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Topological Complexity, Robotics and Social Choice

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Topological complexity is a number that measures how hard it is to plan motions (for robots, say) in terms of a particular space associated to the kind of motion to be planned. This is a burgeoning subject within the wider area of Applied Algebraic Topology. Surprisingly, the same mathematics gives insight into the question of creating social choice functions, which may be viewed as algorithms for making decisions by artificial intelligences.

1 What can topology do for you?

When physicists and engineers study or design physical systems, they must take account of the totality of all states of the system in order to understand questions of, for instance, effectiveness and stability. The set of states is called the *configuration space* of the system. For example, consider a simple pendulum that can only swing in a plane. What are its states? The state of the pendulum is determined by where it is in its circular path; that is, the angle θ (measured counterclockwise starting from the hanging vertical state) tells us everything about the position of the pendulum. The configuration space is the set of all possible angles θ; that is, a circle. In mathematical set notation, this circle, denoted S^1, consists of all angles from $0°$ to $360°$, with $0°$ and $360°$ considered the same angle. We write

$$S^1 = \{\theta \,|\, 0° \leq \theta \leq 360°,\, 0° = 360°\}\,.$$

As another example, take a double pendulum: one pendulum joined to the end of the other. We confine each of the pendula to swing in its own

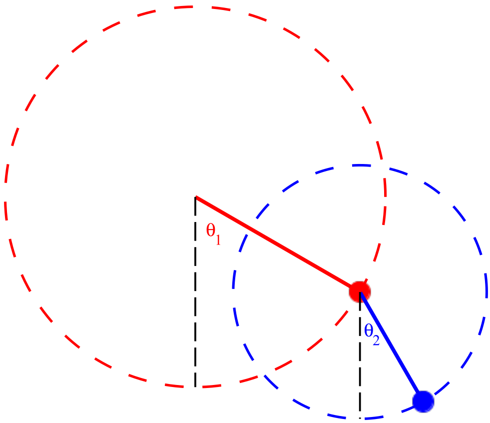

Figure 1: A double planar pendulum

plane, offsetting the two planes so that the pendula do not collide. What is the configuration space? The positions of the rods is what we are after and they are determined by the angles of the first and second pendula, θ_1 and θ_2 (see Figure 1). These are independent of each other, so we can write the set of states as a *product space*: that is, a set of pairs of angles where the first angle always comes from the first pendulum and the second angle comes from the second pendulum. We write

$$S^1 \times S^1 = \{(\theta_1, \theta_2) \,|\, 0 \leq \theta_1 \leq 360°, 0° \leq \theta_2 \leq 360°, 0° = 360°\}$$

where the first circle S^1 of $S^1 \times S^1$ corresponds to the first pendulum and the second to the second.

How can we picture this product space? First take one circle, then attach the center of the second circle at a point (perpendicular to the plane of the first circle for easier visualization). Now move the second circle around the first by simply moving the attached center of the second circle around the first circle. This gives us a space that looks like a hollow bagel — it is called a (2-dimensional) *torus* (see Figure 2).

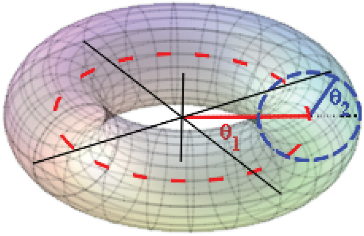

Figure 2: The configuration space of a double pendulum: a torus

Another way in which we can extend our first example is by allowing universal joints connecting the rods of a pendulum, so their motion is not confined to a plane. The configuration space of a single arm on a universal joint is the surface of a sphere — the radius of the sphere is the length of the arm.

As we did with the planar pendulum, we can form a double pendulum with two rods connected by a universal joint (we must idealize, to avoid issues with self-intersections). Furthermore, we could mix-and-match planar and universal joints on our double pendulum, to further extend the range of examples.

But why stop at a double pendulum? By the same reasoning as above, a triple planar pendulum has a configuration space

$$S^1 \times S^1 \times S^1 = \{(\theta_1, \theta_2, \theta_3)\}$$

which is called a 3-dimensional torus. We can keep doing this forever, but we shorten the notation (using T for torus); a pendulum with n arms has configuration space

$$T^n = \{(\theta_1, \theta_2, \ldots, \theta_n)\}.$$

Of course we can see $T^1 = S^1$ and $T^2 = S^1 \times S^1$, but where do these higher dimensional tori live?

S^1 lives in the plane $\mathbb{R}^2$; $T^2 = S^1 \times S^1$ lives naturally in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ (four-dimensional space); $T^3 = S^1 \times S^1 \times S^1$ lives naturally in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^6$ (six-dimensional space); and so on. Actually, as we saw with T^2 above, it is sometimes possible to fit the configuration space T^n into a Euclidean space of dimension less than $2n$. Generally speaking, however, we will need to use Euclidean spaces of arbitrarily high dimension to accommodate the state spaces of these examples. For instance, it is not possible to embed T^3 in ordinary three-dimensional space.

Notice how naturally we have entered the realm of higher-dimensional spaces. We must distinguish between the *work space* of the robot, that is, the setting in which the task is carried out, and its configuration space. Whereas the work space is ordinary three-dimensional space, the configuration space is usually (much) higher dimensional, and thus difficult to visualize or analyze.

Human beings can't visualize higher dimensions very well, so we need help to study spaces of high dimensions. This is where *algebraic topology* comes in. Modern algebraic topology attempts to describe shapes or properties of spaces in terms of algebra. It does this by assigning to a space various algebraic *invariants* that can be calculated (or sometimes just estimated). Knowledge of these invariants allows us to compare spaces, perhaps to say that one configuration space is different from another in a fundamental way. Let's see what algebraic topology can help us understand about robot arms.

A simple type of robot arm can be thought of as a collection of pendula, perhaps some with universal joints, so its configuration space is a product of circles and spheres.

The following problem is fundamental if we hope to be able to use robots to carry out tasks autonomously — that is, without human intervention. For each system, we have the *motion planning problem (MPP)*, as follows:

Problem 1.1 (Motion Planning Problem) *Given any initial state A and any final state B, produce a path in the configuration space of the system. The outputs must be continuous with respect to the inputs.*

That is, we want an algorithm which takes in any pair of states and returns a path, in a *continuous* way, that is, so that a small change in the initial and final states results in only a small change in the resulting path.

To see more precisely what motion-planning looks like, consider just a single arm on a universal joint, whose configuration space is a sphere S^2.

Example 1.2 *Given x and y, two points on S^2, what path would you choose from x to y? There are several different possibilities for how x and y could be related.*

If x and y are not antipodal (that is, directly opposite the sphere from one another), then x, y and $(0, 0, 0)$ define a unique plane in $\mathbb{R}^3$ that slices the sphere S^2 in a circle with the same radius as the sphere: a great circle. *On this circle there is a unique shortest path from x to y (this is where we use the fact that x and y are not antipodal). Always choose this path in this situation, as illustrated in Figure 3.*

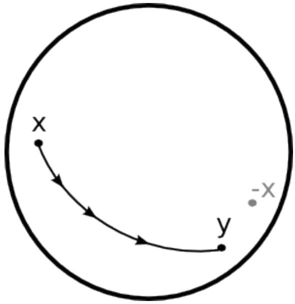

Figure 3: First motion planner: shortest distance

So for any pair (x, y) *in the set*

$$U_1 = \{(x, y) \,|\, y \neq -x\},$$

we have a choice of motion.

What if x *and* y are *antipodal? We need a mathematical fact. You may have heard the theorem that, at any instant of time, there is at least one place on the Earth where the wind does not blow.*[1] *In particular we can find (wind) directions at every point on Earth continuously except for one special point, call it* A*.*[2]

Consider the set of antipodal pairs (excluding A *and its antipode):*

$$U_2 = \{(x, -x) \,|\, x \neq A\}.$$

We'll give a motion-planning rule for such pairs. At x *take the wind direction and use it to get a plane through* x *and* $(0, 0, 0)$ *with the wind direction vector in the plane. The intersection of the plane and* S^2 *is a circle containing* x *and* $-x$*. Use the direction vector to move along the circle from* x *to* $-x$*, as illustrated in Figure 4.*

Thus we have accounted for all pairs except for the one pair $(A, -A)$ *consisting of the no-wind point and its antipode. For this pair we simply choose any path from* A *to* $-A$*.*

We have thus given a way to plan motions from any x to any y on the sphere S^2. It is not so very hard to check, however, that the scheme described above

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[1] This fact is more widely known as the Hairy Ball Theorem, since it says that you cannot neatly comb the hair on, say, a Tribble, except if there is one spot with no hair at all.

[2] The technical phrase is: we have chosen a continuous tangent vector field which vanishes only at A.

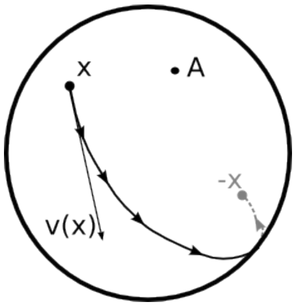

Figure 4: Second motion planner: “Gone With the Wind”

is not continuous as we vary x and y: starting from a non-antipodal pair (x, y), if we slide both x and y toward being antipodal, we must change regimes, and this may result in a large change in the path that the scheme suggests for us. It turns out that we cannot really do better.

The rest of this section and the next will focus on this MPP, and the extent to which we can solve it. Before we can discuss the feasibility of solving the Motion Planning Problem, we need a few mathematical ideas which lie at the heart of modern topology. First, mathematicians understand processes in terms of *mappings*. A mapping f has inputs in a space X and outputs in a space Y; we denote it by $f\colon X \to Y$. A mapping can be thought of as a manufacturing process: something goes in and a different thing (perhaps a different kind of thing) comes out. Two mappings $f\colon X \to Y$ and $g\colon X \to Y$ can be similar even if they do not produce the exact same outputs for the same inputs. The relevant form of similarity we will need is called *homotopy*. Mappings f and g are *homotopic* if f can be continuously deformed to g. Formally, this is when there is a mapping $H\colon X \times [0, 1] \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We write $f \sim g$ to denote this relation.

We observe that homotopy really is a kind of ‘similarity’: every map is homotopic to itself; if $f \sim g$ then $g \sim f$; and if $f \sim g$ and $g \sim h$, we can conclude that $f \sim h$.[3]

Here is a crucial example of homotopic maps.

Example 1.3 *Any mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is homotopic to the mapping* $c_0 : \mathbb{R}^n \to \mathbb{R}^n$ *which sends all points to the origin* $0 \in \mathbb{R}^n$*. We see this by using the homotopy* $H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ *given by* $H(x, t) = (1 - t)f(x)$*.*

[3] Relations with these properties are called *equivalence relations*.

When a space X has this property, we say that X is *contractible*. The idea behind this term is that the entire space X can be continuously collapsed to a single point. This is a very restrictive condition on a space; it means that, viewed through the filter of homotopy, contractible spaces are just like single points — a not-too-interesting class of spaces. Spheres and tori are *not* contractible, although it is not so easy to prove this without using some tools from algebraic topology, such as homology or homotopy groups.

Equipped with the notion of contractibility, we can state the following sobering result.

Theorem 1.4 *Let X be the configuration space of a system. There is a solution to the MPP (Problem 1.1) on X exactly when X is contractible.*

Since many configuration spaces of interest, such as the torus, are *not* contractible, we will often find ourselves in the situation in which a *global* continuous motion planning algorithm (that is, a motion-planning regime which applies to *all* pairs of initial and final point) is not possible. This disappointing result affords an opportunity for topology to introduce some order into the chaos. A major approach to studying the topology of spaces is a kind of molecular approach to spaces: break them apart into understandable pieces and then analyze how the pieces are glued together. Now, how can we use this method to obtain a next-best-solution to the motion planning problem? In order to see this, we need a more precise mathematical formulation of the MPP.

2 Topological Complexity

In the MPP, we are interested in finding a path in a configuration space X from any point $x \in X$ to any other point $y \in X$. A path from x to y is just a mapping $\gamma\colon I = [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We denote the set of all paths in X by X^I. There is a special map from X^I to the product space $X \times X$ denoted $\pi\colon X^I \to X \times X$ and defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$, which associates to each path its endpoints. A motion planning algorithm would then be a mapping $s\colon X \times X \to X^I$ such that $\pi(s(x,y)) = (x,y)$. That is, $s(x,y)$ is a path in X from x to y, the given starting and ending points in X. Such a map s is called a *section* of π. But now we see that Theorem 1.4 says that a section of π exists only when X is contractible. For other spaces, what can we do? We can measure how complicated a space is by asking for the minimal number of pieces into which $X \times X$ can be decomposed, such that for each of these pieces there *is* a motion planning algorithm inside X.

Definition 2.1 *The* topological complexity *[6] of a space X, denoted $TC(X)$, is the least number n of subsets*[4] *$U_1, \ldots, U_n$ of $X \times X$ such that the union of the sets U_j is all of $X \times X$ and for each U_j, there exists a map $s_j \colon U_j \to X^I$ with $\pi s_j \sim i_j \colon U_j \to X \times X$, where $i_j \colon U_j \to X \times X$ denotes the inclusion.*

Example 1.2 shows that $TC(S^2) \leq 3$. In fact, an extra algebraic calculation shows that $\mathrm{TC}(S^2) = 3$.

This means that a robot arm consisting of a single arm with universal articulation needs at least three motion planning algorithms to automate movement. Thus, topology is giving crucial information to engineers about what *not* to waste time doing: namely, trying to use only one or two motion planners.

From just the small amount of mathematical detail given above, we can now give a proof of Theorem 1.4.

Proof. First note that, by the definition of TC, a motion planner exists on the whole space X exactly when $\mathrm{TC}(X) = 1$. So we will show that $\mathrm{TC}(X) = 1$ is equivalent to X being contractible.

Suppose X is contractible. Then there is a homotopy $H \colon X \times I \to X$ with $H(x, 0) = x$ and $H(x, 1) = x_0$ (for some fixed point $x_0 \in X$). Define a section $s \colon X \times X \to X^I$ (recalling that $s(x, y)$ is a path in X) by

$$s(x_1, x_2)(t) = \begin{cases} H(x_1, 2t) & for\ 0 \leq t \leq \frac{1}{2} \\ H(x_2, 2(1-t)) & for\ \frac{1}{2} \leq t \leq 1. \end{cases}$$

This definition is just saying that the path $s(x_1, x_2)$ first goes from x_1 to the fixed point x_0 at twice the usual speed (hence the $2t$ in $H(x_1, 2t)$) and then goes from x_0 to x_2 at twice the usual speed. Thus a contracting homotopy on X provides well-defined paths from any point to any other — a motion planner.

Now suppose that a motion planner $s \colon X \times X \to X^I$ exists (that is, $\pi \circ s \sim \mathrm{id} \colon X \times X$). We can replace s by a map $\widehat{s}$ with $\pi \circ \widehat{s} = \mathrm{id}_X$. Choose any point $x_0 \in X$ and define a homotopy $H \colon X \times I \to X$ by $H(x, t) = \widehat{s}(x, x_0)(t)$. Note then that $H(x, 0) = x = \mathrm{id}_X(x)$ and $H(x, 1) = x_0 = c_{x_0}(x)$ since these are the endpoints of the motion planner $\widehat{s}$. This homotopy H is then a contracting homotopy and we see that X is contractible. **QED**

There are many examples for which the value of $\mathrm{TC}(X)$ is known. For instance (see [6]), we have: $\mathrm{TC}(S^m) = 3$ if m is an even number; $\mathrm{TC}(S^m) = 2$ if m is an odd number; and $\mathrm{TC}(T^n) = n + 1$. Generally, however, it is difficult to

[4] To rule out pathologies, the subsets have to satisfy an extra condition such as being open or being Euclidean Neighborhood retracts, for instance.

compute the value of TC(X); often, we are able to determine a range of possible values. For instance, it is relatively easy to see that the topological complexity of the *Klein bottle* must be either 3 or 4. Which of these it is, however, remains an open problem at the time of writing.

3 Social Choice

An amazing thing about mathematics is that ideas developed for one reason often find use in some completely unrelated context. Who would have thought that the eigenvalues of linear algebra would turn out to be the observables in quantum mechanics? Or that the differential geometry of Riemann would be the setting 50 years later of Einstein's General Theory of Relativity? Or that Radon's transform would be the basis for CT scans a half-century after being defined? Here, let's look at a completely different subject that uses the ideas we have developed in Section 2.

In Social Theory and Economics, a main goal is to understand how societal decisions are made when individuals have different preferences. This is precisely what is done in the limited context of game theory when we find Von Neumann-Morgenstern and Nash equilibria. Here we want to broaden the scope by considering *preference spaces* X which, for a given societal decision problem, contain all the possible preferences of the individuals involved in coming to the societal decision. This space is a continuous (non-discrete) topological space akin to the utility functions of Economics in the sense that individuals have a continuous range of preferences in many "directions". Somehow, in order to obtain a compromise decision, the preferences of the individuals must be aggregated. This leads to the following.

Definition 3.1 *Suppose there are k individuals each of which has preference space X. A* social choice function *is a continuous function $f\colon X^k \to X$ (where $X^k = X \times \cdots \times X$ (k-times)) satisfying two properties.*

- *(**Unanimity**) If all individuals have the same preference, then this is also the societal preference. Mathematically, this is saying that $f(x, x, \ldots, x) = x$.*
- *(**Anonymity**) It is the set of input preferences to the function that determine the output decision, not the individuals who have certain preferences. That is, any shuffling of k fixed preferences among the individuals produces the same outcome. Mathematically, this is saying that*

$$f(x_1, x_2, \ldots, x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}),$$

where σ is any permutation of the integers $\{1, 2, \ldots, k\}$.

These types of functions were actually studied in the 1950's under the name of generalized means. Economists [2, 3] re-discovered the main results about means given in [4] (which, in fact, were also mathematically re-discovered in [9]). Also, we note that there is a beautiful topological approach to the related topic of Arrow's Impossibility Theorem in [1]. Here is the main result (which looks a lot like Theorem 1.4).

Theorem 3.2 *Let X be a preference space*[5] *and $k \geq 2$. Then a social choice function $f\colon X^k \to X$ exists for some $k \geq 2$ exactly when X is contractible.*

See [5, 9] for a proof. Theorem 3.2 says that, unless the preference space is very special (i.e. contractible), there is no algorithmic way to achieve compromise.

Yet it is unlikely that the preference space is contractible, so what can be done? Let's take a cue from topological complexity and make a *wild* proposal. Let's split X^k into pieces where (local) social choice functions can be given.

We start by looking at the case in which there are just two choices to aggregate. Let's think of a couple who need to make a joint decision, based on their preferences from a preference space X. Here we seek a social choice function $f\colon X \times X \to X$ that satisfies

- $f(x, x) = x$, and
- $f(x_2, x_1) = f(x_1, x_2)$ for all x_1, x_2.

As we have seen, this will not be possible unless X is a contractible space. So, we proceed as follows:

Definition 3.3 *The* social choice complexity *of a space X, denoted* $\mathrm{sc}(X)$, *is defined to be the least integer n for which $X \times X$ is the union of n sets $U_1, \ldots, U_n$, each of which is "symmetric," in that we have $T(U_j) = U_j$, and on each of which we have a continuous map $s_j\colon U_j \to X$ that satisfies:*

(i) $s_j(x, x) = x$ *whenever* $x \in U_j \cap D_X(X)$;
(ii) $s_j(x_2, x_1) = s_j(x_1, x_2)$;
(iii) $D_X \circ s_j \sim i_j\colon U_j \hookrightarrow X \times X$ *while leaving* $U_j \cap D_X(X)$ *fixed throughout the homotopy. Here* $D_X\colon X \to X \times X$ *is given by* $D_X(x) = (x, x)$.

In this definition, conditions (i) and (ii) correspond to unanimity and anonymity, respectively, applied to the restricted set of preferences U_j. Condition (iii) is a separate condition that we have added. We call this condition

[5] We always suppose that spaces are a nicely-behaved sort of space known as finite CW complexes.

"close compromise," as it expresses in some way that the social choice should be close to each of the individual chosen preferences. As with topological complexity, $\mathrm{sc}(X)$ gives us a way to measure the necessary degree of discontinuity in a social choice function $f\colon X \times X \to X$ that is determined by the topology of the preference space X. It is clear that there is some similarity between the two notions. We can make this similarity more precise by bringing into the discussion a variant of topological complexity, which was introduced in [7] and is known as *symmetric topological complexity.*

Recall that a motion planner is a section $\sigma\colon X \times X \to X^I$ of the map $\pi\colon X^I \to X \times X$ that assigns to a path its endpoints. Rather than accepting any old path $\sigma(x, y)$ starting at x and ending at y, we can place two reasonable restrictions on the acceptable paths: (1) If the initial and final states agree, then the path is constant, that is, we have $\sigma(x, x) = c_x$; and (2) if we interchange initial and final states, then the path should be the time-reverse of the original, that is, we have $\sigma(x, y)(t) = \sigma(y, x)(1 - t)$ for $t \in [0, 1]$. We call such a motion planner a *symmetric motion planner.* Since a symmetric motion planner is a special kind of motion planner, only a contractible space will admit a global continuous one. As usual, by now, we will decompose $X \times X$ into pieces, each of which allows a local symmetric motion planner. Then we have the following connection between symmetric motion planners and social choice functions.

Theorem 3.4 *Suppose that $X \times X$ is the union of n sets $U_1, \ldots, U_n$ that satisfy $T(U_j) = U_j$, and on each of which we have a (local) symmetric motion planner in X, namely a continuous map $\sigma_j\colon U_j \to X^I$ that satisfies $\pi \circ \sigma_j = i_j\colon U_j \to X \times X$, as well as:*

(i) $\sigma_j(x, x) = c_x$ *whenever* $x \in U_j \cap \Delta(X)$*;*
(ii) $\sigma_j(x, y)(t) = \sigma_j(y, x)(1 - t)$ *for* $t \in [0, 1]$.

Let $r\colon X^I \to X$ be the map that assigns to each path in X its midpoint (in time), namely, $r(\alpha) = \alpha(1/2)$. Then each $r \circ \sigma_j\colon U_j \to X$ is a local social choice function in X, and we have $\mathrm{sc}(X) \leq n$.

In [7], the authors define an invariant, denoted $\mathrm{TC}^S(X)$ and called the symmetric topological complexity, and show that, if $\mathrm{TC}^S(X) = n$, then there is a decomposition of $X \times X$ and local symmetric motion planners, that together satisfy the conditions of Theorem 3.4 (see Lemma 8, Corollary 9, and the discussion in-between, of [7]). Combining this with Theorem 3.4, we see that $\mathrm{sc}(X) \leq \mathrm{TC}^S(X)$.

In fact,

Theorem 3.5 *We have the following inequalities:*

$$\mathrm{TC}(X) \leq \mathrm{sc}(X) \leq \mathrm{TC}^S(X).$$

As with $\mathrm{TC}(X)$, the invariants $\mathrm{sc}(X)$ and $\mathrm{TC}^S(X)$ are generally difficult to compute. Indeed, very little is known about $\mathrm{TC}^S(X)$, and no in-depth study of $\mathrm{sc}(X)$ has yet been undertaken. For instance, in [7, Cor.18]), it is shown that, for a circle, we have $\mathrm{TC}^S(S^1) = 3$. On the other hand, we have $\mathrm{TC}(S^1) = 2$. At present, we do not even know whether $\mathrm{sc}(S^1)$ equals 2 or 3! (It must be one or the other, by Theorem 3.5.)

We should say a few words about social choice functions where more than two choices must be aggregated—after all, we want to consider decisions made in *society* and not simply within *couples*. We indicate the ideas here, but omit details. Suppose we seek a social choice function $f\colon X^k \to X$ that satisfies unanimity and anonymity. A continuous such function will only exist when X is contractible, by Theorem 3.2. Therefore, we decompose X^k into pieces, on each of which there is a local social choice function in X. We adapt Definition 3.3 by replacing $X \times X$ with X^k, the interchange map T with a more general permutation of k coordinates, and making all concomitant changes. We keep the condition (iii) of "close compromise" in this setting, too. The result is an invariant which we denote by $\mathrm{sc}_k(X)$, and call the *k-social choice complexity* of the preference space X.

Now there is another variant of topological complexity, denoted by $\mathrm{TC}_k(X)$, for $k \geq 3$, and called the *k-th (higher) topological complexity.* This is introduced in [8], and is defined as follows:

Definition 3.6 *$TC_k(X)$ is the least number n of subsets of X^k, $U_1, \ldots, U_n$ such that the union of the U_j is all of X^k and for each U_j, there exists a map $s_j\colon U_j \to X^I$ with $\pi_k s_j$ equal to the inclusion $i_j\colon U_j \to X^k$., where $\pi_k\colon X^I \to X^k$ is defined by*

$$\pi_k(\gamma) = (\gamma(0), \gamma(1/(k-1)), \gamma(2/(k-1)), \ldots, \gamma((k-2)/(k-1)), \gamma(1)).$$

This is the mathematical formulation of a motion planning problem with k total points to be visited in the configuration space. (Think of a robot that has to pick up materials in various spots before it goes to its final configuration.) For example (see [8]), we have: $\mathrm{TC}_k(S^m) = k+1$ if m is an even number; $\mathrm{TC}_k(S^m) = k$ if m is an odd number; and $\mathrm{TC}_k(T^n) = n(k-1)+1$. Furthermore, for any k, we have $\mathrm{TC}(X) \leq \mathrm{TC}_k(X) \leq \mathrm{TC}_{k+1}(X)$.

Similar reasoning to that used above now gives inequalities as follows:

Theorem 3.7 *For each $k \geq 3$, we have the following inequalities:*

$$\mathrm{TC}(X) \leq \mathrm{TC}_k(X) \leq \mathrm{sc}_k(X).$$

We omit details of this result. We note that there is not, at present, a suitable notion of "symmetric higher topological complexity" that might be placed as an upper bound on $\mathrm{sc}_k(X)$ here as there was in Theorem 3.5.

Social choice complexity sc_k is a measure of the "social homogeneity" of the individuals with preference space X. That is, if sc_k is small, then there are a small number of sets that X^k can be split into so that, within *these* preference sets, compromise can be obtained algorithmically. Of course, the pieces U_j of X^k in this case must be large in order to cover all of X^k. This means that compromise may be found among vast arrays of preferences. This must be due to the topology of the preference space itself. When sc_k is large, then the set of preferences is complicated and only generically small sets of preferences lead to agreeable compromise.

4 Conclusion

Topology is, more and more, finding a place in Applied Mathematics. But this is not necessarily the applied mathematics of the 19th or 20th centuries. It is a new applied mathematics that attempts to provide principles to guide the investigations of 21st century social scientists and biologists as well as physical scientists and engineers.

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References

[1] Y. M. Baryshnikov, *Topological and discrete social choice: in a search of a theory*, Social Choice and Welfare **14** (1997), no. 2, 199–209.

[2] G. Chichilnisky, *Social choice and the topology of spaces of preferences*, Advances in Mathematics **37** (1980), no. 2, 165–176.

[3] G. Chichilnisky and G. Heal, *Necessary and sufficient conditions for a resolution of the social choice paradox*, Journal of Economic Theory **31** (1983), no. 1, 68–87.

[4] B. Eckmann, *Räume mit Mittelbildungen*, Commentarii Mathematici Helvetici **28** (1954), 329–340.

[5] ______, *Social choice and topology: a case of pure and applied mathematics*, Expositiones Mathematicae **22** (2004), no. 4, 385–393.

[6] M. Farber, *Topological complexity of motion planning*, Discrete and Computational Geometry **29** (2003), no. 2, 211–221.

[7] M. Farber and M. Grant, *Symmetric motion planning*, Topology and robotics, Contemp. Math., vol. 438, American Mathematical Society, Providence, RI, 2007, pp. 85–104.

[8] Y. Rudyak, *On higher analogs of topological complexity*, Topology and Its Applications **157** (2010), no. 5, 916–920.

[9] S. Weinberger, *On the topological social choice model*, Journal of Economic Theory **115** (2004), no. 2, 377–384.

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