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ANTOINE MARNAT AND NIKOLAY G. MOSHCHEVITIN

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation

Antoine Marnat* and Nikolay G. Moshchevitin†

Abstract

We provide a lower bound for the ratio between the ordinary and uniform exponents of both simultaneous Diophantine approximation to n real numbers and Diophantine approximation for one linear form in n variables. This question was first considered in the 50's by V. Jarník who solved the problem for two real numbers and established certain bounds in higher dimension. Recently different authors reconsidered the question, solving the problem in dimension three with different methods. Considering a new concept of parametric geometry of numbers, W. M. Schmidt and L. Summerer conjectured that the optimal lower bound is reached at regular systems. It follows from a remarkable result of D. Roy that this lower bound is then optimal. In the present paper we give a proof of this conjecture by W. M. Schmidt and L. Summerer.

1 Introduction

In the 50's, V. Jarník [6, 7, 8] considered exponents of Diophantine approximation, and in particular the ratio between ordinary and uniform exponent. An optimal lower bound expressed as a function of the uniform exponent was established for simultaneous approximation to two real numbers and for one linear form in two variables. The question was reconsidered recently by different authors [11, 15, 16, 24, 5, 3]. The optimality of V. Jarník's inequalities for two numbers was shown by M. Laurent [11]. The inequality for simultaneous approximation to three real numbers was obtained by the second named author [15]. Introducing parametric geometry of numbers [24, 23], W. M. Schmidt and L. Summerer considered recently a new method to obtain the optimal lower bounds for the approximation to three numbers (both in the cases of simultaneous approximation and approximation for one linear form in three variables), and improve the general lower bound in any dimension. They conjectured in this context that the general lower bound in the problem of approximation to n real numbers arise

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from co-called *regular systems*. The goal of the present paper is to prove this conjecture. To do this we use Schmidt's inequality on heights [21] applied to a well-chosen subsequence of best approximation vectors. Our main result is stated in Theorem 1 below. The optimality of our bound follows from a recent breakthrough paper by D. Roy [20].

Throughout this paper, the integer $n \geq 1$ denotes the dimension of the ambient space, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ denotes an n-tuple of real numbers such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent.

Given $n \geq 1$ and $\boldsymbol{\theta} \in \mathbb{R}^n$, we consider the irrationality measure function

$$\psi(t) = \min_{q \in \mathbb{Z}_+, q \leq t} \max_{1 \leq j \leq n} \|q\theta_j\|,$$

which gives rise to the ordinary exponent of simultaneous Diophantine approximation

$$\lambda(\boldsymbol{\theta}) = \sup\{\lambda : \liminf_{t \to +\infty} t^{\lambda}\psi(t) < +\infty\}$$

and the uniform exponent of simultaneous Diophantine approximation

$$\hat{\lambda}(\boldsymbol{\theta}) = \sup\{\lambda : \limsup_{t \to +\infty} t^{\lambda}\psi(t) < +\infty\}.$$

The irrationality measure function

$$\varphi(t) = \min_{\boldsymbol{q} \in \mathbb{Z}^n, 0 < \max_{1 \leq j \leq n} |q_j| \leq t} \|q_1\theta_1 + \cdots + q_n\theta_n\|$$

gives rise to the ordinary exponent of Diophantine approximation by one linear form

$$\omega(\boldsymbol{\theta}) = \sup\{\omega : \liminf_{t \to +\infty} t^{\omega}\varphi(t) < +\infty\}$$

and the uniform exponent of Diophantine approximation by one linear form

$$\hat{\omega}(\boldsymbol{\theta}) = \sup\{\omega : \limsup_{t \to +\infty} t^{\omega}\varphi(t) < +\infty\}.$$

These exponents were first introduced and studied by A. Khintchine [9, 10] and V. Jarník [6]. Dirichlet's *Schubfachprinzip* ensures that for any $\boldsymbol{\theta}$ with $\mathbb{Q}$-linearly independent coordinate with 1

$$\omega(\boldsymbol{\theta}) \geq \hat{\omega}(\boldsymbol{\theta}) \geq n \text{ and } \lambda(\boldsymbol{\theta}) \geq \hat{\lambda}(\boldsymbol{\theta}) \geq 1/n.$$

Indeed, exponents of Diophantine approximation are about investigating specific $\boldsymbol{\theta}$ for which Dirichlet's *Schubfachprinzip* can be improved. Ordinary exponents question whether Dirichlet's *Schubfachprinzip* can be improved for approximation vectors of arbitrarily large size t, while uniform exponents question whether it can be improved for any sufficiently large upper bound t for the size of approximation vectors. The aim of this paper is to provide a lower bound for the ratios $\lambda(\boldsymbol{\theta})/\hat{\lambda}(\boldsymbol{\theta})$ and $\omega(\boldsymbol{\theta})/\hat{\omega}(\boldsymbol{\theta})$ as a function of $\hat{\lambda}(\boldsymbol{\theta})$ and $\hat{\omega}(\boldsymbol{\theta})$ respectively, in any dimension. In dimension $n=1$ simultaneous approximation and approximation by one linear form coincide. Khintchine [10] observed that the uniform exponent for an irrational $\boldsymbol{\theta}$ always takes the value 1 and it follows from Dirichlet's *Schubfachprinzip* that the ordinary exponent satisfy $\omega(\theta)=\lambda(\theta)\geq 1=\hat{\omega}(\theta)=\hat{\lambda}(\theta)$. In dimension $n=2$, Jarník proved in [7, 8] the inequalities

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})}\geq\frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}, \tag{1}$$

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})}\geq\hat{\omega}(\boldsymbol{\theta})-1. \tag{2}$$

These inequalities are optimal by a result of M. Laurent [11]. In [15], Moshchevitin proved the optimal bound for simultaneous approximation:

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})}\geq\frac{\hat{\lambda}(\boldsymbol{\theta})+\sqrt{4\hat{\lambda}(\boldsymbol{\theta})-3\hat{\lambda}(\boldsymbol{\theta})^2}}{2(1-\hat{\lambda}(\boldsymbol{\theta}))}=\frac{1}{2}\left(\frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}+\sqrt{\left(\frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}\right)^2+\frac{4\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}}\right). \tag{3}$$

Schmidt and Summerer provided an alternative proof using parametric geometry of numbers in [25], and the following bound for approximation by one linear form:

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})}\geq\frac{\sqrt{4\hat{\omega}(\boldsymbol{\theta})-3}-1}{2}. \tag{4}$$

A simple proof of this bound was given in [16]. In [8], Jarník also provided a lower bound in arbitrary dimension $n\geq 2$.

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \;\geq\; \hat{\omega}(\boldsymbol{\theta})^{1/(n-1)}-3,\ \text{provided that } \hat{\omega}(\boldsymbol{\theta})>(5n^2)^{n-1}, \tag{5}$$

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \;\geq\; \frac{\hat{\lambda}(\boldsymbol{\theta})}{1-\hat{\lambda}(\boldsymbol{\theta})}. \tag{6}$$

In fact, these bounds also apply in a more general setting of simultaneous Diophantine approximation by a set of linear forms.

Using their new tools of parametric geometry of numbers, Schmidt and Summerer [23] provided the first general improvement valid for the whole admissible interval of values of the uniform exponents $\hat{\omega}$ and $\hat{\lambda}$.

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})}\geq\frac{(n-2)(\hat{\omega}(\boldsymbol{\theta})-1)}{1+(n-3)\hat{\omega}(\boldsymbol{\theta})}, \tag{7}$$

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})}\geq\frac{\hat{\lambda}(\boldsymbol{\theta})+n-3}{(n-2)(1-\hat{\lambda}(\boldsymbol{\theta}))}. \tag{8}$$

Here relation (8) is sharper than relation (6). Relation (7) is valid for the whole interval of possible values of $\hat{\omega}(\boldsymbol{\theta})$, but Jarník's asymptotic relation (5) is better for large $\hat{\omega}(\boldsymbol{\theta})$. A simple proof of (8) was given in [5].

In [25] Schmidt and Summerer conjecture that, as in dimension $n = 3$, the general optimal lower bound is reached at *regular systems*. In this paper we show that this conjecture holds. Let us first introduce some notation.

For given $n \geq 1$ and $1/n \leq \alpha < 1$, we consider the polynomial

$$R_{n,\alpha}(x) = x^{n-1} - \frac{\alpha}{1-\alpha}(x^{n-2} + \cdots + x + 1) \tag{9}$$

and denote by $G(n, \alpha)$ its unique real positive root. For $\alpha^* \geq n$ we denote by $1/G^*(n, \alpha^*)$ the unique positive root of $R_{n,1/\alpha^*}(x)$.

Theorem 1. *For $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent, one has*

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq G(n, \hat{\lambda}(\boldsymbol{\theta})) \quad \text{and} \quad \frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \geq G^*(n, \hat{\omega}(\boldsymbol{\theta})). \tag{10}$$

Furthermore, for any $\hat{\omega} \geq n$ and any $C \geq G^*(n, \hat{\omega})$, there exists infinitely many $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent and

$$\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \omega(\boldsymbol{\theta}) = C\hat{\omega}$$

and for any $1/n \leq \hat{\lambda} \leq 1$ and any $C \geq G(n, \hat{\lambda})$, there exists infinitely many $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent and

$$\hat{\lambda}(\boldsymbol{\theta}) = \hat{\lambda} \quad \text{and} \quad \lambda(\boldsymbol{\theta}) = C\hat{\lambda}.$$

It follows from Roy's theorem [20] applied to Schmidt-Summerer's *regular systems* [25] [19] that the lower bound is reached and thus optimal. The second part of Theorem 1 refines this observation. Note that for any $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent, we have $\hat{\omega}(\boldsymbol{\theta}) \geq n$ and $\hat{\lambda}(\boldsymbol{\theta}) \in [1/n, 1]$, (see [4], [12]) hence the constraint on $\hat{\lambda}$ and $\hat{\omega}$ is not restrictive.

We can reformulate these lower bounds by the inequalities

$$1 + \omega(\boldsymbol{\theta}) - \hat{\omega}(\boldsymbol{\theta}) \geq \left(\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})}\right)^n \quad \text{and} \quad 1 - \frac{1}{\hat{\lambda}(\boldsymbol{\theta})} + \frac{1}{\lambda(\boldsymbol{\theta})} \leq \left(\frac{\hat{\lambda}(\boldsymbol{\theta})}{\lambda(\boldsymbol{\theta})}\right)^n.$$

In these formulae appears clearly the natural symmetry property of spectra of Diophantine approximation pointed out by Schmidt and Summerer [23]. It is even more obvious in the proof, as the very same geometric analysis applies to both cases.

The main part of Theorem 1 is the lower bound. The proof uses determinants of best approximation vectors, following the idea of [15]. It deeply relies on an inequality of Schmidt [21] applied inductively to a well chosen subsequence of best approximation vectors. The second part of Theorem 1 is a consequence of the parametric geometry of numbers, and is proved independently in Section 6.

In the next section, we define the main tools needed for the proof: best approximation vectors and their properties. With examples of approximation to 3 and 4 numbers in Section 3, we then provide a proof of Theorem 1 in the important case of simultaneous approximation (Section 4). In Section 5, we explain how an hyperbolic rotation reduces the case of approximation by one linear form to the case of simultaneous approximation.

2 Main tools

2.1 Sequences of best approximations

We denote by $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ the sequence of best approximations (or minimal points) to $\boldsymbol{\theta} \in \mathbb{R}^n$. This notion was introduced by Voronoi [26] as minimal points in lattices, it was first defined in our context by Rogers [18]. It has been used implicitly or explicitly in many proofs concerning exponents of Diophantine approximation. Many important properties of best approximation vectors are discussed in a survey by Chevallier [1].

Let $k \geq 1$ be an integer. Let L and N be two applications from $\mathbb{Z}^k$ to $\mathbb{R}_+$, where N represent the size of an approximation vector in $\mathbb{Z}^k$ and L represent the approximation error. We call a sequence of best approximation vectors $(\boldsymbol{z}_l)_{l\geq 0} \in (\mathbb{Z}^k)^{\mathbb{N}}$ to L with respect to N a sequence such that

- $N(\boldsymbol{z}_l)$ is a strictly increasing sequence with lower bound 1,
- $L(\boldsymbol{z}_l)$ is a strictly decreasing sequence with upper bound 1,
- for any approximation vector $\boldsymbol{z} \in \mathbb{Z}^k$, if $N(\boldsymbol{z}) < N(\boldsymbol{z}_{l+1})$ then $L(\boldsymbol{z}) \geq L(\boldsymbol{z}_l)$.

In general we do not have uniqueness of such a sequence, and existence follows if L reaches a minimum on sets of the form

$$E_B = \{X \in \mathbb{Z}^k | N(X) \leq B\},$$

where B is any real bound.

In the context of best approximation vectors for simultaneous Diophantine approximation for $\mathbb{Q}$-independent numbers $1, \theta_1, ..., \theta_n$ the sequence is unique, and we can write

$$\boldsymbol{z}_l = (q_l, a_{1,l}, a_{2,l}, \dots, a_{n,l}) \in \mathbb{Z}^{n+1}, l \in \mathbb{N} \ \text{ with } q_l > 0.$$

Set

$$L_\lambda(\boldsymbol{z}_l) := \xi_l = \max_{1 \le i \le n} |q_l \theta_i - a_{i,l}| \quad \text{and } N_\lambda(\boldsymbol{z}_l) := q_l.$$

By definition of best approximations

$$1 < q_1 < q_2 < \dots < q_l < q_{l+1} < \cdots \quad \text{and} \quad 1 > \xi_1 > \xi_2 > \dots > \xi_l > \xi_{l+1} > \cdots \tag{11}$$

We may also assume that q_1 is large enough so that for every $l \ge 1$

$$\xi_l \le q_{l+1}^{-\alpha}, \tag{12}$$

where $\alpha < \hat{\lambda}(\boldsymbol{\theta})$.

In the context of best approximation vector for approximation by one linear form, we can write

$$\boldsymbol{z}_l = (q_{1,l}, q_{2,l}, \dots, q_{n,l}, a_l) \in \mathbb{Z}^{n+1}, l \in \mathbb{N}.$$

Set

$$L_\omega(\boldsymbol{z}_l) := L_l = q_{1,l}\theta_1 + \dots + q_{n,l}\theta_n - a_l \quad \text{and} \quad N_\omega(\boldsymbol{z}_l) := M_l = \max_{1 \le j \le n} |q_j|.$$

Due do the symmetry we may assume that $L_l > 0$. In the $\mathbb{Q}$-independent case this defines vectors $\boldsymbol{z}_l$ uniquely. By definition of best approximations

$$1 < M_1 < M_2 < \dots < M_l < M_{l+1} < \cdots \quad \text{and} \quad 1 > L_1 > L_2 > \dots > L_l > L_{l+1} > \cdots$$

We may also assume that M_1 is large enough so that for every $l \ge 1$

$$L_l \le M_{l+1}^{-\alpha^*}. \tag{13}$$

where $\alpha^* < \hat{\omega}(\boldsymbol{\theta})$.

In the context of simultaneous Diophantine approximation, provided that $1, \theta_1, \dots, \theta_n$ are linearly independent over $\mathbb{Q}$, it is known that a sequence of best approximation vectors ultimately spans the whole space $\mathbb{R}^{n+1}$. However in the context of approximation by one linear form, the situation is different: it may happen that vectors of best approximation span a strictly lower dimensional subspace of $\mathbb{R}^{n+1}$. See the surveys [13, 14] by Moshchevitin and the paper [1] by Chevallier for more details. Fortunately, if best approximation vectors do not span the whole space $\mathbb{R}^{n+1}$ we get a sharper result, since $G(n, \alpha)$ is a decreasing function of n. Thus, we may assume without loss of generality that in both contexts best approximation vectors ultimately span the whole space $\mathbb{R}^{n+1}$.

Whenever $1, \theta_1, \ldots, \theta_n$ are linearly dependent over $\mathbb{Q}$, consider $\tilde{\boldsymbol{\theta}} = (\theta_{i_1}, \ldots, \theta_{i_p})$ a largest subset of the components of $\boldsymbol{\theta}$ which satisfy the linear independence property over $\mathbb{Q}$ with 1. It is easy to check that $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$ have the same exponents, and thus results of lower dimension apply. Thus, we may assume without loss of generality that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Q}$.

Using sequences of best approximations vectors, proving that

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq G$$

is equivalent to showing that there exists arbitrarily large indices k with $q_{k+1} \gg q_k^G$. Similarly, proving that

$$\frac{\omega(\boldsymbol{\theta})}{\hat{\omega}(\boldsymbol{\theta})} \geq G$$

is equivalent to showing that there exists arbitrarily large indices k with $M_{k+1} \gg M_k^G$. This observation relies on the expression of exponents of Diophantine approximation in terms of best approximation vectors

$$\omega = \limsup_{k \to \infty} \left(-\frac{\log(L_k)}{\log(M_k)} \right) \quad , \quad \hat{\omega} = \liminf_{k \to \infty} \left(-\frac{\log(L_k)}{\log(M_{k+1})} \right),$$

$$\lambda = \limsup_{k \to \infty} \left(-\frac{\log(\xi_k)}{\log(q_k)} \right) \quad , \quad \hat{\lambda} = \liminf_{k \to \infty} \left(\frac{\log(\xi_k)}{\log(q_{k+1})} \right).$$

The proofs in the case of simultaneous approximation and approximation by one linear form rely on the same geometric analysis. The idea is to consider an arbitrarily large index k, and construct a pattern of best approximation vectors in which at least one pair of successive best approximation vectors satisfies

$$q_{k+1} \gg q_k^G \quad \text{or} \quad M_{k+1} \gg M_k^G \tag{14}$$

for the required G. Here and below, the Vinogradov symbols $\ll$, $\gg$ and $\asymp$ refer to constants depending on $\boldsymbol{\theta}$ but not the index k.

Given a sublattice $\Lambda \subset \mathbb{Z}^{n+1}$, we denote by $\det(\Lambda)$ the fundamental volume of the lattice Λ in the linear subspace $\langle \Lambda \rangle_{\mathbb{R}}$. We recall well known facts about best approximation vectors and fundamental determinants of the related lattices.

Lemma 1. *Two consecutive best approximation vectors $\boldsymbol{z}_i$ and $\boldsymbol{z}_{i+1}$ are $\mathbb{Q}$-linearly independent and form a basis of the integer points of the rational subspace they span.*

$$\langle \boldsymbol{z}_i, \boldsymbol{z}_{i+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_i, \boldsymbol{z}_{i+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^{n+1}.$$

See for example [2, Lemma 2].

Lemma 2. *For any $l \geq 1$, consider Λ_l the lattice with basis $\boldsymbol{z}_l, \boldsymbol{z}_{l+1}$ and the lattice Γ_l with basis $\boldsymbol{z}_{l-1}, \boldsymbol{z}_l, \boldsymbol{z}_{l+1}$. In the context of simultaneous approximation we have the estimates of their fundamental volumes*

$$\det(\Lambda_l) \asymp \xi_l q_{l+1}, \tag{15}$$

$$\det(\Gamma_l) \ll \xi_{l-1}\xi_l q_{l+1}, \tag{16}$$

In the context of approximation by one linear form, we do not have directly such estimates. In section 5 we explain how hyperbolic rotation provides an helpful analogue.

The proof of Lemma 2 is well known, see for example [1] or [14]. For the sake of completeness, and because we want to adapt the proof for the case of approximation by one linear form, we provide a detailed proof. The upper bounds rely on the following lemma (see [22, Lemma 1]), while the lower bounds comes from Minkowski's first convex body theorem.

Lemma 3. *Assume $X_1, \dots, X_m$ are vectors of an Euclidean space E^n, and have coordinates $X_t = (x_{t,1}, \dots, x_{t,m})$ for $1 \leq t \leq m$ in some Cartesian coordinate-system of E^n. Then $\det^2(X_1, \dots, X_m)$ is the sum (with $\binom{m}{n}$ summands) of the squares of the absolute values of the determinants of the $(m \times m)$-submatrices of the matrix $(x_{t,j})_{1\leq t\leq m, 1\leq j\leq n}$.*

Proof of Lemma 2. The proof relies on the geometric fact that the best approximation $\boldsymbol{z}_l = (q_l, a_{1,l}, a_{2,l}, \dots, a_{n,l}) \in \mathbb{Z}^{n+1}$ satisfy (11). We first prove the upper bounds.

Consider the 2-dimensional fundamental volume of the lattice spanned by $\boldsymbol{z}_l, \boldsymbol{z}_{l+1}$. The coordinates of these vectors form the matrix

$$\begin{pmatrix} q_l & a_{1,l} & \cdots & a_{n,l} \\ q_{l+1} & a_{1,l+1} & \cdots & a_{n,l+1} \end{pmatrix}.$$

However it is not convenient to use this matrix to apply Lemma 3. We consider a special choice of Cartesian coordinates. We take the system of orthogonal unit vector $(e_0, e_1, \dots, e_n)$ in the following way: e_0 is parallel to $(1, \theta_1, \dots, \theta_n)$ and $e_1, \cdots, e_n$ are arbitrary. Then, in our new coordinates

$$\boldsymbol{z}_l = (Z_l, \Xi_{1,l}, \dots, \Xi_{n,l})$$

where $Z_l \asymp q_l$ and $|\Xi_{i,l}| \ll \xi_l$.
Now we consider the $2 \times (n+1)$ matrix

$$\begin{pmatrix} Z_l & \Xi_{1,l} & \cdots & \Xi_{n,l} \\ Z_{l+1} & \Xi_{1,l+1} & \cdots & \Xi_{n,l+1} \end{pmatrix}.$$

If $M_{i,j}$ is the (2×2) minor of index i,j, we have by Lemma 3

$$\det(\Lambda_l)^2 = \sum_{0\le i<j\le n} M_{i,j}^2 \ll \max_{0\le i<j\le n} M_{i,j}^2 \ll |Z_{l+1}|^2 \max_{1\le i\le n} |\Xi_{i,l}|^2 \ll (\xi_l q_{l+1})^2.$$

Consider the 3-dimensional fundamental volume $\det(\Gamma_l)$ of the lattice spanned by $\boldsymbol{z}_{l-1}, \boldsymbol{z}_l, \boldsymbol{z}_{l+1}$. Denote by $M_{i,j,k}$ the 3×3 minors of the matrix

$$\begin{pmatrix} Z_{l-1} & \Xi_{1,l-1} & \cdots & \Xi_{n,l-1} \\ Z_l & \Xi_{1,l} & \cdots & \Xi_{n,l} \\ Z_{l+1} & \Xi_{1,l+1} & \cdots & \Xi_{n,l+1} \end{pmatrix}.$$

By Lemma 3 we have

$$\det(\Gamma_l)^2 \ll \sum_{0\le i<j<k\le n} M_{i,j,k}^2 \ll \max_{0\le i<j<k\le n} M_{i,j,k}^2 \ll |Z_{l+1}\Xi_l\Xi_{l-1}|^2 \ll |q_{l+1}\xi_l\xi_{l-1}|^2.$$

We now prove the lower bound for $\det(\Lambda_l)$. Consider the symmetric convex body

$$\Pi = \{\boldsymbol{z} \mid |z_0| < q_{l+1}, \max_{1\le j\le n} |z_0\theta_i - z_i| < \xi_l\}$$

and its intersection P with the plan generated by $\langle \boldsymbol{z}_l, \boldsymbol{z}_{l+1}\rangle_{\mathbb{R}}$. The intersection $P\cap\langle \boldsymbol{z}_l, \boldsymbol{z}_{l+1}\rangle_{\mathbb{Z}}$ is reduced to zero by definition of the best approximation. Hence Minkowski's first convex body theorem ensures that the 2-dimensional volumes satisfy

$$\det{}_2(P) \le 4\det(\Lambda_l).$$

But the volume $\det_2 P$ is larger than the 2-dimensional volume Δ of the projection of P on the plan spanned by $(1,\boldsymbol{\theta})$ and $\boldsymbol{z}_l$. The result follow from $\Delta \gg q_{l+1}\xi_l$.

□

Notation We denote by calligraphic letter $\mathcal{S}$ sets of best approximation vectors $\{\boldsymbol{z}_k,\dots,\boldsymbol{z}_m\}$. Given such a set $\mathcal{S}$, we denote by greek letters $\Gamma = \langle \boldsymbol{z}_k,\dots,\boldsymbol{z}_m\rangle_{\mathbb{Z}}$ the lattice spanned by its elements, and by bold roman letters $\mathbf{S} = \langle \boldsymbol{z}_k,\dots,\boldsymbol{z}_m\rangle_{\mathbb{R}}$ the rational subspace spanned over $\mathbb{R}$. Finally, we denote with gothic letters $\mathfrak{S}$ the underlying lattice of integer points $\mathfrak{S} = \mathbf{S}\cap\mathbb{Z}^n$. Note that $\Gamma\subset\mathfrak{S}$. If our objects are 2-dimensional, we rather use the letters $\mathcal{L}, \Lambda, \mathbf{L}$ and $\mathfrak{L}$.

We sometimes call a set $\mathcal{S}$ *pattern*. By pattern we mean a set of triples of best approximation vectors that is described by indices. For example if $\mathcal{S}$ is the set $\{\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}, \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\}$, its associated indices are ν and k. If a pattern $\mathcal{S}$ is the union of say four patterns $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and $\mathcal{S}_4$, we denoted it by

$$\mathcal{S} \quad : \quad \mathcal{S}_1 - \mathcal{S}_2 - \mathcal{S}_3 - \mathcal{S}_4.$$

If moreover the two patterns $\mathcal{S}_2$ and $\mathcal{S}_3$ generate the same rational subspace, we denoted by

$$\mathcal{S} \quad : \quad \mathcal{S}_1 - \mathcal{S}_2 \equiv \mathcal{S}_3 - \mathcal{S}_4.$$

Finally, if the rational subspaces generated by $\mathcal{S}_1$ and $\mathcal{S}_2$ have intersection $\mathbf{Q}$ and $\mathfrak{Q} = \mathbf{Q} \cap \mathbb{Z}^n$ is its lattice of integer points, we denote it by either

$$\mathcal{S}_1 \underset{\mathbf{Q}}{-} \mathcal{S}_2 \quad \text{or} \quad \mathcal{S}_1 \underset{\mathfrak{Q}}{-} \mathcal{S}_2.$$

2.2 Key lemma

Lemma 4 ($\Gamma_- \underset{\Lambda}{-} \Gamma_+$)**.** *In the context of simultaneous Diophantine approximation, consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ the sequence of best approximations to the point $\boldsymbol{\theta} \in \mathbb{R}^n$. Suppose that $k > \nu$ and triples*

$$\mathcal{S}_- := \{\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\} \quad \textit{and} \quad \mathcal{S}_+ := \{\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\}$$

consist of linearly independent consecutive best approximation vectors. Consider the three-dimensional lattices

$$\mathfrak{S}_- = \langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\rangle_\mathbb{R} \cap \mathbb{Z}^n, \quad \textit{and} \quad \mathfrak{S}_+ = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\rangle_\mathbb{R} \cap \mathbb{Z}^n$$

and suppose that

$$\langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\rangle_\mathbb{Z} = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k\rangle_\mathbb{Z} =: \Lambda. \tag{17}$$

Suppose that for positive s and t the following estimate holds

$$(\det \mathfrak{S}_-)^s \, (\det \mathfrak{S}_+)^t \gg \det \Lambda. \tag{18}$$

Suppose that the index of our vectors are large enough so that for $\alpha < \hat{\lambda}(\boldsymbol{\theta})$.

$$\xi_j \leq q_{j+1}^{-\alpha} \quad \textit{for} \quad j = \nu-1, \nu, k-1, k. \tag{19}$$

Define

$$g(s,t) = \frac{\alpha s}{(1-\alpha)(s - w(s,t))} = \frac{\alpha(t + w(s,t) - 1) - w(s,t) + 1}{(1-\alpha)t}. \tag{20}$$

where the second equality comes from $w(s,t) \in (0,1)$ being the root of the equation

$$w^2 - \left(s + 1 + \frac{\alpha}{1-\alpha} t\right) w + s = 0. \tag{21}$$

Then

$$\textit{either} \quad q_{\nu+1} \gg q_\nu^{g(s,t)} \quad \textit{or} \quad q_{k+1} \gg q_k^{g(s,t)}. \tag{22}$$

When the parameters are $s = t = 1$, this lemma provides directly the result for the approximation to 3 numbers (Proof from [15], see subsection 3.1 for details). Parameters s and t are needed in higher dimension. We exhibit a range of pairs of triples of consecutive best approximation vectors, denoted by an index, satisfying conditions of Lemma 4. Parameters s and t appear with values depending on dimension and the geometry of best approximation vectors that need to be optimize with respect to $g(s,t)$. To prove Theorem 1, we show inductively that the optimized parameter $g(s,t)$ is root of the polynomial R_n defined by (9).

Proof of Lemma 4. We use Lemma 2. Substituting (15) in (18) in light of (17), since $\langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_{\nu}, \boldsymbol{z}_{\nu+1}\rangle_{\mathbb{Z}} \subset \mathfrak{S}_-$ and $\langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\rangle_{\mathbb{Z}} \subset \mathfrak{S}_+$ it follows that

$$(\xi_{\nu-1}\xi_\nu q_{\nu+1})^s(\xi_{k-1}\xi_k q_{k+1})^t \gg (\xi_\nu q_{\nu+1})^{w(s,t)}(\xi_{k-1}q_k)^{1-w(s,t)}.$$

This means that either

$$(\xi_{\nu-1}\xi_\nu q_{\nu+1})^s \gg (\xi_\nu q_{\nu+1})^{w(s,t)}$$

or

$$(\xi_{k-1}\xi_k q_{k+1})^t \gg (\xi_{k-1}q_k)^{1-w(s,t)}.$$

Now we take into account (19). We have either

$$q_\nu^{s\alpha} \ll q_{\nu+1}^{(1-\alpha)(s-w(s,t))}$$

or

$$q_k^{1-w(s,t)+\alpha(t+w(s,t)-1)} \ll q_{k+1}^{t(1-\alpha)}.$$

Hence (22) by definition of g.

$\square$

Our proof relies on Schmidt's inequality on height (see [21], in fact this inequality was already used in the last section in [15]). It provides the setting to apply Lemma 4 simultaneously for different parameters s, t.

Proposition 1 (Schmidt's inequality). *Let A, B be two rational subspaces in $\mathbb{R}^n$, we have*

$$H(A+B)\cdot H(A\cap B) \ll H(A)\cdot H(B). \tag{23}$$

where the height $H(A)$ is the fundamental volume of the lattice of integer points $\det(\mathfrak{A}) = \det(A\cap\mathbb{Z}^n)$.

2.3 Properties of the polynomial R_n and the optimized g

In this subsection, we state various properties needed for the proof.

The polynomial R_n defined in (9) can be defined inductively the following way:

$$\begin{cases} R_2(X) = X - \beta, \\ R_{n+1}(X) = XR_n(X) - \beta, \end{cases} \tag{24}$$

where β is $\frac{\alpha}{1-\alpha}$ for simultaneous approximation. To obtain the polynomial $R_{n,1/\alpha^*}$ for the linear form setting, one may put $\beta = \frac{1}{\alpha^*-1}$ in (24).

From (20) and (21), we see that g satisfies the equation

$$g^2 - \left(\beta + \frac{1-s}{t}\right) g - \frac{s\beta}{t} = 0. \tag{25}$$

In particular, we can use this equation to compute the optimal value of either s or t when the other parameter is 1. Namely,

$$\begin{aligned}
s &= \frac{g^2 - \beta g - g}{\beta - g}, \quad \text{for} \quad g = g(s,1) &(26)\\
t &= \frac{\beta}{g(g-\beta)}, \quad \text{for} \quad g = g(1,t) &(27)\\
s &= \frac{g^2 - \beta g - \beta}{g - \beta} = \frac{R_3(g)}{g-\beta}, \quad \text{for} \quad g = g(1-s,1), &(28)\\
t &= \frac{g^2 - \beta g - \beta}{g(g-\beta)} = \frac{R_3(g)}{g(g-\beta)}, \quad \text{for} \quad g = g(1,1-t). &(29)
\end{aligned}$$

3 Examples: simultaneous approximation to three and four numbers.

In this section, we describe in details the proofs in the cases of simultaneous approximation to three and four numbers. The aim is to provide concrete examples of the construction of patterns of best approximation vectors on simple examples before moving to arbitrary dimension in Section 4. An example for approximation by one linear form will be presented in Section 5.3.1.

3.1 Simultaneous approximation to three numbers

Consider $\boldsymbol{\theta} \in \mathbb{R}^3$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations vectors to $\boldsymbol{\theta}$. Recall that as we consider simultaneous approximation, the sequence $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ spans the whole space $\mathbb{R}^4$.

Lemma 5. *For arbitrarily large indices k_0, there exists indices $k > \nu > k_0$ and triples of consecutive best approximation vectors*

$$\mathcal{S}_- := \{\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\} \ \text{ and } \ \mathcal{S}_+ := \{\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\}$$

consisting of linearly independent vectors. Setting

$$\mathfrak{S}_- := \langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\rangle_{\mathbb{R}} \cap \mathbb{Z}^n \quad \text{and} \quad \mathfrak{S}_+ := \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\rangle_{\mathbb{R}} \cap \mathbb{Z}^n$$

we have

$$\mathfrak{S}_- \cap \mathfrak{S}_+ =: \Lambda = \langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}\rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k\rangle_{\mathbb{Z}} \ \text{and} \ \langle \mathfrak{S}_- \cup \mathfrak{S}_+\rangle_{\mathbb{R}} = \mathbb{R}^4. \tag{30}$$

This was proved in [15].

Denote by $\mathcal{S}_4$ the pattern of best approximation vectors described in Lemma 5 (see Figure 1). Lemma 5 ensures that the pattern $\mathcal{S}_4$ enables to apply Lemma 4 for arbitrarily large indices.

Here we chose k_0 sufficiently large for (19) to hold. Schmidt's inequality (23) provides (18) with parameters $s = t = 1$.

Lemma 4 provides that for any $\alpha < \hat{\lambda}(\boldsymbol{\theta})$,

$$q_{l+1} \gg q_l^{g_\alpha}$$

for $l = \nu$ or k, where g_α is solution of the equation (25) with $s = t = 1$. Namely

$$g_\alpha^2 - \beta g_\alpha - \beta = R_3(g_\alpha) = 0$$

which provides

$$g_\alpha = \frac{\beta + \sqrt{\beta^2 + 4\beta}}{2} = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2(1-\alpha)}.$$

Hence for every $\alpha < \lambda(\boldsymbol{\theta})$, we have

$$\frac{\lambda(\boldsymbol{\theta})}{\hat{\lambda}(\boldsymbol{\theta})} \geq g_\alpha = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2(1-\alpha)}.$$

We deduce the lower bound (3).

We now explain how to obtain the pattern of best approximation vectors in Lemma 5. It is the basic step for a more general construction in higher dimension.

Proof of Lemma 5. Figure 1 may be usefull to understand the construction.
Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximation vectors to $\boldsymbol{\theta} \in \mathbb{R}^3$, and an arbitrarily large index k_0. Since $(\boldsymbol{z}_l)_{l\geq k_0}$ spans a 4-dimensional subspace, we can define k to be the smallest index such that

$$\dim\langle \boldsymbol{z}_{k_0}, \boldsymbol{z}_{k_0+1}, \ldots, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\rangle_{\mathbb{R}} = 4.$$

Note that by minimality, $\boldsymbol{z}_{k+1}$ is not in the 3-dimensional subspace spanned by $(\boldsymbol{z}_l)_{k_0\leq l\leq k}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}$ are linearly independent. Set $\nu > k_0$ to be the largest index such that

$$\dim\langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \ldots, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}\rangle_{\mathbb{R}} = 4.$$

Note that by maximality, $\boldsymbol{z}_{\nu-1}$ is not in the 3-dimensional subspace spanned by $(\boldsymbol{z}_l)_{\nu \leq l \leq k+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}$ are linearly independent. Moreover, combining both observations we deduce that the lattice

$$\Lambda := \langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}, \ldots, \boldsymbol{z}_{k-1}, \boldsymbol{z}_k \rangle_{\mathbb{R}} \cap \mathbb{Z}^4 = \langle \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{k-1}, \boldsymbol{z}_k \rangle_{\mathbb{Z}}$$

is 2-dimensional, and spanned by two consecutive best approximation vectors (see Lemma 1). Hence, the considered indices ν and k provide 6 best approximation vectors satisfying Lemma 5.

$\square$

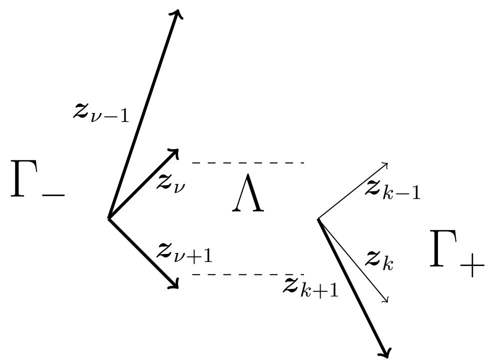

Figure 1: All best approximation vectors with index between ν and k lie in the 2-dimensional lattice Λ. The four bold vectors are linearly independent and span the whole space.

3.2 Simultaneous approximation to four numbers

In the case of simultaneous approximation to four numbers, we select a pattern $\mathcal{S}_5$ of best approximation vectors that combines two patterns $\mathcal{S}_4$ coming from Lemma 5. This is the first step of the induction for arbitrary dimension, where we combine two patterns of lower dimension. Thus, it is an enlightening example. Note that in this simple case, a proper choice of parameters was made in [3, equalities after formula (13) from the case $\mathfrak{i}(\Theta) = 1$]. The example of approximation by one linear form for 4 numbers is presented in Section 5.3.1

Consider $\boldsymbol{\theta} \in \mathbb{R}^4$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(\boldsymbol{z}_l)_{l \in \mathbb{N}}$ a sequence of best approximation vectors to $\boldsymbol{\theta}$.

Lemma 6. *Let k_0 be an arbitrarily large index. There exists indices $k_0 < r_0 < r_1 \leq r_2 < r_3$ such that the following holds.*

1. *The triples of consecutive best approximation vectors*

$$\mathcal{S}_{r_i} := \{\boldsymbol{z}_{r_i-1}, \boldsymbol{z}_{r_i}, \boldsymbol{z}_{r_i+1}\}, \quad 0 \le i \le 3$$

   consist of linearly independent vectors spanning a 3*-dimensional subspace* $\mathbf{S}_{3,i} := \langle \mathcal{S}_{r_i} \rangle_{\mathbb{R}}$*.*

2. *The two triples of consecutive best approximation vectors* $\mathcal{S}_{r_1}$ *and* $\mathcal{S}_{r_2}$ *generate the same rational subspace*

$$\mathbf{Q} := \mathbf{S}_{3,1} = \mathbf{S}_{3,2}.$$

3. *The pairs of consecutive best approximation vectors* $\boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}$ *and* $\boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}$ *span the same* 2*-dimensional lattice*

$$\Lambda_0 := \langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1} \rangle_{\mathbb{Z}} = \mathbf{S}_{3,0} \cap \mathbf{S}_{3,1} \cap \mathbb{Z}^5.$$

4. *The pairs of consecutive best approximation vectors* $\boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1}$ *and* $\boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3}$ *span the same* 2*-dimensional lattice*

$$\Lambda_1 := \langle \boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3} \rangle_{\mathbb{Z}} = \mathbf{S}_{3,2} \cap \mathbf{S}_{3,3} \cap \mathbb{Z}^5.$$

5. *Both quadruples of best approximation* $\{\boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \boldsymbol{z}_{r_1+1}\}$ *and* $\{\boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1}\}$ *consist of linearly independent vectors.*

6. *The whole space* $\mathbb{R}^5$ *is spanned by*

$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \boldsymbol{z}_{r_1+1}, \boldsymbol{z}_{r_3+1} \rangle_{\mathbb{R}} = \langle \mathbf{S}_{3,0} \cup \mathbf{Q} \cup \mathbf{S}_{3,2} \rangle_{\mathbb{R}} = \mathbb{R}^5.$$

We discuss the meaning of the lemma, and apply it to the proof of the main result for simultaneous approximation to four numbers. The proof is postponed at the end of the section.

The 5-dimensional pattern described in Lemma 6 is denoted by

$$\mathcal{S}_5 \quad : \quad \mathcal{S}_{3,0} \underset{\Lambda_0}{-} \mathcal{S}_{3,1} \equiv \mathcal{S}_{3,2} \underset{\Lambda_1}{-} \mathcal{S}_{3,3}.$$

Note that it consists of two 4-dimensional patterns

$$\mathcal{S}_{4,0} \quad : \quad \mathcal{S}_{3,0} \underset{\Lambda_0}{-} \mathcal{S}_{3,1}$$

given by indices $\nu = r_0$ and $k = r_1$ in Lemma 5 and

$$\mathcal{S}_{4,1} \quad : \quad \mathcal{S}_{3,2} \underset{\Lambda_1}{-} \mathcal{S}_{3,3}$$

given by indices $\nu = r_2$ and $k = r_3$ in Lemma 5. These two 4-dimensional patterns $\mathcal{S}_{4,0}$ and $\mathcal{S}_{4,1}$ intersect on the 3-dimensional subspace $\mathbf{Q}$. Thus,

$$\mathcal{S}_5 \quad : \quad \mathcal{S}_{4,0} \underset{\mathbf{Q}}{-} \mathcal{S}_{4,1}.$$

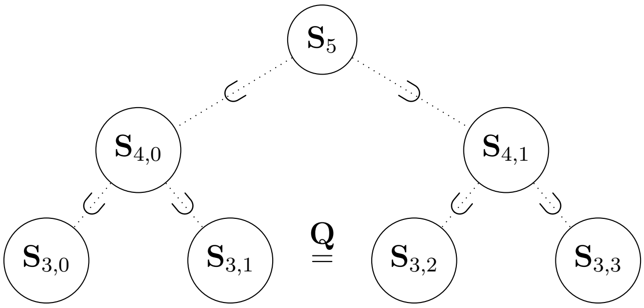

Figure 2: Binary tree sketching the situation described in Lemma 6.

For the pattern $\mathcal{S}_5$, Schmidt's inequality (23) provides

$$\det \mathfrak{S}_{3,0} \det \mathfrak{Q} \det \mathfrak{S}_{3,3} \gg \det \Lambda_0 \det \Lambda_1$$

where $\mathfrak{S}_{i,j} = \mathbf{S}_{i,j} \cap \mathbb{Z}^5$ and $\mathfrak{Q} = \mathfrak{S}_{3,1} = \mathfrak{S}_{3,2}$. It can be rewritten as

$$\frac{\det \mathfrak{S}_{3,0} (\det \mathfrak{S}_{3,1})^x}{\det \Lambda_0} \cdot \frac{(\det \mathfrak{S}_{3,2})^{1-x} \det \mathfrak{S}_{3,3}}{\det \Lambda_2} \gg 1 \tag{31}$$

with arbitrary $x \in (0,1)$. This means that

$$\text{either } \frac{\det \mathfrak{S}_{3,0} (\det \mathfrak{S}_{3,1})^x}{\det \Lambda_0} \gg 1 \text{ or } \frac{(\det \mathfrak{S}_{3,2})^{1-x} \det \mathfrak{S}_{3,3}}{\det \Lambda_2} \gg 1.$$

Applying Lemma 4 two times with parameters $(s,t) = (1,x)$ and $(s,t) = (1-x,1)$ we get the lower bound

$$\frac{\lambda}{\hat{\lambda}} \geq g$$

where g is given by the optimization equations

$$g = g(1,x) = g(1-x,1). \tag{32}$$

From (27), (28) we have

$$x = \frac{\beta}{g(g-\beta)} = \frac{R_3(g)}{g-\beta}$$

and so g satisfies the equation

$$R_4(g) = gR_3(g) - \beta = 0.$$

This proves first part of Theorem 1 for simultaneous approximation to four numbers. $\square$

Here, there is one parameter x to optimize. In higher dimension, we have many more, and need to compute the optimization of these parameters inductively.

Proof of Lemma 6. Figure 3 may be usefull to understand the construction.
Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximation vectors to $\boldsymbol{\theta} \in \mathbb{R}^4$, and an arbitrarily large index k_0. Set r_3 to be the smallest index such that

$$\dim\langle \boldsymbol{z}_{k_0}, \boldsymbol{z}_{k_0+1}, \ldots, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1}\rangle_{\mathbb{R}} = 5.$$

Note that by minimality, $\boldsymbol{z}_{r_3+1}$ is not in the 4-dimensional subspace spanned by $(\boldsymbol{z}_l)_{k_0\leq l\leq r_3}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1}$ are linearly independent and span a 3-dimensional lattice denoted by Γ_3. Set $r_0 > k_0$ to be the largest index such that

$$\dim\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \ldots, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1}\rangle_{\mathbb{R}} = 5.$$

Note that by maximality, $\boldsymbol{z}_{r_0-1}$ is not in the 4-dimensional subspace spanned by $(\boldsymbol{z}_l)_{r_0\leq l\leq r_3+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}$ are linearly independent and span a 3-dimensional lattice denoted by Γ_0. Moreover, combining both observations we deduce that

$$\mathbf{Q} := \langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \ldots, \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3}\rangle_{\mathbb{R}}$$

is a 3-dimensional rational subspace.
Now appears the **induction step**: we apply the same procedure in lower dimension to the two 4-dimensional subspaces

$$\mathbf{S}_{4,0} := \langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \ldots, \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3}\rangle_{\mathbb{R}} \quad \text{and} \quad \mathbf{S}_{4,1} := \langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \ldots, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1}\rangle_{\mathbb{R}}.$$

Note that it gives a proof of Lemma 5.
Set r_1 to be the smallest index such that

$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \ldots, \boldsymbol{z}_{r_1}, \boldsymbol{z}_{r_1+1}\rangle_{\mathbb{R}} = \mathbf{S}_{4,0}.$$

Note that by minimality, $\boldsymbol{z}_{r_1+1}$ is not in the 3-dimensional subspace $\mathbf{S}_{3,0}$ spanned by $(\boldsymbol{z}_l)_{r_0-1\leq l\leq r_1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}, \boldsymbol{z}_{r_1+1}$ are linearly independent and span a 3-dimensional lattice Γ_1 included in $\mathbf{Q} = \mathbf{S}_{3,1}$. By construction, r_0 is already the largest index such that

$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \ldots, \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}\rangle_{\mathbb{R}} = \mathbf{S}_{4,0}.$$

Hence, $\langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \ldots, \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}\rangle_{\mathbb{Z}} =: \Lambda_0$ is a 2-dimensional lattice spanned by either $\langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}\rangle_{\mathbb{Z}}$ or $\langle \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}\rangle_{\mathbb{Z}}$, and is the intersection $\mathbf{S}_{3,0} \cap \mathbf{S}_{3,1} \cap \mathbb{Z}^5$ (see Lemma 1).
Set r_2 to be the largest index such that

$$\langle \boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2}, \ldots, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1}\rangle_{\mathbb{R}} = \mathbf{S}_{4,1}.$$

Note that $\boldsymbol{z}_{r_2-1}$ is not in the 3-dimensional subspace $\mathbf{S}_{3,3}$ spanned by $(\boldsymbol{z}_l)_{r_2\leq l\leq r_3+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1}$ are linearly independent and span a 3-dimensional lattice Γ_2 included in $\mathbf{Q} = \mathbf{S}_{3,1}$. By construction, r_3 is already the smallest index such that

$$\langle \boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2}, \ldots, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1} \rangle_{\mathbb{R}} = \mathbf{S}_{4,1}.$$

Hence, $\langle \boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1}, \ldots, \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3} \rangle_{\mathbb{Z}} =: \Lambda_1$ is a 2-dimensional lattice spanned by $\langle \boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1} \rangle_{\mathbb{Z}}$ or $\langle \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3} \rangle_{\mathbb{Z}}$, and is the intersection $\mathbf{Q} \cap \mathbf{S}_{3,3} \cap \mathbb{Z}^5$ (see Lemma 1).

□

Note that we may have $r_1 = r_2$. Lattices Γ_1 and Γ_2 may not coincide, but are both sub-lattice of $\mathfrak{Q} = \mathbf{Q} \cap \mathbb{Z}^5$.

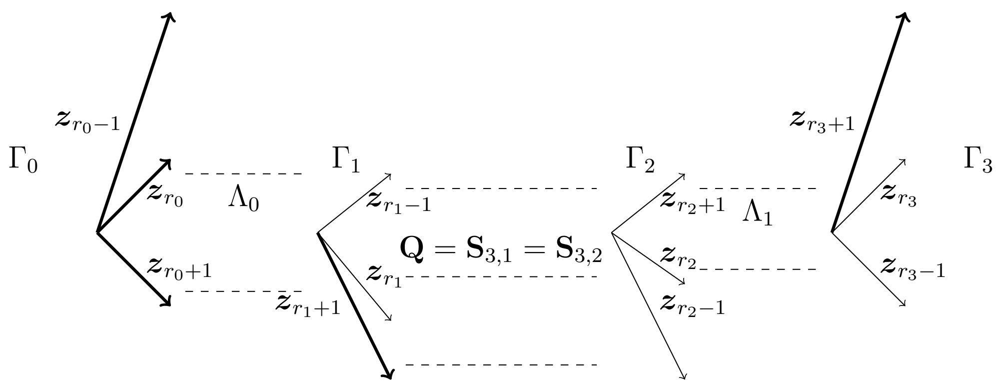

Figure 3: Selected sequence of best approximation vectors.

In Figure 3, the dashed lines should be interpreted as follows. The best approximation vectors $(\boldsymbol{z}_l)_{r_0 \le l \le r_1}$ generate the 2-dimensional lattice Λ_0. The best approximation vectors $(\boldsymbol{z}_l)_{r_2 \le l \le r_3}$ generate the 2-dimensional lattice Λ_1. The best approximation vectors $(\boldsymbol{z}_l)_{r_1-1 \le l \le r_2+1}$ generate the 3-dimensional rational subspace $\mathbf{Q} = \mathbf{S}_{3,1} = \mathbf{S}_{3,2}$. The five bold vectors span the whole space $\mathbb{R}^5$.

4 Arbitrary dimension

Consider $\boldsymbol{\theta} \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(\boldsymbol{z}_l)_{l \in \mathbb{N}}$ a sequence of best approximation vectors to $\boldsymbol{\theta}$.

Lemma 7. *Let k_0 be an arbitrarily large index. There exists 2^{n-2} indices $k_0 < r_0 < r_1, \ldots, r_{2^{n-2}-2} < r_{2^{n-2}-1}$ such that the following holds.*

1. *The triples of consecutive best approximation vectors*
$$\mathcal{S}_{3,l} = \{\boldsymbol{z}_{r_l-1}, \boldsymbol{z}_{r_l}, \boldsymbol{z}_{r_l+1}\}, \quad 0 \leq l \leq 2^{n-2} - 1$$
consist of linearly independent vectors spanning a 3*-dimensional rational subspace* $S_{3,l}$.
2. *For* $4 \leq k \leq n+1$ *and* $0 \leq l \leq 2^{n-k+1} - 1$, *denote by* $\mathcal{S}_{k,l}$ *the set of best approximation vectors*
$$\mathcal{S}_{k,l} = \cup_{\nu=0}^{2^{k-3}-1} \mathcal{S}_{3,2^{k-3}l+\nu}.$$
$\mathcal{S}_{k,l}$ *spans the* k*-dimensional rational subspace* $\mathbf{S}_{k,l}$.
3. *The rational subspaces* $\mathbf{S}_{k,l}$ *satisfy the relations*
$$\begin{aligned} \mathbf{S}_{k,2l} \cup \mathbf{S}_{k,2l+1} &= \mathbf{S}_{k+1,l} && (33) \\ \mathbf{S}_{k,2l} \cap \mathbf{S}_{k,2l+1} &= \mathbf{S}_{k-1,4l+1} = \mathbf{S}_{k-1,4l+2} =: \mathbf{Q}_{k-1,l}. && (34) \end{aligned}$$
In particular, $\mathbf{Q}_{2,l}$ *is spanned by both* $\boldsymbol{z}_{r_{4l+1}}, \boldsymbol{z}_{r_{4l+1}+1}$ *and* $\boldsymbol{z}_{r_{4l+2}-1}, \boldsymbol{z}_{r_{4l+2}}$.
4. *The full space* $\mathbb{R}^{n+1}$ *is spanned by*
$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \boldsymbol{z}_{r_1+1}, \boldsymbol{z}_{r_2+1}, \ldots, \boldsymbol{z}_{r_{2^{n-3}-1}+1} \rangle_{\mathbb{R}} = \langle \cup_{l=0}^{2^{n-k+1}-1} \mathbf{S}_{k,l} \rangle_{\mathbb{R}}, \quad 3 \leq k \leq n+1.$$
In particular, $\mathbf{S}_{n+1,0} = \mathbb{R}^{n+1}$.

Here, the first index always denote the dimension of the considered object. For a given dimension k, there is 2^{n-k+1} subspaces $\mathbf{S}_{k,l}$ and 2^{n-k-1} subspaces $\mathbf{Q}_{k,l}$ of dimension k.

A weaker pattern of best approximation vectors was already considered for any dimension in [17, §2.3].

Lemma 7 coincide with Lemma 5 for the approximation to three numbers and with Lemma 6 for the approximation to four numbers. In the later case, we have $\Lambda_j \sim \mathfrak{Q}_{2,j}$ for $0 \leq j \leq 1$.

We can partially describe the situation with the binary tree from Figure 4, where each child is included in its parent. In particular, the parent of a given rational subspace $\mathbf{S}_{k,l}$ is $\mathbf{S}_{k+1,\sigma(l)}$ where σ is the usual shift on the binary expansion.

We may write the recursive step of the construction of patterns as follows:
$$\mathcal{S}_{n+1,0} \quad : \quad \mathcal{S}_{n,0} \underset{\mathbf{Q}_{n-1,0}}{-} \mathcal{S}_{n,1}$$
where $\mathbf{Q}_{n-1,0}$ is a $n-1$ dimensional subspace. For $\mathbf{S}_{n,0}$, $\mathbf{S}_{n,1}$ and $\mathbf{Q}_{n-1,0}$ the rational subspaces and their underlying lattices $\mathfrak{S}_{n,0}$, $\mathfrak{S}_{n,1}$ and $\mathfrak{Q}_{n-1,0}$, Schmidt's inequality (23) provides
$$\frac{\det \mathfrak{S}_{n,0} \cdot \det \mathfrak{S}_{n,1}}{\det \mathfrak{Q}_{n-1,0}} \gg 1. \tag{35}$$
This relation enables us to shift the optimization equation in the next dimension as obtained in the next lemma.

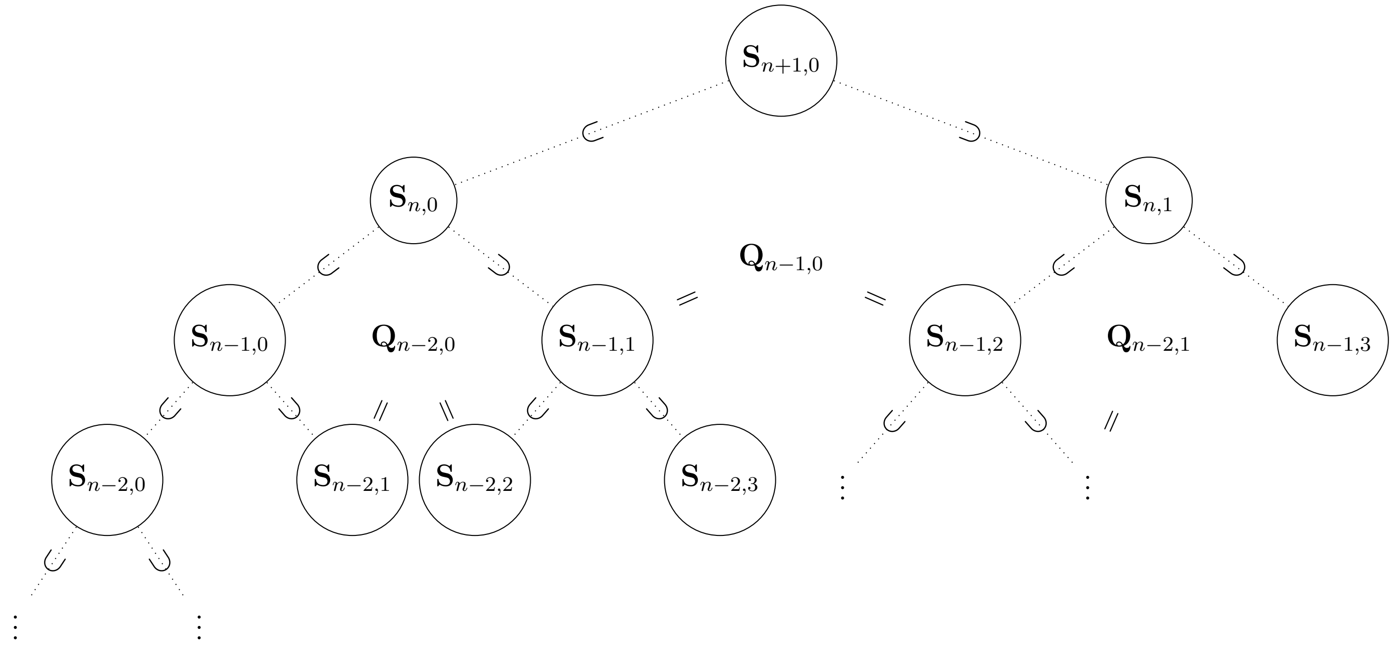

Figure 4: Binary tree sketching the situation described in Lemma 7.

Lemma 8. *Consider the pattern of best approximation vectors $\mathcal{S}_{n+1,0}$ and its sub-patterns given by Lemma 7. Here as before, $\mathfrak{S}_{k,l} = \mathbf{S}_{k,l} \cap \mathbb{Z}^{n+1}$ and $\mathfrak{Q}_{k,l} = \mathbf{Q}_{k,l} \cap \mathbb{Z}^{n+1}$ are the integer points lattices of the rational subspace $\mathbf{S}_{k,l}$ and $\mathbf{Q}_{k,l}$. Then*

$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\det\left(\mathfrak{S}_{3,4l}\right) \det\left(\mathfrak{Q}_{3,l}\right)^{1-y_{n-4}}}{\det\left(\mathfrak{Q}_{2,2l}\right)} \right)^{w_{n-4,l}} \cdot \prod_{l=0}^{2^{n-4}-1} \left(\frac{\det\left(\mathfrak{Q}_{3,l}\right)^{1-z_{n-4}} \det\left(\mathfrak{S}_{3,4l+3}\right)}{\det\left(\mathfrak{Q}_{2,2l+1}\right)} \right)^{w'_{n-4,l}} \gg 1, \tag{36}$$

where the parameters $w_{k,l}, w'_{k,l}, y_k$ and z_k are defined inductively by

$$0 = y_0 + z_0 - 1 \tag{37}$$

$$(y_{k+1}, z_{k+1}) = F(y_k, z_k) = \left(\frac{y_k}{y_k + z_k - y_k z_k}, \frac{z_k}{y_k + z_k - y_k z_k} \right) \tag{38}$$

$$1 = w_{0,0} = w'_{0,0} \tag{39}$$

$$w_{k+1,2l} = w_{k,l}, \quad w_{k+1,2l+1} = (1 - z_k) w'_{k,l}, \quad w'_{k+1,2l} = (1 - y_k) w_{k,l} \quad \textit{and} \quad w'_{k+1,2l+1} = w'_{k,l}. \tag{40}$$

Furthermore, the parameters satisfy the relations

$$\sum_{l=0}^{2^{n-4}-1} (2-y_{n-4})w_{n-4,l} + (2-z_{n-4})w'_{n-4,l} \quad = n-1, \tag{41}$$

$$\sum_{l=0}^{2^{n-4}-1} w_{n-4,l} + w'_{n-4,l} \quad = n-2. \tag{42}$$

We prove Lemma 7 and then Lemma 8 at the end of this section. Here we first finish the proof of Theorem 1 in the case of simultaneous approximation.

We do not need to compute explicitly the values of $w_{k,l}$ or $w'_{k,l}$. From formula (36), we deduce that there exists an index $0 \le l \le 2^{n-4}-1$ such that either

$$\frac{\det(\mathfrak{S}_{3,4l})\det(\mathfrak{Q}_{3,l})^{1-y_{n-4}}}{\det(\mathfrak{Q}_{2,2l})} \gg 1 \quad \text{or} \quad \frac{\det(\mathfrak{Q}_{3,l})^{1-z_{n-4}}\det(\mathfrak{S}_{3,4l+3})}{\det(\mathfrak{Q}_{2,2l+1})} \gg 1$$

Applying Lemma 4 twice, the optimization constant g is given by

$$g = g(1, 1-z_{n-4}) = g(1-y_{n-4}, 1)$$

where $(y_0, z_0) = F^{-n+4}(y_{n-4}, z_{n-4})$ satisfies $y_0 + z_0 - 1 = 0$.

By formulae (28) and (29), we get

$$y_{n-4} = \frac{R_3(g)}{R_2(g)} \quad \text{and} \quad z_{n-4} = \frac{R_3(g)}{gR_2(g)}.$$

Then, the recurrence formula (38) provides that for $4 \le k \le n$

$$y_{n-k} = \frac{R_{k-1}(g)}{R_{k-2}(g)} \quad \text{and} \quad z_{n-k} = \frac{R_{k-1}(g)}{gR_{k-2}(g)} \tag{43}$$

if k is odd and

$$y_{n-k} = \frac{R_{k-1}(g)}{gR_{k-2}(g)} \quad \text{and} \quad z_{n-k} = \frac{R_{k-1}(g)}{R_{k-2}(g)} \tag{44}$$

if k is even.

Indeed we prove (43) and (44) by induction.
Suppose $y_{n-k} = \frac{R_{k-1}(g)}{R_{k-2}(g)}$ and $z_{n-k} = \frac{R_{k-1}(g)}{gR_{k-2}(g)}$. Since

$$F^{-1}(y_{n-k}, z_{n-k}) = \left(\frac{y_{n-k}+z_{n-k}-1}{z_{n-k}}, \frac{y_{n-k}+z_{n-k}-1}{y_{n-k}}\right)$$

we compute

$$\begin{aligned} y_{n-k-1} &= \frac{y_{n-k}+z_{n-k}-1}{z_{n-k}} = \frac{R_{k-1}(g)/R_{k-2}(g)+R_{k-1}(g)/gR_{k-2}(g)-1}{R_{k-1}(g)/gR_{k-2}(g)} = \frac{R_k(g)}{R_{k-1}(g)}, \\ z_{n-k-1} &= \frac{y_{n-k}+z_{n-k}-1}{y_{n-k}} = \frac{R_k(g)}{gR_{k-1}(g)}. \end{aligned}$$

Hence we obtain the formulae (28) and (29) by symmetry and initialization for $k = 4$. □

In particular, $(y_0, z_0) = \left(\frac{R_{n-1}(g)}{gR_{n-2}(g)}, \frac{R_{n-1}(g)}{R_{n-2}(g)}\right)$ or $\left(\frac{R_{n-1}(g)}{R_{n-2}(g)}, \frac{R_{n-1}(g)}{gR_{n-2}(g)}\right)$ depending on the parity of n. This leads to

$$0 = y_0 + z_0 - 1 = \frac{gR_{n-1}(g)+R_{n-1}(g)-gR_{n-2}(g)}{gR_{n-2}(g)} = \frac{R_n(g)}{gR_{n-2}(g)}.$$

That is, $R_n(g) = 0$. So we proved the bound (14) holds for arbitrary large indices, and in arbitrary dimension n, for the required g.

This proves first part of Theorem 1 for simultaneous approximation. □

Proof of Lemma 7. Figure 4 may be usefull to understand the construction.
Let $k_0 \gg 1$. We prove the lemma by induction in the dimension n. Suppose that we can construct a pattern $\mathcal{S}_{m,0}$ of 2^{m-3} triples of consecutive best approximation vectors given by indices $k_0 < r_0 < r_1, \dots, r_{2^{m-3}-2} < r_{2^{m-3}-1}$ spanning a m-dimensional rational space. Such a construction for $m = 4, 5$ holds via Lemmas 5 and 6. This provides the initialization.

Consider $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximation vectors spanning a $(m+1)$-dimensional rational space $\mathbf{S}_{m+1}$. Set $r_{2^{m-2}-1}$ to be the smallest index such that

$$\langle \boldsymbol{z}_{k_0}, \boldsymbol{z}_{k_0+1}, \dots, \boldsymbol{z}_{r_{2^{m-2}-1}}, \boldsymbol{z}_{r_{2^{m-2}-1}+1}\rangle_{\mathbb{R}} = \mathbf{S}_{m+1}.$$

Note that $\boldsymbol{z}_{r_{2^{m-2}-1}+1}$ is not in the m-dimensional subspace spanned by $(\boldsymbol{z}_l)_{k_0\le l\le r_{2^{m-2}-1}}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_{2^{m-2}-1}-1}, \boldsymbol{z}_{r_{2^{m-2}-1}}, \boldsymbol{z}_{r_{2^{m-2}-1}+1}$ are linearly independent and span a 3-dimensional subspace denoted by $\mathbf{S}_{3,2^{m-2}-1}$. Set $r_0 > k_0$ to be the largest index such that

$$\langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \dots, \boldsymbol{z}_{r_{2^{m-2}-1}}, \boldsymbol{z}_{r_{2^{m-2}-1}+1}\rangle_{\mathbb{R}} = \mathbf{S}_{m+1}.$$

Note that $\boldsymbol{z}_{r_0-1}$ is not in the m-dimensional subspace spanned by $(\boldsymbol{z}_l)_{r_0\le l\le r_{2^{m-1}-1}+1}$. In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $\boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}$ are linearly independent and span a 3-dimensional subspace denoted by $\mathbf{S}_{3,0}$. Moreover, combining both observations we get that

$$\mathbf{Q}_{m-1,0} := \langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \dots, \boldsymbol{z}_{r_{2^{m-2}-1}-1}, \boldsymbol{z}_{r_{2^{m-2}-1}}\rangle_{\mathbb{R}}$$

is a $m-1$-dimensional subspace.
We use the induction hypothesis for the two m-dimensional subspaces

$$\mathbf{S}'_m := \langle \boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \ldots, \boldsymbol{z}_{r_{2^{m-2}-1}-1}, \boldsymbol{z}_{r_{2^{m-2}-1}} \rangle_{\mathbb{R}} \quad \text{and} \quad \mathbf{S}''_m := \langle \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}, \ldots, \boldsymbol{z}_{r_{2^{m-2}-1}}, \boldsymbol{z}_{r_{2^{m-2}-1}+1} \rangle_{\mathbb{R}}$$

for $k'_0 = r_0 - 1$ and $k''_0 = r_0$ respectively. This provides two patterns $\mathcal{S}'_m$ and $\mathcal{S}''_m$ of triples of best approximation vectors defined by indices $r_0 \leq r'_0 < r'_1, \ldots, r'_{2^{m-3}-2} < r'_{2^{m-3}-1}$ and $r_0 + 1 \leq r''_0 < r''_1, \ldots, r''_{2^{m-3}-2} < r''_{2^{m-3}-1}$ satisfying the conditions of Lemma 7. A key observation is that by definition of r_0, we necessarily have $r'_0 = r_0$. Similarly, by definition of $r_{2^{m-2}-1}$, we necessarily have $r_{2^{m-2}-1} = r''_{2^{m-3}-1}$. It follows that both sub-patterns $\mathcal{S}'_{m-1,1}$ and $\mathcal{S}''_{m-1,0}$ span the rational subspace $\mathbf{Q}_{m-1,0}$. Hence, the pattern $\mathcal{S}$ defined by the triples given by indices

$$r_i = r'_i \quad \text{and} \quad r_{i+2^{m-3}} = r''_i \quad \text{for } 0 \leq i \leq 2^{m-3} - 1$$

combining the two sub-patterns $\mathcal{S}'_m$ and $\mathcal{S}''_m$ satisfies the required properties at the rank $m+1$.

$$\mathcal{S} \quad : \quad \mathcal{S}'_m \underset{\mathbf{Q}_{m-1,0}}{-} \mathcal{S}''_m.$$

Since $(\boldsymbol{z}_l)_{l \in \mathbb{N}}$ a sequence of best simultaneous approximation vectors to $\boldsymbol{\theta} \in \mathbb{R}^n$ spans the whole space $\mathbb{R}^{n+1}$, Lemma 7 follows. □

Remark. *Note that the proof provides a m-dimensional pattern for $\boldsymbol{\theta} \in \mathbb{R}^n$ where m is the dimension of the space spanned by its best approximation vectors. Furthermore, note that this construction holds for both simultaneous approximation and approximation by one linear form.*

Proof of Lemma 8. We prove by induction a more general formula

$$\prod_{l=0}^{2^{k-1}-1} \left(\frac{\det(\mathfrak{S}_{n-k,4l}) \det(\mathfrak{S}_{n-k,4l+1})^{1-y_{k-1}}}{\det(\mathfrak{S}_{n-k-1,8l+1})} \right)^{w_{k-1,l}} \times \prod_{l=0}^{2^{k-1}-1} \left(\frac{\det(\mathfrak{S}_{n-k,4l+2})^{1-z_{k-1}} \det(\mathfrak{S}_{n-k,4l+3})}{\det(\mathfrak{S}_{n-k-1,8l+3})} \right)^{w'_{k-1,l}} \gg 1. \tag{45}$$

If we write it in terms of $\mathbf{Q}_{i,j} = \mathbf{S}_{i,4j+1} = \mathbf{S}_{i,4j+2}$, we have

$$\prod_{l=0}^{2^{k-1}-1} \left(\frac{\det(\mathfrak{S}_{n-k,4l}) \det(\mathfrak{Q}_{n-k,l})^{1-y_{k-1}}}{\det(\mathfrak{Q}_{n-k-1,2l})} \right)^{w_{k-1,l}} \times \prod_{l=0}^{2^{k-1}-1} \left(\frac{\det(\mathfrak{Q}_{n-k,l})^{1-z_{k-1}} \det(\mathfrak{S}_{n-k,4l+3})}{\det(\mathfrak{Q}_{n-k-1,2l+1})} \right)^{w'_{k-1,l}} \gg 1. \tag{46}$$

Lemma 8 is the latter formula for $k = n-3$.

We call factors of the first product, of the form

$$\left(\frac{\det\left(\mathfrak{S}_{n-k,4l}\right)\det\left(\mathfrak{Q}_{n-k,l}\right)^{1-y_{k-1}}}{\det\left(\mathfrak{Q}_{n-k-1,2l}\right)}\right)^{w_{k-1,l}}$$

factors of *Type I*, and factors of the second product of the form

$$\left(\frac{\det\left(\mathfrak{Q}_{n-k,l}\right)^{1-z_{k-1}}\det\left(\mathfrak{S}_{n-k,4l+3}\right)}{\det\left(\mathfrak{Q}_{n-k-1,2l+1}\right)}\right)^{w'_{k-1,l}}$$

factors of *Type II*.

Initialization follows the steps of approximation to four numbers. Namely, Schmidt's inequality (23) provides

$$\begin{cases}\det(\mathfrak{S}_{n,0})\det(\mathfrak{S}_{n,1}) & \gg \det(\mathfrak{Q}_{n-1,0})\det(\mathfrak{S}_{n+1,0})\ ,\\ \det(\mathfrak{S}_{n-1,0})\det(\mathfrak{S}_{n-1,1}) & \gg \det(\mathfrak{Q}_{n-2,0})\det(\mathfrak{S}_{n,0})\ ,\\ \det(\mathfrak{S}_{n-1,2})\det(\mathfrak{S}_{n-1,3}) & \gg \det(\mathfrak{Q}_{n-2,1})\det(\mathfrak{S}_{n,1})\ .\end{cases} \tag{47}$$

Since $\mathcal{S}_{n+1,0}$ spans the whole space $\mathbb{R}^{n+1}$, we have $\det\mathfrak{S}_{n+1,0} = 1$ and using the fact that $\det\mathfrak{Q}_{n-1,0} = \det\mathfrak{S}_{n-1,1} = \det\mathfrak{S}_{n-1,2}$ (by (34)), we get the formula

$$\frac{\det(\mathfrak{S}_{n-1,0})\det(\mathfrak{Q}_{n-1,0})\det(\mathfrak{S}_{n-1,3})}{\det(\mathfrak{Q}_{n-2,0})\det(\mathfrak{Q}_{n-2,1})} \gg 1.$$

Setting $w_{0,0} = w'_{0,0} = 1$ and y_0 and z_0 such that $y_0 + z_0 - 1 = 0$, we can rewrite

$$\left(\frac{\det(\mathfrak{S}_{n-1,0})\det(\mathfrak{Q}_{n-1,0})^{1-y_0}}{\det(\mathfrak{Q}_{n-2,0})}\right)^{w_{0,0}}\left(\frac{\det(\mathfrak{Q}_{n-1,0})^{1-z_0}\det(\mathfrak{S}_{n-1,3})}{\det(\mathfrak{Q}_{n-2,1})}\right)^{w'_{0,0}} \gg 1.$$

This establishes the expected formula for $k = 1$. In the inductive step, Schmidt's inequality (23) splits each term of the product in two terms involving rational subspaces of lower dimension, and shift the values of the parameters y_k and z_k.

Indeed, for $3 \le i \le n+1$ and $0 \le j \le 2^{n+1-i} - 1$, Schmidt's inequality provides

$$\frac{\det(\mathfrak{S}_{i-1,2j})\det(\mathfrak{S}_{i-1,2j+1})}{\det(\mathfrak{Q}_{i-2,j})} \gg \det(\mathfrak{S}_{i,j}). \tag{48}$$

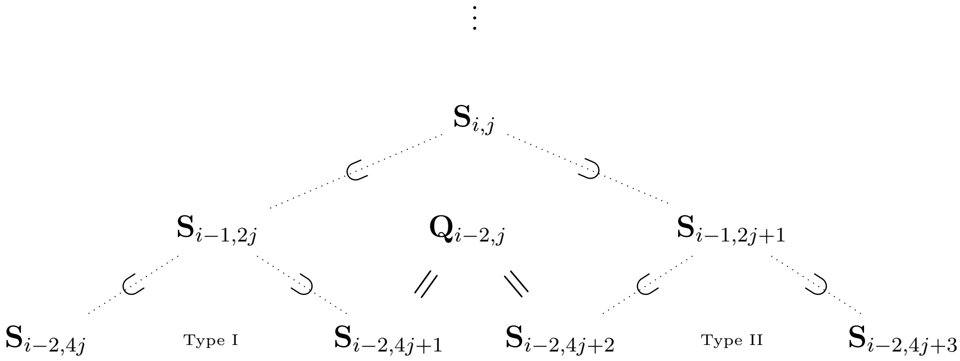

Figure 5: Situation to apply Schmidt's inequality.

Inductive step. Assume that (45) holds for some $1 \leq k < n-3$. In the product (45), there are two types of factors: factors of Type I and of Type II. Each of these factors splits into two factors, one of Type I and one of Type II. We first deal with factors of Type I. For every $0 \leq l \leq 2^{k-1}-1$, we can apply Schmidt's inequality (48) with parameters $i = n-k$ and $j = 4l$ and $j = 4l+1$ respectively to split

$$\begin{aligned} &\left(\frac{\det\left(\mathfrak{S}_{n-k,4l}\right) \det\left(\mathfrak{S}_{n-k,4l+1}\right)^{1-y_k}}{\det\left(\mathfrak{Q}_{n-k-1,2l}\right)} \right)^{w_{k,l}} \\ &\ll \left(\frac{\left(\frac{\det(\mathfrak{S}_{n-k-1,8l}) \det(\mathfrak{S}_{n-k-1,8l+1})}{\det(\mathfrak{Q}_{n-k-2,4l})} \right) \left(\frac{\det(\mathfrak{S}_{n-k-1,8l+2}) \det(\mathfrak{S}_{n-k-1,8l+3})}{\det(\mathfrak{Q}_{n-k-2,4l+1})} \right)^{1-y_k}}{\det\left(\mathfrak{Q}_{n-k-1,2l}\right)} \right)^{w_{k,l}}. \end{aligned} \tag{49}$$

Considering that $\mathbf{Q}_{n-k,2l} = \mathbf{S}_{n-k,8l+1} = \mathbf{S}_{n-k,8l+2}$, for any $u \in (0,1)$ we can write

$$\begin{aligned} &\left(\frac{\det(\mathfrak{S}_{n-k-1,8l}) \det(\mathfrak{S}_{n-k-1,8l+1})^{u(1-y_k)}}{\det(\mathfrak{Q}_{n-k-2,4l})} \right)^{w_{k,l}} \times \\ &\quad \left(\frac{\det(\mathfrak{S}_{n-k-1,8l+2})^{1-u} \det(\mathfrak{S}_{n-k-1,8l+3})}{\det(\mathfrak{Q}_{n-k-2,4l+1})} \right)^{(1-y_k)w_{k,l}} \gg (49) \end{aligned} \tag{50}$$

Similarly, for factors of Type II, for any $v \in (0,1)$, using (48) with $i = n-k$ and $j = 4l+2$ and $j = 4l+3$ respectively, and the fact that $\mathbf{Q}_{n-k,l+1} = \mathbf{S}_{n-k,8l+5} = \mathbf{S}_{n-k,8l+6}$ we get

$$\left(\frac{\det\left(\mathfrak{S}_{n-k,4l+2}\right)^{1-z_{k-1}}\det\left(\mathfrak{S}_{n-k,4l+3}\right)}{\det\left(\mathfrak{Q}_{n-k-1,2l+1}\right)}\right)^{w'_{k,l}} \ll \left(\frac{\det(\mathfrak{S}_{n-k-1,8l+4})\det(\mathfrak{S}_{n-k-1,8l+5})^{1-v}}{\det(\mathfrak{Q}_{n-k-2,4l+2})}\right)^{(1-z_k)w'_{k,l}}$$
$$\times\left(\frac{\det(\mathfrak{S}_{n-k-1,8l+6})^{v(1-z_k)}\det(\mathfrak{S}_{n-k-1,8l+7})}{\det(\mathfrak{Q}_{n-k-2,4l+3})}\right)^{w'_{k,l}} \tag{51}$$

Combining the splitting of Type I (50) and Type II (51) factors in the induction hypothesis (45), it appears that we should define the parameters (y_{k+1}, z_{k+1}) to be solutions of the system in variables (u, v)

$$u(1-y_k) = 1-v \quad \text{and} \quad 1-u = v(1-z_k).$$

That is

$$y_k = \frac{y_{k+1}+z_{k+1}-1}{z_{k+1}} \quad \text{and } z_k = \frac{y_{k+1}+z_{k+1}-1}{y_{k+1}}$$

or equivalently

$$y_{k+1} = \frac{y_k}{y_k+z_k-y_kz_k} \quad \text{and } z_{k+1} = \frac{z_k}{y_k+z_k-y_kz_k}.$$

The last equality coincide with the definition $F(y,z)$ in (38).

This and the parameters (40) establish formula (45) for $k+1$.

We now prove the relation (41) and (42) by descending induction, showing that for any $4 \le k \le n$

$$\sum_{l=0}^{2^{n-k}-1} (2-y_{n-k})w_{n-k,l} + (2-z_{n-k})w'_{n-k,l} \quad = n-k+3, \tag{52}$$
$$\sum_{l=0}^{2^{n-k}-1} w_{n-k,l} + w'_{n-k,l} \quad = n-k+2. \tag{53}$$

First, note that

$$w_{0,0} + w'_{0,0} = 2, \quad \text{and} \quad w_{0,0}(2-y_0) + w'_{0,0}(2-z_0) = 3,$$

hence we have initialization at $k = n$.

Assume that for some $4 \le k \le n$ (52) and (53) holds. The two sums represent the number of determinants that appears respectively at the numerator and at the denominator in (36). The key is to observe the splitting in (49) : the new sum for the denominator is the sum from the previous numerator, while at the numerator, the previous denominator is doubled but we have a simplification with one denominator. Namely, using the recurrence formula (38) and (40) for the parameters

$$\begin{aligned}
\sum_{l=0}^{2^{n-k+1}-1} w_{n-k+1,l} + w'_{n-k+1,l} &= \sum_{l=0}^{2^{n-k}-1} w_{n-k+1,2l} + w_{n-k+1,2l+1} + w'_{n-k+1,2l} + w'_{n-k+1,2l+1} \\
&= \sum_{l=0}^{2^{n-k+1}-1} w_{n-k,l} + w'_{n-k,l}(1-z_{n-k}) + w'_{n-k,l} + w_{n-k,l}(1-y_{n-k}) \\
&= \sum_{l=0}^{2^{n-k}-1} (2-y_{n-k})w_{n-k,l} + (2-z_{n-k})w'_{n-k,l} \\
&= n-k+3,
\end{aligned}$$

$$\begin{aligned}
\sum_{l=0}^{2^{n-k+1}-1} w_{n-k+1,l}(2-y_{n-k+1}) + w'_{n-k+1,l}(2-z_{n-k+1}) &= \sum_{l=0}^{2^{n-k}-1} (w_{n-k+1,2l} + w_{n-k+1,2l+1})(2-y_{n-k+1}) \\
&\quad + \sum_{l=0}^{2^{n-k}-1} (w'_{n-k+1,2l} + w'_{n-k+1,2l+1})(2-z_{n-k+1}) \\
&= \sum_{l=0}^{2^{n-k}-1} w_{n-k,l}(3-2y_{n-k}) + w'_{n-k,l}(3-2z_{n-k}) \\
&= 2(n-k+3) - (n-k+2) = n-k+4.
\end{aligned}$$

Hence the result by descending induction. □

5 Approximation by one linear form

In this section, we explain how the very same geometry of a subsequence of best approximation vectors provides Theorem 1 for approximation by one linear form. We need to consider a hyperbolic rotation to get a suitable analogue of the estimates in Lemma 2. For this, we use Schmidt's inequalities on heights in a slightly larger context than rational subspaces.

5.1 About Schmidt's inequalities on heights

As stated in Proposition 1, Schmidt's inequality applies to rational subspaces and the lattice of integer points. Let $\Lambda \subset \mathbb{R}^d$ be a complete lattice, that plays the role of integer points. Let $\mathbf{M} \subset \mathbb{R}^{\mathbf{d}}$ be a subspace, it is called Λ-rational if the lattice

$$\mathfrak{M} = \mathbf{M} \cap \Lambda$$

is complete, i.e. if $\langle \mathfrak{M} \rangle_{\mathbb{R}} = \mathbf{M}$.

Lemma 9. *The intersection of two Λ-rational subspaces is Λ-rational.*

The proof is the same as for rational subspaces, and use the description of subspaces by their orthogonal vectors.

Definition. Given a fixed complete lattice Λ, we define the height H_Λ of a Λ-rational subspace $\mathbf{M}$ to be the fundamental volume

$$H_\Lambda(\mathbf{M}) = \det(\mathfrak{M}) = \det(\mathbf{M} \cap \Lambda)$$

of the Λ-points of $\mathbf{M}$.

Proposition 2 (Schmidt's inequality)**.** *Let Λ be a complete lattice. Let A, B be two Λ-rational subspaces in $\mathbb{R}^n$, we have*

$$H_\Lambda(\boldsymbol{A} + \boldsymbol{B}) \cdot H_\Lambda(\boldsymbol{A} \cap \boldsymbol{B}) \ll H_\Lambda(\boldsymbol{A}) \cdot H_\Lambda(\boldsymbol{B}). \tag{54}$$

Proof. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two Λ-rational subspaces. Denote their Λ-rational intersection $\mathbf{N} = \mathbf{M}_1 \cap \mathbf{M}_2$. Let $\mathcal{E}$ be a basis of $\mathbf{N} \cap \Lambda$. We complete $\mathcal{E}$ to a basis of $\mathbf{M}_i \cap \Lambda$ by a collection of vectors μ_i in Λ, for $i = 1, 2$. Consider the volume $\mathrm{vol}(\mathcal{E}, \mu_1, \mu_2)$ of the parallelepiped generated by the basis vectors $(\mathcal{E}, \mu_1, \mu_2)$. Then,

$$\mathrm{vol}(\mathcal{E}, \mu_1, \mu_2) \geq \mathrm{vol}((\mathbf{M}_1 + \mathbf{M}_2) \cap \Lambda) = H_\Lambda(\mathbf{M}_1 + \mathbf{M}_2).$$

Let μ_1^{**} denote de projection of μ_1 on $\mathbf{M}_2$ parallel to $\mathbf{N}$. Then $\det(\mu_1^{**}) = H_\Lambda(\mathbf{M}_1)/H_\Lambda(\mathbf{N})$ and

$$\det(\mathcal{E}, \mu_1, \mu_2) = \det(\mathcal{E}, \mu_2) \det(\mu_1^{**}) = H_\Lambda(\mathbf{M}_2) \cdot H_\Lambda(\mathbf{M}_1)/H_\Lambda(\mathbf{N}).$$

Hence we obtain the formula

$$H_\Lambda(\mathbf{M}_1 + \mathbf{M}_2) \cdot H_\Lambda(\mathbf{M}_1 \cap \mathbf{M}_2) \ll H_\Lambda(\mathbf{M}_1) \cdot H_\Lambda(\mathbf{M}_2)$$

for the underlying complete lattice Λ. □

5.2 Hyperbolic rotation

Given $(\boldsymbol{z}_l)_{l \in \mathbb{N}} = (q_{1,l}, \ldots, q_{n,l}, a_l)$ a sequence of best approximations to a point $\boldsymbol{\theta} \in \mathbb{R}^n$ for the approximation by one linear form, we can extract a subsequence satisfying Lemma 7. For approximation by one linear form, it may happen that the sequence of best approximation vectors spans a subspace of dimension $m < n + 1$ in $\mathbb{R}^{n+1}$ (see [1]). In this case, Theorem 1 holds with the stronger lower bound $G^*(m, \hat{\omega}(\boldsymbol{\theta}))$ instead of $G^*(n, \hat{\omega}(\boldsymbol{\theta}))$ (see Remark after Proof of Lemma 7). In the sequel, we suppose that the best approximation vectors span the full space. In particular the coordinates $\theta_1, \ldots, \theta_n$ are linearly independent with 1.

Consider the matrix

$$L = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \theta_1 & \cdots & \theta_n & 1 \end{pmatrix}.$$

We can consider the sequence of best approximation as points of the lattice $\mathcal{L} = L.\mathbb{Z}^{n+1}$ with

$$(\tilde{\boldsymbol{z}}_l)_{l\in\mathbb{N}} = L.(\boldsymbol{z}_l)_{l\in\mathbb{N}} \in \mathcal{L}.$$

Here, we simply replace the last coordinate a_l by the error of approximation L_l.

Consider a large parameter T, and the hyperbolic rotation

$$\mathcal{G}_T = \begin{pmatrix} T^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & T^{-1} & 0 \\ 0 & \cdots & 0 & T^n \end{pmatrix}.$$

The lattice $\mathcal{L}' = \mathcal{G}_T\mathcal{L}$ is complete since the determinants of L and $\mathcal{G}_T$ are 1.

Consider the sequence $(\boldsymbol{z}'_l)_{l\in\mathbb{N}} \in \mathcal{L}'$ defined by

$$(\boldsymbol{z}'_l)_{l\in\mathbb{N}} = \mathcal{G}_T L(\boldsymbol{z}_l)_{l\in\mathbb{N}} \in \mathcal{L}'$$

where

$$\begin{aligned} \boldsymbol{z}'_l &= (z'_{1,l}, \dots, z'_{n,l}, z'_{n+1,l}) \\ &= (T^{-1}q_{1,l}, \dots, T^{-1}q_{n,l}, T^n L_l). \end{aligned}$$

For best approximation by one linear form we defined $M_l = \max_{1\le i\le n} |z_{i,l}|$, and after hyperbolic rotation we have

$$\max_{1\le i\le n} |z'_{i,l}| \le M_l T^{-1}.$$

Since we assume that the best approximation vectors $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ span the full space $\mathbb{R}^{n+1}$, we can apply Lemma 7 to $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ and obtain a set of indices $(r_k)_{0\le 2^{n-2}-1}$. Denote

$$\mathcal{S}'_{3,l} = \{\boldsymbol{z}'_{r_l-1}, \boldsymbol{z}'_{r_l}, \boldsymbol{z}'_{r_l+1}\} = \mathcal{G}_T L\mathcal{S}_{3,l}, \quad 0 \le l \le 2^{n-2}-1$$

and for $4 \le k \le n+1$ and $0 \le l \le 2^{n-k+1}-1$, denote by $\mathcal{S}'_{k,l}$ the set of best approximation vectors

$$\mathcal{S}'_{k,l} = \cup_{\nu=0}^{2^{k-3}-1} \mathcal{S}'_{3,2^{k-3}l+\nu} = \mathcal{G}_T L\mathcal{S}_{k,l}.$$

Since $\mathcal{G}_T$ and $\mathcal{L}$ have determinant 1, these sets satisfies the properties of linear independence and inclusion listed in Lemma 7.

Further in the proof of Theorem 1, we need an estimate of the fundamental volumes of the lattices $\Lambda'_k = \langle \boldsymbol{z}'_k, \boldsymbol{z}'_{k+1}\rangle_{\mathbb{Z}}$ and $\Gamma'_k = \langle \boldsymbol{z}'_{k-1}, \boldsymbol{z}'_k, \boldsymbol{z}'_{k+1}\rangle_{\mathbb{Z}}$ spanned by consecutive independent vectors $\boldsymbol{z}'_l$.

For large T, we can follow a similar proof as Lemma 2.

Lemma 10. *Fix an index k. Let T be large enough so that*

$$T > M_{k+1} \quad \text{and} \quad T > L_{k-1}^{-1/n} \tag{55}$$

Given two consecutive and linearly independent best approximation vectors $\boldsymbol{z}_k, \boldsymbol{z}_{k+1}$, the fundamental volume $\det \Lambda'_k$ satisfies

$$\det \Lambda'_k \asymp L_k T^n M_{k+1} T^{-1} = L_k M_{k+1} T^{n-1}. \tag{56}$$

Given three consecutive and linearly independent best approximation vectors $\boldsymbol{z}_{k-1}, \boldsymbol{z}_k, \boldsymbol{z}_{k+1}$, the fundamental volume $\det \Gamma'_k$ satisfies

$$\det \Gamma'_k \ll L_{k-1} T^n M_k T^{-1} M_{k+1} T^{-1} = L_{k-1} M_k M_{k+1} T^{n-2}. \tag{57}$$

Proof. For T satisfying (55), we see that $\boldsymbol{z}'_l = (T^{-1}q_{1,l}, \dots, T^{-1}q_{n,l}, T^n L_l)$ satisfies

$$|T^n L_l| > 1 \quad \text{and} \quad |T^{-1} q_{i,l}| < 1 \quad \text{for } 1 \le i \le n. \tag{58}$$

Consider the $2 \times (n+1)$ matrix

$$\begin{pmatrix} z'_{1,k} & \cdots & z'_{n,k} & z'_{n+1,k} \\ z'_{1,k+1} & \cdots & z'_{n,k+1} & z'_{n+1,k+1} \end{pmatrix} = \begin{pmatrix} T^{-1}q_{1,k} & \cdots & T^{-1}q_{n,k} & T^n L_k \\ T^{-1}q_{1,k+1} & \cdots & T^{-1}q_{n,k+1} & T^n L_{k+1} \end{pmatrix}.$$

and the $3 \times (n+1)$ matrix

$$\begin{pmatrix} z'_{1,k-+1} & \cdots & z'_{n,k-1} & z'_{n+1,k-1} \\ z'_{1,k} & \cdots & z'_{n,k} & z'_{n+1,k} \\ z'_{1,k+1} & \cdots & z'_{n,k+1} & z'_{n+1,k+1} \end{pmatrix} = \begin{pmatrix} T^{-1}q_{1,k-1} & \cdots & T^{-1}q_{n,k-1} & T^n L_{k-1} \\ T^{-1}q_{1,k} & \cdots & T^{-1}q_{n,k} & T^n L_k \\ T^{-1}q_{1,k+1} & \cdots & T^{-1}q_{n,k+1} & T^n L_{k+1} \end{pmatrix}.$$

Using Lemma 3, the squares of the fundamental volumes $\det \Lambda'_k$ and $\det \Gamma'_k$ are sums of squares of 2×2 and 3×3 minors. As for Lemma 2, the conditions (58) provide the upper bounds $L_k T^n M_{k+1} T^{-1} = L_k M_{k+1} T^{n-1}$ for the 2×2 minors and $L_{k-1} T^n M_k T^{-1} M_{k+1} T^{-1} = L_{k-1} M_k M_{k+1} T^{n-2}$ for the 3×3 minors. The upper bound for the fundamental volumes follows.

The lower bound of $\det \Lambda'_k$, follows from Minkowski's first convex body theorem as well, considering the symmetric convex body $\mathcal{G}_T \mathcal{L} \Pi$ and its intersection with Λ'_l reduced to zero. $\square$

Here, we need a large parameter T to obtain a good upper bound for the minors. If $T = 1$, such upper bound are false.

Remark. *In the case of a lattice generated by both*

$$\Lambda := \langle \boldsymbol{z}'_\nu, \boldsymbol{z}'_{\nu+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}'_{k-1}, \boldsymbol{z}'_k \rangle_{\mathbb{Z}}$$

we have

$$\det \Lambda \asymp L_\nu M_{\nu+1} T^{n-1} \asymp L_{k-1} M_k T^{n-1}. \tag{59}$$

5.3 Proof of the main theorem for approximation by one linear form

The proof in the case of approximation by one linear form follow the same steps as in the case of simultaneous approximation. Considering $(\boldsymbol{z}_l)_{l\in\mathbb{N}}$ a sequence of best approximations to a point $\boldsymbol{\theta} \in \mathbb{R}^n$, we obtain via Lemma 7 a set of indices satisfying good properties. Suppose that k_0 is large enough so that for $\alpha^* < \hat{\omega}(\boldsymbol{\theta})$.

$$L_j \leq M_{j+1}^{-\alpha^*}, \quad \text{for} \quad j \geq k_0. \tag{60}$$

The hyperbolic rotation $(\boldsymbol{z}'_l)_{l\in\mathbb{N}} = \mathcal{G}_T\mathcal{L} \cdot (\boldsymbol{z}_l)_{l\in\mathbb{N}}$, where $T \gg 1$ is fixed, preserves the property of linear independence, and hence the structure of the pattern of best approximation vectors constructed in Lemma 7. We consider the rotated sets $\mathcal{S}'_{k,l} = \mathcal{G}_T L \cdot \mathcal{S}_{k,l}$, $\mathcal{Q}'_{k,l} = \mathcal{G}_T L \cdot \mathcal{Q}_{k,l}$ from the sets $\mathcal{S}_{k,l}$ and $\mathcal{Q}_{k,l}$ defined in Lemma 7. We denote respectively by $\mathfrak{S}'_{k,l}$ and $\mathfrak{Q}'_{k,l}$ the lattices of their $\mathcal{G}_T\mathcal{L}$-points.

We apply Schmidt's inequalities for heights to the $\mathcal{G}_T\mathcal{L}$-rational subspaces $\mathcal{S}'_{k,l} = \mathcal{G}_T L \cdot \mathcal{S}_{k,l}$, to obtain the lower bounds of Lemma 8

$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\det\left(\mathfrak{S}'_{3,4l}\right) \det\left(\mathfrak{Q}'_{3,l}\right)^{1-y_{n-4}}}{\det\left(\mathfrak{Q}'_{2,2l}\right)} \right)^{w_{n-4,l}} \cdot \prod_{l=0}^{2^{n-4}-1} \left(\frac{\det\left(\mathfrak{Q}'_{3,l}\right)^{1-z_{n-4}} \det\left(\mathfrak{S}'_{3,4l+3}\right)}{\det\left(\mathfrak{Q}'_{2,2l+1}\right)} \right)^{w'_{n-4,l}} \gg 1 \tag{61}$$

where the parameters $y_{n-4}, z_{n-4}, w_{n-4,l}$ and $w'_{n-4,l}$ satisfy (38).

As for the proof of Lemma 4, we want to split the denominators. Indeed,

$$\mathfrak{Q}'_{2,2l} = \langle \boldsymbol{z}'_{r_{4l}}, \boldsymbol{z}'_{r_{4l}+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}'_{r_{4l+1}-1}, \boldsymbol{z}'_{r_{4l+1}} \rangle_{\mathbb{Z}} \tag{62}$$

$$\mathfrak{Q}'_{2,2l+1} = \langle \boldsymbol{z}'_{r_{4l+2}}, \boldsymbol{z}'_{r_{4l+2}+1} \rangle_{\mathbb{Z}} = \langle \boldsymbol{z}'_{r_{4l+3}-1}, \boldsymbol{z}'_{r_{4l+3}} \rangle_{\mathbb{Z}} \tag{63}$$

and we can split both $\det\left(\mathfrak{Q}'_{2,2l}\right)$ and $\det\left(\mathfrak{Q}'_{2,2l+1}\right)$ with their two expressions coming from (59). Now we should consider the following parameters for $0 \leq s, t \leq 1$

$$g^*(s,t) = \frac{(1-\alpha^*)s}{(1-\alpha^*)s - w^*(s,t)} = \frac{(1-\alpha^*)(1-w^*(s,t)-t)}{t}. \tag{64}$$

where the second equality comes from $w^*(s,t) \in (0,1)$ being the root of the equation

$$w^{*2} - \left(1 - t - \frac{s}{\alpha^* - 1}\right) w^* + \frac{s}{\alpha^* - 1} = 0. \tag{65}$$

Note that it is analogous to $g(s,t)$ and $w(s,t)$ defined in (21) (20) (Lemma 4).

Applying the estimates of Lemma 10, and the splittings of denominator (59) with parameters $w^*(1, 1 - z_{n-4})$ and $w^*(1 - y_{n-4}, 1)$ we get

$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\left(L_{r_{4l}-1} M_{r_{4l}} M_{r_{4l}+1} T^{n-2}\right) \left(L_{r_{4l+1}-1} M_{r_{4l+1}} M_{r_{4l+1}+1} T^{n-2}\right)^{1-y_{n-4}}}{\left(L_{r_{4l}} M_{r_{4l}+1} T^{n-1}\right)^{w^*(1-y_{n-4},1)} \left(L_{r_{4l+1}-1} M_{r_{4l+1}} T^{n-1}\right)^{1-w^*(1-y_{n-4},1)}} \right)^{w_{n-4,l}} \cdot$$
$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\left(L_{r_{4l+2}-1} M_{r_{4l+2}} M_{r_{4l+2}+1} T^{n-2}\right)^{1-z_{n-4}} \left(L_{r_{4l+3}-1} M_{r_{4l+3}} M_{r_{4l+3}+1} T^{n-2}\right)}{\left(L_{r_{4l+2}} M_{r_{4l+2}+1} T^{n-1}\right)^{w^*(1,1-z_{n-4})} \left(L_{r_{4l+3}-1} M_{r_{4l+3}} T^{n-1}\right)^{1-w^*(1,1-z_{n-4})}} \right)^{w_{n-4,l}} \gg 1$$

Furthermore, by (41) and (42), T has the same power $(n-1)(n-2)$ at numerator and denominator and can be simplified.

$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\left(L_{r_{4l}-1} M_{r_{4l}} M_{r_{4l}+1}\right) \left(L_{r_{4l+1}-1} M_{r_{4l+1}} M_{r_{4l+1}+1}\right)^{1-y_{n-4}}}{\left(L_{r_{4l}} M_{r_{4l}+1}\right)^{w^*(1-y_{n-4},1)} \left(L_{r_{4l+1}-1} M_{r_{4l+1}}\right)^{1-w^*(1-y_{n-4},1)}} \right)^{w_{n-4,l}} \cdot$$
$$\prod_{l=0}^{2^{n-4}-1} \left(\frac{\left(L_{r_{4l+2}-1} M_{r_{4l+2}} M_{r_{4l+2}+1}\right)^{1-z_{n-4}} \left(L_{r_{4l+3}-1} M_{r_{4l+3}} M_{r_{4l+3}+1}\right)}{\left(L_{r_{4l+2}} M_{r_{4l+2}+1}\right)^{w^*(1,1-z_{n-4})} \left(L_{r_{4l+3}-1} M_{r_{4l+3}}\right)^{1-w^*(1,1-z_{n-4})}} \right)^{w_{n-4,l}} \gg 1$$

Hence, at least *one* of the following four inequalities holds:

$$\begin{aligned}
L_{r_{4l}-1} M_{r_{4l}} M_{r_{4l}+1} &\gg \left(L_{r_{4l}} M_{r_{4l}+1}\right)^{w^*(1-y_{n-4},1)}, \\
\left(L_{r_{4l+1}-1} M_{r_{4l+1}} M_{r_{4l+1}+1}\right)^{1-y_{n-4}} &\gg \left(L_{r_{4l+1}-1} M_{r_{4l+1}}\right)^{1-w^*(1-y_{n-4},1)}, \\
\left(L_{r_{4l+2}-1} M_{r_{4l+2}} M_{r_{4l+2}+1}\right)^{1-z_{n-4}} &\gg \left(L_{r_{4l+2}} M_{r_{4l+2}+1}\right)^{w^*(1,1-z_{n-4})}, \\
L_{r_{4l+3}-1} M_{r_{4l+3}} M_{r_{4l+3}+1} &\gg \left(L_{r_{4l+3}-1} M_{r_{4l+3}}\right)^{1-w^*(1,1-z_{n-4})}.
\end{aligned}$$

Following the proof of Lemma 4, from (60) and the definitions of g^* (64) and w^*(65) we deduce

$$M_{\nu+1} \gg M_\nu^{g^*(s,t)} \tag{66}$$

for either $\nu = r_{4l}, r_{4l+1}, r_{4l+2}$ or r_{4l+3}, and $(s,t) = (1, 1 - z_{n-4})$ or $(1 - y_{n-4}, 1)$ respectively.

Using (64) and (65), we see that $1/g^*$ satisfies the same equation as g (25) up to symmetry in s and t

$$1/g^{*2} - \left(\beta + \frac{1-t}{s}\right) \cdot 1/g^* - \frac{t\beta}{s} = 0. \tag{67}$$

In particular, we can use this equation to compute the optimal value of either s or t when the other parameter is 1. Namely,

$$t = \frac{(g^*)^{-2} - \beta(g^*)^{-1} - \beta}{(g^*)^{-1} - \beta} = \frac{R_3((g^*)^{-1})}{(g^*)^{-1} - \beta}, \quad \text{for} \quad g^* = g^*(1-t, 1), \tag{68}$$

$$s = \frac{(g^*)^{-2} - \beta(g^*)^{-1} - \beta}{(g^*)^{-1}((g^*)^{-1} - \beta)} = \frac{R_3((g^*)^{-1})}{(g^*)^{-1}((g^*)^{-1} - \beta)}, \quad \text{for} \quad g^* = g^*(1, 1-s). \tag{69}$$

Similarly to the case of simultaneous approximation, we have

$$g^* = g^*(1, 1 - z_{n-4}) = g^*(1 - y_{n-4}, 1)$$

and the properties of g^* and z_{n-4}, y_{n-4} provide that $R_n(1/g^*) = 0$. □

5.3.1 Example of approximation to 4 numbers

To make the ideas of the proof clearer, we give an example in the simple case of approximation to 4 numbers.

Consider a sequence of best approximation vectors to $\boldsymbol{\theta} \in \mathbb{R}^4$ by one linear form. We may assume that it spans $\mathbb{R}^5$. For an index $k_0 \gg 1$ we apply Lemma 6. It provides a pattern of best approximation vectors

$$\boldsymbol{z}_{r_0-1}, \boldsymbol{z}_{r_0}, \boldsymbol{z}_{r_0+1}; \quad \boldsymbol{z}_{r_1-1}, \boldsymbol{z}_{r_1}, \boldsymbol{z}_{r_1+1}; \quad \boldsymbol{z}_{r_2-1}, \boldsymbol{z}_{r_2}, \boldsymbol{z}_{r_2+1}, \quad \boldsymbol{z}_{r_3-1}, \boldsymbol{z}_{r_3}, \boldsymbol{z}_{r_3+1};$$

of linearly independent triples satisfying properties of Lemma 6. Consider T such that $T > M_{r_3+1}$ and $T > L_{r_3-1}^{-1/n}$, we apply the hyperbolic rotation to the integer vectors $\boldsymbol{z}_j$ to get

$$\boldsymbol{z}'_j = \mathcal{G}_T L \cdot \boldsymbol{z}_j$$

for $j = r_0 - 1, r_0, r_0 + 1, r_1 - 1, r_1, r_1 + 1, r_2 - 1, r_2, r_2 + 1, r_3 - 1, r_3, r_3 + 1$.

For $0 \le i \le 3$ we consider the subspace

$$\mathbf{S}_{3,i} = \langle \boldsymbol{z}'_{r_i-1}, \boldsymbol{z}'_{r_i}, \boldsymbol{z}'_{r_i+1} \rangle_{\mathbb{R}}$$

and its lattice of $\mathcal{G}_T\mathcal{L}$ points

$$\mathfrak{S}_{3,i} = \mathbf{S}_{3,i} \cap \mathcal{G}_T\mathcal{L}.$$

We recall that

$$\mathbf{S}_{3,1} = \mathbf{S}_{3,2} = \mathbf{Q}$$

Consider the 2-dimensional lattices

$$\Lambda_0 := \langle \boldsymbol{z}'_{r_0}, \boldsymbol{z}'_{r_0+1}\rangle_{\mathbb{Z}} = \langle \boldsymbol{z}'_{r_1-1}, \boldsymbol{z}'_{r_1}\rangle_{\mathbb{Z}} = \mathbf{S}_{3,0} \cap \mathbf{S}_{3,1} \cap \mathcal{G}_T\mathcal{L}$$

and

$$\Lambda_1 := \langle \boldsymbol{z}'_{r_2}, \boldsymbol{z}'_{r_2+1}\rangle_{\mathbb{Z}} = \langle \boldsymbol{z}'_{r_3-1}, \boldsymbol{z}'_{r_3}\rangle_{\mathbb{Z}} = \mathbf{S}_{3,2} \cap \mathbf{S}_{3,3} \cap \mathcal{G}_T\mathcal{L}.$$

We apply Schmidt's inequality (Propositon 2) with underlying lattice $\mathcal{G}_T\mathcal{L}$ to obtain the analogue of (31)

$$\frac{\det \mathfrak{S}_{3,0}(\det \mathfrak{S}_{3,1})^x}{\det \Lambda_0} \cdot \frac{(\det \mathfrak{S}_{3,2})^{1-x}\det \mathfrak{S}_{3,3}}{\det \Lambda_1} \gg 1.$$

By Lemma 10, we get

$$\frac{L_{r_0-1}M_{r_0}M_{r_0+1}T^2(L_{r_1-1}M_{r_1}M_{r_1+1}T^2)^x}{L_{r_0}M_{r_0+1}T^3} \cdot \frac{(L_{r_2-1}M_{r_2}M_{r_2+1}T^2)^{1-x}L_{r_3-1}M_{r_3}M_{r_3+1}T^2}{L_{r_3-1}M_{r_3}T^3} \gg 1.$$

Here, T disappears as it has power 6 at numerator and denominator :

$$3+3 = 6 = 2+2x+2(1-x)+2.$$

We deduce

$$\frac{L_{r_0-1}M_{r_0}M_{r_0+1}(L_{r_1-1}M_{r_1}M_{r_1+1})^x}{L_{r_0}M_{r_0+1}} \cdot \frac{(L_{r_2-1}M_{r_2}M_{r_2+1})^{1-x}L_{r_3-1}M_{r_3}M_{r_3+1}}{L_{r_3-1}M_{r_3}} \gg 1.$$

Since the lattices Λ_0 and Λ_1 are both generated by two distinct pairs of consecutive best approximation vectors, we deduce from (59) that

$$L_{r_0}M_{r_0+1} \asymp L_{r_1-1}M_{r_1} \text{ and } L_{r_2}M_{r_2+1} \asymp L_{r_3-1}M_{r_3}.$$

Hence we can replace

$$L_{r_0}M_{r_0+1} \text{ by } (L_{r_0}M_{r_0+1})^{w^*(1,x)}(L_{r_1-1}M_{r_1})^{1-w^*(1,x)}$$

and

$$L_{r_3-1}M_{r_3} \text{ by } (L_{r_2}M_{r_2+1})^{w^*(1-x,1)}(L_{r_3-1}M_{r_3})^{1-w^*(1-x,1)},$$

where $w^*(s,t)$ is defined by (65).

We deduce that at least one of the four following inequalities holds

$$\begin{cases} L_{r_0-1}M_{r_0}M_{r_0+1} & \gg (L_{r_0}M_{r_0+1})^{w^*(1,x)} \\ (L_{r_1-1}M_{r_1}M_{r_1+1})^x & \gg (L_{r_1-1}M_{r_1})^{1-w^*(1,x)} \\ (L_{r_2-1}M_{r_2}M_{r_2+1})^{1-x} & \gg (L_{r_2}M_{r_2+1})^{w^*(1-x,1)} \\ L_{r_3-1}M_{r_3}M_{r_3+1} & \gg (L_{r_3-1}M_{r_3})^{1-w^*(1-x,1)} \end{cases}.$$

We deduce from the definition (64) of $g^* = g^*(1,x) = g^*(1-x,1)$ that

$$M_{\nu+1} \gg M_\nu^{g^*}$$

for either $\nu = r_0, r_1, r_2$ or r_3. From (69) we have

$$x = \frac{\beta}{1/g^*(1/g^* - \beta)} = \frac{R_3(1/g^*)}{1/g^* - \beta}$$

and so g^* satisfies the equation

$$R_4(1/g^*) = 1/g^* R_3(1/g^*) - \beta = 0.$$

□

6 Construction of points with given ratio

In this last section, we prove the second part of Theorem 1. To construct points with given ratio, we place ourselves in the context of parametric geometry of numbers introduced by Schmidt and Summerer in [24]. We refer the reader to [12, §2] for the notation used in this paper and the presentation of the parametric geometry of numbers. We use the notation introduced by D. Roy in [20] which is essentially dual to the one of W. M. Schmidt and L. Summerer [23]. We believe we should denote *generalized n-systems* by *Roy-systems*. We fully use Roy's theorem [20] as stated in [12, Theorem 5] to deduce the existence of a point with expected properties from an explicit family of Roy-systems with three parameters. The construction shows how the values $G(n,\alpha)$ and $G^*(n,\alpha^*)$ appear naturally in the context of parametric geometry of numbers, and why they are reached at *regular systems*.

Fix the dimension $n \geq 2$, and consider the case of approximation by one linear form. Fix the three parameters $\hat{\omega} \geq n$, $\rho = G^*(n,\hat{\omega})$ and $c \geq 1$. Consider the Roy-system $\boldsymbol{P}$ on the interval $[1, c\rho]$ depending on these parameters whose combined graph is given below by Figure 6, where

$$P_1(1) = \frac{1}{1+\hat{\omega}}, \quad P_k(1) = \rho^{k-2} P_1(1) \text{ for } 2 \leq k \leq n{+}1 \text{ and } P_k(c\rho) = c\rho P_k(1) \text{ for } 1 \leq k \leq n{+}1.$$

The fact that all coordinates sum up to 1 for $q = 1$ follows from $1/\rho$ being the root of the polynomial $R_{n,1/\hat{\omega}}$ (9). On each interval between two consecutive division points, there is only one line segment with slope 1. On $[1, q_0]$, there is one line segment of slope 1 starting from the value $\frac{1}{1+\hat{\omega}}$ and reaching the value $\frac{c\rho^n}{1+\hat{\omega}}$. Then, each component P_k increases from $\frac{\rho^{k-1}}{1+\hat{\omega}}$ to $\frac{c\rho^{k-1}}{1+\hat{\omega}}$ with slope 1 where k decreases from $k = n$ down to $k = 2$.

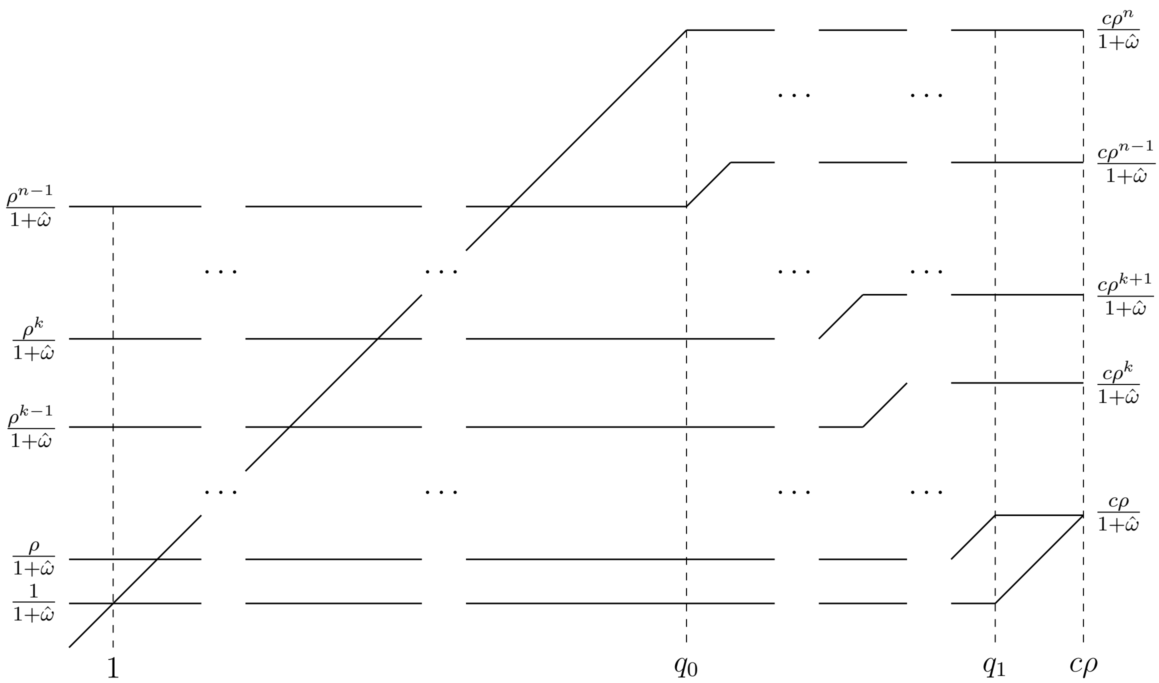

Figure 6: Pattern of the combined graph of $\boldsymbol{P}$ on the fundamental interval $[1, c\rho]$

We extend $\boldsymbol{P}$ to the interval $[1, \infty)$ by self-similarity. This means, $\boldsymbol{P}(q) = (c\rho)^m \boldsymbol{P}((c\rho)^{-m} q)$ for all integers m. In view of the value of $\boldsymbol{P}$ and its derivative at 1 and $c\rho$, one sees that the extension provides a Roy-system on $[1, \infty)$.

Note that for $c = 1$, the parameter q_0 and q_1 coincide and we constructed a *regular system*.

Roy's Theorem [20] provides the existence of a point $\boldsymbol{\theta}$ in $\mathbb{R}^n$ such that

$$\frac{1}{1+\hat{\omega}(\boldsymbol{\theta})} = \limsup_{q \to +\infty} \frac{P_1(q)}{q},$$
$$\frac{1}{1+\omega(\boldsymbol{\theta})} = \liminf_{q \to +\infty} \frac{P_1(q)}{q}.$$

Here, self-similarity ensures that the lim sup (resp. lim inf) is in fact the maximum (resp. the minimum) on the interval $[1, c\rho[$. Thus,

$$\frac{1}{1+\hat{\omega}(\boldsymbol{\theta})} = \max_{[1,c\rho[} \frac{P_1(q)}{q} = \frac{P_1(1)}{1} = \frac{1}{1+\hat{\omega}},$$
$$\frac{1}{1+\omega(\boldsymbol{\theta})} = \min_{[1,c\rho[} \frac{P_1(q)}{q} = \frac{P_1(q_0)}{q_0} = \frac{1}{c\rho\hat{\omega}+1}$$

where

$$q_0 = \frac{c(\rho^n + \cdots + \rho^2 + \rho) + 1}{1+\hat{\omega}} = \frac{c(\rho\hat{\omega}) + 1}{1+\hat{\omega}}.$$

Hence,

$$\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \omega(\boldsymbol{\theta}) = c\rho\hat{\omega}.$$

and we constructed the required points since $c \geq 1$ and $\rho = G^*(n, \hat{\omega})$.

Consider the case of simultaneous approximation. Fix the three parameters $1 \geq \hat{\lambda} \geq 1/n$, $\rho = G(n, \hat{\lambda})$ and $c \geq 1$. Consider the Roy-system $\boldsymbol{P}$ on the interval $[1, c\rho]$ depending on these parameters whose combined graph is given below by Figure 7, where

$$P_{n+1}(1) = \frac{\hat{\lambda}}{1+\hat{\lambda}}, \quad P_k(1) = \rho^{n-k}P_1(1) \text{ for } 1 \leq k \leq n \text{ and } P_k(c\rho) = c\rho P_k(1) \text{ for } 2 \leq k \leq n+1.$$

The fact that all coordinates sum up to 1 for $q = 1$ follows from ρ being the root of the polynomial $R_{n,\hat{\lambda}}$ (9). Up to change of origin and rescaling, this is the same pattern as shown by Figure 6. We extend $\boldsymbol{P}$ to the interval $[1, \infty)$ by self-similarity. This means, $\boldsymbol{P}(q) = (c\rho)^m \boldsymbol{P}((c\rho)^{-m}q)$ for all integers m. In view of the value of $\boldsymbol{P}$ and its derivative at 1 and $c\rho$, one sees that the extension provides a Roy-system on $[1, \infty)$.

For $c = 1$, the parameter q_0 and q_1 coincide and we constructed a *regular system.*

Roy's Theorem [20] provides the existence of a point $\boldsymbol{\theta}$ in $\mathbb{R}^n$ such that

$$\frac{\hat{\lambda}(\boldsymbol{\theta})}{1+\hat{\lambda}(\boldsymbol{\theta})} = \liminf_{q\to+\infty} \frac{P_{n+1}(q)}{q},$$
$$\frac{\lambda(\boldsymbol{\theta})}{1+\lambda(\boldsymbol{\theta})} = \limsup_{q\to+\infty} \frac{P_{n+1}(q)}{q}.$$

Here, self-similarity ensures that the lim sup (resp. lim inf) is in fact the maximum (resp. the minimum) on the interval $[1, c\rho[$. Thus,

$$\frac{\hat{\lambda}(\boldsymbol{\theta})}{1+\hat{\lambda}(\boldsymbol{\theta})} = \min_{[1,c\rho[} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(1)}{1} = \frac{\hat{\lambda}}{1+\hat{\lambda}},$$
$$\frac{\lambda(\boldsymbol{\theta})}{1+\lambda(\boldsymbol{\theta})} = \max_{[1,c\rho[} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(q_1)}{q_1} = \frac{c\rho\hat{\lambda}}{1+c\rho\hat{\lambda}}$$

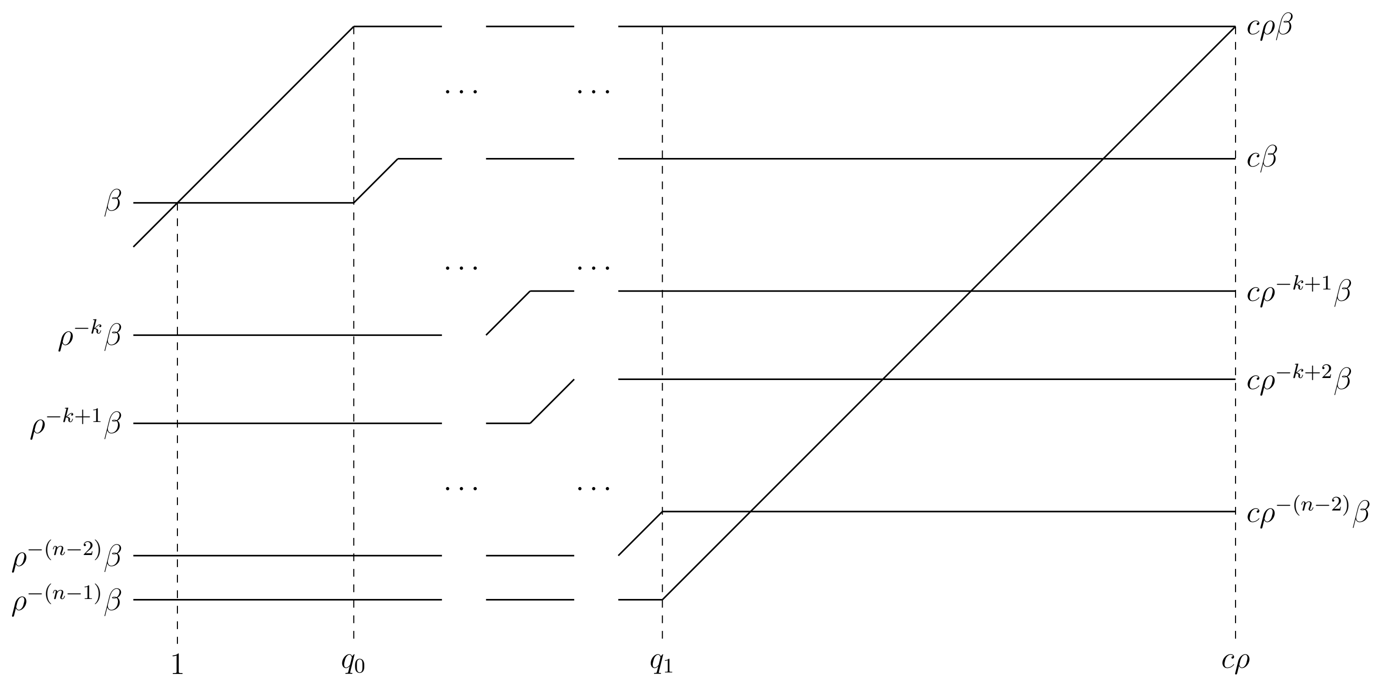

Figure 7: Pattern of the combined graph of $\boldsymbol{P}$ on the fundamental interval $[1, c\rho]$, where $\beta = \frac{\hat{\lambda}}{1+\hat{\lambda}}$.

where

$$q_1 = \frac{\hat{\lambda}(c(\rho^n + \cdots + \rho^2 + \rho) + 1)}{1 + \hat{\lambda}} = \frac{\hat{\lambda}(c\rho + 1/\hat{\lambda})}{1 + \hat{\lambda}}.$$

Hence,

$$\hat{\lambda}(\boldsymbol{\theta}) = \hat{\lambda} \quad \text{and} \quad \lambda(\boldsymbol{\theta}) = c\rho\hat{\lambda}.$$

and we constructed the required points since $c \geq 1$ and $\rho = G(n, \hat{\lambda})$.

Such self-similar Roy-systems provide infinitely many distinct points $\boldsymbol{\theta} \in \mathbb{R}^n$ via Roy's theorem with $\mathbb{Q}$-linearly independent coordinates with 1, as explained in [12, end of §3]. The $\mathbb{Q}$-linear independence comes from $P_1(q) \to \infty$ when $q \to \infty$. The construction of infinitely many points follows from a change of origin with the same pattern and self-similarity. □

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