Mathematisches
Forschungsinstitut
Oberwolfach

Mitglied der Leibniz Gemeinschaft

Oberwolfach Preprints

OWP 2018 - 16

Richard Cleyton, Andrei Moroianu and
Uwe Semmelmann

Metric Connections with Parallel Skew-Symmetric Torsion

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2018-16

METRIC CONNECTIONS WITH PARALLEL SKEW-SYMMETRIC TORSION

RICHARD CLEYTON, ANDREI MOROIANU, UWE SEMMELMANN

Abstract. A geometry with parallel skew-symmetric torsion is a Riemannian manifold carrying a metric connection with parallel skew-symmetric torsion. Besides the trivial case of the Levi-Civita connection, geometries with non-vanishing parallel skew-symmetric torsion arise naturally in several geometric contexts, e.g. on naturally reductive homogeneous spaces, nearly Kähler or nearly parallel G_2-manifolds, Sasakian and 3-Sasakian manifolds, or twistor spaces over quaternion-Kähler manifolds with positive scalar curvature. In this paper we study the local structure of Riemannian manifolds carrying a metric connection with parallel skew-symmetric torsion. On every such manifold one can define a natural splitting of the tangent bundle which gives rise to a Riemannian submersion over a geometry with parallel skew-symmetric torsion of smaller dimension endowed with some extra structure. We show how previously known examples of geometries with parallel skew-symmetric torsion fit into this pattern, and construct several new examples. In the particular case where the above Riemannian submersion has the structure of a principal bundle, we give the complete local classification of the corresponding geometries with parallel skew-symmetric torsion.

1. Introduction

1.1. **Motivation.** Metric connections with torsion on Riemannian manifolds have been studied recently in many articles. Such connections usually arise in special geometric situations and then come with further properties. Besides the (torsion free) Levi-Civita connection, which is, for obvious reasons, the central object in Riemannian geometry, the next most natural class to consider is the one of metric connections with totally skew-symmetric and parallel torsion. These connections have the same geodesics as the Levi-Civita connection. Moreover their curvature tensor is still pair symmetric and satisfies the second Bianchi identity. There are several important geometries admitting metric connections with parallel skew-symmetric torsion as we will now explain.

1.2. **Examples.** The first example are the naturally reductive spaces with their canonical homogeneous connection which turns out to have parallel skew-symmetric torsion and also parallel curvature (see [15]). Another important class of examples are Sasakian and 3-Sasakian

Date: June 30, 2018.

2010 *Mathematics Subject Classification.* 53B05, 53C25.

Key words and phrases. parallel skew-symmetric torsion, nearly Kähler structures, Sasakian structures, naturally reductive homogeneous spaces.

manifolds (see [9]). In even dimensions, nearly Kähler manifolds with their canonical Hermitian connection provide another class of examples (see [6]). Finally, in dimension 7, every manifold with a nearly parallel G_2-structure carries a canonical connection with parallel skew-symmetric torsion (see [9]).

1.3. **Previous results.** Although not directly related to our topic, let us first mention that the possible holonomy groups of arbitrary torsion-free connections (not necessarily metric) have been classified, under the irreducibility assumption, by Merkulov and Schwachhöfer [17].

The classification of metric connections with parallel torsion and *irreducible* holonomy representation was obtained by Cleyton and Swann in [7]. They show that a Riemannian manifold admitting a metric connection with non-trivial parallel torsion is locally isometric to a non-symmetric isotropy irreducible homogeneous space, or to one of the irreducible symmetric spaces $(G \times G)/G$ or G^{C}/G, or the manifold is nearly Kähler in dimension 6, or has a nearly parallel G_2-structure in dimension 7. The homogeneous spaces in the first case are naturally reductive if the torsion is assumed to be skew-symmetric. For the other three cases the torsion is automatically skew-symmetric.

The reducible case turns out to be much more involved, and it is the purpose of the present article to describe a classification scheme in the case of connections with parallel skew-symmetric torsion whose holonomy representation is reducible.

Further classification results only occur in in special geometric situations or in low dimensions: Alexandrov, Friedrich and Schoenemann [3] have shown that if the canonical Hermitian connection of a Hermitian manifold has parallel torsion and holonomy in $\mathrm{Sp}(n)\mathrm{U}(1)$ then the manifold is locally isomorphic to a twistor space over a quaternion-Kähler manifold of positive scalar curvature. Partial classifications of 6-dimensional almost Hermitian manifolds admitting a canonical Hermitian connection with parallel torsion are obtained by Alexandrov [4] and Schoenemann [20]. Similarly, cocalibrated G_2-manifolds with a characteristic connection of parallel torsion are studied by Friedrich [8]. Moreover, Agricola, Ferreira and Friedrich [1] obtained classification results in low dimensions, up to 6.

Finally, we would like to mention the recent work of Storm [21], [22] and that of Agricola and Dileo [2]. In his thesis, Storm describes a new construction method for naturally reductive spaces and gives classification results in dimensions 7 and 8. He also has a general result on metric connections with skew-symmetric and "reducible" parallel torsion (see Thm. 1.3.5 in [22]), similar to our Lemma 3.2 below. Agricola and Dileo introduce in [2] a new classes of almost 3-contact metric manifolds called $3-(\alpha,\delta)$-Sasakian manifolds (including as special cases 3-Sasakian manifolds and quaternionic Heisenberg groups). They show that these spaces admit a canonical metric connection with skew-symmetric and parallel torsion (see Thm. 4.4.1 in [2]).

Several of the examples mentioned above are total spaces of Riemannian submersions over manifolds without torsion, e.g. Sasakian manifolds locally fiber over Kähler manifolds, 3-Sasakian manifolds locally fiber over quaternion-Kähler manifolds and the twistor spaces are

S^2-fibrations over quaternion-Kähler manifolds. We will see that this is a general phenomenon which characterizes connections with parallel skew-symmetric torsion with reducible holonomy.

1.4. **Overview.** We now turn to the main results contained in this paper. Let ∇^τ be a metric connection with parallel skew-symmetric torsion on a Riemannian manifold (M, g^M). As already mentioned, the case where the holonomy representation of ∇^τ is irreducible has been dealt with in [7], so one can always assume that the holonomy representation is reducible. In contrast to the Riemannian (torsion free) situation, this by no means gives a reduction of the manifold as a Riemannian product (unless the geometry is decomposable, see Definition 3.1), since the de Rham theorem does not apply.

Our first achievement is to show that among all possible parallel distributions of the tangent bundle, there is a particular one which we denote with $\mathcal{V}M$ and which has the remarkable property that its leaves are totally geodesic, and define locally a Riemannian submersion $M \to N$, which we call the standard submersion. Even more strikingly, the restriction of the curvature of τ to the leaves is ∇^τ-parallel, so each leaf is a locally homogeneous space by the Ambrose-Singer theorem [5].

The next step is to show that the fibration $M \to N$ can be obtained as the quotient of a principal bundle over N carrying a connection with *parallel curvature* by a subgroup of its structure group. This is achieved as follows. One shows that the holonomy bundle Q of ∇^τ over M with group $K = \mathrm{Hol}(\nabla^\tau)$ can be viewed as a principal bundle over the base N of the standard submersion, with a larger structure group L containing K, and such that L/K is isomorphic to the fibers of the standard submersion. This fact can be interpreted as a Ambrose-Singer-like theorem for families, and reduces to the usual Ambrose-Singer theorem when the base N is a point.

The proof, explained in §4.2, is based on the following idea: the horizontal lift to Q of the parallel distribution $\mathcal{V}M$ defined above, together with the vertical distribution of Q, define a new vertical distribution of Q as principal bundle over the base N. Moreover, this bundle $Q \to N$ has a natural connection, whose connection form is just the sum of the initial connection form of $Q \to M$, and some component of the initial soldering form of $Q \to M$. Since the curvature of this connection is determined by some component of the torsion of ∇^τ, and by the curvature of ∇^τ, one can expect it to be parallel, after doing some further reduction. This is achieved in §4.3, where we show that Q reduces to a principal bundle P over N with parallel curvature, together with some further properties. We call it a *geometry with parallel curvature*. Conversely, this geometry with parallel curvature over N still contains enough information in order to recover the whole structure of M, as shown in §5.

In §6 we study an important particular case of geometries with parallel curvature, called parallel $\mathfrak{g}$-structures. This corresponds to the case where the standard submersion is a principal bundle, and already contains most of the examples of geometries with parallel skew-symmetric torsion available in the literature. In Theorem 7.1 we give the local classification of manifolds with parallel $\mathfrak{g}$-structures, which is thus a first step towards the classification of geometries with parallel skew-symmetric torsion.

Acknowledgment. This work was completed during a "Research in Pairs" stay at the Mathematisches Forschungsinstitut, Oberwolfach, Germany in 2018. We warmly thank the institute for the excellent research conditions offered. We would like thank Ilka Agricola for comments on our work and in particular for pointing out recent research related to our results.

2. Preliminaries

2.1. **Connections with parallel skew-symmetric torsion.** Let (M, g) be a Riemannian manifold. We will identify as usual vectors and 1-forms or skew-symmetric endomorphisms and 2-forms via the metric g.

In the sequel, if A is a skew-symmetric endomorphism of $\mathrm{T}M$, we will denote by $A\cdot$ the action of A on exterior forms as derivation, given by

$$A \cdot \omega := \sum_i Ae_i \wedge e_i \lrcorner\, \omega, \qquad \forall \omega \in \Lambda^* \mathrm{T}M \ , \tag{1}$$

in every local orthonormal basis $\{e_i\}$ of $\mathrm{T}M$. Note that if B is another skew-symmetric endomorphism, identified with a 2-form via the metric, then $A \cdot B$ is the 2-form corresponding to the commutator of A and B, whence:

$$A \cdot B = [A, B] = -B \cdot A \ . \tag{2}$$

Every 3-form τ on M can be identified with a tensor of type $(2, 1)$ by writing for every $x \in M$

$$\tau(X, Y, Z) = g(\tau_X Y, Z), \qquad \forall\ X, Y, Z \in \mathrm{T}_x M \ .$$

In this way τ_X can be viewed as a skew-symmetric endomorphism of $\mathrm{T}_x M$ for every tangent vector $X \in \mathrm{T}_x M$.

Definition 2.1. A *geometry with parallel skew-symmetric torsion* on M is a Riemannian metric g with Levi-Civita connection ∇^g and a 3-form $\tau \in \Omega^3(M)$ which is parallel with respect to the metric connection $\nabla^\tau := \nabla^g + \tau$, i.e. $\nabla^\tau \tau = 0$.

Of course, since τ is a 3-form, ∇^τ has skew-symmetric torsion $T^\tau = 2\tau$.

Writing $\nabla^g = \nabla^\tau - \tau$ and using the fact that τ is ∇^τ-parallel, we readily see that the curvature R^τ satisfies

$$R^g \ = \ R^\tau + \tau^2 \qquad \text{with} \qquad (\tau^2)_{X,Y} Z \ = \ [\tau_X, \tau_Y] Z \ - \ \tau_{\tau_X Y} Z \ + \ \tau_{\tau_Y X} Z \ . \tag{3}$$

Taking the scalar product with a vector W in this formula we obtain

Lemma 2.2. *Let $\nabla^\tau = \nabla^g + \tau$ be a connection with parallel skew-symmetric torsion τ. Then*

$$R^g(X, Y, Z, W) \ = \ R^\tau(X, Y, Z, W) \ - \ g(\tau_Y Z, \tau_X W) \ + \ g(\tau_X Z, \tau_Y W) \ - \ 2g(\tau_X Y, \tau_Z W)\cdot$$

In particular the curvature R^τ is pair symmetric: $R^\tau(X, Y, Z, W) = R^\tau(Z, W, X, Y)$.

2.2. **Examples.** As mentioned in the introduction, there are several known families of geometries with parallel skew-symmetric torsion. We review here the most important ones.

Example 2.3. A homogeneous space $M = G/K$ is called reductive if the Lie algebra $\mathfrak{g}$ of G admits an $\mathrm{Ad}\,(K)$-invariant splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ can be identified with the tangent space to M in the origin o. The canonical homogeneous connection on the K-principal bundle $G \to G/K$ is defined as the projection onto the Lie algebra $\mathfrak{k}$, i.e. its connection 1-form $\alpha \in \Omega^1(G, \mathfrak{k})$ is given by $\alpha(X) = X_{\mathfrak{k}}$ for any vector $X \in \mathfrak{g}$. The connection α induces the canonical homogeneous connection on the tangent bundle of M. Its torsion is given by $T(X,Y)_o = -[X,Y]_{\mathfrak{m}}$ for vectors $X, Y \in \mathfrak{m}$. A reductive homogeneous space M equipped with a G-invariant metric g corresponding to a $\mathrm{Ad}\,(K)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ is called *naturally reductive* if the torsion of the canonical homogeneous connection is skew-symmetric, i.e. if $\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle = 0$ holds for all vectors $X, Y, Z \in \mathfrak{m}$. It is well-known that the canonical homogeneous connection has parallel torsion (see [15], Ch. X, Thm. 2.6). In this situation not only the torsion but also the curvature is parallel. Conversely, the theorem of Ambrose and Singer [5] shows that if a metric connection on a Riemannian manifold has parallel skew-symmetric torsion and parallel curvature, then the manifold is locally homogeneous and naturally reductive. There are many examples of naturally reductive spaces, e.g. all homogeneous spaces G/K of a compact semi-simple Lie group G equipped with the metric induced by the Killing form of G.

Example 2.4. Nearly Kähler manifolds are almost Hermitian manifolds (M, g, J) where the almost complex structure J satisfies the condition $(\nabla_X J)X = 0$ for all tangent vectors X. The canonical Hermitian connection $\bar\nabla$ with $\bar\nabla J = 0 = \bar\nabla g$ is defined by $\bar\nabla_X Y = \nabla^g_X Y - \frac{1}{2} J(\nabla^g_X J)Y$. Hence the nearly Kähler condition directly implies that the canonical Hermitian connection has skew-symmetric torsion. The torsion is also $\bar\nabla$-parallel as was first shown in [13] (see [6] for a short proof). Important examples of nearly Kähler manifolds in any dimension $4k + 2$ are provided by the twistor spaces of quaternion-Kähler manifolds of positive scalar curvature. Another class of examples are the naturally reductive 3-symmetric spaces (see [11], Prop. 5.6). In dimension 6 one has the spaces $S^6, S^3 \times S^3, \mathbb{C}P^3$ and the flag manifold $F(1, 2)$. In [19] it is proved that any strict nearly Kähler manifold is locally isometric to a product with factors either 6-dimensional, or homogeneous of a certain type, or a twistor space of a quaternion-Kähler manifold of positive scalar curvature.

Example 2.5. A *Sasakian* structure on a Riemannian manifold (M, g) is given by a unit length Killing vector field ξ satisfying the condition $\nabla^g_X d\xi = -2X \wedge \xi$ for all tangent vectors X. In this situation $\bar\nabla = \nabla^g + \frac{1}{2}\xi \wedge d\xi$ defines a metric connection with skew-symmetric torsion preserving the Sasakian structure. It is easy to show that its torsion $T = \xi \wedge d\xi$ is $\bar\nabla$-parallel (see [9]). There are many examples of Sasakian structures, e.g. on S^1-fibre bundles over compact Kähler manifolds with integer fundamental class (so-called Hodge manifolds).

In the case of 3-*Sasakian* manifolds one has 3 unit length Killing vector fields satisfying the $\mathfrak{so}(3)$-commutator relations and such that each vector field defines a Sasakian structure. Examples for 3-Sasakian manifolds are given e.g. as the total space of certain SO(3)-bundles over quaternion-Kähler manifolds of positive scalar curvature.

Example 2.6. A G_2-structure on a 7-dimensional manifold is defined by a stable 3-form ω, i.e. for any $x \in M$ the form ω_x lies in an open orbit of the $\mathrm{GL}(T_xM)$-action on the space of 3-forms $\Lambda^3 T_xM$. Then ω defines a structure group reduction to $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and in particular it induces a Riemannian metric g on M. *Nearly parallel* G_2*-manifolds* are defined by the condition that the Hodge dual $*\omega$ is proportional to $d\omega$. Then there is a metric connection $\bar\nabla$ preserving the G_2-structure. Its torsion is skew-symmetric and proportional to ω. It is shown in [9] that the torsion is $\bar\nabla$-parallel. There are several examples of homogeneous nearly parallel G_2-manifolds, e.g. $\mathrm{SO}(5)/\mathrm{SO}(3)$, where the embedding of $\mathrm{SO}(3)$ into $\mathrm{SO}(5)$ is given by the 5-dimensional irreducible representation of $\mathrm{SO}(3)$ or the Aloff-Wallach spaces $\mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$. Moreover one can show that on any 7-dimensional 3-Sasakian manifold there exists a second Einstein metric defined by a nearly parallel G_2-structure (see [10]).

3. The standard decomposition

In contrast to the Riemannian case, there are two different notions of reducibility for geometries with parallel skew-symmetric torsion, which we will explain now.

Definition 3.1. A geometry with parallel skew-symmetric torsion (M, g, τ) is called:

- *reducible* if the holonomy representation of ∇^τ is reducible, and *irreducible* otherwise.
- *decomposable* if the tangent bundle of M decomposes in a (non-trivial) orthogonal direct sum of ∇^τ-parallel distributions $\mathrm{T}M = T_1 \oplus T_2$ such that the torsion form satisfies $\tau = \tau_1 + \tau_2 \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$, and *indecomposable* otherwise.

Of course, irreducibility implies indecomposability, but as we will see, there are many examples of indecomposable but reducible geometries with parallel skew-symmetric torsion.

If (M_1, g_1, τ_1) and (M_2, g_2, τ_2) are geometries with parallel skew-symmetric torsion, then their Riemannian product $(M_1 \times M_2, g_1 + g_2, \tau_1 + \tau_2)$ is again a decomposable geometry with parallel skew-symmetric torsion. Conversely, the next result shows that a decomposable geometry with parallel skew-symmetric torsion is always locally a Riemannian product:

Lemma 3.2. *Assume that (M, g, τ) is decomposable, with $\mathrm{T}M = T_1 \oplus T_2$ and $\tau = \tau_1 + \tau_2 \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$. Then (M, g, τ) is locally isometric to a product of two manifolds with parallel skew-symmetric torsion (M_i, g_i, τ_i).*

Proof. For every vector fields $X \in \Gamma(T_1)$ and $Y \in \Gamma(\mathrm{T}M)$ the assumptions in Definition 3.1 yield $\nabla^\tau_Y X \in \Gamma(T_1)$ and $\tau(X, Y) \in \Gamma(T_1)$, whence

$$\nabla^g_Y X = \nabla^\tau_Y X - \tau(Y, X) \in \Gamma(T_1),$$

thus showing that T_1 is ∇^g-parallel. Similarly, T_2 is ∇^g-parallel. By the local de Rham theorem, (M, g) is locally isometric to a Riemannian product $(M_1, g_1) \times (M_2, g_2)$.

Moreover, for every $X \in \Gamma(T_1)$ we have

$$\nabla^g_X \tau_2 = \nabla^\tau_X \tau_2 - \tau_X \cdot \tau_2 = -\tau_X \cdot \tau_2 = -(\tau_1)_X \cdot \tau_2 = 0,$$

and similarly $\nabla^g_Z \tau^1 = 0$ for every vector field $Z \in \Gamma(T_2)$. This shows that τ_1 and τ_2 are projectable on M_1 and M_2 respectively. It is now clear that τ_i is parallel with respect to $\nabla^{\tau_i} := \nabla^{g_i} + \tau_i$, so (M_i, g_i) are Riemannian manifolds with metric connection ∇^{τ_i} with parallel skew-symmetric torsion for $i = 1, 2$.

□

Now write $\mathfrak{k}$ for the holonomy algebra of ∇^τ and n for the dimension of M. The representation of $\mathfrak{k}$ on $\mathbb{R}^n$ decomposes into an orthogonal sum of irreducible $\mathfrak{k}$-modules $\mathfrak{h}_\alpha$ and $\mathfrak{v}_j$ such that each summand $\mathfrak{h}_\alpha$ satisfies $\mathfrak{k}_\alpha := \mathfrak{so}(\mathfrak{h}_\alpha) \cap \mathfrak{k} \neq 0$ and each summand $\mathfrak{v}_j$ satisfies $\mathfrak{so}(\mathfrak{v}_j) \cap \mathfrak{k} = 0$. We define $\mathfrak{h} := \oplus_\alpha \mathfrak{h}_\alpha$ and $\mathfrak{v} := \oplus_j \mathfrak{v}_j$. Note that $\mathfrak{k}_\alpha$ is an ideal of $\mathfrak{k}$ for every α.

Lemma 3.3. *For every index α, the representation of $\mathfrak{k}$ on $\mathfrak{h}_\alpha \otimes \Lambda^2 \mathfrak{h}_\alpha^\perp$ has no trivial subspace.*

Proof. Consider the space

$$E_\alpha := \{v \in \mathfrak{h}_\alpha \mid Av = 0,\ \forall A \in \mathfrak{k}_\alpha\}.$$

Since $\mathfrak{k}_\alpha$ is an ideal of $\mathfrak{k}$, for every $v \in E_\alpha$, $A \in \mathfrak{k}_\alpha$ and $B \in \mathfrak{k}$ we get:

$$A(Bv) = [A, B]v + B(Av) = 0,$$

thus showing that E_α is a $\mathfrak{k}$-invariant subspace of $\mathfrak{h}_\alpha$, so by the irreducibility of $\mathfrak{h}_\alpha$ we deduce that either $E_\alpha = 0$ or $E_\alpha = \mathfrak{h}_\alpha$. The latter case is impossible by the very definition of $\mathfrak{k}_\alpha$. Thus $E_\alpha = 0$.

Suppose that $u \in \mathfrak{h}_\alpha \otimes \Lambda^2 \mathfrak{h}_\alpha^\perp$ satisfies $Bu = 0$ for every $B \in \mathfrak{k}$. We write $u = \sum_i v_i \otimes w_i$ where w_i is a basis of $\Lambda^2 \mathfrak{h}_\alpha^\perp$. By definition, every $A \in \mathfrak{k}_\alpha$ acts trivially on $\mathfrak{h}_\alpha^\perp$. We thus get $0 = Au = \sum_i (Av_i) \otimes w_i$ for every $A \in \mathfrak{k}_\alpha$, whence

$$Av_i = 0, \qquad \forall i,\ \forall A \in \mathfrak{k}_\alpha \ . \tag{4}$$

Thus $v_i \in E_\alpha = 0$ for all i, so finally $u = 0$.

□

Decomposing $\Lambda^3 \mathbb{R}^n$ according to the decomposition $\mathbb{R}^n = (\oplus_\alpha \mathfrak{h}_\alpha) \oplus (\oplus_j \mathfrak{v}_j) = \mathfrak{h} \oplus \mathfrak{v}$ and using the above result, we readily obtain that every $\mathfrak{k}$-invariant element of $\Lambda^3 \mathbb{R}^n$ is contained in the following sub-space:

$$(\Lambda^3 \mathbb{R}^n)^{\mathfrak{k}} \subset (\oplus_\alpha \Lambda^3 \mathfrak{h}_\alpha) \oplus (\oplus_\alpha \Lambda^2 \mathfrak{h}_\alpha \otimes \mathfrak{v}) \oplus \Lambda^3 \mathfrak{v} \subset \Lambda^3 \mathfrak{h} \oplus (\Lambda^2 \mathfrak{h} \otimes \mathfrak{v}) \oplus \Lambda^3 \mathfrak{v}. \tag{5}$$

Corresponding to the decomposition $\mathbb{R}^n = (\oplus_\alpha \mathfrak{h}_\alpha) \oplus (\oplus_j \mathfrak{v}_j) = \mathfrak{h} \oplus \mathfrak{v}$ we obtain an orthogonal ∇^τ-parallel decomposition of the tangent bundle of M as

$$\mathrm{T}M = (\oplus_\alpha \mathcal{H}_\alpha M) \oplus (\oplus_j \mathcal{V}_j M) = \mathcal{H}M \oplus \mathcal{V}M \ . \tag{6}$$

Definition 3.4. The above defined decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ will be referred to as the *standard decomposition* of the tangent bundle of a manifold with parallel skew-symmetric torsion.

We now decompose the torsion as a sum of ∇^τ-parallel tensors

$$\tau = \tau^{\mathfrak{h}} + \tau^m + \tau^{\mathfrak{v}} ,$$

where $\tau^{\mathfrak{h}} \in \Lambda^3\mathcal{H}M$, $\tau^{\mathfrak{v}} \in \Lambda^3\mathcal{V}M$ and $\tau^m \in (\Lambda^2\mathcal{H}M \otimes \mathcal{V}M) \oplus (\Lambda^2\mathcal{V}M \otimes \mathcal{H}M)$. By (5) we obtain directly:

Lemma 3.5. *The projection of τ onto $\Lambda^2\mathcal{V}M \otimes \mathcal{H}M$ vanishes, i.e. $\tau^m \in \Lambda^2\mathcal{H}M \otimes \mathcal{V}M$. Moreover, $\tau^{\mathfrak{h}}$ and τ^m have further decompositions*

$$\tau^{\mathfrak{h}} = \sum_\alpha \tau^{\mathfrak{h}_\alpha} \in \bigoplus_\alpha \Lambda^3\mathcal{H}_\alpha M, \qquad \tau^m \in \left(\bigoplus_\alpha \Lambda^2\mathcal{H}_\alpha M\right) \otimes \mathcal{V}M.$$

This has an immediate consequence which we will discuss now. We will assume for the rest of this section that (M, g, τ) is a geometry with parallel skew-symmetric torsion and standard decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$.

Lemma 3.6. *The distribution $\mathcal{V}M$ is the vertical distribution of a locally defined Riemannian submersion $(M, g) \xrightarrow{\pi} (N, g^N)$ with totally geodesic fibers.*

Proof. Lemma 3.5 shows that for U, V in $\mathcal{V}M$ we have $\nabla^g_U V = \nabla^\tau_U V - \tau_U V \in \mathcal{V}M$. Thus $\mathcal{V}M$ is a totally geodesic involutive distribution. We need to show that the restriction $g^{\mathfrak{h}}$ of the metric g to $\mathcal{H}M$ is constant along the leaves of $\mathcal{V}M$. The Lie derivative

$$(\mathcal{L}_U g^{\mathfrak{h}})(A, B) = U(g^{\mathfrak{h}}(A, B)) \;-\; g^{\mathfrak{h}}([U, A], B) \;-\; g^{\mathfrak{h}}(A, [U, B])$$

clearly vanishes if A or B is a section of $\mathcal{V}M$. For X, Y sections of $\mathcal{H}M$ we have:

$$\begin{aligned}
(\mathcal{L}_U g^{\mathfrak{h}})(X, Y) &= U(g^{\mathfrak{h}}(X, Y)) \;-\; g^{\mathfrak{h}}([U, X], Y) \;-\; g^{\mathfrak{h}}(X, [U, Y]) \\
&= g(\nabla^g_X U, Y) + g(X, \nabla^g_Y U) \\
&= g(\nabla^\tau_X U, Y) \;+\; g(X, \nabla^\tau_Y U) = 0 \;.
\end{aligned}$$

This shows that if N denotes the space of leaves of $\mathcal{V}M$ in some neighbourhood of M, the restriction of g to the distribution $\mathcal{H}M$ projects to a Riemannian metric g^N on N.

□

Definition 3.7. The Riemannian submersion $(M, g) \to (N, g^N)$ defined in Lemma 3.6 will be called the *standard submersion* of a manifold with parallel skew-symmetric torsion.

The next result is the crucial step for showing that the horizontal part of the torsion is a projectable tensor with respect to the standard submersion:

Lemma 3.8. *For every vector field V in $\mathcal{V}M$ one has $\tau_V \cdot \tau^{\mathfrak{h}} = 0$.*

Proof. Employing a local orthonormal frame $\{e_i\}$ of $\mathrm{T}M$ we write:

$$\mathrm{d}V = \sum_i e_i \wedge \nabla^g_{e_i} V = \sum_i e_i \wedge (\nabla^\tau_{e_i} V - \tau_{e_i} V) = \sum_i e_i \wedge \nabla^\tau_{e_i} V + 2\tau_V,$$

whence

$$
\begin{aligned}
0 &= \mathrm{d}^2 V = \sum_j e_j \wedge \nabla^g_{e_j} \mathrm{d}V = \sum_j e_j \wedge (\nabla^\tau_{e_j} \mathrm{d}V - \tau_{e_j} \cdot \mathrm{d}V) \\
&= \sum_{i,j} e_j \wedge (\nabla^\tau_{e_j}(e_i \wedge \nabla^\tau_{e_i} V)) + 2\sum_j e_j \wedge (\nabla^\tau_{e_j} \tau_V) - \sum_{i,j} e_j \wedge (\tau_{e_j} \cdot (e_i \wedge \nabla^\tau_{e_i} V)) \\
&\quad -2\sum_i e_i \wedge \tau_{e_i} \cdot \tau_V \qquad (7)\\
&= \sum_{i,j} e_j \wedge (\nabla^\tau_{e_j}(e_i \wedge \nabla^\tau_{e_i} V)) + 2\sum_j e_j \wedge \tau_{\nabla^\tau_{e_j} V} - \sum_{i,j} e_j \wedge \tau_{e_j} e_i \wedge \nabla^\tau_{e_i} V \\
&\quad -\sum_{i,j} e_j \wedge e_i \wedge \tau_{e_j} \nabla^\tau_{e_i} V - 2\sum_i e_i \wedge \tau_{e_i} \cdot \tau_V
\end{aligned}
$$

On the other hand, using (2) and the fact that τ_V is a derivation of $\Lambda^* \mathrm{T}M$, the last summand in (7) reads

$$
\begin{aligned}
\sum_i e_i \wedge \tau_{e_i} \cdot \tau_V &= -\sum_i e_i \wedge \tau_V \cdot \tau_{e_i} = -\sum_i (\tau_V \cdot (e_i \wedge \tau_{e_i}) - \tau_V e_i \wedge \tau_{e_i}) \qquad (8)\\
&= -3\tau_V \cdot \tau + \tau_V \cdot \tau = -2\tau_V \cdot \tau.
\end{aligned}
$$

Moreover, using the skew-symmetry of τ we obtain

$$\sum_{i,j} e_j \wedge e_i \wedge \tau_{e_j} \nabla^\tau_{e_i} V = \sum_{i,j} e_i \wedge e_j \wedge \tau_{\nabla^\tau_{e_i} V} e_j = 2\sum_i e_i \wedge \tau_{\nabla^\tau_{e_i} V},$$

so the second and fourth terms in (7) cancel each other. From (7) and (8) we thus get

$$\sum_{i,j} e_j \wedge (\nabla^\tau_{e_j}(e_i \wedge \nabla^\tau_{e_i} V)) - \sum_{i,j} e_j \wedge \tau_{e_j} e_i \wedge \nabla^\tau_{e_i} V + 4\tau_V \cdot \tau = 0.$$

Since ∇^τ preserves $\mathcal{V}M$, the projection onto $\Lambda^3\mathcal{H}M$ of the first two terms in this last equation vanishes. Moreover, by Lemma 3.5 we have $\tau_V \in \Lambda^2\mathcal{V}M \oplus \Lambda^2\mathcal{H}M$, so the action of τ_V as endomorphism on $\mathrm{T}M$ preserves the splitting $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$. We thus obtain

$$0 = \pi_{\Lambda^3\mathcal{H}M}(\tau_V \cdot \tau) = \tau_V \cdot \pi_{\Lambda^3\mathcal{H}M}(\tau) = \tau_V \cdot \tau^{\mathfrak{h}}.$$

□

Using Lemma 3.8 we can now prove the announced result:

Lemma 3.9. *The horizontal part $\tau^{\mathfrak{h}}$ of the torsion τ is projectable to the base N of the standard submersion.*

Proof. It suffices to show that $\mathcal{L}_V \tau^{\mathfrak{h}} = 0$ for all vector fields V in $\mathcal{V}M$. The torsion τ is ∇^τ-parallel, and so are its components $\tau^{\mathfrak{h}}$, τ^m and $\tau^{\mathfrak{v}}$. In particular, for every $X \in \mathrm{T}M$ we have $0 = \nabla^\tau_X \tau^\mathfrak{h} = \nabla^g_X \tau^\mathfrak{h} + \tau_X \cdot \tau^\mathfrak{h}$. Moreover, for $V \in \mathcal{V}M$ we have $V \lrcorner \tau^\mathfrak{h} = 0$, so we can compute in a local orthonormal basis $\{e_i\}$ of $\mathrm{T}M$ using Lemma 3.8:

$$\begin{aligned}
\mathcal{L}_V \tau^\mathfrak{h} &= V \lrcorner \mathrm{d}\tau^\mathfrak{h} = \sum_i V \lrcorner (e_i \wedge \nabla^g_{e_i} \tau^\mathfrak{h}) = -\sum_i V \lrcorner (e_i \wedge \tau_{e_i} \cdot \tau^\mathfrak{h}) \\
&= -\tau_V \cdot \tau^\mathfrak{h} + \sum_i e_i \wedge (V \lrcorner (\tau_{e_i} \cdot \tau^\mathfrak{h})) = \sum_i e_i \wedge (\tau_{e_i} \cdot (V \lrcorner \tau^h) - \tau_{e_i} V \lrcorner \tau^\mathfrak{h}) \\
&= \sum_i e_i \wedge (\tau_V e_i \lrcorner \tau^\mathfrak{h}) = -\sum_i \tau_V e_i \wedge (e_i \lrcorner \tau^\mathfrak{h}) = -\tau_V \cdot \tau^\mathfrak{h} = 0.
\end{aligned}$$

□

Remark 3.10. By Lemma 3.9 there is a 3-form σ on N such that $\tau^\mathfrak{h} = \pi^*\sigma$. If ∇^{g^N} denotes the Levi-Civita connection of the metric g^N, Lemma 3.6 shows that $\nabla^\sigma := \nabla^{g^N} + \sigma$ is a connection with parallel skew-symmetric torsion $T^\sigma = 2\sigma$ on N. Indeed, the fact that $\nabla^\sigma \sigma = 0$ follows immediately from the O'Neill formulas for Riemannian submersions, together with the fact that for every vector X in $\mathcal{H}M$ we have

$$0 = \nabla^\tau_X \tau^\mathfrak{h} = \nabla^g_X \tau^\mathfrak{h} + \tau_X \cdot \tau^\mathfrak{h} = \nabla^g_X \tau^\mathfrak{h} + \tau^\mathfrak{h}_X \cdot \tau^\mathfrak{h}.$$

We will now make another crucial observation, which will give more information about the fibers of the standard submersion. Since by Lemma 3.6 every fiber F of the standard submersion is totally geodesic, the Levi-Civita connection of M restricts to the Levi-Civita connection of F, and the connection ∇^τ restricts to a connection ∇^F with parallel, skew-symmetric torsion $T^F := 2\tau^\mathfrak{v}|_F$.

Proposition 3.11. *The composition of the curvature tensor $R^\tau : \Lambda^2 \mathrm{T}M \to \Lambda^2\mathcal{V}M \oplus \Lambda^2\mathcal{H}M$ with the projection on the first summand $\Lambda^2\mathcal{V}M$ is ∇^τ-parallel. In particular, ∇^F has parallel curvature and parallel skew-symmetric torsion, so F is (locally) a naturally reductive homogeneous space.*

Proof. In order to keep the notation simple, we will identify here $\mathfrak{k}$-representations with the associated bundles over M, and notice that every $\mathfrak{k}$-invariant object corresponds to a ∇^τ-parallel section on M. Since $\mathfrak{so}(\mathfrak{v}_j) \cap \mathfrak{k}$ is trivial for each irreducible summand $\mathfrak{v}_j$ of $\mathfrak{v}$, [7, Thm. 4.4] shows by immediate induction on the number of components $\mathfrak{v}_j$ that the space $\mathcal{K}(\mathfrak{k})$ of algebraic curvature tensors with values in $\mathfrak{k}$ satisfies

$$\mathcal{K}(\mathfrak{k}) \subset \mathrm{Sym}^2(\mathfrak{k}) \cap \mathrm{Sym}^2(\mathfrak{so}(\mathfrak{h})) \subset \mathrm{Sym}^2(\mathfrak{so}(\mathfrak{h})). \tag{9}$$

Decompose $\mathrm{Sym}^2(\mathfrak{k})$ orthogonally into $\mathcal{K}(\mathfrak{k}) \oplus \mathcal{K}(\mathfrak{k})^\perp$ and write $R^\tau = R^\tau_0 + R^\tau_1$ for the corresponding decomposition of the curvature tensor of ∇^τ. The Bianchi map $\mathfrak{b} : \mathrm{Sym}^2(\mathfrak{k}) \to \Lambda^4(\mathfrak{h}\oplus\mathfrak{v})$ is of course $\mathfrak{k}$-invariant. Since its kernel is $\mathcal{K}(\mathfrak{k})$, there exists a $\mathfrak{k}$-invariant isomorphism called $\mathfrak{b}^{-1}$ from $\mathfrak{b}(\mathrm{Sym}^2(\mathfrak{k}))$ to $\mathcal{K}(\mathfrak{k})^\perp$, and (3) shows that $R^\tau_1 = -\mathfrak{b}^{-1}\mathfrak{b}(\tau^2)$. Consequently, $\nabla^\tau R^\tau_1 = 0$ since τ ist ∇^τ-parallel.

Consider now the projection $\pi_{\mathfrak{so}(\mathfrak{v})} : \mathfrak{so}(\mathfrak{h}\oplus\mathfrak{v}) \to \mathfrak{so}(\mathfrak{v})$. By (9) we have $\pi_{\mathfrak{so}(\mathfrak{v})} \circ R^\tau = \pi_{\mathfrak{so}(\mathfrak{v})} \circ R^\tau_1$ and therefore $\nabla^\tau(\pi_{\mathfrak{so}(\mathfrak{v})} \circ R^\tau) = 0$. In particular, restricting this equation to F shows that

∇^F is a metric connection for which both the (skew-symmetric) torsion T^F and curvature R^F are parallel, so by the Ambrose-Singer theorem [5] F is (locally) a naturally reductive homogeneous space.

□

Remark 3.12. If one of the summands in the standard decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ is trivial, then either $\mathcal{H}M = 0$ and (M, g) is locally a naturally reductive homogeneous space by Proposition 3.11, or $\mathcal{V}M = 0$, in which case Lemma 3.2 and Lemma 3.5 show that (M, g) is locally a product of irreducible geometries with torsion. By [7], each factor is either naturally reductive homogeneous, or has a nearly Kähler structure in dimension 6, or a nearly parallel G_2-structure in dimension 7.

We will thus implicitly assume from now on that the standard decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ is non-trivial.

4. Geometries with parallel curvature

We have seen that the base space N of the standard submersion (Definition 3.7) $M \to N$ of a manifold M with parallel skew-symmetric torsion carries again a geometry with parallel skew-symmetric torsion. In this section we will show that this geometry carries additional structure.

4.1. **Connections on principal bundles.** For the convenience of the reader we collect here some notation and well known formulas about principal bundles and connections. The reader familiar with this topic can skip to §4.2.

Let K be a Lie group, M a manifold of dimension n and $\pi : Q \to M$ a K-principal fibre bundle over M. We consider a connection 1-form $\alpha \in \Omega^1(Q, \mathfrak{k})$ with curvature form $\Omega^\alpha \in \Omega^2(Q, \mathfrak{k})$, defined as the horizontal part of $\mathrm{d}\alpha$. The curvature form is given by

$$\Omega^\alpha \;=\; \mathrm{d}\alpha \;+\; \tfrac{1}{2}\,\alpha \wedge \alpha \;, \tag{10}$$

where $(\alpha \wedge \alpha)(U, V) := 2[\alpha(U), \alpha(V)]$ for $U, V \in \mathrm{T}Q$.

We assume from now on that g is a Riemannian metric on M and that Q is a sub-bundle of the orthonormal frame bundle of M (in particular K is a subgroup of the orthogonal group $\mathrm{O}(n)$). Then points $u \in Q$ can be considered as linear isomorphisms $u : \mathbb{R}^n \to \mathrm{T}_{\pi(u)}M$. The canonical (or soldering) 1-form $\theta \in \Omega^1(Q, \mathbb{R}^n)$ is defined by $\theta_u(U) = u^{-1}(\pi_* U)$. Similar to the curvature, the torsion form $\Theta \in \Omega^2(Q, \mathbb{R}^n)$ satisfies the equation

$$\Theta = \mathrm{d}\theta + \alpha \wedge \theta \;, \tag{11}$$

where $(\alpha \wedge \theta)(U, V) := \alpha(U)(\theta(V)) - \alpha(V)(\theta(U))$. Equations (10) and (11) are called *structure equations.*

We define the horizontal and vertical distributions on Q by $\mathrm{T}^{hor}_u Q := \ker(\alpha_u)$ and $\mathrm{T}^{\mathfrak{k}}_u Q := \ker(\pi_*)$, so that $\mathrm{T}Q = \mathrm{T}^{hor}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$. A vector field X on M induces a unique vector field

$\tilde{X}$ on Q tangent to $\mathrm{T}^{hor}Q$ such that X and $\tilde{X}$ are π-related. This vector field is called the horizontal lift of X.

To every vector field X on M one can also associate a $\mathbb{R}^n$-valued function $\hat{X}$ on Q defined by $\hat{X}(u) := u^{-1}X_{\pi(u)}$. The connection α induces a covariant derivative ∇^α on $\mathrm{T}M$ which satisfies

$$(\nabla^\alpha_X Y)_{\pi(u)} = u(\tilde{X}(\hat{Y})) \tag{12}$$

for every vector fields X, Y on M and $u \in Q$ (see [14], Lemma on p. 115).

Comparing ∇^α to the Levi-Civita connection ∇^g of g, we can write $\nabla^\alpha_X Y = \nabla^g_X Y + \tau_X Y$ on every vector fields X, Y on M for some $(2,1)$ tensor τ. Since ∇^α is actually determined by the tensor τ, we will from now on denote it by ∇^τ.

The curvature and torsion of ∇^τ defined by

$$R^\tau_{X,Y}Z := \nabla^\tau_X\nabla^\tau_Y Z - \nabla^\tau_Y\nabla^\tau_X Z - \nabla^\tau_{[X,Y]}Z, \qquad T^\tau_X Y := \nabla^\tau_X Y - \nabla^\tau_Y X - [X,Y].$$

for every vector fields X, Y, Z on M, are related to the curvature and torsion forms of α by the standard formulas (see [14], Prop. 5.2 Ch. III):

$$\Omega^\alpha_u(\tilde{X},\tilde{Y})(u^{-1}Z) \;=\; u^{-1}(R^\tau_{X,Y}Z), \qquad \Theta_u(\tilde{X},\tilde{Y}) \;=\; u^{-1}(T^\tau(X,Y)), \qquad \forall u \in Q\ . \tag{13}$$

More generally, if V is a representation space of K and VM is the associated vector bundle, every element $u \in Q$ defines tautologically a linear isomorphism between V and $VM_{\pi(u)}$ and every section σ of VM determines a V-valued function $\hat{\sigma}$ on Q defined by $\hat{\sigma}(u) := u^{-1}\sigma_{\pi(u)}$. The covariant derivative on VM induced by α satisfies

$$(\nabla^\alpha_X\sigma)_{\pi(u)} = u(\tilde{X}(\hat{\sigma})), \qquad \forall u \in Q,\ \forall X \in\ \mathrm{T}_{\pi(u)}M\ , \tag{14}$$

and the curvature tensor of ∇^α is related to the curvature form Ω^α by the classical formula

$$\Omega^\alpha_u(\tilde{X},\tilde{Y}) \;=\; u^{-1}R^\alpha_{X,Y}, \qquad \forall u \in Q,\ \forall X, Y \in\ \mathrm{T}_{\pi(u)}M\ . \tag{15}$$

Any Lie algebra element $A \in \mathfrak{k}$ induces a vertical vector field A^* on Q defined at $u \in Q$ by $A^*_u := \frac{d}{dt}\big|_{t=0}(u \cdot \exp(tA))$. By definition of the connection we have $\alpha(A^*) = A$. The vector field A^* is also called *fundamental vector field*. As Q is a subbundle of the frame bundle of M, then any $\xi \in \mathbb{R}^n$ induces a horizontal vector field ξ^* defined at $u \in Q$ by $\xi^*_u := \widetilde{u\xi}$. Note that $\theta(\xi^*) = \xi$. The vector field ξ^* is called *standard horizontal vector field*. It is easy to check that for $A, B \in \mathfrak{k}$ and $\xi \in \mathbb{R}^n$ we have

$$[A^*, B^*] = [A,B]^*, \qquad [A^*, \xi^*] = (A\,\xi)^*\ . \tag{16}$$

Note that for every $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}^n$ and $u \in Q$, formulas (13) read

$$\Omega^\alpha_u(\zeta^*_1,\zeta^*_2)(\zeta_3) \;=\; u^{-1}(R^\tau_{u\zeta_1,u\zeta_2}u\zeta_3), \qquad \Theta_u(\zeta^*_1,\zeta^*_2) \;=\; u^{-1}(T^\tau(u\zeta_1,u\zeta_2))\ , \tag{17}$$

or equivalently

$$\Omega^\alpha(\zeta^*_1,\zeta^*_2)(\zeta_3) \;=\; \hat{R^\tau}_{\zeta_1,\zeta_2}\zeta_3, \qquad \Theta(\zeta^*_1,\zeta^*_2) \;=\; \hat{T^\tau}(\zeta_1,\zeta_2)\ . \tag{18}$$

Lemma 4.1. *For every $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}^n$ and $u \in Q$, the covariant derivative of the torsion and curvature tensors of ∇^τ satisfy*

$$(\nabla^\tau_{u\zeta_1} T^\tau)(u\zeta_2, u\zeta_3) \;=\; u(\zeta_1^*(\Theta(\zeta_2^*, \zeta_3^*))) \tag{19}$$

$$(\nabla^\tau_{u\zeta_1} R^\tau)_{u\zeta_2, u\zeta_3} u\zeta_4 \;=\; u(\zeta_1^*(\Omega^\alpha(\zeta_2^*, \zeta_3^*)(\zeta_4))) \;. \tag{20}$$

Proof. Applying (14) to $\sigma = T^\tau$ and $\sigma = R^\tau$ and using (17) we readily obtain

$$(\nabla^\tau_{u\zeta_1} T^\tau)(u\zeta_2, u\zeta_3) = u(\zeta_1^*(\hat{T}^\tau))(u\zeta_2, u\zeta_3) = u(\zeta_1^*(\hat{T}^\tau)(\zeta_2, \zeta_3))) = u(\zeta_1^*(\Theta(\zeta_2^*, \zeta_3^*))) \;,$$

and

$$(\nabla^\tau_{u\zeta_1} R^\tau)_{u\zeta_2, u\zeta_3} u\zeta_4 = u(\zeta_1^*(\hat{R}^\tau))_{u\zeta_2, u\zeta_3} u\zeta_4 = u(\zeta_1^*(\hat{R}^\tau)_{\zeta_2, \zeta_3}\zeta_4)) = u(\zeta_1^*(\Omega^\alpha(\zeta_2^*, \zeta_3^*)(\zeta_4))) \;.$$

□

4.2. **The geometry of the standard submersion.** Let us now return to a geometry with parallel skew-symmetric torsion (M, g^M, τ). We fix some orthonormal frame u on M and denote with K the holonomy group of ∇^τ at u, with $\mathfrak{k}$ its Lie algebra, and with $\pi_M : Q \to M$ the reduction of the frame bundle of M to a principal K-fibre bundle. From Remark 3.12 we can assume that the representation of K on $\mathbb{R}^n$ is reducible. We denote like before by $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ the standard decomposition of the tangent bundle of M (which is a ∇^τ-parallel and orthogonal splitting) and denote correspondingly by $\mathbb{R}^n = \mathfrak{h} \oplus \mathfrak{v}$ the $\mathfrak{k}$-invariant decomposition of $\mathbb{R}^n$.

Our first aim is to define a Lie algebra structure on $\mathfrak{l} := \mathfrak{k} \oplus \mathfrak{v}$ induced from the Lie algebra structure on the space of vector fields on Q by the injective map $\Phi : \mathfrak{l} = \mathfrak{k} \oplus \mathfrak{v} \to \Gamma(\mathrm{T}Q)$, $A + \xi \mapsto A^* + \xi^*$, for $A \in \mathfrak{k}$ and ξ in $\mathfrak{v}$.

Lemma 4.2. *The image of the map Φ is closed under the bracket of vector fields.*

Proof. Since $[A^*, B^*] = [A, B]^*$ for $A, B \in \mathfrak{k}$ and $[A^*, \xi^*] = (A\,\xi)^*$ for $A \in \mathfrak{k}$ and $\xi \in \mathfrak{v}$, it only remains to consider the bracket of the fundamental vertical vector fields induced by $\xi_1, \xi_2 \in \mathfrak{v}$. Denoting by $\alpha \in \Omega^1(Q, \mathfrak{k})$ the connection 1-form induced by ∇^τ, the structure equations (10) and (11) imply

$$\theta_u([\xi_1^*, \xi_2^*]) \;=\; -\,\Theta_u(\xi_1^*, \xi_2^*) \qquad \text{and} \qquad \alpha_u([\xi_1^*, \xi_2^*]) \;=\; -\,\Omega^\alpha_u(\xi_1^*, \xi_2^*), \qquad \forall\, \xi_1, \xi_2 \in \mathfrak{v} \;. \tag{21}$$

Using the fact that the torsion of τ is ∇^τ-parallel, together with (19), we see that for every $\zeta_1, \zeta_2 \in \mathbb{R}^n = \mathfrak{h} \oplus \mathfrak{v}$, the function $u \mapsto \Theta_u(\zeta_1^*, \zeta_2^*)$ is constant along horizontal curves in Q, thus constant on Q since every two points of Q can be joined by a horizontal curve. Consequently, there exists an element $T \in \Lambda^3\mathbb{R}^n$ such that

$$\Theta_u(\zeta_1^*, \zeta_2^*) \;=\; T(\zeta_1, \zeta_2), \qquad \forall\, \zeta_1, \zeta_2 \in \mathbb{R}^n,\ \forall\, u \in Q \;. \tag{22}$$

Moreover, by Lemma 3.5 we have $T \in \Lambda^3\mathfrak{h} \oplus (\Lambda^2\mathfrak{h} \otimes \mathfrak{v}) \oplus \Lambda^3\mathfrak{v} \subset \Lambda^3\mathbb{R}^n$.

Similarly, using the fact that the projection of the curvature tensor R^τ to $\Lambda^2(\mathcal{V}M)$ is ∇^τ-parallel (Proposition 3.11), together with the pair symmetry of R^τ (Lemma 2.2) and (20), we obtain the existence of elements $R_1 \in \Lambda^2\mathfrak{v} \otimes \mathfrak{k}$ and $R_2 \in \Lambda^2\mathbb{R}^n \otimes \mathfrak{so}(\mathfrak{v})$ such that

$$\Omega^\alpha_u(\xi_1^*, \xi_2^*) \ = \ R_1(\xi_1, \xi_2) \qquad \forall\ \xi_1, \xi_2 \in \mathfrak{v},\ \forall\ u \in Q\ . \tag{23}$$

and

$$\rho_*\Omega^\alpha_u(\zeta_1^*, \zeta_2^*) \ = \ R_2(\zeta_1, \zeta_2), \qquad \forall\ \zeta_1, \zeta_2 \in \mathbb{R}^n,\ \forall\ u \in Q\ , \tag{24}$$

where $\rho_* : \mathfrak{k} \to \mathfrak{so}(\mathfrak{v})$ is the differential of the Lie group morphism $\rho : K \to \mathrm{SO}(\mathfrak{v})$ obtained by restricting to $\mathfrak{v}$ the standard representation of the holonomy group K of ∇^τ on $\mathbb{R}^n = \mathfrak{h} \oplus \mathfrak{v}$.

The above equations (21), (22) and (23) yield

$$[\xi_1^*, \xi_2^*] \ = \ -T(\xi_1, \xi_2)^* \ - \ R_1(\xi_1, \xi_2)^*\ , \tag{25}$$

thus proving the lemma.

□

Hence we can define the Lie bracket on $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{v}$ extending the one on $\mathfrak{k}$ by

$$[A, \xi] \ := \ A\xi \qquad \text{and} \qquad [\xi_1, \xi_2] \ := \ -T(\xi_1, \xi_2) \ - \ R_1(\xi_1, \xi_2)\ . \tag{26}$$

The Euclidean scalar product on $\mathbb{R}^n$ induces scalar products on $\mathfrak{v} \subset \mathbb{R}^n$ and on $\mathfrak{k} \subset \mathfrak{so}(n)$, as well as a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{l}$ making the direct sum $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{v}$ orthogonal. This scalar product is clearly $\mathfrak{k}$-invariant. Moreover, since T is skew-symmetric, (26) yields

$$\langle[\xi_1, \xi_2], \xi_3\rangle + \langle\xi_2, [\xi_1, \xi_3]\rangle = 0 \tag{27}$$

for every $\xi_1, \xi_2, \xi_3 \in \mathfrak{v}$.

The decomposition $\mathrm{T}Q = \mathrm{T}^{hor}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$ of the tangent bundle of Q given by the connection α can now be refined as

$$\mathrm{T}Q = \mathrm{T}^{\mathfrak{h}}Q \oplus \mathrm{T}^{\mathfrak{v}}Q \oplus \mathrm{T}^{\mathfrak{k}}Q\ ,$$

where $\mathrm{T}^{\mathfrak{h}}Q_u = \{\eta_u^* \mid \eta \in \mathfrak{h}\}$, $\mathrm{T}^{\mathfrak{v}}Q_u = \{\xi_u^* \mid \xi \in \mathfrak{v}\}$ and $\mathrm{T}^{\mathfrak{k}}Q_u = \ker(\pi_M)_* = \{\mathrm{A}_u^* \mid A \in \mathfrak{k}\}$. The map $\Phi : \mathfrak{l} \to \Gamma(\mathrm{T}Q)$ is by definition of the Lie algebra structure on $\mathfrak{l}$ a Lie algebra homomorphism, i.e. it defines a structure of an infinitesimal $\mathfrak{l}$-principal bundle on Q over some locally defined manifold N, whose fibers are the leaves of the integrable distribution $\Phi(\mathfrak{l}) = \mathrm{T}^{\mathfrak{v}}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$ of Q. Since $(\pi_M)_*^{-1}(\mathcal{V}M) = \mathrm{T}^{\mathfrak{v}}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$, this locally defined manifold N is the same as the locally defined manifold N introduced in the previous section.

Lemma 3.6 shows that the metric of M projects to a metric g^N on N with Levi-Civita covariant derivative denoted by ∇^{g^N}. Moreover, Lemma 3.9 shows that the horizontal part $\tau^{\mathfrak{h}}$ of τ projects to a 3-form σ on N defining a covariant derivative $\nabla^\sigma := \nabla^{g^N} + \sigma$ with parallel skew-symmetric torsion.

We will now introduce a connection on the principal bundle Q over N. By definition, the connection 1-form α of the K-principal bundle Q over M is K-equivariant, i.e. $R_a^*\alpha = \mathrm{Ad}_{\,a^{-1}}(\alpha)$ for every $a \in K$. Differentiating this at the identity we get

$$(\mathcal{L}_{A^*}\alpha)(U) = -[A, \alpha(U)], \qquad \forall\ A \in \mathfrak{k},\ \forall\ U \in \Gamma(\mathrm{T}Q)\ . \tag{28}$$

Let $\theta = \theta^{\mathfrak{h}} + \theta^{\mathfrak{v}}$ be the decomposition of the canonical 1-form θ with respect to the decomposition $\mathbb{R}^n = \mathfrak{h} \oplus \mathfrak{v}$. Since θ is K-equivariant, and the representation of K on $\mathbb{R}^n = \mathfrak{h} \oplus \mathfrak{v}$ preserves the decomposition, we obtain that $\theta^{\mathfrak{v}}$ is K-equivariant too, whence

$$(\mathcal{L}_{A^*}\theta^{\mathfrak{v}})(U) = -A(\theta^{\mathfrak{v}}(U)), \qquad \forall\, A \in \mathfrak{k},\ \forall\, U \in \Gamma(\mathrm{T}Q)\ . \tag{29}$$

Let $\Theta^{\mathfrak{v}} \in \Omega^2(Q, \mathfrak{v})$ and $\Theta^{\mathfrak{h}} \in \Omega^2(Q, \mathfrak{h})$ denote the projections of Θ on $\mathfrak{v}$ and $\mathfrak{h}$, so that $\Theta = \Theta^{\mathfrak{h}} + \Theta^{\mathfrak{v}}$. The K-invariance of $\mathfrak{v}$ and $\mathfrak{h}$ together with the structure equation (11) shows that

$$\Theta^{\mathfrak{v}} = \mathrm{d}\theta^{\mathfrak{v}} + \alpha \wedge \theta^{\mathfrak{v}}, \qquad \Theta^{\mathfrak{h}} = \mathrm{d}\theta^{\mathfrak{h}} + \alpha \wedge \theta^{\mathfrak{h}}\ . \tag{30}$$

Lemma 4.3. *The* 1*-form* $\beta := \alpha + \theta^{\mathfrak{v}} \in \Omega^1(Q, \mathfrak{l})$ *is a connection form on* Q *with respect to the infinitesimal* $\mathfrak{l}$*–principal bundle structure, i.e. it satisfies* $\beta(B^*) = B$ *for every* $B \in \mathfrak{l}$ *and*

$$(\mathcal{L}_{B^*}\beta)(U) = -[B, \beta(U)], \qquad \forall\, B \in \mathfrak{l},\ \forall\, U \in \Gamma(\mathrm{T}Q)\ . \tag{31}$$

Proof. The relation $\beta(B^*) = B$ is tautological from the definition of the infinitesimal action of $\mathfrak{l}$ on Q. Indeed, if $B = A + \xi \in \mathfrak{k} \oplus \mathfrak{v}$ then

$$\beta(B) = (\alpha + \theta^{\mathfrak{v}})(A^* + \xi^*) = \alpha(A^*) + \theta^{\mathfrak{v}}(\xi^*) = A + \xi = B.$$

Equation (31) follows from (28) and (29) when $B \in \mathfrak{k}$. It remains to check it when $B = \xi$ is a vector in $\mathfrak{v}$. By tensoriality and linearity, it is sufficient to consider three cases: when $U = A^*$ for some $A \in \mathfrak{k}$, $U = \xi_1^*$ for $\xi_1 \in \mathfrak{v}$ and $U = \eta^*$ for $\eta \in \mathfrak{h}$. Since $\beta(U)$ is constant in each of these cases, we get:

$$\begin{aligned}
(\mathcal{L}_{\xi^*}\beta)(A^*) &= -\beta([\xi^*, A^*]) = -\beta([\xi, A]^*) = -[\xi, A] = -[\xi, \beta(A^*)]\\
(\mathcal{L}_{\xi^*}\beta)({\xi_1}^*) &= -\beta([\xi^*, {\xi_1}^*]) = -\beta([\xi, \xi_1]^*) = -[\xi, \xi_1] = -[\xi, \beta({\xi_1}^*)]\\
(\mathcal{L}_{\xi^*}\beta)(\eta^*) &= -\beta([\xi^*, \eta^*]) = \mathrm{d}\beta(\xi^*, \eta^*)\ .
\end{aligned}$$

As $\beta(\eta^*) = 0$, we have $[\xi, \beta(\eta^*)] = 0$, and it remains to check that $\mathrm{d}\beta(\xi^*, \eta^*) = 0$. From (10) and (30) we get

$$\mathrm{d}\beta(\xi^*, \eta^*) = \mathrm{d}\alpha(\xi^*, \eta^*) + \mathrm{d}\theta^{\mathfrak{v}}(\xi^*, \eta^*) = \Omega^\alpha(\xi^*, \eta^*) + \Theta^{\mathfrak{v}}(\xi^*, \eta^*).$$

For every $u \in Q$ we denote by $X := u(\xi)$ and $Y := u(\eta)$ the corresponding tangent vectors to M. Then by definition $\tilde{X}_u = \xi^*_u$ and $\tilde{Y}_u = \eta^*_u$. Using (17) we infer

$$\Omega^\alpha(\xi^*, \eta^*)_u + \Theta^{\mathfrak{v}}(\xi^*, \eta^*)_u = u^{-1}(R^\tau_{X,Y} + \pi_{\mathcal{V}M} T^\tau(X, Y)),$$

where $\pi_{\mathcal{V}M}$ denotes the orthogonal projection of $\mathrm{T}M$ onto $\mathcal{V}M$. Since $X \in \mathcal{H}M$ and $Y \in \mathcal{V}M$, the right hand side term vanishes from the pair symmetry of R^τ (Lemma 2.2) and the fact that T^τ has no component in $\Lambda^2\mathcal{V}M \otimes \mathcal{H}M$ (Lemma 3.5).

□

Since β is a connection form on Q over N, it follows that its curvature form Ω^β is equivariant, i.e.

$$(\mathcal{L}_{B^*}\Omega^\beta)(U, V) = -[B, \Omega^\beta(U, V)], \qquad \forall\, B \in \mathfrak{l},\ \forall\, U, V \in \Gamma(\mathrm{T}Q)\ . \tag{32}$$

As $\beta = \alpha + \theta^{\mathfrak{v}}$, (10) and (30) yield the following decomposition of the curvature form Ω^β:

$$\Omega^\beta = \mathrm{d}\beta + \tfrac{1}{2}\beta\wedge\beta = (\mathrm{d}\alpha + \tfrac{1}{2}\,\alpha\wedge\alpha) + (\mathrm{d}\theta^{\mathfrak{v}} + \alpha\wedge\theta^{\mathfrak{v}}) + \tfrac{1}{2}\,(\theta^{\mathfrak{v}}\wedge\theta^{\mathfrak{v}}) = \Omega^\alpha + \Theta^{\mathfrak{v}} + \tfrac{1}{2}\,(\theta^{\mathfrak{v}}\wedge\theta^{\mathfrak{v}})\ . \tag{33}$$

Since $\theta^{\mathfrak{v}}$ vanishes on $\mathrm{T}^{\mathfrak{h}}Q$, projecting the above formula on $\mathfrak{v}$ yields

$$\Omega^\beta(\eta_1^*, \eta_2^*)^{\mathfrak{v}} = \Theta(\eta_1^*, \eta_2^*)^{\mathfrak{v}}, \qquad \forall\ \eta_1, \eta_2 \in \mathfrak{h}\ . \tag{34}$$

For later use, we note that since the torsion T^τ is skew-symmetric, we have for every $\eta_1, \eta_2 \in \mathfrak{h}$ and $\xi \in \mathfrak{v}$:

$$\langle \Theta(\eta_1^*, \eta_2^*), \xi\rangle = -\langle \Theta(\eta_1^*, \xi^*), \eta_2\rangle = -\langle \mathrm{d}\theta(\eta_1^*, \xi^*), \eta_2\rangle = \langle \theta([\eta_1^*, \xi^*]), \eta_2\rangle\ ,$$

whence

$$\langle \theta([\eta_1^*, \xi^*]), \eta_2\rangle = \langle \Omega^\beta(\eta_1^*, \eta_2^*), \xi\rangle, \qquad \forall\ \eta_1, \eta_2 \in \mathfrak{h},\ \forall\ \xi \in \mathfrak{v}\ . \tag{35}$$

4.3. **The reduction procedure.** It turns out that the connection β on the principal bundle Q over N does not have the properties that we would need: some components of its curvature, viewed as a 2-form with values in the adjoint bundle, are parallel but not all of them. This is because the structure group L is too large, and contains some unnecessary information. We will now apply a reduction procedure, which will eventually lead to the desired construction: a principal fibre bundle over N with parallel curvature (plus some further properties), containing enough information in order to recover the geometry of M.

Consider the linear map

$$\lambda_* : \mathfrak{l} = \mathfrak{k} \oplus \mathfrak{v} \to \mathfrak{so}(\mathfrak{v}) \oplus \mathfrak{v}, \qquad \lambda_* = \rho_* \oplus \mathrm{id} \tag{36}$$

where ρ_* denotes as before the differential of the Lie group morphism $\rho : K \to \mathrm{SO}(\mathfrak{v})$. Clearly the kernel of λ_* is an ideal of the Lie algebra $\mathfrak{l}$ (being equal to the kernel of the Lie algebra morphism ρ_*). Consequently, there is a unique Lie algebra structure on $\mathfrak{g} := \mathrm{im}(\lambda_*)$ making λ a Lie algebra morphism. We denote by L and G the simply connected Lie groups with Lie algebras $\mathfrak{l}$ and $\mathfrak{g}$ respectively, and by $\lambda : L \to G$ the group morphism whose differential at the identity is λ_*. For later use, we denote by $\mathfrak{k}_1 \subset \mathfrak{so}(\mathfrak{v})$ the image of $\mathfrak{k}$ by λ_*:

$$\mathfrak{k}_1 := \lambda_*(\mathfrak{k}) \subset \mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{v}. \tag{37}$$

Consider the associated G-principal bundle

$$P := Q \times_\lambda G \tag{38}$$

over N and the canonical principal bundle morphism $f : Q \to P$ given by $f(u) := [u, 1]$. We clearly have $f(ua) = f(u)\lambda(a)$ for every $u \in Q$ and $a \in L$. At the infinitesimal level, this reads $f_*(A_u^*) = (\lambda_*(A))^*_{f(u)}$ for every $u \in Q$ and $A \in \mathfrak{l}$. In particular, this shows that the vertical vector field A^* on Q is f-related to the vertical vector field $(\lambda_*(A))^*$ on P.

From the general theory of principal bundle morphisms, the connection form $\beta \in \Omega^1(Q, \mathfrak{l})$ induces a connection 1-form called $\gamma \in \Omega^1(P, \mathfrak{g})$. These 1-forms and their corresponding curvature forms are related by

$$f^*\gamma = \lambda_*\beta, \qquad f^*\Omega^\gamma = \lambda_*\Omega^\beta\ . \tag{39}$$

We denote by $\mathrm{ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{g}$ the adjoint bundle associated to P via the adjoint representation of G on $\mathfrak{g}$, by ∇^γ the covariant derivative induced by γ on $\mathrm{ad}(P)$ and by R^γ the section of $\Lambda^2\,\mathrm{T}N \otimes \mathrm{ad}(P)$ corresponding to the equivariant curvature form Ω^γ of P.

Lemma 4.4. *The covariant derivative of R^γ with respect to $\nabla^\sigma \otimes \nabla^\gamma$ is given by:*

$$((\nabla^\sigma \otimes \nabla^\gamma)_X R^\gamma)(Y,Z) = u\left(\tilde{X}(\Omega^\gamma(\tilde{Y},\tilde{Z})) - \Omega^\gamma(\widetilde{\nabla^\sigma_X Y},\tilde{Z}) - \Omega^\gamma(\tilde{Y},\widetilde{\nabla^\sigma_X Z})\right)\ , \tag{40}$$

for every vector fields X, Y, Z on N.

Proof. Every vector fields $Y, Z \in \Gamma(\mathrm{T}N)$ define a section $\sigma := R^\gamma(Y,Z)$ of $\mathrm{ad}(P)$. The corresponding $\mathfrak{g}$-valued map $\hat{\sigma}$ on P defined in §4.1 is $\hat{\sigma} = \Omega^\gamma(\tilde{Y},\tilde{Z})$, since by (15):

$$u\Omega^\gamma(\tilde{Y},\tilde{Z}) = R^\gamma(Y,Z), \qquad \forall u \in P\ . \tag{41}$$

Using (14) and (41) we thus get

$$\begin{aligned}((\nabla^\sigma \otimes \nabla^\gamma)_X R^\gamma)(Y,Z) &= \nabla^\gamma_X(R^\gamma(Y,Z)) - R^\gamma(\nabla^\sigma_X Y, Z) - R^\gamma(Y, \nabla^\sigma_X Z)\\ &= u\left(\tilde{X}(\Omega^\gamma(\tilde{Y},\tilde{Z})) - \Omega^\gamma(\widetilde{\nabla^\sigma_X Y},\tilde{Z}) - \Omega^\gamma(\tilde{Y},\widetilde{\nabla^\sigma_X Z})\right)\ .\end{aligned}$$

□

Corollary 4.5. *The section R^γ of $\Lambda^2\,\mathrm{T}N \otimes \mathrm{ad}(P)$ is parallel with respect to $\nabla^\sigma \otimes \nabla^\gamma$.*

Proof. The morphism $f : Q \to P$ of principal bundles over N maps horizontal lifts to horizontal lifts, so by (39) and Lemma 4.4, the condition $(\nabla^\sigma \otimes \nabla^\gamma)R^\gamma = 0$ is equivalent to the following equation on Q

$$\lambda_*\left(\tilde{X}(\Omega^\beta(\tilde{Y},\tilde{Z})) - \Omega^\beta(\widetilde{\nabla^\sigma_X Y},\tilde{Z}) - \Omega^\beta(\tilde{Y},\widetilde{\nabla^\sigma_X Z})\right) = 0, \qquad \forall\ X, Y, Z \in \Gamma(\mathrm{T}N)\ . \tag{42}$$

Let Y_N be a vector field on N with horizontal lift Y_M to a vector field on M. It is clear that the horizontal lifts of Y_N and Y_M to vector fields on Q coincide. Thus we may write $\tilde{Y}_M = \tilde{Y}_N$ for this horizontal lift. We fix a frame $u \in Q$ and write $\eta_1 = u^{-1}(Y_M), \eta_2 = u^{-1}(Z_M)$. Then $\tilde{Y}_N = \tilde{Y}_M = \widetilde{u\eta_1} = \eta_1^*$ and similarly for $\tilde{Z}_N$.

Recall now the existence of an element $R_2 \in \Lambda^2\mathbb{R}^n \otimes \mathfrak{so}(\mathfrak{v})$ which by (24) applied to vectors $\zeta_i := \eta_i \in \mathfrak{h}$ satisfies

$$R_2(\eta_1,\eta_2)\ =\ (\rho_*\Omega^\alpha)(\eta_1^*,\eta_2^*)\ =\ (\rho_*\Omega^\alpha)(\widetilde{u\eta_1},\widetilde{u\eta_2}), \qquad \forall\ \eta_1,\eta_2 \in \mathfrak{h}\ .$$

This relation can be equivalently written $\rho_*\Omega^\alpha(\tilde{Y}_M,\tilde{Z}_M) = R_2(u^{-1}Y_M, u^{-1}Z_M)$ and taking the derivative in the direction direction of $\tilde{X}_M$ gives

$$\begin{aligned}\tilde{X}_M(\rho_*\Omega^\alpha(\tilde{Y}_M,\tilde{Z}_M)) &= R_2(\tilde{X}_M(u^{-1}Y_M),\, u^{-1}Z_M)\ +\ R_2(u^{-1}Y_M,\, \tilde{X}_M(u^{-1}Z_M))\\ &= R_2(u^{-1}(\nabla^\tau_{X_M}Y_M),\, u^{-1}Z_M)\ +\ R_2(u^{-1}Y_M,\, u^{-1}(\nabla^\tau_{X_M}Z_M))\\ &= \rho_*\Omega^\alpha(\widetilde{\nabla^\tau_{X_M}Y_M},\, \tilde{Z}_M)\ +\ \rho_*\Omega^\alpha(\tilde{Y}_M,\, \widetilde{\nabla^\tau_{X_M}Z_M})\end{aligned}$$

In this equation we may replace the horizontal lifts of X_M, Y_M, Z_M by the horizontal lifts of X_N, Y_N, Z_N. Moreover it follows from O'Neill's formula that $\widetilde{\nabla^\tau_{X_M} Y_M} = \widetilde{\nabla^\sigma_{X_N} Y_N}$. Hence

$$\tilde{X}_N(\rho_*\Omega^\alpha(\tilde{Y}_N, \tilde{Z}_N)) \;-\; \rho_*\Omega^\alpha(\widetilde{\nabla^\sigma_{X_N} Y_N},\, \tilde{Z}_N) \;-\; \rho_*\Omega^\alpha(\tilde{Y}_N,\, \widetilde{\nabla^\sigma_{X_N} Z_N}) \;=\; 0\ . \tag{43}$$

Since the torsion τ is ∇^τ-parallel, a similar argument, followed by a projection on $\mathfrak{v}$, gives the equation

$$\tilde{X}_N(\Theta^{\mathfrak{v}}(\tilde{Y}_N, \tilde{Z}_N)) \;-\; \Theta^{\mathfrak{v}}(\widetilde{\nabla^\sigma_{X_N} Y_N},\, \tilde{Z}_N) \;-\; \Theta^{\mathfrak{v}}(\tilde{Y}_N,\, \widetilde{\nabla^\sigma_{X_N} Z_N}) \;=\; 0\ . \tag{44}$$

Finally, since $\theta^{\mathfrak{v}}$ vanishes on $\mathrm{T}^{\mathfrak{h}}Q$, (33) shows that for every vectors $U, V \in \mathrm{T}^{\mathfrak{h}}Q$ one has $\lambda_*\Omega^\beta(U,V) = \rho_*\Omega^\alpha(U,V) + \Theta^{\mathfrak{v}}(U,V)$, so (42) follows from (43) and (44).

□

Lemma 4.6. *For all vector fields $U, V \in \mathrm{T}^{hor}P := \ker(\gamma)$ and vectors $\xi_1, \xi_2 \in \mathfrak{v}$, the following formula holds:*

$$0 = \langle[\Omega^\gamma(U,V), \xi_2], \xi_1\rangle + \langle\Omega^\gamma_{\xi_1}(V), \Omega^\gamma_{\xi_2}(U)\rangle - \langle\Omega^\gamma_{\xi_1}(U), \Omega^\gamma_{\xi_2}(V)\rangle,$$

where $\Omega^\gamma_{\xi_i}$ is the endomorphism of the horizontal distribution $\mathrm{T}^{hor}P$ defined by

$$\langle\Omega^\gamma(U,V), \xi_i\rangle = (\pi_N^* g^N)(\Omega^\gamma_{\xi_i}(U), V), \qquad \forall\ U, V \in \mathrm{T}^{hor}P\ .$$

Proof. As a consequence of (34) we see that the function $u \mapsto \langle\Omega^\beta(\eta_1^*, \eta_2^*), \xi_1\rangle$ is constant on Q for any $\eta_1, \eta_2 \in \mathfrak{h}$ and $\xi_1 \in \mathfrak{v}$. Taking the Lie derivative with respect to ${\xi_2}^*$ for some $\xi_2 \in \mathfrak{v}$ and using (32) yields

$$0 = {\xi_2}^*(\langle\Omega^\beta(\eta_1^*, \eta_2^*), \xi_1\rangle) = -\langle[\xi_2, \Omega^\beta(\eta_1^*, \eta_2^*)], \xi_1\rangle + \langle\Omega^\beta([{\xi_2}^*, \eta_1^*], \eta_2^*), \xi_1\rangle + \langle\Omega^\beta(\eta_1^*, [{\xi_2}^*, \eta_2^*]), \xi_1\rangle\ .$$

On the other hand, (35) shows that the horizontal part of $[{\xi_2}^*, \eta_i^*]$ equals $-\Omega^\beta_{\xi_2}(\eta_i^*)$, where $\Omega^\beta_{\xi_i}$ is the endomorphism of $\mathrm{T}^{\mathfrak{h}}Q$ defined by

$$\langle\Omega^\beta(X,Y), \xi_i\rangle = \pi_M^* g^M(\Omega^\beta_{\xi_i}(X), Y), \qquad \forall\ X, Y \in \mathrm{T}^{\mathfrak{h}}Q\ .$$

The above formula can thus be rewritten as

$$0 = \langle[\Omega^\beta(\eta_1^*, \eta_2^*), \xi_2], \xi_1\rangle + \langle\Omega^\beta_{\xi_1}(\eta_2^*), \Omega^\beta_{\xi_2}(\eta_1^*)\rangle - \langle\Omega^\beta_{\xi_1}(\eta_1^*), \Omega^\beta_{\xi_2}(\eta_2^*)\rangle, \qquad \forall\ \eta_1, \eta_2 \in \mathfrak{h}\ ,$$

or equivalently

$$0 = \langle[\Omega^\beta(X,Y), \xi_2], \xi_1\rangle + \langle\Omega^\beta_{\xi_1}(Y), \Omega^\beta_{\xi_2}(X)\rangle - \langle\Omega^\beta_{\xi_1}(X), \Omega^\beta_{\xi_2}(Y)\rangle, \qquad \forall\ X, Y \in \mathrm{T}^{\mathfrak{h}}Q\ .$$

The result follows from this formula, together with (39), by noticing that f_* maps each space $\mathrm{T}^{\mathfrak{h}}Q_u$ isomorphically onto $\mathrm{T}^{hor}P_{f(u)}$.

□

In view of the above results it makes sense to introduce the following:

Definition 4.7. A *geometry with parallel curvature* $(N, g^N, \sigma, P, \mathfrak{g}, \gamma, \mathfrak{k}_1)$ is defined by a Riemannian manifold (N, g^N) with Levi-Civita covariant derivative ∇^{g^N}, carrying a metric covariant derivative $\nabla^\sigma := \nabla^{g^N} + \sigma$ with parallel skew-symmetric torsion $T^\sigma = 2\sigma$, and a (locally defined) G-principal bundle $p_N : P \to N$ endowed with a connection form $\gamma \in \Omega^1(P, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, such that the following properties hold:

(i) If ∇^γ denotes the covariant derivative induced by γ on $\mathrm{ad}(P)$, then the section R^γ of $\Lambda^2\,\mathrm{T}N \otimes \mathrm{ad}(P)$ defined by the curvature form Ω^γ of γ is parallel with respect to $\nabla^\sigma \otimes \nabla^\gamma$;

(ii) There exists a Lie sub-algebra $\mathfrak{k}_1 \subset \mathfrak{g}$ of compact type and a $\mathfrak{k}_1$-invariant scalar product $\langle .,. \rangle$ on $\mathfrak{g}$ such that the isotropy representation of $\mathfrak{k}_1$ on $\mathfrak{v} := \mathfrak{k}_1^\perp$ is faithful, and the splitting $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{v}$ is naturally reductive, i.e. $\langle [\xi_1, \xi_2], \xi_3 \rangle + \langle \xi_2, [\xi_1, \xi_3] \rangle = 0$ for every $\xi_1, \xi_2, \xi_3 \in \mathfrak{v}$;

(iii) For every local section u of P, $\xi_1, \xi_2 \in \mathfrak{v}$, and $X, Y \in \Gamma(\mathrm{T}N)$, the following relation holds:

$$g^N(R^\gamma_{u\xi_2}(X), R^\gamma_{u\xi_1}(Y)) - g^N(R^\gamma_{u\xi_2}(Y), R^\gamma_{u\xi_1}(X)) + \langle [u^{-1}R^\gamma_{X,Y}, \xi_2], \xi_1 \rangle = 0 \ , \tag{45}$$

where for every $\xi \in \mathfrak{v}$, $R^\gamma_{u\xi}$ is the endomorphism of $\mathrm{T}N$ defined by

$$\langle u^{-1}R^\gamma_{X,Y}, \xi \rangle =: g^N(R^\gamma_{u\xi}(X), Y), \qquad \forall\, X, Y \in \mathrm{T}N \ . \tag{46}$$

One can summarize the results of this section in the following:

Theorem 4.8. *Let (M^n, g, τ) be a geometry with parallel skew-symmetric torsion (Definition 2.1), with standard decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ (Definition 3.4). Then the base N of the standard submersion (Definition 3.7) carries a geometry with parallel curvature (Definition 4.7) canonically induced by the geometry of M.*

Proof. Consider the (locally defined) standard submersion $\pi : M \to N$. From Lemma 3.6, there exists a unique Riemannian metric g^N on N making π into a Riemannian submersion. Lemma 3.9 and Remark 3.10 show that there exists a unique 3-form σ on N such that $\pi^*\sigma = \tau^\mathfrak{h}$, and the connection $\nabla^\sigma = \nabla^{g^N} + \sigma$ has parallel skew-symmetric torsion $T^\sigma = 2\sigma$. The group G, the G-principal bundle P over N, the connection γ on P and the Lie sub-algebra $\mathfrak{k}_1$ of $\mathfrak{g}$ were constructed in (36), (38), (39) and (48) respectively. The properties $(i) - (iii)$ from Definition 4.7 follow from Corollary 4.5, Equation (27) and Lemma 4.6 respectively.

□

5. The inverse construction

The aim of this section is to prove the following converse of Theorem 4.8:

Theorem 5.1. *Let $(N, g^N, \sigma, P, \mathfrak{g}, \gamma, \mathfrak{k}_1)$ be a geometry with parallel curvature, and let $\mathrm{T}^{\mathfrak{k}_1}P$ be the integrable distribution of $\mathrm{T}P$ spanned at each point by fundamental vertical vector fields A^* with $A \in \mathfrak{k}_1$. Then the manifold M, locally defined as the space of leaves of $\mathrm{T}^{\mathfrak{k}_1}P$, carries a geometry with parallel skew-symmetric torsion (g, τ).*

Proof. Let us start by deriving a formula which will be necessary later on. The fact that R^γ is parallel with respect to $\nabla^\sigma \otimes \nabla^\gamma$, together with Lemma 4.4, and the Bianchi identity $\mathrm{d}\Omega^\gamma = -\gamma \wedge \Omega^\gamma$, shows that for all vector fields X, Y, Z on N we have

$$\begin{aligned} 0 &= \mathfrak{S}_{XYZ}\left(\tilde{X}(\Omega^\gamma(\tilde{Y},\tilde{Z})) - \Omega^\gamma(\widetilde{[X,Y]},\tilde{Z})\right) \\ &= \mathfrak{S}_{XYZ}\left(\Omega^\gamma(\widetilde{\nabla^\sigma_X Y},\tilde{Z}) + \Omega^\gamma(\tilde{Y},\widetilde{\nabla^\sigma_X Z}) - \Omega^\gamma(\widetilde{[X,Y]},\tilde{Z}\right) \\ &= \mathfrak{S}_{XYZ}\left(\Omega^\gamma(\widetilde{\nabla^\sigma_X Y},\tilde{Z}) + \Omega^\gamma(\tilde{Z},\widetilde{\nabla^\sigma_Y X}) - \Omega^\gamma(\widetilde{[X,Y]},\tilde{Z}\right) \\ &= 2\,\mathfrak{S}_{XYZ}\,\Omega^\gamma(\widetilde{\sigma(X,Y)},\tilde{Z}) \end{aligned}$$

as $\nabla^\sigma_X Y - \nabla^\sigma_Y X - [X,Y] = \nabla^{g^N}_X Y - \nabla^{g^N}_Y X - [X,Y] + 2\sigma(X,Y) = 2\sigma(X,Y)$. We thus obtain

$$\mathfrak{S}_{XYZ}\,\Omega^\gamma(\widetilde{\sigma(X,Y)},\tilde{Z}) = 0, \qquad \forall X,Y,Z \in \mathrm{T}N\ . \tag{47}$$

Step 1. We define a Riemannian metric g^P on the total space of P by

$$g^P(U,V) = (p_N^* g^N)(U,V) + \langle \gamma(U), \gamma(V)\rangle\ .$$

In this way, the projection $p_N : P \to N$ becomes a Riemannian submersion. The tangent bundle $\mathrm{T}P$ splits into a g^P-orthogonal direct sum of distributions $\mathrm{T}P = \mathrm{T}^{hor}P \oplus \mathrm{T}^{\mathfrak{v}}P \oplus \mathrm{T}^{\mathfrak{k}_1}P$, where $\mathrm{T}^{hor}P := \ker(\gamma)$ is spanned at each point by horizontal lifts $\tilde{X}$ of vector fields X on N, and $\mathrm{T}^{\mathfrak{v}}P$ and $\mathrm{T}^{\mathfrak{k}_1}P$ are spanned at each point by fundamental vertical vector fields A^* with $A \in \mathfrak{v}$ and $A \in \mathfrak{k}_1$ respectively. The Levi-Civita connection of g^P can be easily computed using these adapted vector fields. By definition,

$$g^P(\tilde{X},\tilde{Y}) = g^N(X,Y), \qquad g^P(\tilde{X},A^*) = 0, \qquad g^P(A^*,B^*) = \langle A,B\rangle\ .$$

Moreover, since $\gamma([\tilde{X},\tilde{Y}]) = -\mathrm{d}\gamma(\tilde{X},\tilde{Y}) = -\Omega^\gamma(\tilde{X},\tilde{Y})$, we obtain

$$[A^*,B^*] = [A,B]^*, \qquad [A^*,\tilde{X}] = 0, \qquad [\tilde{X},\tilde{Y}] = \widetilde{[X,Y]} - \Omega^\gamma(\tilde{X},\tilde{Y})^*\ .$$

The Koszul formula immediately implies that the Levi-Civita connection ∇^{g^P} of g^P is given by

$$\nabla^{g^P}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla^{g^N}_X Y} - \tfrac{1}{2}\,\Omega^\gamma(\tilde{X},\tilde{Y})^* \tag{48}$$

$$\nabla^{g^P}_{\tilde{X}} A^* = \nabla^{g^P}_{A^*}\tilde{X} = \tfrac{1}{2}\,\Omega^\gamma_A(\tilde{X}) \tag{49}$$

$$g^P(\nabla^{g^P}_{A^*}B^*,\tilde{X}) = 0 \tag{50}$$

$$g^P(\nabla^{g^P}_{A^*}B^*,C^*) = \tfrac{1}{2}\left(\langle [A,B],C\rangle - \langle [B,C],A\rangle + \langle [C,A],B\rangle\right) \tag{51}$$

where ∇^{g^N} denotes the Levi-Civita covariant derivative of (N,g^N) and $\Omega^\gamma_A(\tilde{X})$ is the horizontal vector field of P satisfying $g^P(\Omega^\gamma_A(\tilde{X}),\tilde{Y}) = \langle \Omega^\gamma(\tilde{X},\tilde{Y}),A\rangle$ for each vector field $Y \in \Gamma(\mathrm{T}N)$. Of course, this definition of $\Omega^\gamma_A(\tilde{X})$ coincides with the one in (46) when $A \in \mathfrak{v}$.

Step 2. We show that the metric g^P projects to a metric g^M on M making the (locally defined) projection $p_M : P \to M$ a Riemannian submersion with totally geodesic fibres tangent to $\mathrm{T}^{\mathfrak{k}_1}P$. The distribution $\mathrm{T}^{\mathfrak{k}_1}P$ is totally geodesic by (50) and (51), and the fact that $[\mathfrak{k}_1,\mathfrak{v}]\subset\mathfrak{v}$.

It remains to show that the restriction h of g^P to $\mathrm{T}^{hor}P\oplus\mathrm{T}^{\mathfrak{v}}P$ is constant in $\mathrm{T}^{\mathfrak{k}_1}P$-directions, that is, $(\mathcal{L}_{A^*}h)(U,V)=0$ for every $A\in\mathfrak{k}_1$ and $U,V\in\Gamma(\mathrm{T}P)$. Note first that

$$h(\nabla^{g^P}_U A^*,V)=U(h(A^*,V))-h(A^*,\nabla^{g^P}_U V)=0.$$

We thus obtain

$$\begin{aligned}(\mathcal{L}_{A^*}h)(U,V) &= A^*(h(U,V))-h([A^*,U],V)-h(U,[A^*,V])\\ &= (\nabla^{g^P}_{A^*}h)(U,V)+h(\nabla^{g^P}_U A^*,V)+h(U,\nabla^{g^P}_V A^*)\\ &= (\nabla^{g^P}_{A^*}h)(U,V).\end{aligned}$$

Since $\mathrm{T}^{\mathfrak{k}_1}P$ is totally geodesic, it is clear that this last term vanishes when U or V are tangent to $\mathrm{T}^{\mathfrak{k}_1}P$. Hence, to check the vanishing of $\nabla^{g^P}_{A^*}h$, it is sufficient to consider the cases $(U,V)=(\tilde X,\tilde Y)$, $(U,V)=(\tilde X,\xi^*)$ and $(U,V)=(\xi^*,{\xi_1}^*)$, where X,Y are vector fields on N and $\xi,\xi_1\in\mathfrak{v}$. Using (48)–(51) and the fact that $A^*(h(U,V))=0$ for the above chosen vector fields (U,V) and $A\in\mathfrak{k}_1$, we get:

$$\begin{aligned}(\nabla^{g^P}_{A^*}h)(\tilde X,\tilde Y) &= -h(\nabla^{g^P}_{A^*}\tilde X,\tilde Y)-h(\tilde X,\nabla^{g^P}_{A^*}\tilde Y)\\ &= -\tfrac12\Big(g^P(\Omega^\gamma_A(\tilde X),\tilde Y)+g^P(\tilde X,\Omega^\gamma_A(\tilde Y))\Big)\\ &= -\tfrac12\Big(\langle\Omega^\gamma(\tilde X,\tilde Y),A\rangle+\langle\Omega^\gamma(\tilde Y,\tilde X),A\rangle\Big)=0\ ,\\ (\nabla^{g^P}_{A^*}h)(\tilde X,\xi^*) &= -h(\nabla^{g^P}_{A^*}\tilde X,\xi^*)-h(\tilde X,\nabla^{g^P}_{A^*}\xi^*)=0\ ,\\ (\nabla^{g^P}_{A^*}h)(\xi^*,{\xi_1}^*) &= -h(\nabla^{g^P}_{A^*}\xi^*,{\xi_1}^*)-h(\xi^*,\nabla^{g^P}_{A^*}{\xi_1}^*)=0\ .\end{aligned}$$

Step 3. We define a 3-form τ^P on P which projects onto a 3-form τ on M. Let $\gamma=\gamma^{\mathfrak{k}_1}+\gamma^{\mathfrak{v}}$ be the decomposition of the connection form γ corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}_1\oplus\mathfrak{v}$. Inspired by formulas (34) and (35) in the previous section, we define

$$\tau^P=\tau_1+\tau_2+\tau_3\ , \tag{52}$$

where

$$\tau_1(U,V,W) := (p_N^*\sigma)(U,V,W)\ , \tag{53}$$

$$\tau_2(U,V,W) := \tfrac12\,\mathfrak{S}_{UVW}\,\langle\Omega^\gamma(U,V),\gamma^{\mathfrak{v}}(W)\rangle\ , \tag{54}$$

$$\tau_3(U,V,W) := -\tfrac12\,\langle[\gamma^{\mathfrak{v}}(U),\gamma^{\mathfrak{v}}(V)],\gamma^{\mathfrak{v}}(W)\rangle\ . \tag{55}$$

Note that the 3-form τ_3 is skew-symmetric because of the natural reductivity of the decomposition $\mathfrak{g}=\mathfrak{k}_1\oplus\mathfrak{v}$. The 3-form τ^P is clearly horizontal with respect to p_M, in the sense that it vanishes whenever one of the entries belongs to $\mathrm{T}^{\mathfrak{k}_1}P$. In order to show that it is projectable onto M, it suffices to show that its Lie derivative with respect to any fundamental vector field A^* with $A\in\mathfrak{k}_1$ vanishes.

First, it is clear that τ_1 is projectable onto N, so $\mathcal{L}_{A^*}\tau_1 = 0$ for every $A \in \mathfrak{g}$. Using the equivariance of γ we have as before

$$(\mathcal{L}_{A^*}\gamma)(U) = -[A, \gamma(U)], \quad (\mathcal{L}_{A^*}\Omega^\gamma)(U,V) = -[A, \Omega^\gamma(U,V)], \quad \forall\, A \in \mathfrak{g},\ \forall\, U,V \in \Gamma(\mathrm{T}P)\ .$$

In particular, when $A \in \mathfrak{k}_1$, the bracket with A preserves the decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{v}$, whence

$$(\mathcal{L}_{A^*}\gamma^{\mathfrak{v}})(U) = -[A, \gamma^{\mathfrak{v}}(U)], \qquad \forall\, A \in \mathfrak{k}_1,\ \forall\, U \in \Gamma(\mathrm{T}P)\ .$$

Using these relations we can compute

$$(\mathcal{L}_{A^*}\tau_2)(U,V,W) = -\tfrac{1}{2}\,\mathfrak{S}_{UVW}\ \left(\langle [A, \Omega^\gamma(U,V)], \gamma^{\mathfrak{v}}(W)\rangle + \langle \Omega^\gamma(U,V), [A, \gamma^{\mathfrak{v}}(W)]\rangle\right) = 0$$

since ad_A is skew-symmetric on $\mathfrak{g}$, and finally, using the Jacobi identity, we get

$$\begin{aligned}
(\mathcal{L}_{A^*}\tau_3)(U,V,W) &= \tfrac{1}{2}\,\langle [[A,\gamma^{\mathfrak{v}}(U)], \gamma^{\mathfrak{v}}(V)], \gamma^{\mathfrak{v}}(W)\rangle + \tfrac{1}{2}\,\langle [\gamma^{\mathfrak{v}}(U), [A, \gamma^{\mathfrak{v}}(V)]], \gamma^{\mathfrak{v}}(W)\rangle \\
&\quad + \tfrac{1}{2}\,\langle [\gamma^{\mathfrak{v}}(U), \gamma^{\mathfrak{v}}(V)], [A, \gamma^{\mathfrak{v}}(W)]\rangle \\
&= \tfrac{1}{2}\,\langle [[\gamma^{\mathfrak{v}}(V), \gamma^{\mathfrak{v}}(U)], A], \gamma^{\mathfrak{v}}(W)\rangle - \tfrac{1}{2}\,\langle [A, [\gamma^{\mathfrak{v}}(U), \gamma^{\mathfrak{v}}(V)]], \gamma^{\mathfrak{v}}(W)]\rangle = 0\ .
\end{aligned}$$

This shows the existence of a 3-form τ on M such that $p_M^*(\tau) = \tau^P$.

Step 4. We check that $\nabla^\tau := \nabla^{g^M} + \tau$ has skew-symmetric parallel torsion, where ∇^{g^M} denotes the Levi-Civita covariant derivative of (M, g^M). Let us denote by $\nabla^{\tau^P} = \nabla^{g^P} + \tau^P$. Since $p_M^*(\tau) = \tau^P$ and since $p_M : (P, g^P) \to (M, g^M)$ is a Riemannian submersion, we have $\nabla^\tau \tau = 0$ if and only if $\nabla^{\tau^P}\tau^P$ vanishes whenever applied to vectors in $\mathrm{T}^{hor}P \oplus \mathrm{T}^{\mathfrak{v}}P$. Since the vector fields of the form $\tilde{X}$ for $X \in \Gamma(\mathrm{T}N)$ span $\mathrm{T}^{hor}P$ and vector fields of the form ξ^* for $\xi \in \mathfrak{v}$ span $\mathrm{T}^{\mathfrak{v}}P$ at each point, we will assume that each of the 4 entries of $\nabla^{\tau^P}\tau^P$ is of one of these types.

First, using (48)–(51) and (53)–(55), we readily compute

$$\nabla^{\tau^P}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla^{\sigma}_X Y} - \tfrac{1}{2}\,(\Omega^\gamma(\tilde{X},\tilde{Y})^{\mathfrak{k}_1})^*\ , \tag{56}$$

$$\nabla^{\tau^P}_{\tilde{X}}\xi^* = 0\ , \tag{57}$$

$$\nabla^{\tau^P}_{\xi^*}\tilde{X} = \Omega^\gamma_\xi(\tilde{X})\ , \tag{58}$$

$$\nabla^{\tau^P}_{\xi_1^*}\xi_2^* = \tfrac{1}{2}\,([\xi_1,\xi_2]^{\mathfrak{k}_1})^*\ , \tag{59}$$

where the superscript $\mathfrak{k}_1$ in (56) and (59) denotes the projection from $\mathfrak{g}$ to $\mathfrak{k}_1$. Now, τ_1 vanishes unless all entries are in $\mathrm{T}^{hor}P$, τ_2 vanishes unless two entries are in $\mathrm{T}^{hor}P$ and one is in $\mathrm{T}^{\mathfrak{v}}P$, and τ_3 vanishes unless all entries are in $\mathrm{T}^{\mathfrak{v}}P$. From (56)–(59) we see that the only possibly non-vanishing terms in $\nabla^{\tau^P}\tau_P$ on vectors of the type $\tilde{X}$ or ξ^* are:

$$\begin{aligned}
(\nabla^{\tau^P}_{\tilde X}\tau_1)(\tilde Y_1,\tilde Y_2,\tilde Y_3) &= (\nabla^\sigma_X\sigma)(Y_1,Y_2,Y_3)\ ,\\
(\nabla^{\tau^P}_{\xi^*}\tau_1)(\tilde Y_1,\tilde Y_2,\tilde Y_3) &= \xi^*(\tau_1(\tilde Y_1,\tilde Y_2,\tilde Y_3)) - \mathfrak{S}_{123}(p_N^*\sigma)(\Omega^\gamma_\xi(\tilde Y_1),\tilde Y_2,\tilde Y_3)\\
&= -\mathfrak{S}_{123}\langle\Omega^\gamma(\tilde Y_1,\widetilde{\sigma(Y_2,Y_3)}),\xi\rangle\ ,\\
(\nabla^{\tau^P}_{\tilde X}\tau_2)(\tilde Y,\tilde Z,\xi^*) &= \tfrac12\Big(\tilde X(\langle\Omega^\gamma(\tilde Y,\tilde Z),\xi\rangle) - \langle\Omega^\gamma(\widetilde{\nabla^\sigma_X Y},\tilde Z),\xi\rangle - \langle\Omega^\gamma(\tilde Y,\widetilde{\nabla^\sigma_X Z}),\xi\rangle\Big)\ ,\\
(\nabla^{\tau^P}_{\xi_1^*}\tau_2)(\tilde Y,\tilde Z,\xi_2^*) &= \tfrac12\Big(\xi_1^*(\langle\Omega^\gamma(\tilde Y,\tilde Z),\xi_2\rangle) - \langle\Omega^\gamma(\Omega^\gamma_{\xi_1}(\tilde Y),\tilde Z),\xi_2\rangle - \langle\Omega^\gamma(\tilde Y,\Omega^\gamma_{\xi_1}(\tilde Z)),\xi_2\rangle\Big)\\
&= \tfrac12\Big(-\langle[\xi_1,\Omega^\gamma(\tilde Y,\tilde Z)],\xi_2\rangle) + \langle\Omega^\gamma_{\xi_1}(\tilde Y),\Omega^\gamma_{\xi_2}(\tilde Z)\rangle - \langle\Omega^\gamma_{\xi_2}(\tilde Y),\Omega^\gamma_{\xi_1}(\tilde Z)\rangle\Big)\ ,\\
(\nabla^{\tau^P}_{\tilde X}\tau_3)(\xi_1^*,\xi_2^*,\xi_3^*) &= -\tfrac12\tilde X(\langle[\xi_1,\xi_2],\xi_3\rangle) = 0\ ,\\
(\nabla^{\tau^P}_{\xi^*}\tau_3)(\xi_1^*,\xi_2^*,\xi_3^*) &= -\tfrac12\xi^*(\langle[\xi_1,\xi_2],\xi_3\rangle) = 0\ .
\end{aligned}$$

The vanishing of the first four expressions follows from the assumption that ∇^σ has parallel torsion on N and from (47), (40), and (45) respectively.

□

6. Parallel $\mathfrak{g}$-structures

We now introduce the following notion, which turns out to be a particular case of geometries with parallel curvature introduced in Definition 4.7 for the case when the sub-algebra $\mathfrak{k}_1$ of $\mathfrak{g}$ vanishes:

Definition 6.1. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. A *parallel* $\mathfrak{g}$*-structure* on a manifold N is given by:

- (i) a Riemannian metric g^N on N;
- (ii) a locally defined G-principal bundle $P\to N$ with adjoint bundle $\mathrm{ad}(P)$;
- (iii) an $\mathrm{ad}_\mathfrak{g}$-invariant scalar product $\langle .,.\rangle$ on $\mathfrak{g}$, thus inducing a scalar product also denoted by $\langle .,.\rangle$ on the fibers of $\mathrm{ad}(P)$;
- (iv) a connection form $\gamma\in\Omega^1(P,\mathfrak{g})$ whose curvature tensor $R^\gamma:\Lambda^2\,\mathrm{T}N\to\mathrm{ad}(P)$ is parallel with respect to the Levi-Civita connection of g^N on $\Lambda^2\,\mathrm{T}N$ and the connection induced by γ on $\mathrm{ad}(P)$;
- (v) a Lie algebra bundle morphism $\psi:\mathrm{ad}(P)\to\Lambda^2\,\mathrm{T}N$ which is the metric adjoint of $-R^\gamma$, in the sense that

$$g^N(\psi(\sigma),X\wedge Y) = -\langle\sigma,R^\gamma_{X,Y}\rangle,\qquad \forall\ X,Y\in\mathrm{T}N,\ \forall\ \sigma\in\mathrm{ad}(P)\ . \tag{60}$$

Remark 6.2. This definition is in many respects similar to the one of parallel even Clifford structures introduced in [18, Def. 2.2]. More precisely, a parallel rank r even Clifford structure on N satisfying the curvature condition in [18, Thm. 3.6 (b)], which up to a factor 2 is exactly

(60) above, defines a parallel $\mathfrak{so}(r)$-structure on N in the sense of Definition 6.1 after rescaling the scalar product on $\mathfrak{so}(r)$ by a factor 2.

We denote by ∇^N and R^N the Levi-Civita covariant derivative and the curvature tensor of g^N. Since by Definition 6.1 (*iv*), $\psi = -(R^\gamma)^*$ is $\nabla^N \otimes \nabla^\gamma$-parallel, we get for every vector fields X, Y on N and local section σ of $\mathrm{ad}(P)$:

$$\psi(\nabla^\gamma_X \sigma) = \nabla^N_X(\psi(\sigma)) \ ,$$

whence after a second covariant derivative and skew-symmetrization:

$$\psi(R^\gamma_{X,Y}\sigma) = R^N_{X,Y}(\psi(\sigma)) = [R^N_{X,Y}, \psi(\sigma)] \ . \tag{61}$$

Here the curvature $R^\gamma_{X,Y}$ acts on σ by the Lie bracket $R^\gamma_{X,Y}\sigma = [R^\gamma_{X,Y}, \sigma]$ of the Lie algebra bundle $\mathrm{ad}(P)$. Since ψ is a Lie algebra bundle morphism, (61) equivalently reads

$$[\psi(R^\gamma_{X,Y}), \psi(\sigma)] = [R^N_{X,Y}, \psi(\sigma)] \tag{62}$$

for all tangent vectors $X, Y \in \mathrm{T}N$ and $\sigma \in \mathrm{ad}(P)$.

There are several types of natural operations that one can make with parallel $\mathfrak{g}$-structures: products, reductions to ideals of the Lie algebra, restrictions to Riemannian factors of the manifold, or Whitney products. We will explain these constructions now.

6.1. **Products of parallel $\mathfrak{g}$-structures.** Clearly, if $(g^{N_i}, P_i, \gamma_i, \psi_i)$ are parallel $\mathfrak{g}_i$-structures on N_i for $i = 1, 2$, and if $\mathfrak{g}$ denotes the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, endowed with the direct sum scalar product, then $(g^{N_1} + g^{N_2}, P_1 \times P_2, \gamma_1 + \gamma_2, \psi_1 + \psi_2)$ is a parallel $\mathfrak{g}$-structure on $N_1 \times N_2$, called the *product* $\mathfrak{g}$*-structure.*

Note that this construction also makes sense in the degenerate cases where N_2 is a point, or when $\mathfrak{g}_2 = 0$ (in which case we call this a 0-structure).

6.2. **Reduction to an ideal of the Lie algebra.** Assume that (P, γ, ψ) is a parallel $\mathfrak{g}$-structure on (N, g^N) and that $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$. Since $\langle ., . \rangle$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant, $\mathfrak{g}$ is a direct sum of Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2 := \mathfrak{g}_1^\perp$. Since everything is local, one can assume that $G = G_1 \times G_2$, such that the Lie algebra of G_i is $\mathfrak{g}_i$.

Lemma 6.3. *Let G_1 and G_2 be compact Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ endowed with bi-invariant scalar products, and let $G := G_1 \times G_2$, with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ endowed with the direct sum scalar product. Then every parallel $\mathfrak{g}$-structure on N with respect to this scalar product, induces in a canonical way parallel $\mathfrak{g}_1$- and $\mathfrak{g}_2$-structures on N.*

Proof. Let (g^N, P, γ, ψ) be a parallel $\mathfrak{g}$-structure on N. One can write the connection form $\gamma = \gamma_1 + \gamma_2$ with $\gamma_i \in \Omega^1(P, \mathfrak{g}_i)$ for $i = 1, 2$. The G-equivariance property of γ

$$g^*\gamma = Ad_{g^{-1}}\gamma, \qquad \forall g \in G$$

shows that γ_i are G_i-equivariant, γ_1 is G_2-invariant and γ_2 is G_1-invariant. Then $P_1 := P/G_2$ and $P_2 := P/G_1$ are G_i-principal bundles over N and γ_i projects to connection forms (also denoted by γ_i) on P_i. The adjoint bundle $\mathrm{ad}(P)$ is naturally identified to $\mathrm{ad}(P_1) \oplus \mathrm{ad}(P_2)$

(and this decomposition is parallel with respect to the covariant derivative induced by γ). For every $X, Y \in \mathrm{T}N$ one has $R^{\gamma}_{X,Y} = R^{\gamma_1}_{X,Y} + R^{\gamma_2}_{X,Y}$. Denoting by ι the natural embedding of $\mathrm{ad}(P_1)$ into $\mathrm{ad}(P)$, we see that the composition $\psi_1 := \psi \circ \iota$ is a parallel Lie algebra bundle morphism from $\mathrm{ad}(P_1)$ to $\Lambda^2 \mathrm{T}N \simeq \mathfrak{so}(\mathrm{T}N)$ which clearly verifies (60).

□

In the sequel, we will say that the parallel $\mathfrak{g}_1$-structure obtained in this way from an ideal $\mathfrak{g}_1$ of $\mathfrak{g}$ is a *reduction* of the initial parallel $\mathfrak{g}$-structure to the ideal $\mathfrak{g}_1$.

6.3. **Restriction to Riemannian factors.**

Lemma 6.4. *Assume that (N, g^N) is the Riemannian product of (N_1, g^{N_1}) and (N_2, g^{N_2}). Then every parallel $\mathfrak{g}$-structure (P, γ, ψ) on (N, g^N) with the property that*

$$\psi(\mathrm{ad}(P)) \subset \Lambda^2 \mathrm{T}N_1 \oplus \Lambda^2 \mathrm{T}N_2 \subset \Lambda^2 \mathrm{T}N \tag{63}$$

induces parallel $\mathfrak{g}$-structures on the factors (N_i, g^{N_i}).

Proof. Every point of N_2 defines an isometric embedding of (N_1, g^{N_1}) into (N, g^N). By pull-back through this embedding one obtains a G-principal bundle P_1 over N_1 with connection γ_1. Moreover, the condition (63) shows that ψ defines by restriction a Lie algebra bundle morphism $\psi_1 : \mathrm{ad}(P_1) \to \Lambda^2 \mathrm{T}N_1$, which is clearly still parallel and satisfies (60). The proof for N_2 is similar.

□

Note that the condition (63) is automatically satisfied for the factors of N in the standard de Rham decomposition, see Lemmas 7.2 and 7.3 below.

6.4. **Whitney products.** As a converse to Lemma 6.3 we have the following:

Lemma 6.5. *Let G_1 and G_2 be compact Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ endowed with bi-invariant scalar products, and let $G := G_1 \times G_2$, with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ endowed with the direct sum scalar product. If (P_1, γ_1, ψ_1) and (P_1, γ_1, ψ_1) are parallel $\mathfrak{g}_1$- and $\mathfrak{g}_2$-structures on (N, g^N) such that $\psi_1(\mathrm{ad}(P_1))$ commutes with $\psi_2(\mathrm{ad}(P_2))$, then the Whitney product $(P_1 \times P_2, \gamma_1 + \gamma_2, \psi_1 + \psi_2)$ is a parallel $\mathfrak{g}$-structure on (N, g^N).*

Proof. Everyhing is tautological, by noticing that the map

$$\psi_1 + \psi_2 : \mathrm{ad}(P_1 \times P_2) = \mathrm{ad}(P_1) \oplus \mathrm{ad}(\mathrm{P}_2) \to \Lambda^2 \mathrm{T}N$$

is a Lie algebra bundle morphism due to the commutation assumption.

□

Definition 6.6. A parallel $\mathfrak{g}$-structure is called *non-degenerate* if for every orthogonal and parallel decomposition $\mathrm{T}N = D_1 \oplus D_2$ and orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i$ Lie sub-algebras of $\mathfrak{g}$ satisfying $\psi(u\xi_1) \in \Lambda^2 D_1$ and $\psi(u\xi_2) \in \Lambda^2 D_2$ for every $u \in P$, $\xi_1 \in \mathfrak{g}_1$ and $\xi_2 \in \mathfrak{g}_2$, then either $D_1 = 0$ and $\mathfrak{g}_1 = 0$, or $D_2 = 0$ and $\mathfrak{g}_2 = 0$.

Equivalently, a parallel $\mathfrak{g}$-structure is non-degenerate if it is not locally a product of parallel $\mathfrak{g}$-structures, as described in §6.1.

Remark 6.7. Note that the morphism ψ of a non-degenerate parallel $\mathfrak{g}$-structure is injective. Indeed, let $\mathfrak{g}_1 \subset \mathfrak{g}$ be defined by $P \times_{\mathrm{Ad}} \mathfrak{g}_1 = \ker \psi$, $\mathfrak{g}_2 := \mathfrak{g}_1^{\perp}$, $D_1 = 0$ and $D_2 = \mathrm{T}N$. The conditions in Definition 6.6 are clearly satisfied, and since $D_2 \neq 0$, we necessarily have $\mathfrak{g}_1 = 0$.

Example 6.8. (i) If (g^N, P, γ, ψ) is a parallel $\mathfrak{u}(1)$-structure on N, $\mathrm{ad}(P)$ has a global parallel section whose image by ψ is a parallel 2-form on N. Conversely, if Ω is a parallel 2-form on (N, g^N), let $\eta \in \Omega^1(N)$ be any locally defined primitive of Ω. We define $P := N \times \mathbb{R}$ (viewed as a principal $\mathbb{R}$-bundle) and $\gamma := \mathrm{d}t + \eta \in \Omega^1(P)$ (everything is locally defined, and we omit writing down the pull-back signs, in order to keep notation simple). The adjoint bundle $\mathrm{ad}(P)$ is trivial, generated by a section called 1. We define the parallel morphism $\psi : \mathrm{ad}(P) \to \Lambda^2 \mathrm{T}N$ by $\psi(1) := -\Omega$. Then (P, γ, ψ) is a non-degenerate $\mathfrak{u}(1)$-structure on (N, g^N). Indeed, R^γ is equal to $\mathrm{d}\gamma = \Omega$, so it is parallel as a section in the trivial bundle $\mathrm{ad}(P)$, and the map ψ satisfies (60) and is clearly a Lie algebra bundle morphism since the fibers of $\mathrm{ad}(P)$ are 1-dimensional. The non-degeneracy of the $\mathfrak{u}(1)$-structure is clearly equivalent to the non-degeneracy of the corresponding 2-form. A non-degenerate parallel $\mathfrak{u}(1)$-structure thus defines a Kähler structure on N, which is moreover unique up to sign when (N, g^N) is irreducible.

(ii) More generally, if (g^N, P, γ, ψ) is a parallel $\mathfrak{u}(1)^m$-structure on N, $\mathrm{ad}(P)$ is spanned by m parallel sections, whose images by ψ are m parallel 2-forms on N whose associated endomorphisms mutually commute. Moreover these endomorphisms have no common kernel if the $\mathfrak{u}(1)^m$-structure is non-degenerate. By diagonalising them simultaneously applying de Rham's decomposition theorem, we see that (N, g^N) is a Riemannian product of Kähler manifolds. Conversely, let $N = N_1 \times \ldots \times N_s$ be a Riemannian product of Kähler manifolds with fundamental 2-forms Ω_α, and let $c_{i\alpha}$ be real numbers for $\alpha \in \{1, \ldots, s\}$ and $i \in \{1, \ldots, m\}$. We consider the parallel forms $F_i := \sum_\alpha c_{i\alpha} \Omega_\alpha$ on N and some locally defined primitive $\eta_i \in \Omega^1(N)$ of Ω_i and denote by $P := N \times \mathbb{R}^m$, and $\gamma_i := \mathrm{d}t_i + \eta_i \in \Omega^1(P)$. Then γ_i are the components of a connection form on P whose curvature form has components F_i. For every scalar product on $\mathfrak{u}(1)^m$ one obtains a parallel $\mathfrak{u}(1)^m$-structure on N by choosing a parallel orthonormal basis $\xi_1, \ldots, \xi_m$ of $\mathrm{ad}(P)$ and defining $\psi(\xi_j) := -F_j$, so that (60) holds.

Once we have fixed Kähler structures on the factors N_α, a parallel $\mathfrak{u}(1)^m$-structure on $N = N_1 \times \ldots \times N_s$ is thus determined by the $m \times s$ real matrix $\{c_{i\alpha}\}$. It is possible to express the non-degeneracy condition in terms of this matrix:

Lemma 6.9. *The above defined parallel $\mathfrak{u}(1)^m$-structure is degenerate if and only if there exists a partition $\{1, \ldots, s\} = A \sqcup B$ and an orthogonal decomposition $\mathbb{R}^m = V_1 \oplus V_2$ with $V_1, V_2 \neq 0$, such that $(c_{1\alpha}, \ldots, c_{m\alpha}) \in V_1$ for every $\alpha \in A$ and $(c_{1\beta}, \ldots, c_{m\beta}) \in V_2$ for every $\beta \in B$.*

Proof. Consider A, B, V_1, V_2 satisfying the above condition. We define

$$D_1 := \oplus_{\alpha\in A} \mathrm{T}N_\alpha, \qquad D_2 := \oplus_{\beta\in B} \mathrm{T}N_\beta, \qquad \mathfrak{g}_j := \{\sum_{i=1}^m x_i\xi_i \mid x \in V_j\},\ j = 1, 2\ .$$

For every $x \in V_1$ we have

$$\psi(\sum_i x_i\xi_i) = \sum_i x_iF_i = \sum_{i=1}^m \sum_{\alpha=1}^s x_ic_{i\alpha}\Omega_\alpha = \sum_{\alpha\in A} \sum_{i=1}^m x_ic_{i\alpha}\Omega_\alpha\ ,$$

showing that $\psi(\mathfrak{g}_1)$ vanishes on D_2, and similarly $\psi(\mathfrak{g}_2)$ vanishes on D_1. Since $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are non vanishing, Lemma 6.6 shows that the $\mathfrak{u}(1)^m$-structure is degenerate. The converse statement can be proved similarly.

□

Corollary 6.10. *If a parallel* $\mathfrak{u}(1)^m$*-structure on* $N = N_1 \times \ldots \times N_s$ *constructed as before is non-degenerate, then* $m \le s$.

Proof. Indeed, if $m > s$ then the s vectors $(c_{1\alpha}, \ldots, c_{m\alpha})$ span a strict subspace V_1 of $\mathbb{R}^n$ (if $V_1 = 0$ we replace it by any proper subspace of $\mathbb{R}^n$). Then $V_2 := V_1^\perp$, $A := \{1, \ldots, s\}$ and $B := \emptyset$ satisfy the condition of Lemma 6.9, so the structure is degenerate. □

Example 6.11. Every quaternion-Kähler manifold (N, g) carries a 3-dimensional parallel sub-bundle B of skew-symmetric endomorphisms locally spanned by almost complex structures satisfying the quaternion relations. The frame bundle P of B was introduced by Konishi [16], who showed that it has a connection (induced from the Levi-Civita connection of N) whose curvature is parallel as section of $\Lambda^2\,\mathrm{T}N \otimes \mathrm{ad}(P)$. If N has positive scalar curvature, then (60) is satisfied after rescaling the metric on N if necessary. We thus have a parallel non-degenerate $\mathfrak{sp}(1)$-structure on N.

Conversely, if (N, g^N) carries a parallel $\mathfrak{sp}(1)$-structure, then by Lemma 7.4 below, the structure is degenerate unless N is irreducible, in which case it is quaternion-Kähler with positive scalar curvature by Proposition 7.6 below.

Example 6.12. Every symmetric space of compact type $N = L/G$ carries a natural parallel $\mathfrak{g}$-structure. Indeed, consider the natural metrics on N and $\mathfrak{g}$ induced by an Ad_L-invariant scalar product on the Lie algebra $\mathfrak{l}$ of L, and define $P := L$, seen as G-principal bundle over N, with the connection γ induced from the Levi-Civita connection of N. If $\mathfrak{m}$ denotes the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{l}$, γ is just the $\mathfrak{m}$-component of the Maurer-Cartan form of L. Then the G-equivariant map $\phi : \mathfrak{g} \to \mathfrak{so}(\mathfrak{m}) \simeq \Lambda^2\mathfrak{m}$, $a \mapsto \mathrm{ad}_a|_\mathfrak{m}$ induces a parallel Lie algebra bundle morphism $\psi : \mathrm{ad}(P) \to \Lambda^2\,\mathrm{T}N$ by $\psi(u\xi) := u\phi(\xi)$ for every $u \in P$ and $\xi \in \mathfrak{g}$. In order to check (60), let u be a local section of P, $x, y \in \mathfrak{m}$ and $X := ux$, $Y := uy$ the corresponding local vector fields on N. Then for every $\xi \in \mathfrak{g}$ we have

$$\begin{aligned}\langle u\xi, R^\gamma_{X,Y}\rangle &= -\langle u\xi, u[x, y]\rangle = -\langle \xi, [x, y]\rangle = \langle [x, \xi], y\rangle = -\langle \mathrm{ad}_\xi(x), y\rangle \\ &= -\langle \phi(\xi), x \wedge y\rangle = -g^N(u\phi(\xi), ux \wedge uy) = -g^N(\psi(u\xi), X \wedge Y)\ .\end{aligned}$$

More generally, according to Lemma 6.3, a symmetric space of compact type $N = L/H$ carries a canonical parallel $\mathfrak{g}$-structure for every ideal $\mathfrak{g}$ of the isotropy Lie algebra $\mathfrak{h}$.

Conversely, we have the following:

Lemma 6.13. *Let $\mathfrak{g}$ be a semi-simple Lie algebra of compact type and let (P, γ, ψ) be a parallel $\mathfrak{g}$-structure on a locally symmetric space $(N = L/H, g^N)$ of compact type with ψ fiberwise injective. Then $\mathfrak{g}$ is an ideal of $\mathfrak{h}$ and the $\mathfrak{g}$-structure on N obtained as in Example 6.12 by reduction of the canonical parallel $\mathfrak{h}$-structure on N to $\mathfrak{g}$.*

Proof. Let us consider as usual the curvature tensors R^γ of (P, γ) and R^N of (N, g^N) as bundle morphisms $R^\gamma : \Lambda^2 \mathrm{T}N \to \mathrm{ad}(P)$ and $R^N : \Lambda^2 \mathrm{T}N \to \Lambda^2 \mathrm{T}N$ by

$$R^\gamma(\omega) := \tfrac{1}{2}\sum_{i,j} \omega(e_i, e_j) R^\gamma_{e_i,e_j}, \qquad R^N(\omega)(X, Y) := \tfrac{1}{2}\sum_{i,j} \omega(e_i, e_j) g^N(R^N_{e_i,e_j} X, Y)\ .$$

Since $N = L/H$ is of compact type, the metric on N is defined by a bi-invariant scalar product on the Lie algebra $\mathfrak{l}$ of L. Let $\mathfrak{h}$ denote the Lie algebra of H and let $\mathfrak{m}$ be its orthogonal complement in $\mathfrak{l}$. The isotropy representation of $\mathfrak{h}$ on $\mathfrak{m}$ defines an embedding of $\mathfrak{h}$ in $\Lambda^2\mathfrak{m}$, and we denote by $\mathfrak{h}^\perp$ its orthogonal complement, so that $\Lambda^2\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Correspondingly, the bundle $\Lambda^2 \mathrm{T}N$ decomposes in an orthogonal direct sum $\Lambda^2 \mathrm{T}N = \mathfrak{h}N \oplus \mathfrak{h}^\perp N$. As an endomorphism of $\Lambda^2 \mathrm{T}N$, the Riemannian curvature tensor R^N takes values in $\mathfrak{h}N$, so by pair symmetry R^N vanishes on $\mathfrak{h}^\perp N$.

We now use (61), which in the present context reads

$$\psi(R^\gamma(\omega)s) = [R^N(\omega), \psi(s)] \tag{64}$$

for every $s \in \mathrm{ad}(P)$ and $\omega \in \Lambda^2 \mathrm{T}N$. Applying this to some $\omega \in \mathfrak{h}^\perp N$ and using the vanishing of R^N on $\mathfrak{h}^\perp N$, together with the injectivity of ψ yields $R^\gamma(\omega)s = 0$ for every $\omega \in \mathfrak{h}^\perp N$ and $s \in \mathrm{ad}(P)$. Moreover, $R^\gamma(\omega)s = [R^\gamma(\omega), s]$ and since $\mathfrak{g}$ is semi-simple, this shows that $R^\gamma(\omega) = 0$ for every $\omega \in \mathfrak{h}^\perp N$. From (60) we thus get that $\psi(\mathrm{ad}(P))$ is orthogonal to $\mathfrak{h}^\perp N$, i.e. $\psi(\mathrm{ad}(P)) \subset \mathfrak{h}N$. Since ψ is a Lie algebra bundle morphism, this shows that $\mathfrak{g}$ is identified with a Lie sub-algebra of $\mathfrak{h}$. Moreover, it is well known that R^N is an isomorphism of $\mathfrak{h}N$, so (64) shows that $\mathfrak{g}$ is actually an ideal of $\mathfrak{h}$.

□

7. Classification of non-degenerate parallel $\mathfrak{g}$-structures

The aim of this section is the following classification result:

Theorem 7.1. *Let $\mathfrak{g}$ be a Lie algebra of compact type and $(g^N, P, \mathfrak{g}, \gamma, \psi)$ a non-degenerate parallel $\mathfrak{g}$-structure on a manifold N. Then either*

- *N is quaternion-Kähler with positive scalar curvature, $\mathfrak{g} = \mathfrak{sp}(1)$ and P is the Konishi bundle like in Example 6.11, or*

- *$N = L/H$ is an irreducible locally symmetric space of compact type, $\mathfrak{g}$ is isomorphic to a semi-simple factor of $\mathfrak{h}$ and the parallel $\mathfrak{g}$-structure is the one described in Example 6.12, or*
- *N is locally a Riemannian product $N = N_1 \times \ldots \times N_p \times S_1 \times \ldots \times S_q$ with N_α Kähler for $\alpha \in \{1, \ldots, p\}$, $S_\beta = L_\beta/\mathrm{U}(1)H_\beta$ Hermitian symmetric of compact type for $\beta \in \{1, \ldots, q\}$, $\mathfrak{g} = \mathfrak{u}(1)^m \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_q$ and $\mathfrak{k}_\beta$ a non-zero factor of $\mathfrak{h}_\beta$. The parallel $\mathfrak{g}$-structure on N is the Whitney product of a parallel $\mathfrak{u}(1)^m$-structure on N like in Example 6.8 (ii) and a parallel $\mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_q$-structure on N which is the Riemannian product of the canonical parallel $\mathfrak{k}_\beta$-structures on S_β (Example 6.12) and the 0-structures on the factors N_α.*

Proof. Let $(N, g^N) = N_0 \times N_1 \times \ldots \times N_s$ be the local de Rham decomposition of N, with N_0 flat and N_i irreducible for $i \geq 1$. We decompose the Lie algebra $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l$, where $\mathfrak{z}$ denotes its center and $\mathfrak{g}_i$ are simple Lie algebras of compact type. Since the scalar product is $\mathrm{ad}_\mathfrak{g}$-invariant, this decomposition can be chosen to be orthogonal. We define the corresponding Lie algebra bundles $\mathfrak{z}P := P \times_{\mathrm{ad}} \mathfrak{z}$ (which is actually trivial) and $\mathfrak{g}_i P := P \times_{\mathrm{ad}} \mathfrak{g}_i$, so that $\mathrm{ad}(P) = \mathfrak{z}P \oplus \mathfrak{g}_1 P \oplus \ldots \oplus \mathfrak{g}_l P$. Recall that by Remark 6.7 the map ψ is injective since the parallel $\mathfrak{g}$-structure is assumed to be non-degenerate.

Lemma 7.2. *If the Lie algebra bundle morphism ψ is injective, then*
$$\psi(\mathfrak{g}_i P) \subset \bigoplus_{\alpha \geq 1} \Lambda^2 \mathrm{T} N_\alpha, \qquad \forall\, i \in \{1, \ldots, l\}\ .$$

Proof. Let $\alpha, \beta \in \{0, \ldots, s\}$ be either different or both equal to 0, and let X_α, X_β be tangent vectors to N_α and N_β respectively. From the symmetries of the Riemannian curvature tensor we obtain $R^N_{X_\alpha, X_\beta} = 0$. Using (61) we get for every $\xi \in \mathfrak{g}$
$$0 = R^N_{X_\alpha, X_\beta}(\psi(u\xi)) = \psi(R^\gamma_{X_\alpha, X_\beta}(u\xi)),$$
whence $R^\gamma_{X_\alpha, X_\beta}(u\xi) = 0$ by the injectivity assumption. On the other hand
$$0 = R^\gamma_{X_\alpha, X_\beta}(u\xi) = [R^\gamma_{X_\alpha, X_\beta}, u\xi]$$
as local sections of $\mathrm{ad}(P)$, so $R^\gamma_{X_\alpha, X_\beta}$ is a section of $\mathfrak{z}P$. This shows that for every $i \geq 1$ and $\xi \in \mathfrak{g}_i$ we have
$$0 = \langle R^\gamma_{X_\alpha, X_\beta}, u\xi \rangle = -g^N(\psi(u\xi), X_\alpha \wedge X_\beta),$$
so finally $\psi(u\xi)$ is orthogonal to the sub-bundles $\Lambda^2 \mathrm{T} N_0$ and to $\mathrm{T} N_\alpha \wedge \mathrm{T} N_\beta$ of $\Lambda^2 \mathrm{T} N$ for all $\alpha \neq \beta$.

∎

Lemma 7.3. $\psi(\mathfrak{z}P) \subset \oplus_{\alpha \geq 0} \Lambda^2 \mathrm{T} N_\alpha$

Proof. Let $\xi_1, \ldots, \xi_m$ be an orthonormal basis of the center $\mathfrak{z}$, inducing a global orthonormal parallel basis $\hat{\xi}_1, \ldots, \hat{\xi}_m$ of $\mathfrak{z}P$. Let $\psi(\hat{\xi}_i) =: F_i$ be the corresponding parallel skew-symmetric endomorphisms of $\mathrm{T}N$.

The de Rham theorem shows that the restricted holonomy group of N is isomorphic to a product $K_0 \times \ldots \times K_s$, and $\mathrm{T}N$ is associated to a representation of this group on a direct sum $\mathfrak{h}_0 \oplus \ldots \oplus \mathfrak{h}_s$, (with $\mathfrak{h}_\alpha$ corresponding to $\mathrm{T}N_\alpha$), such that $K_0 = 0$ and for every $\alpha \geq 1$, K_α acts irreducibly on $\mathfrak{h}_\alpha$ and trivially on $\mathfrak{h}_\beta$ for $\beta \neq \alpha$. Each parallel endomorphism F_i corresponds to an equivariant map of this representation. On the other hand, every equivariant map clearly preserves each summand $\mathfrak{h}_\alpha$, thus showing that $F_i(\mathrm{T}N_\alpha) \subset \mathrm{T}N_\alpha$ for all $\alpha \geq 0$.

□

We now define $I = \{1, \ldots, l + m\}$ and denote by $\mathfrak{g}_{l+i}$ the sub-algebra generated by ξ_i for $1 \leq i \leq m$. Let $\mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ with corresponding ∇^γ-parallel decomposition of $\mathrm{ad}(P) = \oplus_{i \in I} \mathfrak{g}_i P$ of the adjoint bundle. By Lemmas 7.2 and 7.3, the parallel bundle morphism ψ maps $\mathrm{ad}(P)$ to $\oplus_{\alpha \in A} \Lambda^2 \mathrm{T}N_\alpha$, where $A = \{0, \ldots, s\}$. Let π_α be the projection of $\Lambda^2 \mathrm{T}N$ onto the sub-bundle $\Lambda^2 \mathrm{T}N_\alpha$. We use the notation

$$E_{\alpha i} := \pi_\alpha(\psi(\mathfrak{g}_i P))|_{N_\alpha} \tag{65}$$

for the corresponding parallel sub-bundle of $\Lambda^2 \mathrm{T}N_\alpha \to N_\alpha$. In other words, $E_{\alpha i}$ is the parallel sub-bundle of $\Lambda^2 \mathrm{T}N_\alpha$ corresponding to parallel $\mathfrak{g}_i$-structure on N_α obtained by reducing the initial $\mathfrak{g}$-structure to $\mathfrak{g}_i$ (Lemma 6.3) and then restricting to N_α (Lemma 6.4).

The next result can be seen as a generalization to parallel $\mathfrak{g}$-structures of the well known fact that quaternion-Kähler manifolds are irreducible.

Lemma 7.4. *If $E_{\alpha i} \neq 0$ for some $i \in \{1, \ldots, l\}$, then $E_{\beta i} = 0$ for every $\beta \in \{0, \ldots, s\} \setminus \alpha$.*

Proof. The first Bianchi identity and the Riemannian curvature identities show that for every $\alpha \neq \beta$ and tangent vectors $X, Y \in \mathrm{T}N_\alpha$ and $Z \in \mathrm{T}N_\beta$ one has

$$R^N_{X,Y} Z = 0. \tag{66}$$

Let x be any point of N and u an element in the fibre of P over x. The hypothesis gives the existence of two vectors $X, Y \in \mathrm{T}_x N_\alpha$ and some $\xi \in \mathfrak{g}_i$ such that $g^N(\psi(u\xi), X \wedge Y) \neq 0$. By (60) we thus get $\langle u\xi, R^\gamma_{X,Y} \rangle \neq 0$. Since $\mathfrak{g}_i$ is a simple Lie algebra, it has no center, so there exists some $\zeta \in \mathfrak{g}_i$ with $[R^\gamma_{X,Y}, u\zeta] \neq 0$. For every $\beta \in \mathrm{A}$, the map

$$\pi_\beta \circ \psi : \mathrm{ad}(P) \to \Lambda^2 \mathrm{T}N_\beta$$

is a parallel Lie algebra bundle morphism, so using again the fact that $\mathfrak{g}_i$ is simple, $\pi_\beta \circ \psi$ either vanishes identically, or is injective. Assume for a contradiction that $E_{\beta i} \neq 0$ for some $\beta \in \{0, \ldots, s\} \setminus \alpha$. Then from (61) we obtain

$$0 \neq \pi_\beta \circ \psi([R^\gamma_{X,Y}, u\zeta]) = \pi_\beta([\psi(R^\gamma_{X,Y}, \psi(u\zeta)]) = \pi_\beta([R^N_{X,Y}, \psi(u\zeta)]).$$

This shows that there exists $Z \in \mathrm{T}_x N_\beta$ such that

$$[R^N_{X,Y}, \psi(u\zeta)](Z) \neq 0.$$

Denoting by $F := \psi(u\zeta)$, this reads $R^N_{X,Y} F Z \neq F R^N_{X,Y} Z$. On the other hand, $FZ \in \mathrm{T}_x N_\beta$ by Lemma 7.2, so both $R^N_{X,Y} F Z$ and $F R^N_{X,Y} Z$ vanish from (66). This contradiction concludes the proof.

□

Lemma 7.5. *If there are partitions $A = A_1 \sqcup A_2$ and $I = I_1 \sqcup I_2$ of the two index sets I and A such that $E_{\alpha i} = 0$ for all $\alpha \in A_2, i \in I_1$ and for all $\alpha \in A_1, i \in I_2$, then either $A_1 = I_1 = \emptyset$ or $A_2 = I_2 = \emptyset$.*

Proof. The argument is similar to the one used in the proof of Proposition 8.4. Consider such partitions $A = A_1 \sqcup A_2$ and $I = I_1 \sqcup I_2$. For $i = 1, 2$ we define the distributions T_i on $M := P$ spanned by the horizontal lifts of vectors in $\bigoplus_{\alpha \in A_i} \mathrm{T}N_\alpha$ and by fundamental vector fields ξ^* with $\xi \in \bigoplus_{j \in I_i} \mathfrak{g}_j$. We claim that $\tau \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$.

It is enough to show that $\tau(U, V, W) = 0$ whenever two of the vectors U, V, W belong to T_1 and one to T_2, and by multi-linearity, one can assume each of them is either a horizontal lift or a vertical fundamental vector field. Using Lemmas 3.5 and 8.3, we are left with two cases:

a) U, V, W are all vertical, and $U = \xi^*, V = \zeta^* \in T_1$, and $W = \eta^* \in T_2$. Then by (52) we have

$$\tau(U, V, W) = -\tfrac{1}{2}\langle [\xi, \zeta], \eta \rangle = 0,$$

since ξ and ζ belong to the sub-algebra $\bigoplus_{j \in I_1} \mathfrak{g}_j$ of $\mathfrak{g}$, which is orthogonal to $\bigoplus_{j \in I_2} \mathfrak{g}_j$, which contains η.

b) $U = \tilde{X}$ and $V = \tilde{Y}$ are horizontal lifts and $W = \xi^*$, with $X \in \mathrm{T}N_\alpha$, $Y \in \mathrm{T}N_\beta$, $\xi \in \mathfrak{g}_i$, and either $\alpha, \beta \in A_1$, $i \in I_2$, or $\alpha \in A_1$, $\beta \in A_2$, $i \in I_2$. In both cases we have by (52)

$$\tau(U, V, W) = \tfrac{1}{2}\langle \Omega^\gamma(\tilde{X}, \tilde{Y}), \xi \rangle = \tfrac{1}{2}\langle u^{-1} R^\gamma_{X,Y}, \xi \rangle = -\tfrac{1}{2} g^N(\psi(u\xi), X \wedge Y).$$

If $\alpha = \beta \in A_1$, $i \in I_2$, this expression vanishes by the assumption that $E_{\alpha i} := \pi_\alpha(\psi(\mathfrak{g}_i P))|_{N_\alpha}$ vanishes. If $\alpha \neq \beta$, this expression vanishes by Lemmas 7.2 and 7.3.

If T_1 and T_2 are non-vanishing, the decomposition $\mathrm{T}M = T_1 \oplus T_2$ satisfies the decomposability conditions in Definition 3.1. On the other hand, (M, g, τ) is assumed to be indecomposable by Definition 8.1, we necessarily have $T_1 = 0$ or $T_2 = 0$, thus proving that either $A_1 = I_1 = \emptyset$ or $A_2 = I_2 = \emptyset$.

□

We will now restrict our attention to the case where N is irreducible.

Proposition 7.6. *Let (N, g^N) be an irreducible Riemannian manifold with a parallel $\mathfrak{g}$-structure such that the morphism ψ is not identically zero. Then one of the following three cases may occur:*

- *The Lie algebra sub-bundle $\mathrm{Im}(\psi) \simeq \mathrm{ad}(P)/\mathrm{Ker}(\psi)$ of $\Lambda^2 \mathrm{T}N$ is a line bundle, and N is Kähler;*
- *Each fiber of $\mathrm{Im}(\psi)$ is isomorphic to $\mathfrak{sp}(1)$ and N is quaternion-Kähler with positive scalar curvature;*
- *N is locally symmetric of compact type.*

Proof. The sub-bundle $VN := \mathrm{Im}(\psi)$ of $\Lambda^2\,\mathrm{T}N \simeq \mathrm{End}^-(\,\mathrm{T}N)$ is a parallel sub-bundle, closed under the usual bracket of endomorphisms. It corresponds to an invariant subspace V of $\Lambda^2 T$, where T denotes the holonomy representation of N.

Consider the Riemannian curvature tensor R^N of N, also viewed as an endomorphism $R^N : \Lambda^2\,\mathrm{T}N \to \Lambda^2\,\mathrm{T}N$ by

$$g^N(R^N(X\wedge Y), Z\wedge W) = g^N(R^N_{X,Y}Z, W)\ . \tag{67}$$

Using the relation between the curvatures of $\mathrm{ad}(P)$ and N obtained in (62), we see that the endomorphism $R^\perp$ of $\Lambda^2\,\mathrm{T}N$ defined by

$$R^\perp := R^N - \psi\circ R^\gamma \tag{68}$$

takes values in the centralizer of VN.

Let us decompose V in an orthogonal direct sum of irreducible components $V = V_1\oplus\ldots\oplus V_k$ and correspondingly $VN = \oplus_a V_a N$.

Schur's lemma shows that there exist positive real numbers λ_a such that $\psi\circ\psi^* = \sum_a \lambda_a\pi_a$, where π_a denotes the orthogonal projection from $\Lambda^2\,\mathrm{T}N$ to V_aN. On the other hand, (60) shows that $\psi\circ R^\gamma = -\psi\circ\psi^*$, so by (68) we obtain

$$R^N = R^\perp - \sum_a \lambda_a\pi_a\ , \tag{69}$$

where we recall that $R^\perp$ takes values in the centralizer of VN. We introduce the symmetric endomorphisms of $\mathrm{T}N$

$$S_a(X) := \sum_i e_i\lrcorner\pi_a(e_i\wedge X), \qquad S^\perp(X) := \sum_i e_i\lrcorner R^\perp(e_i\wedge X),$$

for every local orthonormal basis $\{e_i\}$ of $\mathrm{T}N$.

We fix some $a\in\{1,\ldots,k\}$ and consider any orthonormal basis $\{A_s\}$ of V_aN (with respect to the natural scalar product on 2-forms induced by g^N). The endomorphism $\sum_s A_s^2$ of $\mathrm{T}N$ is clearly parallel, so by the irreducibility of N, there exists some positive constant b_a such that $\sum_s A_s^2 = -b_a\mathrm{id}$. We thus have for every $X\in\mathrm{T}N$:

$$S_a(X) = \sum_{i,s} g^N(e_i\wedge X, A_s)A_s(e_i) = -\sum_s A_s^2(X) = b_aX. \tag{70}$$

Moreover, since V_a is a simple Lie algebra, the Casimir element of its adjoint representation is a multiple of the identity. Consequently, there exists a positive constant c_a such that $\sum_s[A_s,[A_s,A]] = -c_aA$ for every section A of V_aN. Consequently we have:

$$-c_aA = \sum_s[A_s,[A_s,A]] = \sum_s A_s^2A + A\sum_s A_s^2 - 2\sum_s A_sAA_s, \tag{71}$$

whence

$$\sum_s A_sAA_s = (\tfrac12 c_a - b_a)A, \qquad \forall A\in V_aN. \tag{72}$$

Let A be any section of V_aN. Using (69), (70), the first Bianchi identity for R^N and the fact that A commutes with the images of $R^\perp$ and of π_b for every $b \neq a$, we obtain for every tangent vector X:

$$\begin{aligned}
R^\perp(A)(X) &= R^N(A)(X) + \lambda_a A(X) = \tfrac{1}{2}\sum_i R^N_{e_i,Ae_i}X + \lambda_a A(X)\\
&= -\tfrac{1}{2}\sum_i R^N_{X,e_i}Ae_i - \tfrac{1}{2}\sum_i R^N_{Ae_i,X}e_i + \lambda_a A(X) = \sum_i R^N_{e_i,X}Ae_i + \lambda_a A(X)\\
&= \sum_i R^\perp(e_i \wedge X)(Ae_i) - \sum_{i,b}\lambda_b\pi_b(e_i\wedge X)(Ae_i) + \lambda_a A(X)\\
&= AS^\perp(X) - \sum_{b\neq a}\lambda_b b_b A(X) - \lambda_a\sum_i \pi_a(e_i\wedge X)(Ae_i) + \lambda_a A(X).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_i \pi_a(e_i\wedge X)(Ae_i) &= \sum_{i,s} g^N(A_s, e_i\wedge X)A_sAe_i = -\sum_{i,s} g^N(A_s(X), e_i)A_sAe_i\\
&= -\sum_s A_sAA_s(X),
\end{aligned}$$

so using (72) we obtain

$$R^\perp(A) = AS^\perp + d_aA, \qquad \forall A \in V_aN, \tag{73}$$

where

$$d_a := -\sum_{b\neq a}\lambda_b b_b + \lambda_a(\tfrac{1}{2}c_a - b_a) + \lambda_a = -\sum_b \lambda_b b_b + \lambda_a(1 + \tfrac{1}{2}c_a).$$

Since A and $R^\perp(A)$ are skew-symmetric and $S^\perp$ is symmetric, (73) shows that $S^\perp$ commutes with A for every $A \in VN$. On the other hand, $R^\perp(A)$ commutes with every $B \in VN$, so using (73) again, we obtain

$$(S^\perp + d_a\mathrm{id})\circ[A,B] = 0 \tag{74}$$

for every $A, B \in VN$. Let us denote by DN the parallel sub-bundle of $\mathrm{T}N$ spanned by the images of all endomorphisms of the form $[A,B]$ with $A, B \in VN$.

Then the irreducibility of N implies that either $DN = 0$ or $DN = \mathrm{T}N$.

Case 1: $DN = 0$. In this case VN is an Abelian Lie algebra sub-bundle of $\Lambda^2\,\mathrm{T}N$. Thus ψ vanishes on all sub-bundles $\mathfrak{g}_iP$ for $i \in \{1,\dots,l\}$, so VN is spanned by the parallel commuting endomorphisms $\psi(\hat\xi_1),\dots,\psi(\hat\xi_m)$ defined in the proof of Lemma 7.3. Using again the irreducibility of N we obtain that V has dimension 1, hence VN is generated by a parallel endomorphism whose square is proportional to the identity, and thus N is Kähler.

Case 2: $DN = \mathrm{T}N$. In this case, $S^\perp + d_a\mathrm{id} = 0$ by (74). In particular d_a is independent of a, whence $\lambda_a(1+\frac{c_a}{2}) =: r > 0$ for every a. By (69) we obtain

$$\mathrm{Ric}^N = \sum_a \lambda_a S_a - S^\perp = (\sum_a \lambda_a b_a)\mathrm{id} - (\sum_b \lambda_b b_b - r)\mathrm{id} = r\,\mathrm{id},$$

and thus N is Einstein and has positive scalar curvature. If N is locally symmetric, we are in the last case of the proposition. If N is not locally symmetric, we examine the possible holonomy representations of N given by the Berger-Simons holonomy theorem:

1. If $\mathrm{Hol}_0(N)$ is one of $\mathrm{G}_2 \subset \mathrm{SO}(7)$, $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, $\mathrm{SU}(m) \subset \mathrm{SO}(2m)$ or $\mathrm{Sp}(q) \subset \mathrm{SO}(4q)$, then N is Ricci-flat, so it cannot have positive scalar curvature.

2. If $\mathrm{Hol}_0(N) = \mathrm{SO}(n)$ and $T = \mathbb{R}^n$, then $\Lambda^2 T$ is irreducible and has no center, unless $n = 4$, when $\Lambda^2 T = \Lambda^+ T \oplus \Lambda^- T$. Up to a change of orientation for N one can assume that $VN = \Lambda^+\,\mathrm{T}N$. Let us denote by R^+ the orthogonal projection from $\Lambda^2\,\mathrm{T}N$ onto $\Lambda^+\,\mathrm{T}N$. From (69), the curvature endomorphism of N can be written $R^N = R^\perp - \lambda R^+$, where $\lambda > 0$ and $R^\perp$ takes values in the centralizer of VN, which in the present situation is $\Lambda^-\,\mathrm{T}N$. Using the well known decomposition of the curvature operator in dimension 4 as

$$R^N = -\begin{pmatrix} \frac{\mathrm{Scal}}{12}\mathrm{id} + W^+ & \frac{1}{2}\widetilde{\mathrm{Ric}_0^N} \\ \frac{1}{2}\widetilde{\mathrm{Ric}_0^N} & \frac{\mathrm{Scal}}{12}\mathrm{id} + W^- \end{pmatrix}, \tag{75}$$

(where $\widetilde{\mathrm{Ric}_0^N}$ denotes the Kulkarni-Nomizu product of g^N with the trace-less Ricci tensor of N) we thus obtain that $\mathrm{Scal} = 12\lambda > 0$, $W^+ = 0$ and $\mathrm{Ric}_0^N = 0$, so N is anti-selfdual and Einstein, which corresponds to the quaternion-Kähler condition in dimension 4.

3. If $\mathrm{Hol}_0(N) = \mathrm{U}(m)$ and $T = \mathbb{R}^{2m}$, then the decomposition of $\Lambda^2 T$ in irreducible components reads $\mathfrak{so}(2m) \simeq \Lambda^2 T = \mathfrak{u}(1) \oplus \mathfrak{su}(m) \oplus \mathfrak{m}$, where $\mathfrak{m} \simeq \Lambda^{(2,0)+(0,2)}T$ is isomorphic to the isotropy representation of the symmetric space $\mathrm{SO}(2m)/\mathrm{U}(m)$ (and thus verifies $[\mathfrak{m},\mathfrak{m}] = \mathfrak{u}(1) \oplus \mathfrak{su}(m)$). Consequently, the only $\mathfrak{su}(m)$-invariant Lie sub-algebras of $\mathfrak{so}(2m)$ of dimension larger than 1 are $\mathfrak{su}(m)$ and $\mathfrak{u}(1) \oplus \mathfrak{su}(m)$. Their centralizers in $\mathfrak{so}(2m)$ are both equal to $\mathfrak{u}(1)$. Geometrically, this means that N is a Kähler manifold such that the endomorphism $(\nabla^N_X R^N)_{Y,Z}$ is proportional to the complex structure for every tangent vectors X, Y, Z. This easily implies that $\nabla^N R^N = 0$. Indeed, assume that $(\nabla^N_X R^N)_{Y,Z} = T(X, Y, Z)J$ for some tensor T. Using the second Bianchi identity we obtain for every tangent vectors A, B, C, Y, Z:

$$0 = \mathfrak{S}_{A,B,C}\, g^N((\nabla^N_A R^N)_{Y,Z} B, C) = \mathfrak{S}_{A,B,C}\, T(A, Y, Z) g^N(JB, C).$$

Taking $B = JC$ of unit length and orthogonal to A and JA yields $T(A, Y, Z) = 0$ for every A, Y, Z. Thus $T = 0$, so N is locally symmetric.

4. If $\mathrm{Hol}_0(N) = \mathrm{Sp}(q) \cdot \mathrm{Sp}(1)$ and $T = \mathbb{R}^{4q}$ with $q \geq 2$, then N is quaternion-Kähler. It remains to check that the fibers of $\psi(\mathfrak{g}P)$ are isomorphic to $\mathfrak{sp}(1)$. It is well known that the decomposition of $\Lambda^2 T$ in irreducible summands is $\Lambda^2 T = \mathfrak{sp}(q) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}$. We denote by $\Lambda^2\,\mathrm{T}N = \mathfrak{sp}(q)N \oplus \mathfrak{sp}(1)N \oplus \mathfrak{m}N$ the corresponding decomposition of the bundle of 2-forms on N. We claim that $[\mathfrak{m},\mathfrak{m}]$ contains $\mathfrak{sp}(q) \oplus \mathfrak{sp}(1)$. Indeed, if $[\mathfrak{m},\mathfrak{m}]$ were orthogonal to some

non-zero element in $\mathfrak{sp}(q) \oplus \mathfrak{sp}(1)$, then this element would commute with each element of $\mathfrak{m}$, and this would contradict the fact that the isotropy representation of the symmetric space $\mathrm{SO}(4q)/\mathrm{U}(2q)$ is faithful.

The centralizer of $\mathfrak{sp}(q) \oplus \mathfrak{sp}(1)$ in $\mathfrak{so}(4q)$ clearly vanishes, so we are left with two possibilities: either $V = \mathfrak{sp}(1)$ (in which case we are done), or $V = \mathfrak{sp}(q)$.

We will show that this last case is impossible. Indeed, if $V = \mathfrak{sp}(q)$, (69) reads $R^N = \mathrm{R}^\perp - \lambda R_V$ for some positive constant λ, where R_V denotes the projection on VN and $R^\perp$ is a symmetric endomorphism of $\mathfrak{sp}(1)N$ satisfying the first Bianchi identity. At every point of N one can diagonalize $R^\perp$ in an orthonormal basis ω_1, ω_2, ω_3 of $\mathfrak{sp}(1)N$ so that

$$R^\perp(X \wedge Y) = \tfrac{1}{2q} \sum_a \lambda_a g^N(X \wedge Y, \omega_a)\omega_a.$$

An easy computation then shows that the Bianchi condition $\sum_{i,j} e_i \wedge e_j \wedge R^\perp(e_i \wedge e_j) = 0$ is equivalent to $\sum_a \lambda_a \omega_a \wedge \omega_a = 0$. On the other hand, for $q \geq 2$ the 4-forms ω_a^2 are linearly independent, so $R^\perp = 0$, which shows that R^N is parallel. Since N was assumed to be non locally symmetric, this case is impossible, so the proposition is proved.

□

By Lemma 6.4, together with Lemmas 7.2 and 7.3, we see that every factor N_α of N (including the flat factor N_0) inherits a parallel $\mathfrak{g}$-structure $(P_\alpha, \gamma_\alpha, \psi_\alpha)$. We will distinguish two cases:

Case 1. Assume first that there exists a factor N_α such that the sub-bundles $E_{\alpha j}$ defined in (65) vanish for every $j \in \{l+1, \ldots, l+m\}$.

We consider the partitions $A = A_1 \sqcup A_2$ and $I = I_1 \sqcup I_2$ of the two index sets $I = \{1, \ldots, l+m\}$ and $A = \{0, \ldots, s\}$ defined by

$$A_1 := \{\alpha\}, \qquad A_2 := A \setminus \{\alpha\}, \qquad I_1 = \{i \in I \mid E_{\alpha i} \neq 0\}, \qquad I_2 = \{i \in I \mid E_{\alpha i} = 0\}.$$

By Lemma 7.4 we have that $E_{\beta i} = 0$ for all $\beta \in A_2, i \in I_1$, and by the very definition of I_2 we have $E_{\beta i} = 0$ for all $\beta \in A_1, i \in I_2$. Moreover A_1 is non-empty, so by Lemma 7.5 we must have $A_2 = I_2 = \emptyset$. Thus $N = N_\alpha$ is irreducible. By Proposition 7.6, N is either a non locally symmetric quaternion-Kähler manifold with positive scalar curvature as in Example 6.8 (iii), or a locally symmetric space of compact type L/H. In the latter case, Lemma 6.13 shows that $\mathfrak{g}$ is an ideal of the Lie algebra $\mathfrak{h}$ of H and the parallel $\mathfrak{g}$-structure on N is the reduction of the canonical parallel $\mathfrak{h}$-structure of L/H to $\mathfrak{g}$.

Case 2. For every $\alpha \in A$, there exists $j \in \{l+1, \ldots, l+m\}$ such that $E_{\alpha j} \neq 0$. By Lemmas 6.3 and 6.4, the reduction of the $\mathfrak{g}$-structure to the element of the center of $\mathfrak{g}$ generated by ξ_j defines a non-vanishing parallel 2-form on N_α, so by irreducibility, N_α is Kähler for every $\alpha \in A \setminus \{0\}$. The same holds for N_0, except that here the Kähler structure is not unique (one might need to further decompose N_0 into a product of flat Kähler factors, but we don't want to insist on this). The important fact is that the reduction of the $\mathfrak{g}$-structure to the center $\mathfrak{z}$ of $\mathfrak{g}$ is an Abelian $\mathfrak{g}$-structure on N, which can be written as in Example 6.8 (ii).

We now denote by

$$A' := \{\alpha \in A \mid E_{\alpha i} = 0 \ \forall i \in \{1, \dots, l\}\}, \qquad A'' := \{\alpha \in A \mid \exists i \in \{1, \dots, l\},\ E_{\alpha i} \neq 0\}.$$

By Lemma 7.2, $0 \in A'$. By Proposition 7.6, for each $\alpha \in A''$, the corresponding factor is locally symmetric, $N_\alpha = L_\alpha / H_\alpha$, so being Kähler, it is in fact Hermitian symmetric. By Lemma 6.13, the reduction of the parallel $\mathfrak{g}$-structure on N to the semi-simple part of $\mathfrak{g}$, followed by restriction to N_α is a reduction of the canonical parallel $\mathfrak{h}_\alpha$-structure of L_α/H_α to a semi-simple factor of $\mathfrak{h}_\alpha$.

Finally, the parallel $\mathfrak{g}$-structure on N is the Whitney product (Lemma 6.5) of its reductions to $\mathfrak{z}$ and to the semi-simple part of $\mathfrak{g}$, which is exactly the last case in the theorem.

□

8. Special geometries with torsion

Definition 8.1. A *special geometry with torsion* is an indecomposable geometry with parallel skew-symmetric torsion (M, g, τ) satisfying one of the following equivalent conditions:

- the summand $\mathcal{V}M$ in the standard decomposition (Definition 3.4) is spanned by ∇^τ-parallel vector fields;

- the holonomy representation of $\mathfrak{k} := \mathfrak{hol}(\nabla^\tau)$ on $\mathfrak{v}$ defined in §3 is trivial;

- the Lie algebra $\mathfrak{k}_1$ defined in (37) vanishes.

We assume for the remaining part of this section that (M, g, τ) is a special geometry with torsion. Thus $\mathcal{V}M$ is spanned by an orthonormal frame of ∇^τ-parallel vector fields $\xi_1, \dots, \xi_r$.

Denote as before by $\tau = (\sum_\alpha \tau^{\mathfrak{h}_\alpha}) + \tau^{\mathfrak{m}} + \tau^{\mathfrak{v}}$ the decomposition of τ. We start by deriving some useful properties of ∇^τ-parallel vector fields.

Lemma 8.2. *Let ξ be a ∇^τ-parallel vector field. Then ξ is a Killing vector field and $\tau_\xi \cdot \tau^{\mathfrak{h}_\alpha} = 0$ for every α.*

Proof. From the definition of the standard decomposition it follows that ξ has to be a section of $\mathcal{V}M$. By Lemma 3.8 we thus have $\tau_\xi \cdot \tau^{\mathfrak{h}} = 0$. Moreover, the skew-symmetric endomorphism τ_ξ is a section of the bundle $(\oplus_\alpha \Lambda^2 \mathcal{H}_\alpha M) \oplus \Lambda^2 \mathcal{V}M$. It thus follows that it preserves the sub-bundles $\mathcal{H}_\alpha M$ and $\mathcal{V}M$ of $\mathrm{T}M$, and its action on $\Lambda^3 \mathrm{T}M$ preserves the sub-bundles $\Lambda^3 \mathcal{H}_\alpha M$, so $\tau_\xi \cdot \tau^{\mathfrak{h}_\alpha} = 0$ for every α.

To see that ξ is a Killing vector field, one writes for every tangent vector X

$$0 = \nabla^\tau_X \xi = \nabla^g_X \xi - \tau_X \xi = \nabla^g_X \xi + \tau_\xi X,$$

showing that $X \mapsto \nabla^g_X \xi = -\tau_\xi X$ is skew-symmetric.

□

The torsion decomposes under the action of the holonomy group of ∇^τ as

$$\tau \;=\; \sum_\alpha \tau_\alpha \;+\; \sum_{i,\alpha} \xi_i \otimes F_{i\alpha} \;+\; \sum_{ijk} c_{ijk}\, \xi_i \wedge \xi_j \wedge \xi_k \;,$$

with $\tau_\alpha \in \Lambda^3\mathfrak{h}_\alpha$, $F_{i\alpha} \in \Lambda^2\mathfrak{h}_\alpha$. Since all components are ∇^τ-parallel, it follows that c_{ijk} are constants.

Lemma 8.3. *For every special geometry with torsion, the horizontal part $\tau^{\mathfrak{h}} = \sum_\alpha \tau_\alpha$ of τ vanishes.*

Proof. We compute the action of τ_{ξ_i} on $\tau^{\mathfrak{h}}$. It is clear that $F_{i\alpha}$ acts trivially on the components $\tau^{\mathfrak{h}_\beta}$ for every $\beta \neq \alpha$. From Lemma 3.8 we thus obtain

$$0 = \tau_{\xi_i} \cdot \tau^{\mathfrak{h}} \;=\; \left(\sum_\alpha F_{i\alpha} \;+\; 3 \sum_{j,k} c_{ijk}\, \xi_j \wedge \xi_k \right) \cdot \tau^{\mathfrak{h}} \;=\; \sum_\alpha F_{i\alpha} \cdot \tau^{\mathfrak{h}} = \sum_\alpha F_{i\alpha} \cdot \tau^{\mathfrak{h}_\alpha} \;.$$

This shows that $F_{i\alpha} \cdot \tau_\alpha = 0$ for all α. Note that $F_{i\alpha} \in \Lambda^2\mathcal{H}_\alpha M$ is a ∇^τ-parallel 2-form on the irreducible sub-bundle $\mathcal{H}_\alpha M$, so as an endomorphism it is proportional to a complex structure on $\mathcal{H}_\alpha M$. On the other hand, complex structures act injectively on 3-forms.

Assume that there exists some index α_0 with $\tau^{\mathfrak{h}_{\alpha_0}} \neq 0$. Then the above argument shows that $F_{i\alpha_0} = 0$ for every i, hence the decomposition $\mathrm{T}M = \mathcal{H}_{\alpha_0} M \oplus (\mathcal{H}_{\alpha_0} M)^\perp$ would satisfy the hypothesis of Lemma 3.2, contradicting the indecomposability of M. This shows that $\tau^{\mathfrak{h}_\alpha} = 0$ for every α.

□

We will now explain the correspondence between parallel $\mathfrak{g}$-structures and special geometries with torsion.

Proposition 8.4. *The (locally defined) base N of the standard submersion of a special geometry with torsion (M, g, τ) carries a non-degenerate parallel $\mathfrak{g}$-structure. Conversely, every non-degenerate parallel $\mathfrak{g}$-structure $(g^N, P, \mathfrak{g}, \gamma, \psi)$ on a manifold N induces a special geometry with torsion on the total space P.*

Proof. By Theorem 4.8, N carries a geometry with parallel curvature $(g^N, \sigma, P, \mathfrak{g}, \gamma, \mathfrak{k}_1)$ satisfying Definition 4.7. This shows already that conditions (i) and (ii) in Definition 6.1 hold. By Definition 8.1, we have $\mathfrak{k}_1 = 0$, which by (38) implies that P can be identified with M itself. Lemma 8.3 shows that $\tau^{\mathfrak{h}} = 0$ on M, so $\sigma = 0$ by Remark 3.10, i.e. the connection ∇^σ on N is the Levi-Civita connection of g^N. The Lie algebra $\mathfrak{g}$ is in this special case isomorphic to $\mathfrak{v}$ and the scalar product on $\mathfrak{g}$ from Definition 4.7 (ii) is now $\mathrm{ad}_{\mathfrak{g}}$-invariant. In particular $\mathfrak{g}$ is of compact type, thus proving condition (iii). As for (iv), it is a direct consequence of Definition 4.7 (i).

Finally, (v) follows from Definition 4.7 (iii). Indeed, with the notation introduced in (46), the metric adjoint of $-R^\gamma$, denoted by $\psi : \mathrm{ad}(P) \to \Lambda^2\, \mathrm{T}N \simeq \mathfrak{so}(\mathrm{T}N)$, defined by (60), satisfies $\psi(u\xi) = -R^\gamma_{u\xi}$. Using (45) together with the $\mathrm{ad}_{\mathfrak{g}}$ invariance of the scalar product on

$\mathfrak{g}$ we obtain for every local section u of P, for every elements $\xi_1, \xi_2 \in \mathfrak{v} = \mathfrak{g}$, and for every local vector fields X, Y on N:

$$\begin{aligned} g^N(\psi([u\xi_1, u\xi_2]), X \wedge Y) &= -\langle [u\xi_1, u\xi_2], R^\gamma_{X,Y} \rangle = -\langle u[\xi_1, \xi_2], R^\gamma_{X,Y} \rangle \\ &= -g^N(R^\gamma_{u\xi_2}(X), R^\gamma_{u\xi_1}(Y)) + g^N(R^\gamma_{u\xi_2}(Y), R^\gamma_{u\xi_1}(X)) \\ &= g^N(R^\gamma_{u\xi_1}(R^\gamma_{u\xi_2}(X)), Y) - g^N(R^\gamma_{u\xi_1}(R^\gamma_{u\xi_2}(Y)), X) \\ &= g^N([R^\gamma_{u\xi_1}, R^\gamma_{u\xi_2}], X \wedge Y) = g^N([\psi(u\xi_1), \psi(u\xi_2)], X \wedge Y), \end{aligned}$$

thus showing that ψ is a Lie algebra bundle morphism.

We will now check that the parallel $\mathfrak{g}$-structure on N is non-degenerate. Suppose that there is an orthogonal and parallel decomposition $\mathrm{T}N = D_1 \oplus D_2$ and an orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i$ Lie sub-algebras of $\mathfrak{g}$ satisfying $\psi(u\xi_1) \in \Lambda^2 D_1$ and $\psi(u\xi_2) \in \Lambda^2 D_2$ for every $u \in P$, $\xi_1 \in \mathfrak{g}_1$ and $\xi_2 \in \mathfrak{g}_2$. Consider for $i = 1, 2$ the distributions T_i on $M = P$ spanned by the horizontal lift of D_i and by fundamental vector fields ξ^* with $\xi \in \mathfrak{g}_i$. By (53)–(55), if T_1 and T_2 are non-trivial, (M, g, τ) would be decomposable, according to Definition 3.1. On the other hand (M, g, τ) is assumed to be indecomposable by Definition 8.1, so we necessarily have $T_1 = 0$ or $T_2 = 0$. This shows that the $\mathfrak{g}$-structure of N is non-degenerate.

Conversely, a parallel $\mathfrak{g}$-structure $(g^N, P, \mathfrak{g}, \gamma, \psi)$ on N defines in a tautological way a geometry with parallel curvature on N (Definition 4.7) with $\mathfrak{k}_1 = 0$. By Theorem 5.1, the total space of P carries a geometry with parallel skew-symmetric torsion (g, τ), and the fact that $\mathfrak{k}_1 = 0$ just means that the holonomy of ∇^τ acts trivially on the vertical space $\mathfrak{v} \simeq \mathfrak{g}$ (see Definition 8.1).

It remains to show that if the parallel $\mathfrak{g}$-structure of N is non-degenerate, then (g, τ) is indecomposable (see Definition 3.1). Indeed, assume that $\mathrm{T}P = T_1 \oplus T_2$ is an orthogonal ∇^τ-parallel decomposition of the tangent bundle of P, such that $\tau \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$. We denote by $\mathcal{H} := \ker(\gamma)$ the horizontal distribution of P, and by $\mathcal{V}$ the vertical distribution (tangent to the fibers). Then $\mathrm{T}P = \mathcal{H} \oplus \mathcal{V}$ is another orthogonal ∇^τ-parallel decomposition, and from (52) we have $\tau(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$, $\tau(\mathcal{V}, \mathcal{V}) \subset \mathcal{V}$ and $\tau(\mathcal{V}, \mathcal{H}) \subset \mathcal{H}$. Let $\mathcal{V}'$ be the set of vectors $V \in \mathcal{V}$ such $\tau_V|_{\mathcal{H}}$ is non-degenerate. For $V \in \mathcal{V}'$ we decompose $V = Y_1 + Y_2$ with $Y_1 \in T_1$ and $Y_2 \in T_2$, and then we decompose $Y_i = X_i + V_i$ with $X_i \in \mathcal{H}$ and $V_i \in \mathcal{V}$ for $i = 1, 2$. We thus have $V = V_1 + V_2$ and $X_1 + X_2 = 0$. Since $\tau \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$ we obtain for every $X \in H$:

$$\begin{aligned} 0 &= \tau(Y_1, Y_2, X) = \tau(V_1 + X_1, V_2 - X_1, X) = \tau(V_1, V_2, X) + \tau(X_1, V_2, X) - \tau(V_1, X_1, X) \\ &= -\tau(V_1 + V_2, X_1, X) = -\tau_V(X_1, X). \end{aligned}$$

The assumption that $\tau_V|_{\mathcal{H}}$ is non-degenerate thus shows that $X_1 = 0$, whence $V = V_1 + V_2$. This shows that $(T_1 \cap \mathcal{V}) \oplus (T_2 \cap \mathcal{V})$ contains the set $\mathcal{V}'$, which is dense in $\mathcal{V}$, so

$$(T_1 \cap \mathcal{V}) \oplus (T_2 \cap \mathcal{V}) = \mathcal{V}.$$

The orthogonal complement of $T_i \cap \mathcal{V}$ in T_i is clearly contained in $T_i \cap \mathcal{H}$, so a dimension count immediately shows that

$$(T_1 \cap \mathcal{H}) \oplus (T_2 \cap \mathcal{H}) = \mathcal{H}.$$

Denoting $T_i \cap \mathcal{V} =: \mathcal{V}_i$ and $T_i \cap \mathcal{H} =: \mathcal{H}_i$, we thus get orthogonal ∇^τ-parallel decompositions $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. The assumption $\tau \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$ implies $\tau(\mathcal{V}_1, \mathcal{V}_2) = 0$, whence $\tau(\mathcal{V}_i, \mathcal{V}_i) \subset \mathcal{V}_i$ for $i = 1, 2$. By (55), this means that $\mathcal{V}_i$ are involutive distributions. Recalling that $\mathcal{V} = P \times_{\mathrm{Ad}} \mathfrak{g}$, this shows that the Lie algebra $\mathfrak{g}$ decomposes in an orthogonal direct sum of Lie sub-algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathcal{V}_i = P \times_{\mathrm{Ad}} \mathfrak{g}_i$.

Being ∇^τ-parallel, the distributions $\mathcal{H}_i$ project to parallel distributions D_i on N. Indeed, if V is a section of $\mathcal{V}$ and X_1 is a section of $\mathcal{H}_1$, then the horizontal part of $[V, X_1]$ reads

$$[V, X_1]_{\mathcal{H}} = (\nabla_V X_1 - \nabla_{X_1} V)_{\mathcal{H}} = (\nabla^\tau_V X_1 - \nabla^\tau_{X_1} V + 2\tau(V, X_1))_{\mathcal{H}} = (\nabla^\tau_V X_1 + 2\tau(V, X_1))_{\mathcal{H}} \in \mathcal{H}_1.$$

The fact that $\tau(V_1, X_2) = 0$ for every $V_1 \in \mathcal{V}_1$ and $X_2 \in \mathcal{H}_2$ is equivalent by (54) with the condition $\psi(\mathcal{V}_1) \subset \Lambda^2 D_1$. Similarly, $\psi(\mathcal{V}_2) \subset \Lambda^2 D_2$. From Definition 6.6 we thus obtain that $\mathcal{V}_i = \mathcal{H}_i = 0$ for some $i \in \{1, 2\}$. This shows that $T_i = 0$, so (g, τ) is indecomposable.

$\square$

Remark 8.5. The above result shows that every parallel $\mathfrak{g}$-structure $(g^N, P, \mathfrak{g}, \gamma)$ on N defines a geometry with parallel curvature $(N, g^N, \sigma = 0, P, \mathfrak{g}, \gamma, \mathfrak{k}_1 = 0)$ in the sense of Definition 4.7. More generally, for every sub-algebra $\mathfrak{k}_1 \subset \mathfrak{g}$, it defines a geometry with parallel curvature $(N, g^N, \sigma = 0, P, \mathfrak{g}, \gamma, \mathfrak{k}_1)$. By Theorem 5.1, we thus see that the principal bundle P of a parallel $\mathfrak{g}$-structure on N, as well as each of its quotients by subgroups of G, carry geometries with parallel skew-symmetric torsion.

Example 8.6. A parallel rank r even Clifford structure on N satisfying the curvature condition in [18, Thm. 3.6 (b)] determines a S^{r-1}-fibration $Z \to N$ whose vertical distribution belongs to the curvature constancy of Z. By Remark 6.2, the parallel rank r even Clifford structure also defines a parallel $\mathfrak{so}(r)$-structure on N, whose quotient by $\mathfrak{so}(r-1)$ is exactly Z. Remark 8.5 thus shows that Z also carries a geometry with parallel skew-symmetric torsion.

In particular, when $r = 3$, a parallel even Clifford structure is just a quaternion-Kähler structure on N, and Z is its twistor space. The curvature condition in [18, Thm. 3.6 (b)] is satisfied after rescaling the metric provided that N has positive scalar curvature. We thus recover the well known fact that the twistor spaces of positive quaternion-Kähler manifolds carry a connection with parallel skew-symmetric torsion.

Note that Proposition 8.4 together with the classification of Riemannian manifolds carrying non-degenerate parallel $\mathfrak{g}$-structures (Theorem 7.1), yield the classification of special geometries with torsion.

References

[1] I. Agricola, A.C. Ferreira, Th. Friedrich, *The classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$*, Differential Geom. Appl. **39** (2015), 59–92.

[2] I. Agricola, G. Dileo, *Generalizations of 3-Sasakian manifolds and skew torsion*, arXiv:1804.06700 (2018).

[3] B. Alexandrov, Th. Friedrich, Nils Schoemann, *Almost Hermitian 6-manifolds revisited*, J. Geom. Phys. **53** (2005), 1–30.

[4] B. Alexandrov, *$Sp(n)U(1)$-connections with parallel totally skew-symmetric torsion*, J. Geom. Phys. **57** (2006), 323–337.
[5] W. Ambrose, I.M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.
[6] F. Belgun, A. Moroianu, *Nearly-Kähler 6-manifolds with reduced holonomy*, Ann. Global Anal. Geom. **19** (2001), 307–319.
[7] R. Cleyton, A. Swann, *Einstein metrics via intrinsic or parallel torsion*, Math. Z. **247** (2004), 513–528.
[8] Th. Friedrich, *G_2-manifolds with parallel characteristic torsion*, Differ. Geom. Appl. **25** (2007), 632–648.
[9] Th. Friedrich, S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–335.
[10] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, *On nearly parallel G_2-structures*, J. Geom. Phys. **23** (1997), no. 3-4, 259–286.
[11] A. Gray, *Riemannian manifolds with geodesic symmetries of order 3*, J. Differential Geometry **7** (1972), 343–369.
[12] S. Ivanov, *Connections with torsion, parallel spinors and geometry of Spin(7) manifolds*, Math. Res. Lett. **11** (2004), no. 2-3, 171–186.
[13] V. Kirichenko, *K-spaces of maximal rank*, Matem. Zametki **22** (4) (1977), 465–476.
[14] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol I. Interscience Publishers, John Wiley & Sons, New York-London 1963 xi+329 pp.
[15] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol. II. Interscience Publishers John Wiley & Sons, New York-London 1969 xv+470 pp.
[16] M. Konishi, *On manifolds with Sasakian 3-structure over quaternion Kaehler manifolds*, Kōdai Math. Sem. Rep. **26** (1974/75), 194–200.
[17] S. Merkulov, L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. of Math. (2) **150** (1999), no. 1, 77–149.
[18] A. Moroianu, U. Semmelmann, *Clifford structures on Riemannian manifolds*, Adv. Math. **228** (2011), no. 2, 940–967.
[19] P.-A. Nagy, *Nearly Kähler geometry and Riemannian foliations*, Asian J. Math. **6** (2002), no. 3, 481–504.
[20] N. Schoemann, *Almost Hermitian structures with parallel torsion*, J. Geom. Phys. **57** (2007), 2187–2212.
[21] R. Storm, *A new construction of naturally reductive spaces*, Transform. Groups **23** (2018), no. 2, 527–553.
[22] R. Storm, *The classification of naturally reductive homogeneous spaces in dimension* 7 *and* 8, PhD thesis, Universität Marburg (2017).

Richard Cleyton, Ribe Katedralskole, Puggårdsgade 22, 6760 Ribe, Denmark

E-mail address: `rc@ribekatedralskole.dk`

Andrei Moroianu, Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France

E-mail address: `andrei.moroianu@math.cnrs.fr`

Uwe Semmelmann, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: `uwe.semmelmann@mathematik.uni-stuttgart.de`