ingredients in the detection of the concentration of the curvature in the Milnor fibre of a holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Earlier, Garcia Barroso and Teissier [GT] have used a totally different method in order to detect the concentration of curvature, which is based on Langevin's [La1, La2] study of the total curvature. Both methods are based on

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polar curves, and we show here that they are adapted to the further study of the metric properties of the function germs.

The gradient canyon data. In order to state our main result, we give an account of the canyon data and send to the next sections for the details.

Let γ_* denote some *polar* of $f = f(x,y) : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, i.e. an irreducible curve which is a solution of the equation $f_x = 0$. We consider some *Newton-Puiseux parametrization* of it, i.e. of the form $\alpha(y) = (\tau(y), y)$, which can be obtained by starting from a holomorphic parametrization $\alpha : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ with $\mathrm{ord}_t\alpha_2 \le \mathrm{ord}_t\alpha_1$, and then making the change of parameter $\alpha(y) = (\tau(y), y)$ with $y = \alpha_2(t)$. Then $m := m(\gamma_*) := \min \mathrm{ord}_t\alpha_2$ (minimum over all parametrisations) is the *multiplicity* of the polar γ_*; it is also equal to the total number of parametrizations of γ_* of order m which are *conjugate*.

Let $d_{\mathrm{gr}}(\gamma)$ be the degree for which the order of the gradient is stabilized and let $\mathcal{GC}(\gamma_*)$ be *the gradient canyon of* γ (Definition 2.1). Such a canyon contains one or more polars which turn out to have the same *canyon degree*, as well as all the curves inside the canyon. The *multiplicity of the canyon* $m(\mathcal{GC}(\gamma_*))$ is the sum of the multiplicities of its polars.

While analytic maps do not preserve polars, we prove here the following analytic invariance of the canyons:

Theorem 1.1. *If $f = g \circ \varphi$ with φ subanalytic bi-Lipschitz, then φ transforms canyons into canyons by preserving their degrees and multiplicities.*

It follows that the map φ establishes a bijection between the canyons of f and those of g such that the degrees $d_{\mathrm{gr}}(\gamma_*)$ and the multiplicities $m(\mathcal{GC}(\gamma_*))$ are the same. We say that *f and g have the same canyon data.*

Remark 1.2. The gradient canyons are not topological invariants. For instance, consider

$$f_t(x,y) = x^3 + y^{12} + tx^2y^5,$$

which is a topologically trivial family. At $t = 0$, we have only one double polar with $d = \frac{11}{2}$, hence only one canyon. For $t \neq 0$ there are two disjoint canyons corresponding to the two distinct polars, both having degree $d = 6$.

When we drop the subanalyticity assumption for the bi-Lipschitz map φ, the perspectives are challenging since not only that polar curves are not sent to polar curves, but we cannot prove anymore that gradient canyons are sent to gradient canyons. Up to now, the only result in full generality has been obtained by Henry and Parusiński [HP1, HP2], namely the authors have found that the leading coefficient in the expansion (21), modulo an equivalence relation, is a bi-Lipschitz invariant. More than that, Henry and Parusiński showed that certain zones in the Milnor fibre, which one may characterise by the higher order of the change of the gradient, are preserved by bi-Lipschitz homeomorphisms.

We reveal here a rich set of new bi-Lipschitz invariants. They extend in a certain sense the topological invariants of plane curves, but this time they refer to branches of polar curves instead of branches of the curve $\{f = 0\}$. Our clustering description of the polar curves and their associated zones refines in a multi-scale manner the Henry and Parusiński zone.

We first establish the following faithful correspondence between the concentration of curvature invariants coming from Garcia Barroso and Teissier's interpretation [GT] and the Koike, Kuo and Păunescu interpretation [KKP2] in §5, in particular we prove that (Theorem 5.1) *the contact degree* $d_{\gamma(\tau)}$ *and the gradient canyon* $\mathcal{GC}(\gamma(\tau))$ *do not depend on the direction* τ *of the polar* $\gamma(\tau)$*, for generic* τ*.* This result contributes, among others, to the proof of our main result, Section 6; let us give here a brief account of it.

Let $f = g \circ \varphi$ with φ a bi-Lipschitz homeomorphism. Even if one cannot prove that the image by φ of a canyon is a canyon, our key result says that the *canyon disks* (which are defined as the intersections of the *horn domains* (11) with the Milnor fibre) are sent by φ to canyon disks, establishing a bijection at this level.

We are therefore able to show that φ induces a bijection between the gradient canyons of f and those of g, and moreover, that there are *clusters of canyons* of f which correspond by φ to similar clusters of g. Such clusters are defined in terms of orders of contact (i.e. certain rational integers) which are themselves bi-Lipschitz invariants. Theorem 6.9 states precisely this correspondence; since it needs certain preliminaries, we sent to Section 6 for its formulation.

EXAMPLE 1.3. The function $f_t = z^4 + tz^2w^2 + w^4$ has 3 polars with canyon degrees $d_\gamma = 1$ which collapse at $t = 0$ to a single polar of canyon degree $d = 1$. Even if the number of polars is not constant, there is a single canyon with constant degree 1 and multiplicity 3. This is a bi-Lipschitz invariant, after Theorem 1.1 and Theorem 6.9. The Lipschitz triviality can be checked by using Kuo's trivializing vector field since f_t is homogeneous of degree 4 for any t.

EXAMPLE 1.4. $f_t = z^3 + tz^2w^5 + w^{12}$ has the same Henry-Parusinski invariants [HP1, HP2] but does not have the same canyon data. The family f_t is topologically trivial but not Lipschitz trivial.

2. Gradient canyons

We recall from [KPa] and [KKP2] some of the definitions and results that we shall use.

One calls *holomorphic arc* the image $\alpha_* := \mathrm{Im}(\tilde{\alpha})$ of an irreducible plane curve germ:

$$\tilde{\alpha} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, 0), \quad \tilde{\alpha}(t) = (z(t), w(t)).$$

It has a unique complex tangent line $T(\alpha_*)$ at 0, considered as a point in the projective line, i.e. $T(\alpha_*) \in \mathbb{C}P^1$.

The classical Newton-Puiseux Theorem asserts that the field $\mathbb{F}$ of convergent fractional power series in y is algebraically closed, see e.g. [Walk], [Wall]. A non-zero element of $\mathbb{F}$ is a convergent series with positive rational exponents:

$$\alpha(y) = a_0 y^{n_0/N} + \cdots + a_i y^{n_i/N} + \cdots, \quad n_0 < n_1 < \cdots,$$

where $0 \neq a_i \in \mathbb{C}$, $N, n_i \in \mathbb{N}$, $N > 0$, with $\gcd(N, n_0, n_1, ...) = 1$, $\limsup_i |a_i|^{\frac{1}{n_i}} < \infty$.

The *conjugates* of α are

$$\alpha_{conj}^{(k)}(y) := \sum a_i \theta^{kn_i} y^{n_i/N}, \text{ where } 0 \le k \le N-1 \text{ and } \theta := e^{\frac{2\pi\sqrt{-1}}{N}}.$$

The *order* of α is

$$\mathrm{ord}(\alpha) := \mathrm{ord}_y(\alpha) = \tfrac{n_0}{N} \text{ if } \ \alpha \neq 0 \text{ and } \mathrm{ord}(\alpha) := \infty \text{ if } \alpha = 0,$$

and $m_{puiseux}(\alpha) := N$ is the *Puiseux multiplicity* of α.

For any $\alpha \in \mathbb{F}_1 := \{\alpha | \mathrm{ord}_y(\alpha) \geq 1\}$, the map germ

$$\tilde{\alpha} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, 0), \quad t \mapsto (\alpha(t^N), t^N), \quad N := m_{puiseux}(\alpha),$$

is holomorphic and the holomorphic arc α_* is then well defined.

For such fixed α_*, one defines:

$$\mathcal{D}^{(e)}(\alpha_*; \rho) := \{\beta_* \mid \beta(y) = [J^{(e)}(\alpha)(y) + cy^e] + \text{h.o.t.}, \ |c| \leq \rho\}, \tag{1}$$

where $1 \leq e < \infty$, $\rho \geq 0$, and where $J^{(e)}(\alpha)(y)$ is the *e-jet* of α and "h.o.t." means as usual "higher order terms". Moreover, one defines:

$$\begin{aligned} \mathcal{L}^{(e)}(\alpha_*) &:= \mathcal{D}^{(e)}(\alpha_*; \infty) := \cup_{0<\rho<\infty} \mathcal{D}^{(e)}(\alpha_*; \rho) \\ &= \{\beta_* \mid \beta(y) = [J^{(e)}(\alpha)(y) + cy^e] + \text{h.o.t.}, \ |c| \in \mathbb{R}\}. \end{aligned} \tag{2}$$

Note that in the above definitions (1) and (2), the parameter $\alpha \in \mathbb{F}_1$ runs over all its conjugates.

Consider the Newton-Puiseux factorizations:

$$f(x, y) = u \cdot \prod_{i=1}^{m} (x - \zeta_i(y)), \quad f_x(x, y) = v \cdot \prod_{j=1}^{m-1} (x - \gamma_j(y)), \tag{3}$$

where ζ_i, $\gamma_j \in \mathbb{F}_1$ and u, v are units. Note that all conjugates of roots are also roots. We call *polar* any such root γ_j, as well as its geometric representation γ_{j*}.

If a polar γ is also a root of f, i.e. $f(\gamma(y), y) \equiv 0$, then it is a multiple root of f.

From the Chain Rule it follows:

$$f_z(\alpha(y), y) \equiv f_y(\alpha(y), y) \equiv 0 \implies f(\alpha(y), y) \equiv 0 \tag{4}$$

for any $\alpha \in \mathbb{F}_1$. Let us fix a polar γ with $f(\gamma(y), y) \not\equiv 0$. By (4), γ is not a common Newton-Puiseux root of f_z and f_y. If q is sufficiently large, then one has the equality:

$$\mathrm{ord}_y(\| \mathrm{grad}\, f(\gamma(y), y)\|) = \mathrm{ord}_y(\| \mathrm{grad}\, f(\gamma(y) + uy^q, y)\|), \ \forall\, u \in \mathbb{C}. \tag{5}$$

Definition 2.1. The *gradient degree* $d_{\mathrm{gr}}(\gamma)$ is the smallest number q such that (5) holds for generic $u \in \mathbb{C}$. In the case $f(\gamma(y), y) \equiv 0$, one sets $d_{\mathrm{gr}}(\gamma) := \infty$.

Definition 2.2. Let γ be a polar of gradient degree $d := d_{\mathrm{gr}}(\gamma)$, $1 \leq d \leq \infty$. The *gradient canyon of* γ_* is by definition

$$\mathcal{GC}(\gamma_*) := \mathcal{L}^{(d)}(\gamma_*).$$

One calls $d_{\mathrm{gr}}(\gamma_*) := d_{\mathrm{gr}}(\gamma)$ the *gradient degree of* γ_*, or the *degree of* $\mathcal{GC}(\gamma_*)$.

One says that $\mathcal{GC}(\gamma_*)$ is *minimal* if $d_{\rm gr}(\gamma_*) < \infty$ and if, for any polar γ_i of finite degree, the inclusion $\mathcal{GC}(\gamma_{i*}) \subseteq \mathcal{GC}(\gamma_*)$ implies the equality $\mathcal{GC}(\gamma_{i*}) = \mathcal{GC}(\gamma_*)$.

Definition 2.3. The *multiplicity* of the gradient canyon $\mathcal{GC}(\gamma_*)$ is:

$$m(\mathcal{GC}(\gamma_*)) := \sharp\{j \mid 1 \le j \le m-1,\ \gamma_{j*} \in \mathcal{GC}(\gamma_*)\}. \tag{6}$$

Up to some generic unitary change of coordinates, one has the following presentation:

$$f(x,y) := f_m(x,y) + f_{m+1}(x,y) + \text{h.o.t.}, \tag{7}$$

where f_k denotes a homogeneous k-form, with $f_m(1,0) \neq 0$ and $m = \operatorname{ord}(f)$.

The initial form $f_m(x,y)$ factors as:

$$f_m(x,y) = c(x - x_1 y)^{m_1} \cdots (x - x_r y)^{m_r},\ \ m_i \ge 1,\ \ x_i \neq x_j \text{ if } i \neq j, \tag{8}$$

and $1 \le r \le m$, $m = m_1 + \cdots + m_r$, $c \neq 0$.

We have that $f_m(x,y)$ is degenerate if and only if $r < m$. The following useful result sheds more light over the landscape of gradient canyons:

Theorem 2.4. [KKP2, Theorem B] *Any gradient canyon of degree $1 < d_{\rm gr} < \infty$ is a minimal canyon. The canyons of degrees $1 < d_{\rm gr} \le \infty$ are mutually disjoint.*

There are exactly $r-1$ polars of gradient degree 1, counting multiplicities, and they belong to the unique gradient canyon of degree 1, denoted by $\mathbb{C}_{\rm enr}$.

If $1 < r \le m$, then $\mathbb{C}_{\rm enr}$ is minimal if and only if $f(z,w)$ has precisely r distinct roots ζ_i in (3). In particular, if $f_m(x,y)$ is non-degenerate then $\mathbb{C}_{\rm enr}$ is minimal. □

The horn, the partial Milnor number, and the total curvature of a gradient canyon.

A well-known formula to compute the Milnor number μ_f is the following:

$$\mu_f = \sum_{j=1}^{m-1} \left[\operatorname{ord}_y(f(\gamma_j(y), y)) - 1\right], \tag{9}$$

where the sum runs over all γ_j, i.e. over all polars and their conjugates.

One defines the *Milnor number of f on a gradient canyon* $\mathcal{GC}(\gamma_*)$ with $d_{\rm gr}(\gamma_*) < \infty$, as:

$$\mu_f(\mathcal{GC}(\gamma_*)) := \textstyle\sum_j [\operatorname{ord}_y(f(\gamma_j(y), y)) - 1], \tag{10}$$

where the sum is taken over all j, $1 \le j \le m-1$, such that $\gamma_{j*} \in \mathcal{GC}(\gamma_*)$.

From (10) and (6) one has:

$$\mu_f(\mathcal{GC}(\gamma_*)) + m(\mathcal{GC}(\gamma_*)) = \textstyle\sum_j \operatorname{ord}_y(f(\gamma_j(y), y)),$$

where j runs like in the sum of (10).

Consider an enriched disc $\mathcal{D}^{(e)}(\alpha_*; \rho)$ of finite order $e \ge 1$ and finite radius $\rho > 0$, and a compact ball $B(0;\eta) := \{(x,y) \in \mathbb{C}^2 \mid \sqrt{|x|^2 + |y|^2} \le \eta\}$ with small enough $\eta > 0$ (usually we consider a Milnor ball of f). Let then:

$$\mathrm{Horn}^{(e)}(\alpha_*;\rho;\eta) := \{(x,y)\in D(0;\eta) \mid x=\beta(y)=J^{(e)}(\alpha)(y)+cy^e, |c|\le\rho\} \tag{11}$$

be the *horn domain* associated to $\mathcal{D}^{(e)}(\alpha_*;\rho)$; it is a compact subset of $\mathbb{C}^2$.

The *total asymptotic Gaussian curvature over* $\mathcal{D}^{(e)}(\alpha_*;\rho)$ is then by definition:

$$\mathscr{M}_f(\mathcal{D}^{(e)}(\alpha_*;\rho)) := \lim_{\eta\to 0}\left[\lim_{\lambda\to 0}\int_{\{f=\lambda\}\cap \mathrm{Horn}^{(e)}(\alpha_*;\rho;\eta)} K dS\right], \tag{12}$$

where S is the surface area and K is the Gaussian curvature.

The total asymptotic Gaussian curvature over an enriched line is by definition:

$$\mathscr{M}_f(\mathcal{L}^{(e)}(\alpha_*)) := \lim_{\rho\to\infty}\mathscr{M}_f(\mathcal{D}^{(e)}(\alpha_*;\rho)). \tag{13}$$

The above definitions are easily extended to the case $e=\infty$, so that

$$\mathscr{M}_f(\mathcal{D}^{(\infty)}(\alpha_*)) = \mathscr{M}_f(\mathcal{L}^{(\infty)}(\alpha_*)) = \mathscr{M}_f(\{\alpha_*\}) = 0.$$

Theorem 2.5. [KKP2, Theorem C] *Let γ_* be a polar, $1<d_{\mathrm{gr}}(\gamma_*)\le\infty$. Then*

$$\mathscr{M}_f(\mathcal{GC}(\gamma_*)) = \begin{cases} 2\pi[\mu_f(\mathcal{GC}(\gamma_*)) + m(\mathcal{GC}(\gamma_*)], & 1<d_{\mathrm{gr}}(\gamma_*)<\infty,\\ 0, & d_{\mathrm{gr}}(\gamma_*)=\infty.\end{cases} \tag{14}$$

□

3. The arc valeys

3.1. Arc valeys and gradient canyons.

Let $\alpha:(\mathbb{C},0)\to(\mathbb{C}^2,0)$ be a subanalytic germ of a curve, of order $\mathrm{ord}_y(\alpha)\ge 1$. One may assume, modulo transposition and rescaling, that $\alpha(y)=(\bar\alpha(y),y)$ where $\bar\alpha(y)$ is a fractional power series of order $\mathrm{ord}_y(\alpha)\ge 1$; in the following we identify $\bar\alpha$ with α.

We define the contact degrees:

$$d_{\mathrm{con}}(\alpha) := \inf\{d \mid ||\,\mathrm{grad}\, f(\alpha+cy^d+\text{h.o.t.})|| \sim ||\,\mathrm{grad}\, f(\alpha)||, \text{ for generic } c\in\mathbb{C}^2\}$$

and:

$$\tilde d_{\mathrm{con}}(\alpha) := \inf\{d \mid ||\,\mathrm{grad}\, f(\alpha+cy^d)|| \sim ||\,\mathrm{grad}\, f(\alpha)||, \text{ for generic } c\in\mathbb{C}^2\}.$$

It follows from the definitions that $\tilde d_{\mathrm{con}}(\alpha)\le d_{\mathrm{con}}(\alpha)$. Let us show that we actually have equality.

If $\|\,\mathrm{grad}\, f(\alpha(y)+uy^e)\|^2 := D_{(\alpha,e)}(u)y^{2L_{gr}(\alpha,e)}+\text{h.o.t.}$, $D_{(\alpha,e)}(u)\not\equiv 0$, so for generic $u\in\mathbb{C}^2$ it has the same order $L_{gr}(\alpha,e)$ which is increasing in e, i.e. $\|\,\mathrm{grad}\, f(\alpha(y)+uy^e)\|\sim \|\,\mathrm{grad}\, f(\alpha(y)+vy^r)\|, \forall r>e, v\in\mathbb{C}^2$.

If we set $\beta(y)=\alpha(y)+uy^e$ then $\|\,\mathrm{grad}\, f(\alpha)\|\sim\|\,\mathrm{grad}\, f(\alpha(y)+uy^e)\| = \|\,\mathrm{grad}\, f(\beta)\|\sim \|\,\mathrm{grad}\, f(\beta(y)+vy^r)\| = \|\,\mathrm{grad}\, f(\alpha+uy^d+vy^r)\|$ and so on i.e. the two definitions give the same number (the order of the gradient stabilises, for instance if $r\ge \mathrm{ord}_y(\|\,\mathrm{grad}\, f(\alpha(y))\|^2)$.

Let $\alpha_*\in\mathbb{C}_{\mathrm{enr}}$ be given, $f(\alpha(y),y)\not\equiv 0$, α is not a common Newton-Puiseux root of f_x and f_y. Hence, if q is sufficiently large, then

$$\mathrm{ord}_y(\|\,\mathrm{grad}\, f(\alpha(y),y)\|) = \mathrm{ord}_y(\|\,\mathrm{grad}\, f(\alpha(y)+uy^q,y)\|),\ \forall\, u\in\mathbb{C}. \tag{15}$$

Let d_α denote the *smallest number* q such that (15) holds for *generic* $u \in \mathbb{C}$. This definition gives the same degree as the previous definition and from now on we will use the later notation.

In case $f(\alpha(y), y) \equiv 0$, we set $d_\alpha := \infty$ if α is a multiple root of f.

Definition 3.1. The *valley* of α_* is, by definition:

$$\mathcal{V}(\alpha_*) := \mathcal{L}^{(d_\alpha)}(\alpha_*).$$

We then call d_α the *degree* of $\mathcal{V}(\alpha_*)$, or *the valey degree of* α (since it does not depend of the representative α of α_*).

We say that $\mathcal{V}(\alpha_*)$ is *minimal* if $d_{\alpha_*} < \infty$ and if for every arc β with $d_\beta < \infty$, we have:

$$\mathcal{V}(\beta_*) \subseteq \mathcal{V}(\alpha_*) \Longrightarrow \mathcal{V}(\beta_*) = \mathcal{V}(\alpha_*).$$

Remark 3.2. In the case γ is a polar the construction above gives the notion of gradient canyon and canyon degree ($\mathcal{V}(\gamma_*) = \mathcal{GC}(\gamma_*), d_{\mathrm{gr}}(\gamma) = d_\gamma$) as introduced in [KKP2] and mentioned earlier. If α is in a canyon then its valley coincides with the canyon.

3.2. **Newton polygon.** Let α be a given arc, $d_\alpha < \infty$.

We can apply a unitary transformation, if necessary, so that $\alpha \in \mathbb{F}_{\geq 1}$, $T(\alpha_*) = [0:1]$.

We then change variables (formally):

$$Z := z - \alpha(w), \quad W := w, \quad F(Z, W) := f(Z + \alpha(W), W). \tag{16}$$

Since $\alpha \in \mathbb{F}_{\geq 1}$, i.e., $\mathrm{ord}(\alpha) > 1$, it is easy to see that

$$\| \operatorname{grad}_{z,w} f\| \sim \| \operatorname{grad}_{Z,W} F\|, \quad \Delta_f(z, w) = \Delta_F(Z, W) + \alpha''(W) F_X^3. \tag{17}$$

The Newton polygon $\mathcal{NP}(F)$ is defined in the usual way, as follows. Let us write

$$F(Z, W) = \sum c_{iq} Z^i W^q, \quad c_{iq} \neq 0, \quad (i, q) \in \mathbb{Z} \times \mathbb{Q}.$$

A monomial term with $c_{iq} \neq 0$ is represented by a "*Newton dot*" at (i, q). We shall simply call it a *dot* of F. The *boundary* of the convex hull generated by $\{(i+u, q+v) | u,\ v \geq 0\}$, for all dots (i, q), is the Newton Polygon $\mathcal{NP}(F)$, having edges E_i and angles θ_i, as shown in Fig.1. In particular, E_0 is the half-line $[m, \infty)$ on the Z-axis. [1]

For a line in $\mathbb{R}^2$ joining $(u, 0)$ and $(0, v)$, let us call v/u its *co-slope*. Thus

$$\textit{co-slope of } E_s = \tan \theta_s.$$

Some elementary, but useful, facts are:

- If $i \geq 1$, then (i, q) is a dot of F *if and only if* $(i-1, q)$ is one of F_Z.
- When $f(\alpha(w), w) \not\equiv 0$, we know $F(0, W) \not\equiv 0$. Let us write

$$F(0, W) = aW^h + \text{h.o.t.}, \quad a \neq 0, \ h := \mathrm{ord}_W(F(0, W)) \in \mathbb{Q}. \tag{18}$$

Then $(0, h)$ is a vertex of $\mathcal{NP}(F)$, $(0, h-1)$ is one of $\mathcal{NP}(F_W)$. (See Fig. 2.)

[1] In[KL], [KPar], this is called the Newton Polygon of f *relative to* α, denoted by $\mathcal{NP}(f, \alpha)$.

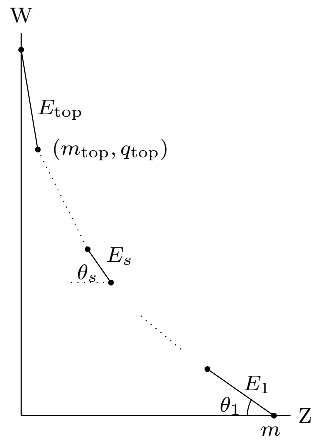

FIGURE 1. $\mathcal{NP}(F)$

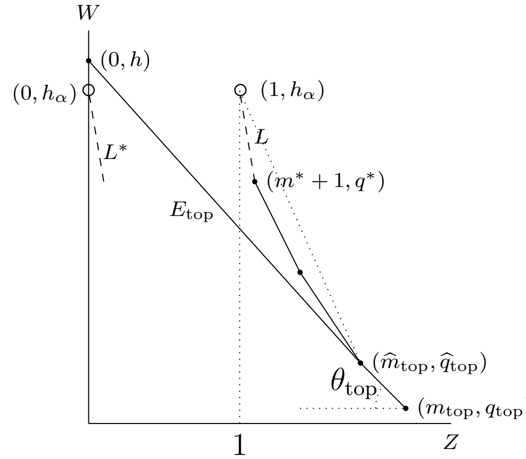

FIGURE 2. $\mathcal{NP}(F)$ vs $\mathcal{NP}(F_Z)$.

NOTATIONS 3.3. In the case $d_\alpha < \infty$, i.e. either $f(\alpha(w), w) \not\equiv 0$ or $f(\alpha(w), w) = 0, f_z(\alpha(y), y) \not\equiv 0$, let E_{top} denote the edge whose left vertex is, in the first case $(0, h)$, h as in (18) or $(1, h')$ in the second case, and right vertex is $(m_{\text{top}}, q_{\text{top}})$, as shown in the figures. We call it the *top* edge; the angle is θ_{top}. In the case $f(\alpha(w), w) \equiv 0$ the top edge E_{top} is not ending on $z = 0$ but on $z = 1$, except α is a multiple root of f in which case we precisely have $d_\alpha = \infty$. However, except in the latter case, we always extend informally the top edge to virtually cut $z = 0$ at $(0, h)$.

Let $(\widehat{m}_{\text{top}}, \widehat{q}_{\text{top}}) \neq (0, h)$ be the dot of F on E_{top} which is *closest* to the left end of E_{top} (which is $(0, h)$ if α not a multiple root of f) . (Of course, $(\widehat{m}_{\text{top}}, \widehat{q}_{\text{top}})$ may coincide with $(m_{\text{top}}, q_{\text{top}})$.) Then, clearly,

$$2 \leq \widehat{m}_{\text{top}} \leq m_{\text{top}}, \quad \frac{h - \widehat{q}_{\text{top}}}{\widehat{m}_{\text{top}}} = \frac{h - q_{\text{top}}}{m_{\text{top}}} = \tan \theta_{\text{top}}.$$

Now we draw a line L through $(1, h_\alpha), h_\alpha \leq h - 1$, with the following two properties (in particular this defines h_α):

(a) If (m', q') is a dot of F_Z, then $(m' + 1, q')$ lies on or above L;
(b) There exists a dot (m^*, q^*) of F_Z such that $(m^*+1, q^*) \in L$. (Of course, (m^*+1, q^*) may coincide with $(\widehat{m}_{\text{top}}, \widehat{q}_{\text{top}})$.)
(c) h_α is the largest with these properties.

Lemma 3.4. *Let σ^* denote the co-slope of L. Then*

$$(i)\ d_\alpha = \sigma^*;\ (ii)\ \sigma^* \geq \tan \theta_{\text{top}};\ (iii)\ \sigma^* = \tan \theta_{\text{top}} \Leftrightarrow (1, h_\alpha) \in E_{\text{top}}. \tag{19}$$

All dots of F_W lie on or above L^*, L^* being the line through $(0, h_\alpha)$ parallel to L.

In the case $\tan \theta_{\text{top}} > 1$, $(0, h - 1)$ may be the only dot of F_W on L^* (exactly when $h_\alpha = h - 1$ and α not a root of f).

NOTATIONS 3.5. Take $e \geq 1$. Let $\omega(e)$ denote the weight system: $\omega(Z) = e$, $\omega(W) = 1$.

Let $G(Z, W^{1/N}) \in \mathbb{C}\{Z, W^{1/N}\}$ be given. Consider its weighted Taylor expansion relative to this weight. We shall denote the *weighted initial form* by $\mathcal{I}_{\omega(e)}(G)(Z, W)$, or simply $\mathcal{I}_\omega(G)$ when there is no confusion.

If $\mathcal{I}_\omega(G) = \sum a_{ij} Z^i W^{j/N}$, the *weighted order* of G is $\operatorname{ord}_\omega(G) := ie + \frac{j}{N}$.

Proof. Note that (ii) and (iii) are clearly true, since $(1, h_\alpha)$ lies on or above E_{top}.

Next, if (i, q) is a dot of F_W, then $(i, q+1)$ is one of F, lying on or above E_{top}. Hence, by (ii), all dots of F_W lie on or above L^*.

It also follows that if $\tan\theta_{\text{top}} > 1$, then $(0, h-1)$ may be the only dot of F_W on L^*.

Let us show (i).

It is easy to see that if $\tan\theta_{\text{top}} = 1$, then $d_\alpha = 1$.

It remains to consider the case $\sigma^* > 1$. By construction,

$$\operatorname{ord}_y(\|\operatorname{grad} f(\alpha(y), y)\|) = h_\alpha. \tag{20}$$

Let us first take weight $\omega := \omega(e)$ where $e \geq \sigma^*$. In this case, since $\sigma^* > 1$:

$$\mathcal{I}_\omega(F_W)(Z, W) = ahW^{h-1}, \ \operatorname{ord}_W(F_Z) = h_\alpha, \text{ if } h_\alpha \leq h-1, \text{ and } > h-1 \text{ otherwise},$$

where a, h are as in (18), $ah \neq 0$. Hence for generic $u \in \mathbb{C}$,

$$\operatorname{ord}_W(F_W(uW^e, W)) = h-1, \quad \operatorname{ord}_W(F_Z(uW^e, W)) = h_\alpha.$$

It follows that $d_\alpha \leq \sigma^*$. It remains to show that $\sigma^* > d_\alpha$ is impossible.

Let us take $\omega(e)$ with $e < \sigma^*$. Note that (m^*, q^*) is a dot of F_Z on L^*, where (m^*+1, q^*) is shown in Figure 2. Hence, for generic u,

$$\operatorname{ord}_W(F_Z(uW^e, W)) < h_\alpha, \quad \operatorname{ord}_W(\|\operatorname{grad} F(uW^e, W)\|) < h_\alpha.$$

Thus, by (20), we must have $d(\alpha) > e$. This completes the proof of Lemma 3.4. ☐

Example 3.6. For $F(Z, W) = Z^4 + Z^3W^{27} + Z^2W^{63} - W^{100}$ and $\gamma = 0$, $\mathcal{NP}(F)$ has only two vertices $(4, 0)$, $(0, 100)$, while $\mathcal{NP}(F_Z)$ has three: $(3, 0)$, $(2, 27)$, $(1, 63)$. The latter two and $(0, 99)$ are collinear, spanning L^*; $h = 100$, $\sigma^* = (99 - 27)/2 = 36$.

4. Proof of Theorem 1.1

Here again we identify $\alpha(y)$ with $(\alpha(y), y)$.

Proposition 4.1. (a) *For any arc α, there is some polar γ of f such that $d_\gamma \geq d_\alpha$.*
(b) *For all $\beta \in \mathcal{GC}(\gamma_*)$ we have $d_\beta = d_{\text{gr}}(\gamma)$.*
(c) *For any $\alpha \in \mathcal{GC}(\gamma_*)$ we have $f(\alpha(y)) = ay^h + \text{h.o.t.}$, where a and h depend only on the canyon.*

Proof. (a). By using the Newton polygon relative to α, $\mathcal{NP}(f, \alpha)$, see Figure 1, we observe that a polar can be obtained by pushing forward along L^*. Namely we construct a root of f_x starting from $\mathcal{NP}(f_x, \alpha))$ by the Newton-Puiseux algorithm. This procedure adds up terms of degree at least d_α, so we end up with at least one polar of the form $\gamma = \alpha + cy^{d_\alpha} + \text{h.o.t.}$, where c is a root of the associated polynomial in x (i.e. the derivative of the de-homogenisation of the polynomial associated to L^*). We then get $\operatorname{ord}_y(\alpha(y) - \gamma(y)) \geq d_\alpha$, hence $d_\alpha \leq d_{\text{gr}}(\gamma)$ for any such polar.

Hence, starting with α one constructs polars by the diagram method and the process is not necessarily unique. Nevertheless, all such polars are clearly in the valley of α.
(b). Our assumption implies that $\beta = \gamma + cy^{d_{\rm gr}(\gamma)} + \text{h.o.t.}$ for some $c \in \mathbb{C}$, hence the Newton polygons $\mathcal{NP}(f,\beta)$ and $\mathcal{NP}(f,\gamma)$ will have the corresponding L parallel, see Figure 1, and thus the same coslope, which is $d_{\rm gr}(\gamma)$.
(c). We have by definition $f(\gamma(y), y) = ay^h + \text{h.o.t.}$ and by our assumption $\alpha(y) = \gamma(y) + cy^d + \text{h.o.t.}$. Thus

$$f(\gamma(y) + cy^d + \text{h.o.t.}, y) = ay^h + \cdots + \alpha(c)y^{d+h-1} + \text{h.o.t.}, \tag{21}$$

where the first terms depend only on the canyon (and not on the perturbation of γ), in particular the dependence of c starts at the degree $d + h - 1$.

□

Remark 4.2. Point (b) above holds for the gradient canyons but it is not necessarily true for arbitrary valleys. More precisely, in case of a valley, the claim that (b) holds only for a generic coefficient of y^d.

Remark 4.3. In general, given $\alpha \in \mathcal{GC}(\gamma_*)$ with $\mathrm{ord}_y(\alpha) = 1$, to put it in the form $(\tilde\alpha(y), y)$ requires a rescaling of y (i.e. replacing y by cy for some $c \neq 0$) and this yields $f(\alpha(y)) = ac^h y^h + \text{h.o.t.}$

Corollary 4.4. *The function $\alpha \mapsto d_\alpha$ has its local maxima at the polars γ of f with $d_\gamma > 1$.* □

Let now $f = g \circ \varphi$, for subanalytic bi-Lipschitz $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. For some arc $(\alpha(y), y)$, we have that $\varphi(\alpha(y), y) = (\varphi_1(\alpha(y), y), \varphi_2(\alpha(y), y))$ and $\mathrm{ord}_y \varphi_2(\alpha(y), y) = 1$, hence we may write $\varphi(\alpha(y), y) = (\beta(\bar y), \bar y)$ for some arc β, where $\bar y := \varphi_2(\alpha(y), y)$. Then we have:

Theorem 4.5. *For any polar γ_f of f there exists a polar γ_g of g such that*

$$\varphi(\mathcal{GC}(\gamma_f)) = \mathcal{GC}(\gamma_g)$$

and the canyon degrees are the same.

Proof. Let us prove first:

Lemma 4.6. *$d_{\varphi(\alpha)} = d_\alpha$ and $\varphi(\mathcal{V}(\alpha)) = \mathcal{V}(\varphi(\alpha(y), y))$.*

Proof. We have $\varphi(\alpha + cy^{d_\alpha}, y)) = \varphi(\alpha) + a(c)\bar y^{d_\alpha} + \text{h.o.t.}$ and the following equivalence:

$$\begin{aligned} \mathrm{grad}\, f(\alpha + cy^{d_\alpha}, y) &\simeq_{ord} \mathrm{grad}\, g(\varphi(\alpha + cy^{d_\alpha}, y)) \\ &= \mathrm{grad}\, g(\varphi(\alpha) + a\bar y^{d_\alpha} + \text{h.o.t.}, \bar y) \end{aligned}$$

for some generic coefficient $c \in \mathbb{C}$ and its transform $a(c)$ which is also generic. From Remark we get the inequality $d_\alpha \geq d_{\varphi(\alpha)}$ and then we apply all this to φ^{-1} and obtain the converse inequality $d_{\varphi(\alpha)} \geq d_{\varphi^{-1}(\varphi(\alpha))} = d_\alpha$, thus our first claim is proved.

Next, we have:

$$\mathrm{ord}_y \|\varphi((\alpha(y), y) - \varphi((\beta(y), y)\| = \mathrm{ord}_y \|(\alpha(y), y) - (\beta(y), y)\|.$$

By using the just proven equality of degrees we get the second claimed equality. □

We have $\varphi(\mathcal{GC}(\gamma)) = \mathcal{V}(\varphi(\gamma(y), y))$ by Lemma 4.6. After Proposition 4.1 we may associate to $\varphi(\gamma(y), y)$ some polar γ_g in the valley of $\varphi(\gamma(y), y)$ with $d_{\gamma_g} \geq d_{\varphi(\gamma)}$, and therefore $\mathcal{V}(\varphi(\gamma(y), y)) \supset \mathcal{V}(\gamma_g) = \mathcal{GC}(\gamma_g)$.

We apply φ^{-1} and get similarly: $\mathcal{GC}(\gamma) \supset \varphi^{-1}(\mathcal{GC}(\gamma_g)) \supset \mathcal{GC}(\gamma_f)$ for some γ_f constructed like in Proposition 4.1, with $d_{\gamma_f} \geq d_{\varphi^{-1}(\gamma_g)}$. According to the minimality principle of polar canyons Theorem 2.4, we must have equality: $\mathcal{GC}(\gamma) = \mathcal{GC}(\gamma_f)$ and $d_{\gamma_f} = d_\gamma$. Consequently we get that $\varphi(\mathcal{GC}(\gamma)) = \mathcal{GC}(\gamma_g)$ and the degrees are equal. □

Theorem 4.5 shows that φ sends a gradient canyon to a gradient canyon by preserving the degree. The preservation of the multiplicity follows from Proposition 4.1(c).

5. Generic polars

Based on Langevin's approach [La1] to the integral of the curvature of the Milnor fibre of a function of n complex variables, Garcia Barroso and Teissier [GT] gave a method to detect the concentration of curvature on the Milnor fibre of a function germ in 2 variables. Using Langevin's exchange formula which interprets the curvature in terms of polar curves, they showed that the intersections of the Milnor fibre with all generic polar curves is concentrated in certain small balls, and hence the curvature too.

More recently, Koike, Kuo and Păunescu [KKP2] adopted a new viewpoint by looking into the curvature formula itself and studying its variation over the space of arcs. Their method uses the gradient canyons and provides sharper localization of the "A'Campo bumps" i.e. maxima of curvature.

We shall find here the relations between the results obtained in [GT] and in [KKP2]. Let γ_0 denote a solution of the equation $f_x(\gamma_0(y), y) = 0$. Let $l_\tau \subset \mathbb{C}^2$ of coordinates (x, y) denote the line $\{y - \tau x = 0\}$, and call it *the line of co-direction* τ. The polars γ_τ are the solutions of the equation:

$$(f_x + \tau f_y)(\gamma_\tau(y), y) = 0. \tag{22}$$

Theorem 5.1. *The gradient canyon $\mathcal{GC}(\gamma_\tau)$ does not depend on the direction $\tau \in \mathbb{C}$, i.e. $\mathcal{GC}(\gamma_0) = \mathcal{GC}(\gamma_\tau)$, $\forall \tau \in \mathbb{C}$. The canyon degree d_{γ_0} is the lowest exponent from which the polar expansions γ_τ start to depend of τ. The multiplicities m_{γ_τ} do not depend of τ.*

Proof. In case $d_{\gamma_0} > 1$ we consider the function $(f_x + \tau f_y)(x + \gamma_0(y), y)$ from which we want to construct a solution of (22) by the method of "edging forward" in the Newton diagram, as explained in §4.

The polars associated to the direction τ are the Newton-Puiseux zeroes of the function $g(x, y) = (f_x + \tau f_y)(x + \gamma_0(y), y)$ translated by γ_0, namely $\gamma_\tau(y) := x(y) + \gamma_0(y)$. The top edge E_{top} of the Newton polygon of g is parallel to the segment L defined as in Figure 2, taking $\alpha = \gamma_0$. Whenever $d_{\gamma_0} > 1$, the segment L has only one dot which depends on $\tau \neq 0$, namely the one corresponding to the monomial $a(\tau)y^{h-1}$ (which comes from the contribution τf_y). Therefore the edging forward process will start with the initial term of the form $c(\tau)y^{d_{\gamma_0}}$ in order to annihilate $a(\tau)y^{h-1}$. Thus the Newton-Puiseux zero of g will be of the form $x(y) = c(\tau)y^{d_{\gamma_0}} + \text{h.o.t.}$, hence $\gamma_\tau(y) = \gamma_0(y) + c(\tau)y^{d_{\tau_0}} + \text{h.o.t.}$ is a polar associated to τ. This shows in particular that the constructed solution γ_τ is in the same gradient canyon as the polar γ_0.

Note that the generic polars that we have constructed $\gamma_\tau(y) = \gamma_0(y) + c(\tau)y^{d_{\tau_0}} + \text{h.o.t.}$ are in the canyon of γ_0 and therefore $f(\gamma_\tau, y) = ay^h + \text{h.o.t.}$, thus the initial term is constant in the canyon, and in particular the exponent h does not depend of τ.

By construction the number of roots $x = x(y)$ of $g(x, y) = 0$ is m_{γ_0}, where $(m_{\gamma_0} + 1, r)$ is the initial (lowest) dot of L. Consequently (m_{γ_0}, r) is the initial dot of E_{top}, hence

$$\frac{h - 1 - r}{m_{\gamma_0}} = d_{\gamma_0}$$

and m_{γ_0} is the multiplicity of the canyon, i.e. the total number of polars in the canyon $\mathcal{GC}(\gamma_0)$, for any $\tau \in \mathbb{C}$.

□

5.1. **Garcia-Barroso and Teissier's approach** [GT].

Let us recall some of the results obtained in [GT] by following their original notations.

(1). Let $P_q(\tau) = (x(t), y(t))$ where $x(t) = t^m$, $y(t) = at^m + \text{h.o.t.}$, be a minimal parametrisation of an irreducible branch of the polar curve with respect to a direction $\tau \in \mathbb{P}^1$. Here $m_q = m_q(\tau)$ is the multiplicity at 0 of $P_q(\tau)$. Teissier had proved that the family $P_q(\tau)$ depending of τ is equisingular for generic τ, thus the multiplicity $m_q(\tau)$ is constant for generic τ. In the following we consider τ in such a generic set.

Barroso and Teissier show in [GT] that the polars fall into subsets called "packets" indexed by the black vertices of the Eggers diagram of f, such that they have the same contact with all branches of the curve $C := \{f = 0\}$. Such a "packet" of polars is the set of polars from a certain union of canyons.

(2). By [GT, Theorem 2.1], the coefficients of $P_q(\tau)$ depend on τ only from a certain well-defined exponent of t. Let g_q denote the first exponent of $y(t)$ the coefficient of which depends of τ. It is shown that all the polars in the same packet have the same exponent g_q and this is denoted by γ_q, cf [GT, pag. 406].

(3). Moreover, in the development of $f(t^{m_q}, y_q(t, \tau))$, the first exponent the coefficient of which depends of τ is $e_q + g_q$. The geometric significance of e_q is given by the identity

$$\text{mult}_0(C, P_q(\tau)) = e_q + m_q,$$

where $e_q = \mu_q(f)$ is a partial Milnor number in the sense that, by Teissier's formula for the polar multiplicity, the sum of e_q's over all polar in the packet and over all packets is equal to the Milnor number μ_f.

(4). The concentration of points of intersection $P_q(\tau) \cap \{f = \lambda\}$ on the Milnor fibre, for all generic τ and as λ approaches 0, is equivalent to the concentration of curvature, according to Langevin's approach [La1]. In order to locate the zones of concentration on the Milnor fibre, i.e. the centers of the balls and their radii, Barroso and Teissier invert the convergent series $\lambda = \lambda(t)$ and expresses the coordinates $(x(t), y(t))$ as functions of λ (see [GT, (5), page 408]).

Let us now see what are the relations between these invariants and those defined in [KKP2] and in our previous sections.

5.2. **A dictionary.**

We have shown that the value $h = \mathrm{ord}_y f(\gamma(y), y)$ is the same for all polars in some canyon (Proposition 4.1). Therefore $m_\gamma h := \mathrm{ord}_y f(\gamma(y^{m_\gamma}), y^{m_\gamma}) = \mathrm{mult}_0(C, \gamma_*)$, where m_γ is the multiplicity of the polar considered with its multiple structure. Note the difference to [KKP2] and the above sections.... where by "polar" we mean with multiplicity 1, and precisely m_γ such polars have the same image γ_*.

On the other hand, by the above (3) from the [GT] viewpoint we have $\mathrm{mult}_0(C, \gamma) = e_\gamma + m_\gamma$. We therefore conclude:

$$e_\gamma = m_\gamma(h - 1) \tag{23}$$

which can be identified with a partial sum of (10).

By Theorem 5.1:

$$\gamma_\tau(y) = \gamma_0(y) + c(\tau)y^{d_{\gamma_0}} + \text{h.o.t.}$$

which implies

$$f(\gamma_\tau(y), y) = ay^h + \cdots + u(\tau)y^{d_{\gamma_0}+h-1} + \text{h.o.t.}$$

where $u(\tau)$ is the first coefficient which depends of τ; thus:

$$f(\gamma_\tau(y^m), y^m) = ay^{mh} + \cdots + u(\tau)y^{md_{\gamma_0}+m(h-1)} + \text{h.o.t.}$$

By [GT, Lemma 2.2]:

$$f(\gamma_\tau(y^m), y^m) = ay^{mh} + \cdots + u(\tau)y^{e_q+g_q} + \text{h.o.t.}$$

using the notations e_q and g_q from the above (c).

We obtain:

$$e_q + g_q = m_q d_\gamma + m_q(h - 1)$$

hence

$$g_q = m_q d_\gamma$$

which shows that the exponent g_q of [GT] reminded at the point (b) above is essentially the same as the degree d_γ of the canyon, i.e. modulo multiplication by the multiplicity m_γ.

6. The correspondence of canyon disks

We consider in this section a gradient canyon $\mathcal{GC}(\gamma_*)$ of degree $d_\gamma > 1$. Let $D^{(e')}_{\gamma_*,\varepsilon}(\lambda)$ be the union of disks in the Milnor fibre $\{f = \lambda\} \cap B(0; \eta)$ of f defined as follows (see (11) for the definition of the Horn):

$$D^{(e')}_{\gamma_*,\varepsilon}(\lambda) := \{f = \lambda\} \cap \mathrm{Horn}^{(e')}(\gamma_*; \varepsilon; \eta),$$

for some rational e' close enough to d_γ, with $1 < e' < d_\gamma$, for some small enough $\varepsilon > 0$, and where by disk we mean an open contractible set. In addition, we ask that $d < e' < d_\gamma$ for any other canyon degree $d < d_\gamma$.

We have:

$$\bigcap_{e' \in \mathbb{Q}, e' \to d_\gamma} D^{(e')}_{\gamma_*,\varepsilon}(\lambda) = \{f = \lambda\} \cap \mathrm{Horn}^{(d_\gamma)}(\gamma_*; \varepsilon; \eta) \tag{24}$$

and we shall write $D_{\gamma_*}(\lambda)$ in the following as shorthand for $D^{(e')}_{\gamma_*,\varepsilon}(\lambda)$, keeping in mind the parameters e', ε.

By [KKP2, Lemma 6.6] we have–see also (12) and (13):

$$\mathscr{M}_f(D^{(e')}(\gamma_*, \varepsilon)) = \mathscr{M}_f(\mathcal{L}^{(e')}(\gamma_*)) = \mathscr{M}_f(\mathcal{GC}(\gamma_*)), \tag{25}$$

and moreover $\mathcal{L}^{(e')}(\gamma_*)$ does not contain any other polar canyon besides $\mathcal{GC}(\gamma_*)$.

This result means that a certain part of the curvature of the Milnor fibre is concentrated in the union $D_{\gamma_*}(\lambda) = \cup_i D_{\gamma_*,i}(\lambda)$, of connected disks $D_{\gamma_*,i}(\lambda)$. The number of disks is the intersection number $\mathrm{mult}_0(\{f = 0\}, \overline{\gamma_*})$, where $\overline{\gamma_*}$ is the truncation of the polar at the order d_γ. Here we have to understand γ_* as image of γ_*, which is thus the same image for all conjugates of γ, and similarly for the truncations. There might be non-conjugate polars in the same canyon, and then the (centres of) the disks are the same.

The centres and the radii of the disks $D_{\gamma_*,i}(\lambda)$ are given more explicitly in [GT, §3.1], as we shall briefly describe in the following.

First, one has to express the coordinates $x = \gamma(y)$ and y in terms of λ. One obtains an expansion $(x(\lambda), y(\lambda))$ with complex coefficients:

$$\left(\sum_{i=m}^{\infty} \alpha_i \lambda^{\frac{i}{mh}}, \sum_{i=m}^{\infty} \beta_i \lambda^{\frac{i}{mh}} \right). \tag{26}$$

For polars γ_τ depending of the generic direction τ, as we have discussed in §5, the first coefficients of (26) which depend of τ are α_{md} and β_{md}, where m is the multiplicity of γ_τ and d is its polar degree, both of which are independent of the generic τ, by Theorem 5.1. Note that h is also independent of τ.

The centers of the disks[2] $D_{\gamma_*,i}(\lambda)$ are then the truncations of (26) up to the order $(md-1)/mh$. They are conjugated by $\lambda \to \omega^i \lambda$, where ω is a primitive root of order mh. The radii of the disks are of the form $r|\lambda|^{d/h} \sim |y|^d$, where $r \in \mathbb{R}_+$ depends on the compact subset of $\mathbb{P}^1$ in which τ varies. The distance between two (centres of) such disks is of order $|\lambda|^{1/h} \sim |y|$.

Theorem 6.1. *Let $f = g \circ \varphi$, where φ is bi-Lipschitz. Then:*

$$\lim_{\lambda \to 0} \frac{1}{2\pi} \int_{\varphi(D_{\gamma_*,i}(\lambda))} K_g \; dS \; \geq 1$$

for any i.

The proof consists in several steps.

Lemma 6.2.

$$\lim_{\lambda \to 0} \frac{1}{2\pi} \int_{D_{\gamma_*,i}(\lambda)} K_f \; dS \; \geq 1.$$

Proof. By (25) the canyon $\mathcal{GC}(\gamma_*)$ concentrates the total curvature:

$$\frac{1}{2\pi} \mathscr{M}_f \mathcal{GC}(\gamma_*) = \sum \mathrm{mult}_0(\{f = 0\}, \gamma'_*),$$

[2]The number of the disks is mh.

where the sum is taken over all polars γ'_* in the canyon $\mathcal{GC}(\gamma_*)$. This is a multiple of the number $\mathrm{mult}_0(\{f=0\},\overline{\gamma_*})$ of disks of the canyon $\mathcal{GC}(\gamma_*)$. These disks contain all the intersections of the Milnor fibre with the polars $\gamma_{\tau *}$, for τ in some dense subset of a compact $K(\lambda)\subset\mathbb{P}^1$ which tends to $\mathbb{P}^1$ when $\lambda\to 0$. On the other hand, as we have seen just above, these disks are conjugate. Therefore, when $\lambda\to 0$, each such disk concentrates the same total curvature, which must be a positive integer (modulo 2π); this proves our lemma. □

We need the interpretation of Lemma 6.2 in terms of the directions τ. By applying Milnor's exchange formula (see Langevin's paper [La1]) we have the equalities:

$$\frac{1}{2\pi}\int_{D_{\gamma_{*,i}}(\lambda)} K_f\ dS = \frac{1}{2\pi}\int_{D_{\gamma_{*,i}}(\lambda)} |\mathrm{Jac}\ \nu_{\mathbb{C}}|^2\ dS \tag{27}$$

where Jac $\nu_{\mathbb{C}}$ denotes the Jacobian determinant of the complex Gauss map. In turn, this is equal, cf [La1], to:

$$\frac{1}{2\pi}\int_{D_{\gamma_{*,i}}(\lambda)} \nu_{\mathbb{C}}^* dp = \frac{1}{2\pi}\int_{\nu_{\mathbb{C}}(D_{\gamma_{*,i}}(\lambda))} \deg(\nu_{\mathbb{C}|D_{\gamma_{*,i}}(\lambda)})\ dp \tag{28}$$

where the last equality follows from the constancy of the degree $\deg(\nu_{\mathbb{C}|D_{\gamma_{*,i}}(\lambda)})$ by Theorem 5.1.

Since 2π represents the volume of $\mathbb{P}^1$, we have proved:

Lemma 6.3. *The image of the disk $D_{\gamma_{*,i}}(\lambda)$ by the Gauss map $\nu_{\mathbb{C}}$, as λ tends to 0, is a dense subset set of $\mathbb{P}^1$, the complementary of which has measure zero.* □

We continue the proof of our theorem. From $f=g\circ\varphi$ we get the relation:

$$\mathrm{grad}\, f(x,y) = \mathrm{grad}\, g(\varphi(x,y))\circ M_\varphi(x,y)$$

where M_φ is a certain matrix, which plays the role of the Jacobian matrix, not everywhere defined but only in almost all points. Let us introduce it. The idea is that even if the partial derivatives of φ do not exist at all points, the limits used to define them are bounded away from 0 in absolute value.

By the bi-Lipschitz property of $\varphi=(\varphi_1,\varphi_2)$ we have, in some ball neighbourhood $B(0,\eta)$ of the origin $(0,0)$, for some $0<m<M$:

$$m\le\frac{\|(\varphi_1,\varphi_2)(x,y_0)-(\varphi_1,\varphi_2)(x_0,y_0)\|}{|x-x_0|}\le M$$

and by taking the limit as $x\to x_0$ we get:

$$m\le\|(\varphi_{1,x},\varphi_{2,x})(x_0,y_0)\|\le M \tag{29}$$

where the notation $\varphi_{1,x}$ suggests partial derivative with respect to x; it has a well-defined value at points where this derivative exists. This limit is not defined elsewhere, but it is however bounded by the values m and M. We shall call *pseudo-derivatives* such bounded quantities $\varphi_{1,x}$ and $\varphi_{2,x}$.

Similarly we get, by taking the limit $y \to y_0$:

$$m \leq \|(\varphi_{1,y}, \varphi_{2,y})(x_0, y_0)\| \leq M. \tag{30}$$

We shall also use the notations $\operatorname{grad} \varphi_i := (\varphi_{i,x}, \varphi_{i,y})$ for $i = 1, 2$.

With these notations we shall prove that the matrix $M_\varphi = \begin{pmatrix} \varphi_{1,x} & \varphi_{1,y} \\ \varphi_{2,x} & \varphi_{2,y} \end{pmatrix}$ is bounded in some neighbourhood of the origin, in a strong sense that we shall define below.

Lemma 6.4. *There exist $r_1, r_2 > 0$ such that:*

$$\| \operatorname{grad} \varphi_1\| \geq r_1 \text{ and } \| \operatorname{grad} \varphi_2\| \geq r_2$$

in some neighbourhood of the origin.

Proof. With the above notations, from $\varphi^{-1} \circ \varphi = \mathrm{id}$ on $B(0, \eta)$ we get:

$$\begin{pmatrix} {\varphi^{-1}}_{1,x} & {\varphi^{-1}}_{1,y} \\ {\varphi^{-1}}_{2,x} & {\varphi^{-1}}_{2,y} \end{pmatrix} \begin{pmatrix} \varphi_{1,x} & \varphi_{1,y} \\ \varphi_{2,x} & \varphi_{2,y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so:

$$\begin{cases} {\varphi^{-1}}_{1,x}\varphi_{1,x} + {\varphi^{-1}}_{1,y}\varphi_{2,x} = 1 \\ {\varphi^{-1}}_{2,x}\varphi_{1,y} + {\varphi^{-1}}_{2,y}\varphi_{2,y} = 1. \end{cases}$$

From this and from (29) we get that $\| \operatorname{grad} {\varphi^{-1}}_1\|$ and $\|\operatorname{grad} {\varphi^{-1}}_2\|$ are bounded away from 0 in some neighbourhood of the origin. By symmetry we get the same conclusion for φ_1 and φ_2, hence our claim is proved. □

Lemma 6.5. *There exists some $m_1 > 0$ such that:*

$$\|\varphi_{2,y}(x, y)\| \geq m_1$$

for any (x, y) belonging to the canyon $\mathcal{GC}(\gamma_*)$.

Proof. We have, by the definition of the canyon, and denoting $d := d_\gamma$:

$$(x, y) \in \mathcal{GC}(\gamma_*) \Rightarrow \|(x, y) - (\gamma(y), y)\| \sim |y|^d.$$

By the bi-Lipschitz property we then have the equivalence:

$$\|\varphi(x, y) - \varphi(\gamma(y), y)\| \sim |y|^d.$$

Since $d > 1$ we get on the one hand that $\|(x, y)\| \sim |y|$ and on the other hand, by dividing with $|y|$, the limit

$$\left\| \frac{\varphi(x, y)}{|y|} - \frac{\varphi(\gamma(y), y)}{|y|} \right\| \to 0 \text{ as } y \to 0.$$

These imply in particular:

$$\left\| \frac{\varphi_2(x, y)}{|y|} - \frac{\varphi_2(\gamma(y), y)}{|y|} \right\| \to 0 \text{ as } y \to 0. \tag{31}$$

We claim that $\|\varphi_2(\gamma(y), y)\| \sim |y|$. From this and from (31) we then get that $\|\varphi_2(\gamma(y), y)/|y|\|$ is bounded away from 0 as $y \to 0$, which means that the pseudo-derivative norm $\|\varphi_{2,y}\|$ is bounded away from 0 in the canyon; this proves our lemma.

Let us now prove the above claim. From the very beginning we choose the coordinates in $\mathbb{C}^2$ such that both f and g are *miniregular*[3], i.e. that the tangent cones of f and g do not contain the direction $[1;0]$. By our assumptions, the polar γ is *tangential*, i.e. its tangent cone is included in the one of $\{f=0\}$. Let us assume without loss of generality that this is the y-axis. This means that γ has contact $k>1$ with some root $(\xi(y),y)$ of $\{f=0\}$. By the bi-Lipschitz property:

$$m\|(\xi(y),y)-(\gamma(y),y)\| \leq \|\varphi(\xi(y),y)-\varphi(\gamma(y),y)\| \leq M\|(\xi(y),y)-(\gamma(y),y)\| \tag{32}$$

and we have the equivalence $\|(\xi(y),y)-(\gamma(y),y)\| \sim |y|^k$. Since by bi-Lipschitz we have $\|\varphi(\gamma(y),y)\| \sim |y|$ then by the above facts we get:

$$\|\varphi(\xi(y),y)\| \sim |y|. \tag{33}$$

Next:

$$\|(\varphi_1(\xi(y),y),\varphi_2(\xi(y),y)\| = |\varphi_2(\xi(y),y)| \left\| \left(\frac{\varphi_1(\xi(y),y)}{\varphi_2(\xi(y),y)}, 1 \right) \right\|$$

Since $f = g\circ\varphi$, the root ξ is sent by φ to some root $\eta = (\eta_1,\eta_2)$ of g, which means that the direction $\left[\frac{\varphi_1(\xi(y),y)}{\varphi_2(\xi(y),y)}, 1\right]$ is the same as the direction $[\frac{\eta_1}{\eta_2}, 1]$. The later tends to the direction of the tangent line to η, which is different from $[1,0]$ by our assumption. Hence this is of the form $[a,1]$, where $a\in\mathbb{C}$. Consequently:

$$\mathrm{ord}_y\varphi_1(\xi(y),y) \geq \mathrm{ord}_y\varphi_2(\xi(y),y).$$

Thus, with help of (33), we get:

$$\mathrm{ord}_y\varphi_2(\xi(y),y) = \mathrm{ord}_y\varphi(\xi(y),y) = 1,$$

which implies $\|\varphi_2(\xi(y),y)\| \sim |y|$ and which, in turn, implies our claim by using again (32) and since $\gamma(y)$ has contact >1 with $\xi(y)$. $\square$

6.1. **Proof of the main theorems.**

Step 1. We claim that $\varphi(D_{\gamma_{*,i}}(\lambda))$ intersects a disk cut out by some horn $\mathrm{Horn}^{(d)}(\gamma_{g,*};\varepsilon;\eta)$ of degree >1 into the fibre $g=\lambda$.

By *reductio ad absurdum*, let us suppose that the image $\varphi(D_{\gamma_{*,i}}(\lambda))$ is disjoint from all horns of f of degree >1, for any λ close enough to 0. Since $\varphi(D_{\gamma_{*,i}}(\lambda))$ is a *valley disk*, by applying a straightforward extension of [KKP2, Lemma 6.7] for valley disks instead of canyon disks, we conclude that:

$$\lim_{\lambda\to 0}\int_{\varphi(D_{\gamma_{*,i}}(\lambda))} K_g\, dS = 0. \tag{34}$$

Then, by using the "exchange formula" (28) and (27) for $\varphi(D_{\gamma_{*,i}}(\lambda))$, it follows that its image in $\mathbb{P}^1$ by the Gauss map $\nu_{\mathbb{C},g} = \frac{\mathrm{grad}\, g}{|\mathrm{grad}\, g|} : B(0;\eta) \to \mathbb{P}^1$ is a contractible set which tends to a measure zero subset $A\subset\mathbb{P}^1$ as $\lambda\to 0$. But we claim more: the image $\nu_{\mathbb{C},g}(\varphi(D_{\gamma_{*,i}}(\lambda)))$ tends to a constant when $\lambda\to 0$.

[3]in the terminology of [HP1, KKP2]

Let $G(\lambda)$ denote what remains from the Milnor fibre $g = \lambda$ after taking out all the horns. The distribution of curvature in $G(\lambda)$ is of order 1, and the integral of curvature over $G(\lambda)$ equals $2\pi(r-1)$, which means that the gradient on $G(\lambda)$ has dense image in $\mathbb{P}^1$ and the degree of this map is $r-1$, hence at least 1 if $r \geq 2$. Nevertheless the concentration of curvature in $G(\lambda)$ is of order 1. Moreover, by the results of Henry and Parusiński [HP1, HP2], the variation of the gradient is of order 1. Hence the curvature and the variation of the gradient are homogeneous of degree 1 and the radius of their distribution is of order $1/h$. A variation of the gradient of order > 1 on $G(\lambda)$ means that the variation of the gradient on $\varphi(D_{\gamma_{*,i}}(\lambda))$ is asymptotically zero, hence the gradient tends to a constant. Indeed our subset $\varphi(D_{\gamma_{*,i}}(\lambda)) \subset G(\lambda)$ belongs to a valley which is tangent to the zero locus $g = 0$ (since φ preserves the tangency to the zero locus and the canyon of γ_* was tangent to the zero locus of f) and therefore cannot contain non-tangential curvature. This ends the proof of our claim.

We are now finishing the proof of Step 1. We have:

$$\operatorname{grad} f(x,y) = \operatorname{grad} g(\varphi(x,y)) \circ M_\varphi(x,y).$$

Since the above claim is true, we obtain:

$$(a,1)\begin{pmatrix}\varphi_{1,x} & \varphi_{1,y}\\ \varphi_{2,x} & \varphi_{2,y}\end{pmatrix} = (a\varphi_{1,x} + \varphi_{2,x}, a\varphi_{1,y} + \varphi_{2,y}) \tag{35}$$

and, by using (30) and Lemma 6.5, we get

$$|a\varphi_{1,y} + \varphi_{2,y}| \geq |\varphi_{2,y}| - |a\varphi_{1,y}| \geq m_1 - |a\varphi_{1,y}| > 0.$$

It then follows from the relation (35) that the modulus of the direction of the gradient vector $\operatorname{grad} f(x,y)$:

$$\frac{\|a\varphi_{1,x} + \varphi_{2,x}\|}{\|a\varphi_{1,y} + \varphi_{2,y}\|}$$

is bounded, since the denominator is bounded away from 0 and the numerator is less or equal to $\max(m_1, M)$. This contradicts Lemma 6.3; thus Step 1 is proved.

Step 2.

We still refer to canyon disks of canyon degree > 1. Let D_f be some disk cut out on the Milnor fibre $f^{-1}(\lambda)$ by a horn $H(\gamma)$ of a canyon $\mathcal{GC}(\gamma)$ the degree of which we shall denote by $\deg D_f$. We recall[4] that the radius of D_f is equal to $k|y^d|$, for some $k > 0$ and that the distance between two such discs is of order $|y|$. If two polar are in the same canyon, then their disks coincide.

By "canyon disk" we shall mean in the following such a disk of radius order d and some constant $k > 0$ which is not specified.

By Step 1, there is some canyon disk D_g of g such that:

$$\varphi(D_f) \cap D_g \neq \emptyset.$$

[4] cf the discussion about radius before the statement of Theorem 6.1

Lemma 6.6. *If $\varphi(D_f) \cap D_g \neq \emptyset$ then:*

$$\deg D_g \geq \deg D_f$$

and moreover $\varphi(D_f)$ includes D_g.

Proof. The diameter of $\varphi(D_f)$ is asymptotically of order[5] equal to $\deg D_f$, since φ is bi-Lipschitz. So if $\deg D_g < \deg D_f$ it follows by [KKP2, Lemma 6.7] – the extension of this result is needed, i.e. not only for canyon disks but also for any "disks" of the same diameter – that the total curvature over $\varphi(D_f)$ must be zero asymptotically (i.e. when $\lambda \to 0$). This yields a contradiction as we have proved above for (34).

Now if $\deg D_g \geq \deg D_f$ then, by the definition of the disks (i.e. with fixed order and arbitrary radius) and since $\varphi(D_f) \cap D_g \neq \emptyset$, it follows that $\varphi(D_f)$ includes D_g for appropriate diameters. □

Lemma 6.7. *$\varphi(D_f)$ intersects a single disk D_g.*

Proof. If D_g and D'_g are two disjoint disks of g which intersect $\varphi(D_f)$, then they are of degree strictly greater than $\deg D_f$, otherwise they must be included one into the other up to rescaling their radii. Hence they are included in $\varphi(D_f)$, by the above lemma.

Next, by applying φ^{-1} we get $\varphi^{-1}(D_g) \subset D_f$ with $\deg D_g > \deg D_f$, hence, by Step 1 and Lemma 6.6, there must exist another disk $D'_f \subset \varphi^{-1}(D_g)$ with $\deg D'_f \geq \deg D_g$. But this means that we have the inclusion $D'_f \subset D_f$ with the inequality $\deg D'_f > \deg D_f$ and this contradicts the theorem that canyons of degree > 1 are disjoint [KKP2]. □

We therefore have a graduate bijection between canyon disks of f and canyon disks of g, respecting the degrees. More precisely, we have shown the following:

Theorem 6.8. *The bi-Lipschitz map φ establishes a bijection between the canyon disks of f and the canyon disks of g by preserving the canyon degree.* □

This key result has several implications resulting in bi-Lipschitz invariants. Let us display them below.

Definition of the multi-layer cluster decomposition.

Let $f = g \circ \varphi$ with φ bi-Lipschitz homeomorphism. Our Theorem 6.8 immediately implies that certain subsets of gradient canyons, defined in terms of the mutual contact, are preserved by φ.

As a matter of fact, even if one cannot prove anymore that the image by φ of a canyon is a canyon, we derive from Theorem 6.8 that φ induces a bijection between the gradient canyons of f and those of g. Moreover, we show that there are well-defined "clusters" of canyons of f which are sent by φ into similar clusters of g, and that such clusters are determined by certain rational integers which are are themselves bi-Lipschitz invariants.

We consider the tangential canyons only, i.e. those of degree more than 1. The canyon of degree 1 is preserved, since it covers the Milnor fibre entirely, together with its multiplicity, or, equivalently, its partial Milnor number, or its total curvature, as follows directly from [GT], [KKP2].

[5] the order is the exponent of $|y|$

Note that the exponent h from (21) is a topological invariant, see e.g. [GT], [KKP2]. We can group the canyons in terms of the essentials bars of the tree of f, namely those canyons departing from an essential bar $B(h)$ corresponding to h, i.e. associated to the polars leaving the tree of f on that bar $B(h)$. Their contact, for distinct canyons, can be greater or equal to the coslope of the corresponding bar, say $\theta_{B(h)}$, but less than their canyon degrees.

The *order of contact*, see [KKP2], between two different holomorphic arcs α and β is well defined as:

$$\max \operatorname{ord}_y(\alpha(y) - \beta(y)) \tag{36}$$

where the maximum is taken over all conjugates of α and of β. Whenever the canyons $\mathcal{GC}(\gamma_{1*}) \ni \alpha_*$ and $\mathcal{GC}(\gamma_{2*}) \ni \beta_*$ are different and both of degree $d > 1$, this order is lower than d and therefore does not depend on the choice of α_* in the first canyon, and of β_* in the second canyon. This yields a well-defined order of contact between two canyons of degree d.

In a similar way we can define the contact of any two canyons as the contact of the corresponding polars in the canyons. In this way all the contacts which are contacts between roots of f are automatically preserved by φ. The more interesting situations appear after the polars leave the tree, namely at a higher level than the coslopes $\theta_{B(h)}$.

Let $G_d(f)$ be the union of gradient canyons of a fixed degree $d > 1$. Let $G_{d,B(h)}(f)$ be the union of canyons the polars of which grow on the same bar B and thus have the same h, for $d > \theta_{B(h)} > 1$, more precisely those canyons of degree d with the same top edge relative to the Newton polygon relative to polar.

One then has the disjoint union decomposition:

$$G_d(f) = \bigsqcup_h G_{d,B(h)}(f). \tag{37}$$

Note that each canyon from $G_{d,B(h)}(f)$ has the same contact, higher than 1, with a unique irreducible component $\{f_i = 0\}$.

Next, each cluster union of canyons $G_{d,B(h)}(f)$ has a partition into unions of canyons according to the mutual order of contact between canyons. More precisely, a fixed gradient canyon $\mathcal{GC}_i(f) \subset G_{d,B(h)}(f)$ has a well defined order of contact $k(i,j)$ with each other gradient canyon $\mathcal{GC}_j(f) \subset G_{d,B(h)}(f)$ from the same cluster; we count also the *multiplicity* of each such contact, i.e. the number of canyons $\mathcal{GC}_j(f)$ from the cluster $G_{d,B(h)}(f)$ which have exactly the same contact with $\mathcal{GC}_i(f)$.

Let then $K_{d,B(h),i}(f)$ be the (un-ordered) set of those contact orders $k(i,j)$ of the fixed canyon $\mathcal{GC}_i(f)$, counted with multiplicity.

Let now $G_{d,B(h),\omega}(f)$ be the union of canyons from $G_{d,B(h)}(f)$ which have exactly the same set $\omega = K_{d,B(h),i}(f)$ of orders of contact with the other canyons from $G_{d,B(h)}(f)$. This defines a partition:

$$G_{d,B(h),\omega}(f) = \bigsqcup_\omega G_{d,B(h)}(f). \tag{38}$$

In this way each canyon has its "identity card" composed of these orders of contact (which are rational numbers), and it belongs to a certain cluster $G_{d,B(h),\omega}(f)$ in the partition of $G_d(f)$. It is possible that two canyons have the same "identity card". We clearly have, by definition, the inclusions:

$$G_d(f) \supset G_{d,B(h)}(f) \supset G_{d,B(h),\omega}(f)$$

for any defined indices.

With these notations, we have:

Theorem 6.9. *The bi-Lipschitz map φ induces a bijection between the gradient canyons of f and those of g. The following are bi-Lipschitz invariants:*

(a) *the set $G_d(f)$ of canyon degrees $d > 1$, and for each fixed degree $d > 1$, each bar B and rational h, the cluster of canyons $G_{d,B(h)}(f)$.*

(b) *the set of contact orders $K_{d,B(h),i}(f)$, and for each such set, the sub-cluster of canyons $G_{d,B(h),K_{d,B(h),i}}(f)$.*

Moreover, φ preserves the contact orders between any two clusters of type $G_{d,B(h),K_{d,B(h),i}}(f)$.

Proof. We know from Theorem 6.8, Lemma 6.6 and Lemma 6.7 that φ induces a bijection between canyon disks since every canyon disk of f is sent by φ to a unique canyon disk of g. The contact between two canyons of degree $d > 1$ translates into an asymptotic order of the distances between the canyon disks in the Milnor fibre. The map φ transforms the Milnor fibre $\{f = \lambda\}$ into the Milnor fibre $\{g = \lambda\}$ and we know that canyon disks are sent to canyon disks of the same degree (Theorem 6.8). In addition to that, the order of the distance between any two disks is preserved by φ since φ is bi-Lipschitz.

We then check the order of the distance between disks corresponding to two different canyons and translate it into the order of contact (36) between these canyons, starting with the lowest orders which are higher than 1. Doing this on the set $G_d(f)$ will have as result the partition (37). Continuing to do this with each cluster of canyons $G_{d,B(h)}(f)$ will have as result the partition (38). This proves (a) and (b).

Our first assertion follows now from the bijective correspondence between the smallest clusters, as follows. In case if one small cluster of type $G_{d,h,K_{d,h,i}}(f)$ contains more than one gradient canyon, the number of canyons is detected by the multiplicity of the cntact order, and this multiplicity is obviously preserved by the bi-Lipschitz map φ.

By the same reasons as above, we get our last claim, that φ preserves the contact orders between any cluster of canyons. □

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