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Applications of BV Type Spaces

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Preface

Variatio delectat (Cicero)

This survey addresses to the working mathematician who wants to get an idea of the theory and applications of functions of bounded variation without getting drowned too much into technicalities. As the title suggests, the scope is more application-oriented, so the central part is Chapter 4 on existence (and in part uniqueness) results for nonlinear integral equations of Hammerstein type. Since such equations are intimately related to boundary value problems, our results may simultaneously serve as source of existence (and in part uniqueness) results for nonlinear differential equations, subject to various types of boundary conditions.

Since we put the main emphasis on Chapter 4, we will give complete proofs in that chapter. In the previous three chapters we will provide the necessary theoretical background. However, as we are not primarily interested in the theory, no proofs will be given in Chapters 1-3 for results published in easily accessible journals. The only exception are very recent results, mostly obtained by the authors themselves, which have not been published yet.

The plan of this survey is as follows. In Chapter 1 we collect the basic properties of various *spaces of functions of bounded variation*, not only in the sense of the classical Jordan variation, but also for the more general Wiener variation, Riesz variation, and Waterman variation. Given a partition $P = \{t_0, t_1, \ldots, t_{m-1}, t_m\}$ ($m \in \mathbb{N}$ variable) of $[0, 1]$, we consider expressions of the form

- $\sum_{j=1}^{m} |x(t_j) - x(t_{j-1})|$ in case of the Jordan variation (Section 1.1),
- $\sum_{j=1}^{m} |x(t_j) - x(t_{j-1})|^p$ in case of the Wiener variation (Section 1.2),
- $\sum_{j=1}^{m} \frac{|x(t_j) - x(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}}$ in case of the Riesz variation (Section 1.3), and
- $\sum_{n=1}^{\infty} \lambda_n |x(b_n) - x(a_n)|$ in case of the Waterman variation (Section 1.4).

Here the precise requirements on p, λ_n, a_n and b_n will be specified later. In the first chapter we will discuss, for each of these variations, the algebraic and analytical properties of the corresponding function spaces BV, WBV_p, RBV_p, and ΛBV, with a particular emphasis on those properties which will be important in subsequent chapters.

Chapter 2 is concerned with some classes of *linear operators* (multiplication, substitution, and integral operators) in such spaces. More precisely, for a given function $\varphi : [0, 1] \to [0, 1]$, by Σ_φ we denote the *substitution operator* defined by

(1) $$\Sigma_\varphi(x)(t) := x(\varphi(t)),$$

while for a given function $\mu : [0, 1] \to \mathbb{R}$, by M_μ we consider the *multiplication operator* defined by

(2) $$M_\mu(x)(t) := \mu(t)x(t).$$

If X is a function space over $[0,1]$, the first problem consists in characterizing all $\varphi : [0,1] \to [0,1]$ such that $\Sigma_\varphi(X) \subseteq X$, and all $\mu : [0,1] \to \mathbb{R}$ such that $M_\mu(X) \subseteq X$. For some spaces X this is easy, for others highly nontrivial.

The definition of integral operators is standard. Given a function $k : [0,1] \times [0,1] \to \mathbb{R}$, by K we denote by

$$K(x)(t) := \int_0^1 k(t,s)x(s)\,ds \tag{3}$$

the *integral operator* defined by k. As before, we are interested in conditions on k, possibly both necessary and sufficient, under which K maps a certain function space X into itself.

All operators studied in Chapter 2 are linear. In Chapter 3 we discuss two classes of *nonlinear operators*, namely composition and superposition operators. The *composition operator* C_f generated by some function $f : \mathbb{R} \to \mathbb{R}$ acting on functions $x : [0,1] \to \mathbb{R}$ is defined by

$$C_f(x)(t) := f(x(t)) \qquad (0 \le t \le 1). \tag{4}$$

More generally, the *superposition operator* S_f generated by some function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ acting on functions $x : [0,1] \to \mathbb{R}$ is defined by

$$S_f(x)(t) := f(t, x(t)) \qquad (0 \le t \le 1). \tag{5}$$

In spite of their simple form, the operators (4) and (5) exhibit a strange and unexpected behaviour even in such simple spaces like those introduced in the first chapter. For instance, while we obtain boundedness of the operator (4) as a "fringe benefit" whenever it maps BV, WBV_p, RBV_p, or ΛBV into itself, to find criteria for its continuity is a very hard problem which remained open for many years. We will collect the most important results and illustrate them by several examples which are scattered over a vast literature.

Finally, the theoretical results of the second and third chapter are applied in Chapter 4, as mentioned above, to several *nonlinear integral equations* involving Hammerstein and Hammerstein-Volterra operators. A *Hammerstein equation* has the form

$$x(t) = g(t) + \lambda \int_0^1 k(t,s) f(x(s))\,ds \qquad (0 \le t \le 1), \tag{6}$$

with $\lambda \in \mathbb{R}$, where $g : [0,1] \to \mathbb{R}$, $k : [0,1] \times [0,1] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given functions, and the function $x : [0,1] \to \mathbb{R}$ is unknown. Using the operators (3) and (4), we may rewrite (6) as operator equation

$$x = g + \lambda K(C_f(x)), \tag{7}$$

and the structure of (7) suggests to apply fixed point principles. More generally, the Hammerstein equation

$$x(t) = g(t) + \lambda \int_0^1 k(t,s) f(s, x(s))\,ds \qquad (0 \le t \le 1), \tag{8}$$

where now $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ depends also on s, leads to the fixed point equation

$$x = g + \lambda K(S_f(x)) \tag{9}$$

involving the superposition operator (5). Occasionally we will also consider *Hammerstein-Volterra equations*, where the upper integration limit 1 in (6) and (8) is replaced by the variable limit t.

It is well known that Hammerstein equations naturally occur in the study of boundary value problems, while Hammerstein-Volterra equations naturally occur in the study of initial value problems. So every existence (and uniqueness) result we obtain for the operator equations (7) and (9) leads to a corresponding existence (and uniqueness) result for boundary value or initial value problems. Such problems will be studied in a forthcoming paper.

This survey is the outcome of several meetings and fruitful discussions of the authors. Our profound gratitude goes to the Mathematical Research Institute in Oberwolfach, Germany, where the authors from Poland and Germany spent a couple of weeks in the framework of the "Research in Pairs" Programme and enjoyed the unique spirit, excellent working facilities, and overwhelming hospitality of Oberwolfach. The fifth author (S.R.) gratefully acknowledges repeated hospitality of the Adam Mickiewicz University of Poznań, Poland, while the first author (J.A.) in addition acknowledges hospitality of the Central University of Venezuela in Caracas. Finally, we are indebted to the Oberwolfach Institute for the opportunity to publish this survey in the MFO Preprint Series.

We do hope that readers who are not experts in the theory and applications of functions of bounded variation but want to get an idea of the developments in the last decades, as well as a glimpse of the diversity in which current research is moving, will find this survey both readable and stimulating.

Caracas, Poznań, Würzburg, January 2019

The authors

Chapter 1. BV Spaces

In this chapter we are going to collect all notions and facts on BV type spaces we need in the sequel. Without loss of generality, we will restrict ourselves to functions $x : [0,1] \to \mathbb{R}$; every result easily carries over to functions on $[a,b]$ through the linear isomorphisms $t \mapsto a + (b-a)t$ which respects, up to a constant, the finiteness of every variation we are going to define.

We will basically be working in the setting of four types of variation. Given a partition $P = \{t_0, t_1, \ldots, t_{m-1}, t_m\}$ ($m \in \mathbb{N}$ variable), such that

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1, \tag{1.0.1}$$

we consider expressions of the form

- $\displaystyle\sum_{j=1}^{m} |x(t_j) - x(t_{j-1})|$ in case of the Jordan variation (Section 1.1),
- $\displaystyle\sum_{j=1}^{m} |x(t_j) - x(t_{j-1})|^p$ in case of the Wiener variation (Section 1.2),
- $\displaystyle\sum_{j=1}^{m} \frac{|x(t_j) - x(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}}$ in case of the Riesz variation (Section 1.3), and
- $\displaystyle\sum_{n=1}^{\infty} \lambda_n |x(b_n) - x(a_n)|$ in case of the Waterman variation (Section 1.4).

Here the precise requirements on p, λ_n, a_n and b_n will be specified later. In this chapter we will discuss, for each of these variations, the algebraic and analytical properties of the correspponding spaces of functions of bounded variation, with a particular emphasis on those properties which will be important in subsequent chapters. The proofs of all statements given in this chapter, together with many more examples and remarks, may be found in the book [ABM].

1.1. Functions of bounded Jordan variation. Before giving the definition of the classical space of functions of bounded variation, we recall the definition of several spaces of *continuous* functions for further use. Since we always consider functions over the interval $[0,1]$, we drop this interval in any notation, i.e., we simply write X instead of $X[0,1]$ for a function space X.

By C we denote the linear spaces of all continuous functions $x : [0,1] \to \mathbb{R}$, equipped with the usual norm

$$\|x\|_C := \max\{|x(t)| : 0 \le t \le 1\}, \tag{1.1.1}$$

and by C^1 the linear subspaces of all continuously differentiable functions $x : [0,1] \to \mathbb{R}$, equipped with either the norm

$$\|x\|_{C^1} := |x(0)| + \|x'\|_C$$

or the (equivalent) norm

$$|||x|||_{C^1} := \|x\|_C + \|x'\|_C.$$

Moreover, we will sometimes need some intermediate spaces between C^1 and C. Recall that a function $x : [0,1] \to \mathbb{R}$ is called *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$|x(s) - x(t)| \le L|s-t| \qquad (0 \le s, t \le 0). \tag{1.1.2}$$

More generally, x is called *Hölder continuous* (or *γ-Lipschitz continuous* for $0 < \gamma \leq 1$) if there exists a constant $L > 0$ such that

$$|x(s) - x(t)| \leq L|s-t|^\gamma \qquad (0 \leq s,t \leq 1). \tag{1.1.3}$$

We denote the set of all Lipschitz continuous functions on $[0,1]$ by Lip, and the set of all γ-Lipschitz continuous functions on $[0,1]$ by Lip_γ. Writing

$$lip(x) = lip(x;[0,1]) := \sup_{s\neq t} \frac{|x(s)-x(t)|}{|s-t|}$$

for the minimal Lipschitz constant L in (1.1.2) and, for $0 < \gamma \leq 1$,

$$lip_\gamma(x) = lip_\gamma(x;[0,1]) := \sup_{s\neq t} \frac{|x(s)-x(t)|}{|s-t|^\gamma}$$

for the minimal Hölder constant L in (1.1.3), one may show that the spaces $Lip = Lip_1$ and Lip_γ, equipped with the norms

$$\|x\|_{Lip} := |x(0)| + lip(x),$$

and

$$\|x\|_{Lip_\gamma} := |x(0)| + lip_\gamma(x),$$

respectively, are Banach spaces. The inclusions

$$C^1 \subseteq Lip \subseteq Lip_\alpha \subseteq Lip_\beta \subseteq C \quad (\alpha \geq \beta) \tag{1.1.4}$$

show that Lipschitz and Hölder continuity is situated "between" continuity and continuous differentiability, and the space Lip_γ becomes smaller if γ increases. In particular, the function $x_\tau(t) := t^\tau$ belongs to $Lip_\tau \setminus Lip$ for $0 < \tau < 1$. All inclusions in (1.1.4) are continuous imbeddings which are strict for $0 < \beta < \alpha < 1$. The last imbedding in (1.1.4) is even compact for $\beta > 0$, since a bounded set in Lip_β is clearly equicontinuous.

Now we define and study two parameter-dependent families of functions which will be quite helpful in what follows to illustrate our abstract results and to construct examples.

Definition 1.1.1. Given $(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$, consider the function $\omega_{\alpha,\beta} : [0,1] \to \mathbb{R}$ defined by

$$\omega_{\alpha,\beta}(t) := \begin{cases} t^\alpha \sin t^\beta & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0. \end{cases} \tag{1.1.5}$$

We will call (1.1.5) the *oscillation function* associated to the pair $(\alpha,\beta) \in \mathbb{R}^2$ in what follows. □

Of course, the function (1.1.5) is "oscillatory" only for $\beta < 0$; however, we keep this name for all values of α and β. It is instructive to determine all values of $(\alpha,\beta) \in \mathbb{R}^2$ for which $\omega_{\alpha,\beta}$ belongs to a certain function space. To begin, we do this for the spaces occurring in the chain of inclusions (1.1.4).

Proposition 1.1.2. *For $\alpha,\beta \in \mathbb{R}$, let $\omega_{\alpha,\beta}$ be defined by* (1.1.5). *Then the following holds.*

(a) $\omega_{\alpha,\beta} \in C$ *if and only if $\alpha > 0$ and β is arbitrary, or $\alpha \leq 0$ and $\beta > -\alpha$.*

(b) $\omega_{\alpha,\beta} \in Lip$ *if and only if α is arbitrary and $\beta \geq 1-\alpha$.*

(c) $\omega_{\alpha,\beta} \in Lip_\gamma$ *if and only if α is arbitrary and $\beta \geq 1-\alpha/\gamma$.*

(d) $\omega_{\alpha,\beta} \in C^1$ *if and only if α is arbitrary and $\beta > 1-\alpha$.*

The proof of Proposition 1.1.2 is straightforward and may be found in [A]. The oscillatory functions (1.1.5) may be used to show that all inclusions in (1.1.4) are strict. For instance, $\omega_{-1,\beta} \in C \setminus Lip_\gamma$ for $1 < \beta < 1 + 1/\gamma$, $\omega_{-1,\beta} \in Lip_\gamma \setminus Lip$ for $\beta = 1 + 1/\gamma$, and $\omega_{-1,\beta} \in Lip \setminus C^1$ for $\beta = 2$.

The second useful family of functions is constructed over the interval $[0, 1]$ as follows.

Definition 1.1.3. Given $\theta > 0$, let

$$c_n := 2^{-n}, \qquad d_n := n^{-\theta}, \tag{1.1.6}$$

and define $\zeta_\theta : [0, 1] \to \mathbb{R}$ by $\zeta_\theta(0) := 0$ and

$$\zeta_\theta(t) := \begin{cases} \sum_{k=1}^{n} (-1)^{k+1} d_k = 1 - \frac{1}{2^\theta} + \frac{1}{3^\theta} - + \ldots + \frac{(-1)^{n+1}}{n^\theta} & \text{for } t = 1 - c_n = \frac{2^n - 1}{2^n}, \\ \text{linear} & \text{otherwise.} \end{cases} \tag{1.1.7}$$

Geometrically, ζ_θ increases linearly by d_1 on the interval $[0, c_1]$ so that $\zeta_\theta(c_1) = d_1$. Afterwards ζ_θ decreases linearly by d_2 on $[c_1, c_1 + c_2]$, increases linearly by d_3 on $[c_1 + c_2, c_1 + c_2 + c_3]$, decreases linearly by d_4 on $[c_1 + c_2 + c_3, c_1 + c_2 + c_3 + c_4]$, and so on. For this reason we call (1.1.7) a *zigzag function* of order θ. □

It follows from the construction and continuity of the zigzag function ζ_θ that

$$\zeta_\theta(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^\theta} < \infty$$

for all $\theta > 0$. Again, it is illuminating to determine all values of $\theta > 0$ for which the zigzag function (1.1.7) belongs to the function classes introduced so far. Of course, the function ζ_θ is always continuous, by construction, but not differentiable at its peaks. So it is only interesting to find all $\theta > 0$ for which ζ_θ is Hölder (in particular, Lipschitz) continuous. Choosing c_n and d_n as in (1.1.6) we obtain

$$\sup \{d_n c_n^{-\gamma} : n = 1, 2, 3, \ldots\} = \sup \{n^{-\theta} 2^{n\gamma} : n = 1, 2, 3, \ldots\} = \infty,$$

since the exponential term $2^{n\gamma}$ grows essentially faster than the power type term n^θ. So we get the somewhat disappointing result that, loosely speaking, the zigzag function ζ_θ does not "feel" the dependence on θ, as the oscillation function $\omega_{\alpha,\beta}$ feels the dependence on α and β. We summarize with the following

Proposition 1.1.4. *The zigzag function* (1.1.7) *belongs to C for all values of $\theta > 0$, but does not belong to Lip_γ for any $\gamma \in (0, 1]$.*

The oscillation function (1.1.5) and the zigzag function (1.1.7) are useful for constructing counterexamples. For further reference, we collect in the following Table 1.1 the values of α, β, and θ, respectively, for which these functions belong to the function spaces occurring in (1.1.4).

	The function $\omega_{\alpha,\beta}$	*The function ζ_θ*
belongs to C if and only if	$\alpha > 0$ or $\alpha \leq 0$ and $\alpha + \beta > 0$	always
belongs to C^1 if and only if	$\alpha + \beta > 1$	never
belongs to Lip if and only if	$\alpha + \beta \geq 1$	never
belongs to Lip_γ if and only if	$\alpha + \beta\gamma \geq \gamma$	never

Table 1.1: Oscillation functions and zigzag functions

An essential extension of this table will be given in Table 1.3 at the end of this chapter.

Let us now recall the definition of the classical space BV which, as far as we know, goes back to Camille Jordan [J] and is our main object of attention in this survey. Throughout the following, we denote by $\mathcal{P}$ the family of all *partitions* of the interval $[0,1]$, i.e., all finite sets $P = \{t_0, t_1, \ldots, t_{m-1}, t_m\}$ ($m \in \mathbb{N}$ variable) satisfying (1.0.1).

Definition 1.1.5. Given a partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}$ and a function $x : [0,1] \to \mathbb{R}$, the nonnegative real number

$$Var(x,P) = Var(x,P;[0,1]) := \sum_{j=1}^{m} |x(t_j) - x(t_{j-1})| \tag{1.1.8}$$

is called the *variation* (or *Jordan variation*) of x on $[0,1]$ with respect to P. Moreover, the (possibly infinite) number

$$Var(x) = Var(x;[0,1]) := \sup\{Var(x,P;[0,1]) : P \in \mathcal{P}\}, \tag{1.1.9}$$

where the supremum is taken over all partitions of $[0,1]$, is called the *total* (Jordan) *variation* of x on $[0,1]$. In case $Var(x) < \infty$ we say that x is a *function of bounded variation* (or *function of bounded Jordan variation*) on $[0,1]$ and write $x \in BV$. □

For further use we collect in the following proposition some important properties of the quantities (1.1.8) and (1.1.9).

Proposition 1.1.6. *The quantities* (1.1.8) *and* (1.1.9) *have the following properties.*

(a) *The variation* (1.1.9) *is subadditive, i.e.,*

$$Var(x+y) \le Var(x) + Var(y)$$

for $x, y : [0,1] \to \mathbb{R}$.

(b) *The variation* (1.1.9) *is homogeneous, i.e.,*

$$Var(\mu x) = |\mu| Var(x)$$

for $\mu \in \mathbb{R}$.

(c) *The estimate*

$$|x(s) - x(t)| \le Var(x;[s,t]) \tag{1.1.10}$$

holds for $0 \le s < t \le 1$.

(d) *Every function* $x \in BV$ *is bounded with*

$$\|x\|_\infty \le |x(0)| + Var(x),$$

where

$$\|x\|_\infty := \sup\{|x(t)| : 0 \le t \le 1\}. \tag{1.1.11}$$

(e) *From* $x \in BV$ *it follows that* $|x| \in BV$ *with*

$$Var(|x|) \le Var(x). \tag{1.1.12}$$

(f) *The variation* (1.1.8) *is monotone with respect to partitions, i.e.,*

$$Var(x,P) \le Var(x,Q)$$

for $P, Q \in \mathcal{P}$ *with* $P \subseteq Q$.

We make some comments on Proposition 1.1.6. The simple example of the Dirichlet type function $x = \chi_{[0,1]\cap\mathbb{Q}} - \chi_{[0,1]\setminus\mathbb{Q}}$ shows that the converse of (e) is not true in general. One can prove, however, that the converse *is* true under an additional hypothesis: if $|x| \in BV$ and x has the intermediate value property (which means that $[f(a), f(b)] \subseteq f([a, b])$ for each subinterval $[a, b] \subseteq [0, 1]$), then $x \in BV$.

Proposition 1.1.6 (a) and (b) show that BV is a linear space. It is easy to see that $x \in BV$ with $x(t) \neq 0$ in general does not imply that $1/x \in BV$. For example, if

$$x(t) := \begin{cases} 1 & \text{for} \quad t = 0, \\ t & \text{for} \quad 0 < t \leq 1, \end{cases}$$

it is clear that $x \in BV$, but $1/x \notin BV$, since $1/x$ is unbounded on $[0, 1]$. If we replace the condition $x(t) \neq 0$ by the stronger condition $|x(t)| \geq \delta$ for some $\delta > 0$, however, then $1/x \in BV$ with

$$Var(1/x) \leq \frac{1}{\delta^2} Var(x). \tag{1.1.13}$$

This fact is useful for studying multiplication operators in BV and related spaces, see Section 2.2.

Equipped with the norm

$$\|x\|_{BV} := |x(0)| + Var(x) \qquad (x \in BV), \tag{1.1.14}$$

the linear space BV is a Banach space which is continuously imbedded into the space $B = B[0, 1]$ of all bounded functions on $[0, 1]$ with norm (1.1.11). Moreover, BV is an algebra with

$$Var(xy) \leq \|x\|_\infty Var(y) + \|y\|_\infty Var(x)$$

for all $x, y \in BV$. Even better, we have

$$\|xy\|_{BV} \leq \|x\|_{BV} \|y\|_{BV} \qquad (x, y \in BV)$$

which means that BV is a *normalized algebra.*

Occasionally, we will also consider BV equipped with one of the norms

$$|||x|||_{BV} := \|x\|_\infty + Var(x) \tag{1.1.15}$$

or

$$|||x|||_{BV} := \|x\|_{L_1} + Var(x), \tag{1.1.16}$$

which are both equivalent to the norm (1.1.14) and so also turn BV into a Banach space.

Clearly, every monotone function $x : [0, 1] \to \mathbb{R}$ belongs to BV with

$$Var(x) = |x(1) - x(0)|.$$

This provides a link between monotone functions and functions of bounded variation. One could ask if this could be in some sense inverted. However, this is very far from being true, because there exist functions of bounded variation which are not monotone on *any* interval (see [ABM, Example 1.4]).

Now comes an important point. Although functions of bounded variation have no monotonicity behavior at all, there is a natural interconnection between bounded variation and monotonicity which is the statement of the classical *Jordan decomposition theorem*:

Theorem 1.1.7 [J]. *A function $x : [0, 1] \to \mathbb{R}$ has bounded variation if and only if it may be represented in the form $x = y - z$, where both y and z are monotonically increasing functions.*

For the proof of Theorem 1.1.7 one takes as y the *variation function* $V_x : [0,1] \to \mathbb{R}$ of x defined by $V_x(t) := Var(x;[0,t])$, which is trivially increasing in t. The fact that also $z := x - y$ is increasing follows from the simple estimate

$$x(t) - x(s) \leq Var(x;[s,t]) = V_x(t) - V_x(s) = y(t) - y(s),$$

which holds by (1.1.10) and the additivity of the variation with respect to intervals. In what follows, we will refer to the representation $x = y - z$ constructed in Theorem 1.1.7 as the *Jordan decomposition* of $x \in BV$.

Obviously, the monotone functions on $[0,1]$ do not form a linear space, as may easily be seen by considering $x(t) = t^2$ and $y(t) = 1 - t$. Proposition 1.1.6 (a) and Theorem 1.1.7 show that BV is the *linear hull* (or *span*) of the monotone functions, i.e., the smallest linear space which contains all monotone functions. This is another way of introducing functions of bounded variation.

Theorem 1.1.7 explains why many "nice" properties of monotone functions carry over to functions of bounded variation. In particular, a function $x \in BV$ has at most countably many points of discontinuity in $[0,1]$, all being of first kind (jumps) or removable. Bounded functions which do not have discontinuities of second kind are usually called *regular*; so we have the inclusions

$$BV \subseteq R \subseteq B, \tag{1.1.17}$$

where $R = R[0,1]$ denotes the linear space of all regular functions $x : [0,1] \to \mathbb{R}$. These inclusions show that, roughly speaking, functions of bounded variation are not "too discontinuous". On the other hand, the following well known example from every first-year calculus course shows that a continuous function need not have bounded variation.

Example 1.1.8. Consider the oscillation function $\omega_{1,-1} : [0,1] \to \mathbb{R}$ from (1.1.5), i.e.,

$$\omega_{1,-1}(t) = \begin{cases} t \sin \dfrac{1}{t} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0. \end{cases}$$

By Proposition 1.1.2 (a), $\omega_{1,-1}$ is continuous on $[0,1]$. However, by choosing partitions of alternating maxima and minima one may easily show that the variation of $\omega_{1,-1}$ is unbounded. □

Note that, if we interchange the role of the amplitude and the frequency in Example 1.1.8, which means that we consider the function

$$\omega_{-1,1}(t) = \begin{cases} \dfrac{1}{t} \sin t & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

we get a discontinuous function with bounded variation. The fact that $\omega_{1,-1} \notin BV$, but $\omega_{-1,1} \in BV$ follows from the following general

Proposition 1.1.9. *The oscillation function* (1.1.5) *with* $\alpha \in \mathbb{R}$ *arbitrary, belongs to* BV *if and only if* $\beta \geq 0$ *and* $\alpha + \beta \geq 0$, *or* $\beta < 0$ *and* $\alpha + \beta > 0$.

A parallel result for zigzag functions reads as follows. Since

$$Var(\zeta_\theta) = \sum_{k=1}^{\infty} \frac{1}{k^\theta}, \tag{1.1.18}$$

the zigzag function (1.1.7) belongs to BV if and only if $\theta > 1$. So the function ζ_θ may serve, for every $\theta \in (0,1]$, as another example of a continuous functions with unbounded variation.

A comparison between Proposition 1.1.2 (a) and Proposition 1.1.9 shows that the set of admissible pairs (α, β) for BV is contained in the corresponding set for C, although a function of bounded variation need of course not be continuous. This is a consequence of the fact that all oscillation functions $\omega_{\alpha,\beta} \in BV$ have the intermediate value property. In general, every function $x \in BV$ which has the intermediate value property is continuous. In fact, having bounded variation, we know that all discontinuities of x, if there are any, are jumps. On the other hand, jumps are excluded by the intermediate value property, and so the claim follows.

Conversely, it is also interesting to study subclasses of C which are contained in BV. The most important such subclass is given in the following

Definition 1.1.10. A function x is called *absolutely continuous* on $[0, 1]$ if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that, given a collection of non-overlapping subintervals $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$ of $[0, 1]$ satisfying

$$\sum_{j=1}^{k} (b_j - a_j) \leq \delta,$$

we have

$$\sum_{j=1}^{k} |x(b_j) - x(a_j)| \leq \varepsilon.$$

As usual, we denote the set of all absolutely continuous functions on $[0, 1]$ by AC. □

Of course, absolute continuity implies continuity. We point out that it is important to choose only mutually non-overlapping subintervals $[a_j, b_j] \subseteq [0, 1]$ in Definition 1.1.10. If we drop this assumption, we end up with the essentially smaller class of Lipschitz continuous functions.

As for the class BV, from $x, y \in AC$ and $\mu \in \mathbb{R}$ it follows that $|f|$, $f + g$, μf and fg also belong to AC. Moreover, in contrast to the class BV here it suffices to require $x(t) \neq 0$ to guarantee that $1/x \in AC$, since a continuous function which is nonzero on a compact interval is bounded away from zero.

The relation of AC with the previously defined function classes is given by the chain of inclusions

$$Lip \subseteq AC \subseteq C \cap BV \subseteq BV. \tag{1.1.19}$$

In the following Example 1.1.11 we show that all inclusions in (1.1.19) are strict.

Example 1.1.11. The function $x(t) := \sqrt{t}$ is absolutely continuous, but not Lipschitz continuous on $[0, 1]$. The well known Cantor function is continuous on $[0, 1]$ and, being monotonically increasing, has also bounded variation, but is not absolutely continuous. Finding a discontinuous function of bounded variation is trivial. □

Absolutely continuous functions are important in two aspects: first, they may be precisely characterized by three different properties and, second, they are precisely primitives of L_1-functions. This is the contents of the following theorem.

Theorem 1.1.12. (a) *A function x is absolutely continuous if and and only if x is continuous, belongs to BV, and has the Luzin property, i.e., maps nullsets into nullsets.*

(b) *A function x is absolutely continuous on $[0, 1]$ if and and only if it may be represented in the form*

$$x(t) = x(0) + \int_0^t y(s)\, ds \qquad (0 \leq t \leq 1),$$

for some function $y \in L_1$; here $y(t) = x'(t)$ a.e. on $[0, 1]$.

Theorem 1.1.12 (a) explains why we have chosen the Cantor function as an example of a function in $(C \cap BV) \setminus AC$ in Example 1.1.11: it is well-known that this function maps the Cantor nullset into a set of positive Lebesgue measure, so it fails to have the Luzin property.

Theorem 1.1.12 (b) suggests to consider the norm

$$\|x\|_{AC} := |x(0)| + \|x'\|_{L_1} \qquad (x \in AC) \tag{1.1.20}$$

on AC; equipped with this norm, AC becomes a Banach space. We will come back to the space AC when we discuss functions of bounded Riesz variation in Section 1.3.

In view of the inclusion $Lip \subseteq BV$ one could ask if the space BV also contains the larger Hölder space Lip_γ for some $\gamma < 1$. This is not true, because one can construct, for fixed $\gamma \in (0,1)$, a function $x \in Lip_\gamma \setminus BV$ (see [ABM, Example 1.23]) and even a function

$$x \in \left(\bigcap_{0<\gamma<1} Lip_\gamma \right) \setminus BV$$

(see [ABM, Example 1.24]). On the other hand, the "dual" construction is also possible: the function

$$x(t) := \begin{cases} \dfrac{1}{\log(2/t)} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

satisfies

$$x \in AC \setminus \left(\bigcup_{0<\gamma<1} Lip_\gamma \right).$$

So far we have studied algebraic properties of BV, as well as relations with other classes of function, but not compositions of function. This will be done in great detail in Sections 2.1 and 3.1. We just confine ourselves to an interesting characterization of BV functions which can be found in Federer's book [F].

Theorem 1.1.13. *A function $z : [0,1] \to \mathbb{R}$ belongs to BV if and only if it may be represented as composition $z = y \circ x$, where $x : [0,1] \to [0,1]$ is increasing and $y \in Lip$ with Lipschitz constant $L = 1$.*

The statement of Theorem 1.1.13 is somewhat surprising: although functions of bounded variation can have infinitely many discontinuities, these discontinuities may be "smoothed out", after a monotone change of variables, by a nonexpansive map. A parallel result to Theorem 1.1.13 with $y \in Lip$ replaced by $y \in Lip_\gamma$ will be given in Theorem 1.2.4 in the next section.

Proposition 1.1.6 (d) shows that a sequence which converges in the BV-norm (1.1.14) is uniformly convergent. The following theorem establishes an important relation between convergence in BV and pointwise convergence.

Theorem 1.1.14. (a) *Let $(x_n)_n$ be a sequence in BV which converges pointwise on $[0,1]$ to some function x. Then*

$$Var(x;[0,1]) \leq \liminf_{n\to\infty} Var(x_n;[0,1]).$$

Consequently, the pointwise limit of a sequence of functions with equibounded variations on the interval $[0,1]$ is a function of bounded variation on $[0,1]$.

(b) *Let $(x_n)_n$ be a bounded sequence in BV with respect to the norm* (1.1.14). *Then $(x_n)_n$ contains a subsequence which converges pointwise on $[0,1]$ to some $x \in BV$.*

Part (b) of Theorem 1.1.14 is known as *Helly's selection theorem* in the literature. It may be considered as analogue to the well-known Arzelà-Ascoli compactness criterion in the space C of continuous functions.

To conclude this section we consider a curious fact which shows that modifying the partitions in the definition of BV may lead to unexpected phenomena. The usual space $BV[0,1]$ is defined by the condition $Var(x;[0,1]) < \infty$, see (1.1.9), where the supremum is taken over all partitions $\{t_0, t_1, \ldots, t_m\}$ of $[0,1]$ with

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1. \tag{1.1.20}$$

Instead, let us define a space $SBV = SBV[0,1]$ to consist of all functions x with the following property: There exists $M > 0$ such that, for all collections $t_0, t_1, \ldots, t_m$ of points $t_j \in [0,1]$ *which are not necessarily ordered* by (1.1.20), we have

$$\sum_{j=1}^{m} |t_j - t_{j-1}| \leq 1 \Rightarrow \sum_{j=1}^{m} |x(t_j) - x(t_{j-1})| \leq M, \tag{1.1.21}$$

and call the elements of SBV *functions of superbounded variation.* Clearly, $SBV \subseteq BV$. The following example shows that the inclusion is strict.

Example 1.1.15. Let $x(t) := \chi_{[0,1/2)}(t)$. Then $x \in BV[0,1]$ with $Var(x) = 1$. However, $x \notin SBV[0,1]$ which may be seen by considering the collection of points

$$t_0 = 0,\ t_1 = \frac{1}{2},\ t_2 = \frac{1}{2} - \frac{1}{4},\ t_3 = \frac{1}{2},\ t_4 = \frac{1}{2} - \frac{1}{8},\ t_5 = \frac{1}{2},\ \ldots\ t_{2k} = \frac{1}{2} - \frac{1}{2^{(k+2)}},\ t_{2k+1} = \frac{1}{2}$$

for k sufficiently large. □

The question arises to characterize the space SBV. A first conjecture could be that $SBV = BV \cap C$ or even $SBV = BV \cap AC$. However, this is false; the next example shows even more.

Example 1.1.16. Let $x(t) := \sqrt{t}$. Then $x \in BV \cap Lip_{1/2} \cap AC$ with $Var(x) = 1$. To show that $x \notin SBV$ we use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Consider the collection of points

$$t_1 = 0,\ t_2 = \frac{3}{\pi^2},\ t_3 = 0,\ t_4 = \frac{3}{\pi^2}\frac{1}{4},\ \ldots,\ t_{2k-1} = 0,\ t_{2k} = \frac{3}{\pi^2}\frac{1}{k^2}.$$

Then

$$\sum_{j=2}^{2k} |t_j - t_{j-1}| = 2\frac{3}{\pi^2}\sum_{j=1}^{2k}\frac{1}{j^2} \leq \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1.$$

Suppose that $x \in SBV$, and choose $M > 0$ according to (1.1.21). Since

$$\sum_{j=1}^{2k} |x(t_j) - x(t_{j-1})| = 2\sum_{j=1}^{2k}\frac{\sqrt{3}}{\pi}\frac{1}{j},$$

we get a contradiction for k sufficiently large. □

The next theorem characterizes the class SBV. Since this result is new, we also sketch the proof.

Theorem 1.1.17. *The equality $SBV = Lip$ holds.*

Proof. The inclusion $Lip \subseteq SBV$ is trivial. So assume that $x \in SBV$, and fix two points a, b with $0 \leq a < b \leq 1$. Let $n = \mathrm{ent}(1/(b-a))$ denote the integer part of $1/(b-a)$; then

$$1 \leq n \leq \frac{1}{b-a} < n+1. \tag{1.1.22}$$

Now we choose the partition

$$t_0 := a,\ t_1 := b,\ t_2 := a,\ t_3 := b,\ \dots,$$

where we repeat a and b so often that the partition contains $n+1$ points. Then

$$\sum_{j=1}^{n} |t_j - t_{j-1}| = n(b-a) \le 1,$$

by (1.1.22). So we find $M > 0$ according to (1.1.21) such that

$$\sum_{j=1}^{n} |x(t_j) - x(t_{j-1})| = n|x(b) - x(a)| \le M.$$

This implies that

$$|x(b) - x(a)| \le \frac{M}{n} = \frac{M}{n}\frac{n+1}{n+1} \le M(b-a)\left(1+\frac{1}{n}\right) \le 2M(b-a),$$

where we again used (1.1.22) for the second $\le$-sign. Since a and b were arbitrary, we conclude that $x \in Lip$ with Lipschitz constant $2M$. ■

1.2. Functions of bounded Wiener variation. Now we consider a certain extension of the spaces BV which was introduced in 1924 by Wiener [Wi].

Definition 1.2.1. Given a real number $p \ge 1$, a partition $P = \{t_0, t_1, \dots, t_m\} \in \mathcal{P}$, and a function $x : [0,1] \to \mathbb{R}$, the nonnegative real number

$$WVar_p(x,P) = WVar_p(x,P;[0,1]) := \sum_{j=1}^{m} |x(t_j) - x(t_{j-1})|^p \tag{1.2.1}$$

is called the *Wiener variation* of x on $[0,1]$ with respect to P, while the (possibly infinite) number

$$WVar_p(x) = WVar_p(x;[0,1]) := \sup\left\{WVar_p(x,P;[0,1]) : P \in \mathcal{P}\right\}, \tag{1.2.2}$$

where the supremum is taken over all partitions of $[0,1]$, is called the *total Wiener variation* of x on $[0,1]$. In case $WVar_p(x) < \infty$ we say that x has *finite Wiener variation* (or *finite p-variation in Wiener's sense*) on $[0,1]$ and write $x \in WBV_p$. □

It is useful to complete this definition by defining WBV_∞ to consist of all regular functions. In the following Proposition 1.2.2 which to some extent is parallel to Proposition 1.1.6 we collect some properties of the quantities (1.2.1) and (1.2.2).

Proposition 1.2.2. *The quantities* (1.2.1) *and* (1.2.2) *have the following properties.*
(a) *The p-th root of the variation* (1.2.2) *is subadditive, i.e.,*

$$WVar_p(x+y)^{1/p} \le WVar_p(x)^{1/p} + WVar_p(y)^{1/p}$$

for $x, y : [0,1] \to \mathbb{R}$.
(b) *The p-th root of the variation* (1.2.2) *is homogeneous, i.e.,*

$$WVar_p(\mu x)^{1/p} = |\mu| WVar_p(x)^{1/p}$$

for $\mu \in \mathbb{R}$.
(c) *The estimate*

$$|x(s) - x(t)| \le WVar_p(x;[s,t])^{1/p} \tag{1.2.3}$$

holds for $0 \le s < t \le 1$.

(d) *Every function* $x \in WBV_p$ *is bounded with*

$$\|x\|_\infty \le |x(0)| + WVar_p(x)^{1/p},$$

where the norm $\|\cdot\|_\infty$ *is given by* (1.1.11).

(e) *From* $x \in WBV_p$ *it follows that* $|x| \in WBV_p$ *with*

$$WVar_p(|x|) \le WVar_p(x).$$

As before, it is not hard to show that the linear space WBV_p equipped with the norm

(1.2.4) $$\|x\|_{WBV_p} := |x(0)| + WVar_p(x)^{1/p} \qquad (x \in WBV_p)$$

or the equivalent norm

$$|||x|||_{WBV_p} := \|x\|_\infty + WVar_p(x)^{1/p}$$

is a Banach algebra.

The reader may have noticed that we did not state an analogue to Proposition 1.1.6 (f) on the monotonicity of variations with respect to partitions. The reason is that for $p > 1$ this does not hold:

Example 1.2.3. Consider the function $x : [0, 1] \to \mathbb{R}$ defined by

$$x(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < 1/2, \\ 1 & \text{for} \quad t = 1/2, \\ 2 & \text{for} \quad 1/2 < t \le 1, \end{cases}$$

and consider the partitions $P := \{0, 1\}$ and $Q := \{0, 1/2, 1\} \supset P$. An easy calculation shows then that $WVar_p(x, P) = 2^p$ and $WVar_p(x, Q) = 2 < 2^p$ for $p > 1$. □

It follows immediately from the definition that $WBV_1 = BV$. Moreover, it is easy to show that the space WBV_p is increasing with respect to p. So we have the chain of inclusions

(1.2.5) $$BV \subseteq WBV_p \subseteq WBV_q \subseteq R \qquad (1 \le p \le q).$$

Furthermore, the inclusion $Lip \subseteq BV$ stated in (1.1.19) becomes here

(1.2.6) $$Lip_{1/p} \subseteq WBV_p.$$

This means that we have to replace Lipschitz continuous functions by Hölder continuous functions when passing from Jordan variation to Wiener variation. Of course, increasing p in (1.2.6) makes both spaces larger, which is reasonable.

Let us mention that the inclusion $WBV_p \subseteq WBV_q$ in (1.2.5) is strict in case $p < q$. To show this, we consider the zigzag function ζ_θ introduced in Definition 1.1.3. Taking into account increasing partitions containing the peaks of ζ_θ one may show that

(1.2.7) $$WVar_p(\zeta_\theta) = \sum_{k=1}^{\infty} \frac{1}{k^{p\theta}} \qquad (1 \le p < \infty).$$

This simple observation allows us to show that $WBV_p \subset WBV_q$ for $p < q$. For $p \ge 1$, consider the function $\zeta_{1/p}$. From (1.2.7) it follows then that $\zeta_{1/p} \in WBV_q$, but $\zeta_{1/p} \notin WBV_p$ for any $q > p$. However, we can actually do better: the same function satisfies of course

$$\zeta_{1/p} \in \left(\bigcap_{q>p} WBV_q \right) \setminus WBV_p.$$

In particular, the zigzag function ζ_1 belongs to WBV_p for all values of $p > 1$, but not to BV.

We may use the zigzag function (1.1.7) as well to show that the inclusion (1.2.6) is strict. In fact, choosing $\theta > 1/p$ arbitrary, by (1.2.7) we conclude that $\zeta_\theta \in WBV_p$. On the other hand, we have already seen in Proposition 1.1.4 that no zigzag function ζ_θ belongs to the Hölder space $Lip_{1/p}$.

To conclude, we give a decomposition result for functions from WBV_p which is parallel to Theorem 1.1.13 and shows again that one has to replace Lipschitz continuous functions by Hölder continuous functions when passing from Jordan variation to Wiener variation.

Theorem 1.2.4. *A function $z : [0,1] \to \mathbb{R}$ belongs to WBV_p if and only if it may be represented as composition $z = y \circ x$, where $x : [0,1] \to [0,1]$ is increasing and $y \in Lip_{1/p}$ with Hölder constant $L = 1$.*

Of course, Theorem 1.1.13 is contained in Theorem 1.2.4 in the special case $p = 1$. Both theorems are similar to the famous *Sierpiński decomposition* of regular functions [Si] which we recall for the sake of completeness.

Theorem 1.2.5. *A function $z : [0,1] \to \mathbb{R}$ belongs to R if and only if it may be represented as composition $z = y \circ x$, where $x : [0,1] \to [0,1]$ is strictly increasing and $y \in C$.*

1.3. Functions of bounded Riesz variation. In this section we study yet another concept of variation which goes back to Riesz [Ri,Ri1], depends on a parameter $p \geq 1$, and also reduces to BV for $p = 1$. Functions of bounded Riesz variation have particularly interesting applications, since they are intimately related to Sobolev spaces, see Theorem 1.3.5 below.

Definition 1.3.1. Given a real number $p \geq 1$, a partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}$, and a function $x : [0,1] \to \mathbb{R}$, the nonnegative real number

$$RVar_p(x, P) = RVar_p(x, P; [0,1]) := \sum_{j=1}^{m} \frac{|x(t_j) - x(t_{j-1})|^p}{(t_j - t_{j-1})^{p-1}} \tag{1.3.1}$$

is called the *Riesz variation* of x on $[0,1]$ with respect to P, while the (possibly infinite) number

$$RVar_p(x) = RVar_p(x; [0,1]) := \sup\left\{RVar_p(x, P; [0,1]) : P \in \mathcal{P}\right\}, \tag{1.3.2}$$

where the supremum is again taken over all partitions of $[0,1]$, is called the *total Riesz variation* of x on $[0,1]$. In case $RVar_p(x) < \infty$ we say that x has *bounded Riesz variation* (or *bounded p-variation in Riesz's sense*) on $[0,1]$ and write $x \in RBV_p$. □

From Hölder's inequality it follows that

$$RVar_p(x) \geq Var(x)^p,$$

which shows that the inclusion $RBV_p \subseteq BV$ holds. Moreover, one may prove, similarly as for BV and WBV_p, the following result for RBV_p.

Proposition 1.3.2. *The set RBV_p, equipped with the norm*

$$\|x\|_{RBV_p} := |x(0)| + RVar_p(x)^{1/p} \qquad (x \in RBV_p) \tag{1.3.3}$$

is a Banach space which is, for p > 1, continuously imbedded into the space C with norm (1.1.1), *as well as into the space BV with norm* (1.1.14). *Moreover, RBV_p is an algebra with*

$$\|xy\|_{RBV_p} \leq \|x\|_\infty \|y\|_{RBV_p} + \|y\|_\infty \|x\|_{RBV_p}$$

for all $x, y \in RBV_p$.

A comparison with Definition 1.2.1 shows that both spaces WBV_p and RBV_p reduce to BV for $p = 1$. However, there are some essential differences. First, in contrast to the space BV (or the spaces WBV_p), all functions $x \in RBV_p$ are *continuous* in case $p > 1$. The question arises if the spaces RBV_p are connected to the space Lip (as BV) or the spaces Lip_γ (as WBV_p). The following result is parallel to the chain of inclusions (1.2.5); in particular, it shows that RBV_p is intermediate between Lip and AC. It also exhibits an important difference between Wiener and Riesz variation: while the spaces WBV_p are increasing with respect to p, the spaces RBV_p are decreasing with respect to p.

Proposition 1.3.3. *The inclusions*

$$Lip \subseteq RBV_p \subseteq RBV_q \subseteq AC \subseteq BV \tag{1.3.4}$$

hold for $1 < q \leq p$.

We illustrate (1.3.4) by means of the zigzag function (1.1.7).

Example 1.3.4. In the last section we have seen that $\zeta_\theta \in WBV_p$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{k^{p\theta}} < \infty,$$

i.e., precisely for $p\theta > 1$. A similar computation shows that

$$RVar_p(\zeta_\theta) = \sum_{k=1}^{\infty} \frac{2^{k(p-1)}}{k^{p\theta}}. \tag{1.3.5}$$

Since $p \geq 1$, elementary convergence criteria show that $\zeta_\theta \in RBV_p$ if and only if $p = 1$ and $\theta > 1$. So zigzag functions only belong to BV, but not to RV_p for any $p > 1$. □

At this point let us collect some of the function classes considered so far, together with relations between them, in the following Table 1.2, where $1 < p < \infty$.

$Lip_{1/p}$	$\subset$	WBV_p	$\supset$	BV
$\cup$		$\cup$		$\cup$
Lip	$\subset$	RBV_p	$\subset$	AC

Table 1.2: Relations between function classes

All inclusions in Table 1.2 are strict for $1 < p < \infty$; we have shown this for $Lip_{1/p} \subset WBV_p$ by ζ_θ for $\theta > 1/p$, for $BV \subset WBV_p$ by ζ_1, and for $AC \subset BV$ in Example 1.1.11. The function $x_\tau(t) := t^\tau$ belongs to $RBV_p \setminus Lip$ for $1 - 1/p < \tau < 1$, to $AC \setminus RBV_p$ for $\tau = 1 - 1/p$, and to $Lip_{1/p} \setminus Lip$ for $\tau = 1/p$.

Since $RBV_p \subseteq AC$ for $p > 1$, see (1.3.4), one cannot expect that the space RBV_p contains some Hölder space Lip_γ for a suitable choice of $\gamma < 1$. In fact, we remarked that there exist functions which belong to *each* Hölder space Lip_γ for $\gamma < 1$, but not to BV. Consequently, such a function cannot belong to the smaller space RBV_p for any $p > 1$.

A comparison of our results on the Wiener space WBV_p and the Riesz space RBV_p shows that these spaces have quite different properties. Let us state them for further reference.

- The space RBV_p is decreasing in p, while the space WBV_p is increasing in p.
- The space RBV_p is contained in C for $p > 1$, while the space WBV_p contains discontinuous functions.

- The space RBV_p is contained in BV, while the space WBV_p contains functions of unbounded Jordan variation.
- The space WBV_p contains all Hölder continuous functions for $\gamma \leq 1/p$, while the space RBV_p contains functions which are not Hölder continuous for any γ.

The next theorem which is due to Riesz [Ri,Ri1] gives a complete characterization of the elements of RBV_p in terms of their derivatives for $p > 1$. It shows that the space RBV_p basically is the same as the well-known Sobolev space $W^{1,p}$.

Theorem 1.3.5 [Ri,Ri1]. *Let $1 < p < \infty$. Then a function x belongs to RBV_p if and only if $x \in AC$ and $x' \in L_p$. Moreover, in this case the equality*

$$RVar_p(x) = \|x'\|_{L_p}^p = \int_0^1 |x'(t)|^p \, dt \tag{1.3.6}$$

holds, where $RVar_p(x)$ denotes the p-variation (1.3.2) *of x in Riesz's sense.*

Clearly, Theorem 1.3.5 is not true for $p = 1$, since a function in $RBV_1 = BV$ is usually not continuous, let alone absolutely continuous.
The equality (1.3.6) sometimes makes the calculation of the Riesz variation much easier. For example, for calculating the Riesz variation of the simple function $x_\tau(t) := t^\tau$ for $0 < \tau < 1$, we should fix a partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}$ and then compute the sum

$$RVar_p(x_\tau, P) = \sum_{j=1}^m \frac{(t_j^\tau - t_{j-1}^\tau)^p}{(t_j - t_{j-1})^{p-1}},$$

which is somewhat cumbersome. On the other hand, with (1.3.6) we immediately get

$$RVar_p(x_\tau) = \int_0^1 |x_\tau'(t)|^p \, dt = \tau^p \int_0^1 t^{(\tau-1)p} \, dt = \frac{\tau^p}{1 - p(1-\tau)}$$

which shows that $x_\tau \in RBV_p$ if and only if $\tau > 1 - 1/p$.

1.4. The Waterman variation. Now we are going to investigate the last of the 4 variations of this chapter, which is somewhat different from the others. It is usually called *Waterman variation* (or Λ-variation) and was introduced and studied in [Wt,Wt1,Wt2].
We denote by $\Sigma = \Sigma[0,1]$ the family of all infinite collections $S = \{[a_n, b_n] : n \in \mathbb{N}\}$ of non-overlapping intervals $[a_n, b_n] \subset [0,1]$.

Definition 1.4.1. A *Waterman sequence* is a decreasing sequence $\Lambda = (\lambda_n)_n$ of positive real numbers such that $\lambda_n \to 0$ as $n \to \infty$ and

$$\sum_{n=1}^\infty \lambda_n = \infty. \tag{1.4.1}$$

Given a function $x : [0,1] \to \mathbb{R}$, a set $S \in \Sigma$, and a Waterman sequence $\Lambda = (\lambda_n)_n$, the positive real number

$$Var_\Lambda(x, S) = Var_\Lambda(x, S; [0,1]) := \sum_{k=1}^\infty \lambda_k |x(b_k) - x(a_k)| \tag{1.4.2}$$

(which may be infinite) is called the *Waterman variation* of x on $[0,1]$ with respect to S, while the (possibly infinite) number

$$Var_\Lambda(x) = Var_\Lambda(x; [0,1]) := \sup \{Var_\Lambda(x, S; [0,1]) : S \in \Sigma\}, \tag{1.4.3}$$

where the supremum is taken over all collections $S \in \Sigma$, is called the *total Waterman variation* of x on $[0,1]$. In case $\mathrm{Var}_\Lambda(x) < \infty$ we say that x has *bounded Waterman variation* (or *bounded Λ-variation in Waterman's sense*) on $[0,1]$ and write $x \in \Lambda BV$. □

The most natural example of a Waterman sequence is of course $\Lambda_q := (n^{-q})_n$ for $0 < q \leq 1$; in this case we write $\Lambda_q BV$ instead of ΛBV. In particular, the elements of the space $\Lambda_1 BV =: HBV$ are called *functions of bounded harmonic variation.* Historically, they have been the starting point and motivation for studying Waterman sequences and spaces.

If we drop the condition $\lambda_n \to 0$ and take $\lambda_n \equiv 1$, the space ΛBV coincides with the classical space BV. The equality (1.4.5) below shows that this is in a certain sense an "extremal" choice for Λ. We also remark that ΛBV and BV coincide if and only if the sequence $(\lambda_n)_n$ is bounded away from zero. This is the reason why we require $\lambda_n \to 0$ as $n \to \infty$ in Definition 1.4.1.

We collect some properties of the Waterman variations (1.4.2) and (1.4.3) for further reference.

Proposition 1.4.2. *The quantities* (1.4.2) *and* (1.4.3) *have the following properties.*

(a) *The variation* (1.4.3) *is subadditive, i.e.,*

$$Var_\Lambda(x+y) \leq Var_\Lambda(x) + Var_\Lambda(y)$$

for $x, y : [0,1] \to \mathbb{R}$.

(b) *The variation* (1.4.3) *is homogeneous, i.e.,*

$$Var_\Lambda(\mu x) = |\mu| Var_\Lambda(x)$$

for $\mu \in \mathbb{R}$.

(c) *The variation* (1.4.2) *is monotone with respect to collections of subintervals, i.e.,*

$$Var_\Lambda(x, S) \leq Var_\Lambda(x, T)$$

for $S, T \in \Sigma$ with $S \subseteq T$.

(d) *If $x \in BV$, then $x \in \Lambda BV$ for every Waterman sequence Λ.*

(e) *If $x \in \Lambda BV$, then x is bounded.*

Properties (d) and (e) in Proposition 1.4.2 may be summarized as inclusions

(1.4.4) $$BV \subseteq \bigcap_\Lambda \Lambda BV, \qquad \bigcup_\Lambda \Lambda BV \subseteq B,$$

where the intersection and union in (1.4.4) are taken over all Waterman sequences Λ. One may even prove the much sharper result

(1.4.5) $$\bigcap_\Lambda \Lambda BV = BV, \qquad \bigcup_\Lambda \Lambda BV = R,$$

where R denotes the space of all regular functions.

Proposition 1.4.2 (a) and (b) show that the set ΛBV is a linear space; as before, one can show that, equipped with the norm

(1.4.6) $$\|x\|_{\Lambda BV} := |x(0)| + Var_\Lambda(x) \qquad (x \in \Lambda BV),$$

ΛBV is a Banach space.

The next result states that, roughly speaking, every continuous function $x : [0,1] \to \mathbb{R}$ is contained in some appropriate Waterman space ΛBV; as Example 1.1.8 shows, this is *not* true for the classical space BV. However, it is not true either that the whole space C is contained in every Waterman space

ΛBV. That is, for every individual continuous function x we have to find an appropriate Waterman sequence Λ such that $x \in \Lambda BV$; this follows from the second equality in (1.4.5).

Proposition 1.4.3. *The inclusion*

$$C \subseteq \bigcup_{\Lambda} \Lambda BV \tag{1.4.7}$$

holds, i.e., every function $x \in C$ is contained in ΛBV for some suitable Waterman sequence $\Lambda = \Lambda(x)$.

The inclusion (1.4.7) complements in some sense the second inclusion in (1.4.4): the union of all Waterman spaces is "intermediate" between continuous and bounded functions, and by (1.4.5) coincides with the space of all regular functions.

Now we are going to compare the spaces ΛBV and ΓBV for two different Waterman sequences $\Lambda = (\lambda_n)_n$ and $\Gamma = (\gamma_n)_n$. This requires a technical definition.

Definition 1.4.4. Given two Waterman sequences $\Lambda = (\lambda_n)_n$ and $\Gamma = (\gamma_n)_n$, we write $\Lambda \prec \Gamma$ if

$$\lim_{n\to\infty} \frac{\lambda_n}{\gamma_n} = 0, \tag{1.4.8}$$

and $\Lambda \preceq \Gamma$ if there exists a constant $c > 0$ such that

$$\sum_{j=1}^{n} \lambda_j \le c \sum_{j=1}^{n} \gamma_j \qquad (n = 1, 2, 3, \ldots). \tag{1.4.9}$$

Since

$$\liminf_{n\to\infty} \frac{\lambda_n}{\gamma_n} \le \liminf_{n\to\infty} \frac{\sum_{j=1}^{n} \lambda_j}{\sum_{j=1}^{n} \gamma_j} \le \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \lambda_j}{\sum_{j=1}^{n} \gamma_j} \le \limsup_{n\to\infty} \frac{\lambda_n}{\gamma_n},$$

and the finiteness of the third term is equivalent to (1.4.9), the condition

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \lambda_j}{\sum_{j=1}^{n} \gamma_j} < \infty \tag{1.4.10}$$

is "intermediate" between (1.4.8) and (1.4.9), inasmuch as (1.4.8) $\Rightarrow$ (1.4.10) $\Rightarrow$ (1.4.9). □

One may show that $\Lambda \preceq \Gamma$ is equivalent to the inclusion $\Gamma BV \subseteq \Lambda BV$, while $\Lambda \prec \Gamma$ implies (but is not equivalent to) the strict inclusion $\Gamma BV \subset \Lambda BV$.

Let us check the conditions (1.4.8) – (1.4.10) by means of the particularly simple Waterman sequence $\Lambda_q := (n^{-q})_n$ for $0 < q \le 1$ introduced at the beginning of this section. Clearly, we have

$$\Lambda_p \prec \Lambda_q \;\Leftrightarrow\; p > q,$$

and

$$\Lambda_p \preceq \Lambda_q \;\Leftrightarrow\; p \ge q.$$

So we get the chain of inclusions

$$BV \subseteq \Lambda_p BV \subseteq \Lambda_q BV \subseteq HBV \qquad (0 < p \le q \le 1), \tag{1.4.11}$$

where HBV is the space of all functions of bounded harmonic variation. To show that all inclusions are strict for $p < q$, we may again use the zigzag functions (1.1.7).

Example 1.4.5. Let ζ_θ be the zigzag function (1.1.7), and let $\Lambda_q = (n^{-q})_n$ $(0 < q \leq 1)$. Then

$$Var_{\Lambda_q}(\zeta_\theta) = \sum_{n=1}^{\infty} \frac{1}{n^{\theta+q}} \tag{1.4.12}$$

and so $\zeta_\theta \in \Lambda_q BV$ if and only if $\theta + q > 1$. So if we choose $\theta := 1 - p$, the corresponding zigzag function ζ_{1-p} belongs to $\Lambda_q BV$ for $q > p$, but not to $\Lambda_p BV$. However, we can again do better: for fixed $p < 1$, the same reasoning shows of course that

$$\zeta_{1-p} \in \left(\bigcap_{q>p} \Lambda_q BV \right) \setminus \Lambda_p BV. \tag{1.4.13}$$

In particular, the zigzag function ζ_1 belongs to $HBV \setminus \Lambda_p BV$ for $0 < p < 1$, and also to $HBV \setminus BV$. □

A comparison of the inclusions (1.2.5) and (1.4.11) shows that both scales of spaces $\{WBV_p : 1 \leq p < \infty\}$ and $\{\Lambda_p BV : 0 < p \leq 1\}$ are increasing with respect to the index p. So it is natural to ask whether or not these two scales are related. This is in fact true.

Let $[a_1, b_1], \ldots, [a_n, b_n] \subset [0, 1]$ be a collection of non-overlapping intervals, and let $\eta_k := |x(b_k) - x(a_k)|$. Suppose that $x \in WBV_p$ for some $p > 1$, which implies that

$$\sum_{k=1}^{n} \eta_k^p \leq WVar_p(x) =: M.$$

Applying the Hölder inequality yields

$$\sum_{k=1}^{n} \frac{\eta_k}{k^q} \leq \left(\sum_{k=1}^{n} \eta_k^p \right)^{1/p} \left(\sum_{k=1}^{n} \frac{1}{k^{qp'}} \right)^{1/p'} \leq M^{1/p} \left(\sum_{k=1}^{n} \frac{1}{k^{qp'}} \right)^{1/p'},$$

where $p' = p/(p-1)$, and the right sum remains bounded for $n \to \infty$ if and only if $q > 1 - 1/p$. We conclude that $WBV_p \subseteq \Lambda_q BV$ for these values of q.

Conversely, assume now that $x \in \Lambda_q BV$, and let η_k be as before, where we assume without loss of generality that $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$. From our assumption

$$\sum_{k=1}^{n} \frac{\eta_k}{k^q} \leq Var_{\Lambda_q}(x) =: M$$

we obtain now

$$M \geq \sum_{j=1}^{k} \frac{\eta_j}{j^q} \geq \sum_{j=1}^{k} \frac{\eta_k}{k^q} = k \frac{\eta_k}{k^q} = k^{1-q} \eta_k \qquad (k = 1, 2, \ldots, n).$$

Consequently, for $q = 1 - 1/p$ we have

$$\eta_k^p = \eta_k^{p-1} \eta_k \leq \eta_k^{pq} \eta_k \leq \frac{M^{pq}}{k^q} \eta_k \qquad (k = 1, 2, \ldots, n),$$

hence

$$\sum_{k=1}^{n} \eta_k^p \leq M^{pq} \sum_{k=1}^{n} \frac{\eta_k}{k^q} \leq M^{p-1} M = M^p.$$

This shows that $x \in WBV_p$ with $WVar_p(x) \leq M^p$. We may summarize our discussion with the following

Proposition 1.4.7. *The inclusion*

$$WBV_p \subseteq \Lambda_q BV \tag{1.4.14}$$

holds for $p > 1$ *and* $1 - 1/p < q \leq 1$, *while the reverse inclusion*

$$\Lambda_q BV \subseteq WBV_p \tag{1.4.15}$$

holds for $p > 1$ *and* $0 < q \leq 1 - 1/p$.

To conclude this section, let us consider again our favorite families of functions which we introduced in Section 1.1, namely the oscillatory functions (1.1.5) and the zigzag functions (1.1.7). In Table 1.3 below, which is an essential completion of Table 1.1, we report the precise values of $(\alpha, \beta) \in \mathbb{R}^2$ and $\theta > 0$ for which the function $\omega_{\alpha,\beta}$ resp. the function ζ_θ belongs to the function spaces we dealt with in this chapter.

We make some comments on the last column in Table 1.3. The table shows that the oscillatory function (1.1.5) has a different behavior in all spaces, but the zigzag function (1.1.7) exhibits a more interesting behavior in spaces of functions of (generalized) bounded variation than in spaces of continuous functions.

If we take $p = 1$ in the row for WBV_p and RBV_p, we get for ζ_θ in both cases the condition $\theta > 1$, which is of course nothing else but the condition for BV. Moreover, if we take, at least formally, $q = 1 - 1/p$ in the row for $\Lambda_q BV$, we get the condition $1 < \theta + q = \theta + 1 - 1/p$, hence $p\theta > 1$, which is the same as the condition for WBV_p; in view of Proposition 1.4.7, this is not surprising.

The zigzag function (1.1.7) may also be used that both inclusions (1.4.14) and (1.4.15) are strict. Indeed, in case $1 - 1/p < q$ we may take $\theta \in (1 - q, 1/p]$ and get $\zeta_\theta \in \Lambda_q BV \setminus WBV_p$, while in case $1 - 1/p > q$ we may take $\theta \in (1/p, 1 - q]$ and get $\zeta_\theta \in WBV_p \setminus \Lambda_q BV$.

Finally, since the zigzag function is always continuous and has the Luzin property, it is clear that we get the same condition in the rows for BV and for AC.

	The function $\omega_{\alpha,\beta}$	*The function* ζ_θ
belongs to C *if and only if*	$\alpha > 0$ or $\alpha \leq 0$ and $\alpha + \beta > 0$	always
belongs to C^1 *if and only if*	$\alpha + \beta > 1$	never
belongs to Lip if and only if	$\alpha + \beta \geq 1$	never
belongs to Lip_γ *if and only if*	$\alpha + \beta\gamma \geq \gamma$	never
belongs to BV *if and only if*	$\beta \geq 0$ and $\alpha + \beta \geq 0$ or $\beta < 0$ and $\alpha + \beta > 0$	$\theta > 1$
belongs to WBV_p *if and only if*	$\beta \geq 0$ and $p\alpha + \beta \geq 0$ or $\beta < 0$ and $p\alpha + \beta > 0$	$p\theta > 1$
belongs to RBV_p *if and only if*	$\beta \geq 0$ and $p\alpha + \beta \geq p - 1$ or $\beta < 0$ and $p\alpha + \beta > p - 1$	$p = 1$ and $\theta > 1$
belongs to AC *if and only if*	$\alpha + \beta > 0$	$\theta > 1$
belongs to $\Lambda_q BV$ *if and only if*	$\beta \geq 0$ and $\alpha + (1-q)\beta \geq 0$ or $\beta < 0$ and $\alpha + (1-q)\beta > 0$	$\theta + q > 1$

Table 1.3: Oscillation functions and zigzag functions

We remark that there exist other generalizations of the Jordan variation which, however, we will consider in this survey only marginally. The following definition, due to Young [Y] generalizes the Wiener variation (1.2.1) and imitates the transition from Lebesgue spaces to Orlicz spaces.

Recall that a *Young function* is a strictly increasing convex function $\phi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{u\to 0+} \frac{\phi(u)}{u} = 0, \qquad \lim_{u\to\infty} \frac{\phi(u)}{u} = \infty,$$

and $\phi(0) = 0$.

Definition 1.4.8. Given a partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}$, and a function $x : [0,1] \to \mathbb{R}$, the nonnegative real number

$$YVar_\phi(x,P) = YVar_\phi(x,P;[0,1]) := \sum_{j=1}^m \phi(|x(t_j) - x(t_{j-1})|)$$

is called the *Young variation* of x on $[0,1]$ with respect to P. If the expression

$$YVar_\phi(\lambda x) = YVar_\phi(\lambda x;[0,1]) := \sup\,\{YVar_\phi(\lambda x, P;[0,1]) : P \in \mathcal{P}\}$$

is finite for some $\lambda > 0$, where the supremum is taken over all partitions of $[0,1]$, it is called the *total Young variation* of λx on $[0,1]$. In this case we say that x has *finite Young variation* (or *finite ϕ-variation in Young's sense*) on $[0,1]$ and write $x \in YBV_\phi$. □

Of course, the simplest choice $\phi(u) := u^p$ for $1 < p < \infty$ gives the total Wiener variation (1.2.2). However, more sophisticated choices like $\phi(u) := \exp(u) - 1$ or $\phi(u) := (1+u)\log(u+1)$ lead to new applications. A natural norm on $YBV_\phi = YBV_\phi[0,1]$ is

$$\|x\|_{YBV_\phi} := |x(0)| + \inf\,\{\mu > 0 : YVar_\phi(x/\mu) \le 1\}, \tag{1.4.16}$$

which is modelled on the usual Orlicz space norm and in case $\phi(u) = u^p$ coincides with (1.2.4).

The space YBV_ϕ has been studied in detail in [MO] and [LO]. Here we only sketch a connection with the Waterman space ΛBV. Let ϕ be a Young function and $\Lambda_q = (n^{-q})_n$, for $0 < q < 1$ be the special Waterman sequence introduced in Section 1.4. Then the condition

$$\int_0^1 \frac{du}{\phi(u)^{1-q}} < \infty \tag{1.4.17}$$

implies that $YBV_\phi \subseteq \Lambda_q BV$. A similar finiteness condition with the integral replaced by a series reads as follows. If

$$\sum_{n=1}^\infty \psi(\lambda_n) < \infty, \tag{1.4.18}$$

where

$$\psi(u) := \sup\,\{uv - \phi(v) : v \ge 0\}$$

denotes the conjugate Young function to ϕ, then $YBV_\phi \subseteq \Lambda BV$. In case $\phi(u) = u^p$ $(1 < p < \infty)$ condition (1.4.17) becomes

$$\int_0^1 \frac{du}{u^{p(1-q)}} < \infty,$$

while condition (1.4.18) becomes

$$\sum_{n=1}^\infty \frac{1}{n^{pq/(p-1)}} < \infty,$$

and both condition are equivalent to $p(1-q) < 1$.

All spaces of functions of bounded variation considered in this chapter are continuously imbedded into the space $B = B[0,1]$ of bounded functions on $[0,1]$ with norm (1.1.11). For further reference we collect in a table upper estimates for the sharp imbedding constants

$$c(X,B) := \inf\{c > 0 : \|x\|_X \leq c\|x\|_\infty\},$$

where X stands for one of the spaces BV, WBV_p, RBV_p, ΛBV, $\Lambda_q BV$, or YBV_ϕ.

$X =$	$BV[0,1]$	$WBV_p[0,1]$	$RBV_p[0,1]$	$\Lambda BV[0,1]$	$\Lambda_q BV[0,1]$	$YBV_\phi[0,1]$
$c(X,B) \leq$	1	1	1	$\max\{1, 1/\lambda_1\}$	1	$\phi(1)$

Table 1.4: Some imbedding constants

The imbedding constant $c(BV,B) = 1$ follows from Proposition 1.1.6 (d), the imbedding constant $c(WBV_p, B) = 1$ from Proposition 1.2.2 (d).
To calculate $c(\Lambda BV, B)$ it suffices to observe that

$$\begin{aligned} |x(t)| &\leq |x(0)| + |x(t) - x(0)| = |x(0)| + \frac{1}{\lambda_1}\lambda_1|x(t) - x(0)| \\ &\leq |x(0)| + \frac{1}{\lambda_1}Var_\Lambda(x) \leq \max\{1, 1/\lambda_1\}\|x\|_{\Lambda BV}. \end{aligned}$$

The estimate for $c(YBV_\phi, B)$ follows from the fact that $YVar_\phi(x) \leq \phi(Var(x))$, where $Var(x)$ is the Jordan variation (1.1.9).

Chapter 2. Linear Operators in BV Spaces

In this chapter we study basically three different types of linear operators in the space BV and related spaces, namely *substitution operators, multiplication operators*, and *integral operators*.

Given a function $\varphi : [0,1] \to [0,1]$, by Σ_φ we denote the (linear) *substitution operator* defined by

$$\Sigma_\varphi(x)(t) := x(\varphi(t)) \qquad (0 \leq t \leq 1). \tag{2.0.1}$$

If X is a function space over $[0,1]$, the first problem consists in characterizing all $\varphi : [0,1] \to [0,1]$ such that $\Sigma_\varphi(X) \subseteq X$. In other words, we want to find the largest possible class of "changes of variable" $s = \varphi(t)$ for which the composition $x \circ \varphi$ remains in the space X if we take x from X.

Given a function $\mu : [0,1] \to \mathbb{R}$, by M_μ we denote the (linear) *multiplication operator* defined by

$$M_\mu(x)(t) := \mu(t)x(t) \qquad (0 \leq t \leq 1). \tag{2.0.2}$$

Again, if X is a function space over $[0,1]$, the first problem consists in characterizing all $\mu : [0,1] \to \mathbb{R}$ such that $M_\mu(X) \subseteq X$. In other words, we want to find the largest possible class of "multipliers" μ for which the product μx remains in the space X if we take x from X.

Given a function $k : [0,1] \times [0,1] \to \mathbb{R}$, by K we denote the (linear) *integral operator* defined by

$$K(x)(t) := \int_0^1 k(t,s)x(s)\,ds \qquad (0 \leq t \leq 1). \tag{2.0.3}$$

As before, if X is a function space over $[0,1]$, the first problem consists in characterizing all $k : [0,1] \times [0,1] \to \mathbb{R}$ such that $K(X) \subseteq X$. In other words, we want to find the largest possible class of "kernel functions" k for which after integration of $k(t,s)x(s)$ with respect to s we get a function of t which remains in the space X if we take x from X.

To be specific, we are interested in the following questions for the space BV.

1. Under what conditions on φ we have $\Sigma_\varphi(BV) \subseteq BV$?
2. Under what conditions on φ is the operator Σ_φ bounded in BV?
3. Under what conditions on φ is the operator Σ_φ compact in BV?
4. Under what conditions on μ we have $M_\mu(BV) \subseteq BV$?
5. Under what conditions on μ is the operator M_μ bounded in BV?
6. Under what conditions on μ is the operator M_μ compact in BV?
7. Under what conditions on k we have $K(BV) \subseteq BV$?
8. Under what conditions on k is the operator K bounded in BV?
9. Under what conditions on k is the operator K compact in BV?

Of course, similar questions arise for other spaces than BV. It turns out that each of these operators exhibits several surprising features. We will discuss them and illustrate them with examples in the following sections.

2.1. Substitution operators. Let us start with analyzing the properties of the substitution operator (2.0.1) in the space BV. Since the identity has bounded variation on $[0,1]$, it is clear that $\varphi \in BV$

is a *necessary* condition for the inclusion $\Sigma_\varphi(BV) \subseteq BV$. The following example shows that this condition is not sufficient.

Example 2.1.1. Define $\varphi : [0,1] \to [0,1]$ and $x : [0,1] \to \mathbb{R}$ by

$$\varphi(t) = \omega^2_{1,-1}(t) = \begin{cases} t^2 \sin^2 \dfrac{1}{t} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

see (1.1.5), and $x(s) = \sqrt{s}$, respectively. Being monotone, we have $x \in BV$. Moreover, it is not hard to see that $\varphi \in BV$, since φ' exists and is bounded on $[0,1]$. However, Example 1.1.8 (or Proposition 1.1.9) shows that $\Sigma_\varphi(x) = x \circ \varphi \notin BV$. □

The problem of characterizing all functions φ for which $\Sigma_\varphi(BV) \subseteq BV$ was completely solved by Josephy in [Jo]. To describe the result, let $\mathcal{J}_n$ ($n \in \mathbb{N}$) denote the class of all functions $\varphi : [0,1] \to [0,1]$ with the property that the preimage $\varphi^{-1}([a,b])$ of any interval $[a,b] \subseteq [0,1]$ can be written as union of exactly n intervals. We call a function $\varphi : [0,1] \to [0,1]$ *pseudo-monotone* if $\varphi \in \mathcal{J}_n$ for some n. Clearly, every monotone function is pseudo-monotone, since it belongs to $\mathcal{J}_1$, and every pseudo-monotone function has bounded variation. The converse is not true:

Example 2.1.2. By Proposition 1.1.9, the function $\varphi := \omega^2_{1,-1}$ belongs to BV. However, this function is not pseudo-monotone, since

$$\varphi^{-1}(\{0\}) = \left\{0, \tfrac{1}{\pi}, \tfrac{1}{2\pi}, \tfrac{1}{3\pi}, \ldots\right\}.$$

To find a pseudo-monotone function which is not monotone is of course trivial. □

Theorem 2.1.3 [Jo]. *The inclusion $\Sigma_\varphi(BV) \subseteq BV$ holds precisely for all pseudo-monotone functions φ.*

Theorem 2.1.3 explains, combined with Example 2.1.2, why the function φ from Example 2.1.1 does not generate a substitution operator Σ_φ in BV. The following result shows that, if the operator Σ_φ maps BV into itself, it is always continuous.

Theorem 2.1.4. *If $\varphi : [0,1] \to [0,1]$ is pseudo-monotone, the corresponding operator $\Sigma_\varphi : BV \to BV$ is bounded in BV.*

Let us briefly sketch the proof of this result, where we use the norm (1.1.15). If $x : [0,1] \to \mathbb{R}$ is increasing, a straightforward calculation shows that $x \circ \varphi \in \mathcal{J}_n$ for $\varphi \in \mathcal{J}_n$; moreover,

$$Var(x \circ \varphi) \leq 4(n+1)\|x \circ \varphi\|_\infty \leq 4(n+1)\|x\|_\infty$$

in this case. If $x \in BV$ is arbitrary, we use the Jordan decomposition $x = y - z$ described in Theorem 1.1.7. Then

$$|||\Sigma_\varphi(x)|||_{BV} = |||x \circ \varphi|||_{BV} = |||(y-z) \circ \varphi|||_{BV} = |||y \circ \varphi - z \circ \varphi|||_{BV}$$
$$\leq |||y \circ \varphi|||_{BV} + |||z \circ \varphi|||_{BV} = \|y \circ \varphi\|_\infty + Var(y \circ \varphi) + \|z \circ \varphi\|_\infty + Var(z \circ \varphi)$$
$$(4(n+1)+1)(\|y\|_\infty + \|z\|_\infty) \leq (4n+5)(2\|y\|_\infty + \|x\|_\infty) \leq (8n+10)|||x|||_{BV}.$$

Here we have used the fact that $\|y\|_\infty = \|Var(x;[0,\cdot])\|_\infty = Var(x)$ and $\|z\|_\infty \leq \|y\|_\infty + \|x\|_\infty$. Observe that our upper bound for the norm of the linear operator Σ_φ cannot be chosen universally for all pseudo-monotone functions, but depends on the class $\mathcal{J}_n$ where the substitution function φ belongs to. In particular, we get the (not optimal) upper estimate $|||\Sigma_\varphi|||_{BV \to BV} \leq 18$ for increasing functions φ.

To conclude, let us discuss some mapping properties of the substitution operator (2.0.1). We will present an analogous discussion in Section 3.1 for the so-called composition operator

$$C_f(x)(t) := f(x(t)) \qquad (0 \leq t \leq 1) \tag{2.1.1}$$

generated by some function $f : \mathbb{R} \to \mathbb{R}$. This operator may be considered as some kind of "twin brother" of Σ_φ, inasmuch as the outer function f is fixed, but the inner function x varies over some function space. There is one essential difference, however: the operator Σ_φ in (2.0.1) is linear, while the operator C_f in (2.1.1) is nonlinear (which makes its study pretty complicated).

Although this is not the main focus in this survey, let us analyze the mapping properties of the operator (2.0.1) first in the space C of continuous functions $x : [0,1] \to \mathbb{R}$. To this end, we suppose that $\varphi : [0,1] \to [0,1]$ is continuous, and we are interested in the question how the properties of the "change of variables" φ are reflected in analogous properties of the operator Σ_φ. First of all, it is clear that Σ_φ is bounded in the space C. Here the computation of its norm is trivial, since

$$\|\Sigma_\varphi\|_{C\to C} = \|\Sigma_\varphi e\|_C = 1,$$

where $e(t) \equiv 1$. The following example shows that the injectivity or surjectivity of φ does not imply the injectivity resp. surjectivity of Σ_φ.

Example 2.1.5. Let $\varphi(t) := t/2$. Then $\varphi : [0,1] \to [0,1]$ is injective, but $\Sigma_\varphi : C \to C$ is not, because the function defined by $x(t) := t$ for $0 \le t \le 1/2$ is mapped into the function $y(t) = t/2$ for $0 \le t \le 1$, no matter how we define x on $(1/2, 1]$.

On the other hand, let $\varphi(t) := 4t(1-t)$. Then $\varphi : [0,1] \to [0,1]$ is surjective, but $\Sigma_\varphi : C \to C$ is not, because the function $y(t) = t$ is not in the range of Σ_φ. □

Surprisingly, we get a correct result when we interchange the role of injectivity and surjectivity.

Proposition 2.1.6. *Let $\varphi : [0,1] \to [0,1]$ be continuous. With Σ_φ given by* (2.0.1), *the following is true.*

(a) *The operator $\Sigma_\varphi : C \to C$ is surjective if and only if the corresponding function $\varphi : [0,1] \to [0,1]$ is injective.*

(b) *The following three assertions are equivalent.*

(i) *The function $\varphi : [0,1] \to [0,1]$ is surjective.*

(ii) *The operator $\Sigma_\varphi : C \to C$ is an isometry, i.e.,*

(2.1.2) $$\|\Sigma_\varphi x\|_C = \|x\|_C \qquad (x \in C)$$

(iii) *The operator $\Sigma_\varphi : C \to C$ is injective.*

Proof. (a) If φ is injective, then the set $K := \varphi([0,1]) \subseteq [0,1]$ is a compact interval, and the map $\varphi : [0,1] \to K$ is a homeomorphism. Given $y \in C$, the function $y \circ \varphi^{-1} : K \to \mathbb{R}$ is therefore continuous, and by the Tietze-Uryson Theorem we may find a continuous function $x : [0,1] \to \mathbb{R}$ with $x(t) = (y \circ \varphi^{-1})(t)$ for $t \in K$, hence $y = \Sigma_\varphi x$.

Conversely, suppose that Σ_φ is surjective, and fix $s_0, s_1 \in [0,1]$ with $s_0 \neq s_1$. The map $y : [0,1] \to \mathbb{R}$ defined by

(2.1.3) $$y(t) := \frac{|t - s_0|}{|t - s_0| + |t - s_1|} \qquad (0 \le t \le 1)$$

is then well-defined, continuous, and satisfies $0 \le y(t) \le 1$, $y(s_0) = 0$ and $y(s_1) = 1$. Since Σ_φ is surjective, there exists $x \in C$ such that $\Sigma_\varphi x = y$. In particular,

$$x(\varphi(s_0)) = y(s_0) = 0 \neq 1 = y(s_1) = x(\varphi(s_1)),$$

which implies $\varphi(s_0) \neq \varphi(s_1)$ and shows that φ is injective.

(b) If φ is surjective, we have $K = \varphi([0,1]) = [0,1]$, hence

$$\|\Sigma_\varphi x\|_C = \max\{|x(\varphi(s))| : 0 \le s \le 1\} = \max\{|x(t)| : 0 \le t \le 1\} = \|x\|_C$$

for every $x \in C$, which means that Σ_φ is an isometry. The fact that (ii) implies (iii) is of course trivial. So suppose now that Σ_φ is injective, and assume that there exists $t_0 \in [0,1] \setminus K$. Since K is compact, the function $x : [0,1] \to \mathbb{R}$ defined by

$$x(t) := \frac{\text{dist}\,(t, K)}{\text{dist}\,(t, K) + |t - t_0|} \qquad (0 \leq t \leq 1)$$

is well-defined, continuous, and satisfies $\Sigma_\varphi x(t) \equiv 0$. So by assumption we also have $x(t) \equiv 0$, because Σ_φ is a linear operator, contradicting $x(t_0) = 1$. This shows that $[0,1] \setminus K = \emptyset$, i.e., φ is surjective. ■

Observe that the "crossover" between surjectivity and injectivity in our proposition is not only perfectly symmetric, but in (b) we even get the isometry property of Σ_φ as a fringe benefit. Also, this proposition shows that it was not accidental that the first function φ in Example 2.1.5 is not surjective, while the second one is not injective.

Our aim is now to see how we may imitate the proof to get a similar result in the space BV, assuming that φ is pseudomonotone.

Proposition 2.1.7. *Let $\varphi : [0,1] \to [0,1]$ be pseudomonotone. With Σ_φ given by* (2.0.1)*, the following is true.*

(a) *If the operator $\Sigma_\varphi : BV \to BV$ is surjective then the corresponding function $\varphi : [0,1] \to [0,1]$ is injective.*

(b) *The operator $\Sigma_\varphi : BV \to BV$ is injective if and only if the corresponding function $\varphi : [0,1] \to [0,1]$ is surjective.*

Proof. (a) If φ is not injective there exist $\sigma, \tau \in [0,1]$ with $\sigma \neq \tau$ but $\varphi(\sigma) = \varphi(\tau)$. The function $y := \chi_{\{\tau\}}$ belongs to BV, and by the surjectivity of Σ_φ we find $x \in BV$ such that $\Sigma_\varphi x = y$. But this implies

$$1 = y(\tau) = \Sigma_\varphi x(\tau) = x(\varphi(\tau)) = x(\varphi(\sigma)) = \Sigma_\varphi x(\sigma) = y(\sigma) = 0,$$

a contradiction.

(b) Assume first that $\varphi : [0,1] \to [0,1]$ is surjective, and let $x \in BV$ satisfy $\Sigma_\varphi x(t) \equiv 0$. For fixed $t \in [0,1]$ we find by assumption $s \in [0,1]$ such that $\varphi(s) = t$. It follows that

$$x(t) = x(\varphi(s)) = \Sigma_\varphi x(s) = 0,$$

i.e., $x(t) \equiv 0$, since t was arbitrary. Conversely, assume now that $\varphi : [0,1] \to [0,1]$ is not surjective, and choose $\tau \in [0,1] \setminus \varphi([0,1])$. The function $x := \chi_{\{\tau\}}$ belongs to BV and is not identically zero, but satisfies

$$\Sigma_\varphi x(t) = x(\varphi(t)) \equiv 0 \qquad (0 \leq t \leq 1),$$

contradicting the injectivity of Σ_φ. ■

Closer scrutiny of the proof of Proposition 2.1.7 shows that we did not use the special structure of the space BV, but only the fact that BV contains all characteristic functions of singletons. Examples of other spaces with this property are, e.g., WBV_p and ΛBV. So we may use the same reasoning to prove the following more general

Proposition 2.1.8. *Suppose that the operator Σ_φ given by* (2.0.1) *maps some function space X into some function space Y. Then the following is true.*

(a) *If $\varphi : [0,1] \to [0,1]$ is surjective, then $\Sigma_\varphi : X \to Y$ is injective.*

(b) *If $\Sigma_\varphi : X \to Y$ is injective, and the space X contains all characteristic functions of singletons, then $\varphi : [0,1] \to [0,1]$ is surjective.*

(c) *If $\Sigma_\varphi : X \to Y$ is surjective, and the space Y contains all characteristic functions of singletons, then $\varphi : [0,1] \to [0,1]$ is injective.*

Observe that the additional hypothesis on the space X in (b) cannot be dropped, as the space $X = Y = \mathbb{R}$ of all constant functions shows: in this case Σ_φ is *always* injective, no matter how we choose the function φ. The problem whether or not the injectivity of φ implies the surjectivity of Σ_φ in BV (or some similar space) seems to be open.

2.2. Multiplication operators. Apart from the substitution operator (2.0.1), the multiplication operator (2.0.2) constitutes another important example of linear operators in BV and many other function spaces. If X is an algebra, i.e., closed under multiplication, then for every $\mu \in X$ we have $M_\mu(X) \subseteq X$. Conversely, if the constant function $x(t) \equiv 1$ belongs to X, the inclusion $M_\mu(X) \subseteq X$ implies $\mu \in X$. This fact may also be restated in the following form, even in the more general framework of different spaces.

Let X and Y two function spaces over $[0,1]$. Following [BaRw], we call the set

$$Y/X := \{z : [0,1] \to \mathbb{R} : zx \in Y \text{ for all } x \in X\} \tag{2.2.1}$$

the *multiplier space* of Y over X. An interesting problem consists in calculating this space explicitly for given spaces X and Y, and in some cases this is easy. For example, the classical Hölder inequality implies that, for $1 \leq p, q \leq \infty$,

$$L_q/L_p = \begin{cases} L_{pq/(p-q)} & \text{for} \quad p > q, \\ L_\infty & \text{for} \quad p = q, \\ \{0\} & \text{for} \quad p < q. \end{cases}$$

It is also straightforward to show that $C/C = C$, $B/B = B$, $B/C = B$, and $C/B = \{0\}$, where B denotes the space of all bounded functions with the sup norm. Other less trivial multiplier spaces have been calculated in the recent papers [BaRw,BaRw1]. Recall that the *support* of a function $\mu : [0,1] \to \mathbb{R}$ is defined by

$$\operatorname{supp}(\mu) := \{t : 0 \leq t \leq 1, \mu(t) \neq 0\}. \tag{2.2.2}$$

We point out that this definition of support is somewhat unusual: in the theory of PDEs one usually takes the closure of (2.2.2). Also, note that a continuous function can never have a countable support, by the permanence theorem.

Occasionally we will also consider the set

$$\operatorname{supp}_\delta(\mu) := \{t : 0 \leq t \leq 1, |\mu(t)| > \delta\} \qquad (\delta \geq 0). \tag{2.2.3}$$

So $\operatorname{supp}_\delta(\mu) \supseteq \operatorname{supp}_{\delta'}(\mu)$ if $\delta < \delta'$, and $\operatorname{supp}_0(\mu) = \operatorname{supp}(\mu)$. Moreover, if $\operatorname{supp}_\delta(\mu)$ is finite for all $\delta > 0$, then $\operatorname{supp}(\mu)$ is at most countable, since

$$\operatorname{supp}(\mu) = \bigcup_{n=1}^{\infty} \operatorname{supp}_{1/n}(\mu).$$

Given a function space X, in what follows we denote by X_f the subspace of all functions $\mu \in X$ with finite support, and by X_c the subspace of all functions $\mu \in X$ with countable support. With this terminology, the following results are proved in [BaRw] for the spaces D of all Darboux functions and the space Δ of all functions which have a primitive:

$$C/\Delta = D/B = C/D = \Delta/D = \Delta/B = \{0\}, \quad B/D = B/\Delta = B_f.$$

The multiplier spaces Δ/C and Δ/Δ are also known, but their description is more technical.

Now, the relation between multiplier spaces and the multiplication operator (2.0.2) is clear: we have $M_\mu(X) \subseteq Y$ if and only if $\mu \in Y/X$. In the paper [BaRw1] the authors also calculate the multiplier space for various spaces of BV type, which are the main topics of interest for us. We summarize with the following Theorem 2.2.1. For Waterman spaces ΛBV and ΓBV, we refer to (1.4.9) for the meaning of the notation $\Lambda \preceq \Gamma$.

Theorem 2.2.1 [BaRw,BaRw1]. *The following equalities hold for multiplier spaces* (2.2.1) *with* $X, Y \in \{BV, WBV_p, RBV_p, \Lambda BV\}$.

$$BV/BV = BV; \quad BV/C = BV_c; \quad C/BV = \{0\}; \quad BV/B = BV_c; \quad B/BV = B;$$

$$WBV_q/WBV_p = WBV_q \text{ if } p \leq q; \quad WBV_q/WBV_p = (WBV_q)_c \text{ if } p > q;$$

$$RBV_q/RBV_p = RBV_q \text{ if } p \geq q \geq 1; \quad RBV_q/RBV_p = \{0\} \text{ if } 1 \leq p < q;$$

$$\Lambda BV/\Gamma BV = \Lambda BV \text{ if } \Lambda \preceq \Gamma; \quad \Lambda BV/\Gamma BV = \Lambda BV_c \text{ if } \Lambda \not\preceq \Gamma.$$

So Theorem 2.2.1 implies, in particular, that the operator (2.0.2) maps $X \in \{BV, WBV_p, RBV_p, \Lambda BV\}$ *into itself* if and only if $\mu \in X$. This is of course what one expects for function algebras.

Apart from the mere acting condition $M_\mu(X) \subseteq Y$, mapping properties of M_μ are of course of intereset, the most imporant ones being injectivity, surjectivity, and bijectivity. Here one encounters some surprising asymmetry between conditions for injectivity and surjectivity which have been proved and illustrated in the recent paper [KRw]. It turns out that all such conditions may be expressed by means of the support of μ.

Since M_μ is a linear operator, it is clear that $M_\mu : X \to Y$ is injective if and only if for each $x \in X\setminus\{0\}$ there is some $t \in [0, 1]$ such that both $x(t) \neq 0$ and $\mu(t) \neq 0$. In particular, M_μ is certainly injective if $\text{supp}(\mu) = [0, 1]$. For practical purposes, however, this criterion is too general. To be more explicit, we introduce some technical terminology.

Let us say that a space X of functions $x : [0, 1] \to \mathbb{R}$ *separates points* if for each $t \in [0, 1]$ we can find some $x \in X$ such that $x(t) \neq 0$, *strongly separates points* if X contains all characteristic functions of sigletons, and *uniformly separates points* if $X \subseteq C$ and for each $t \in [0, 1]$ and each $\delta > 0$ we can find some $x \in X$ such that $t \in \text{supp}(x) \subseteq [t - \delta, t + \delta]$.

It is easy to see that strong or uniform separation implies separation. The converse is not true; for example, the space of constant functions separates points, but neither strongly nor uniformly. Moreover, there is no relation between strong and uniform separation: the spaces B, BV, WBV_p and ΛBV strongly separate points, but not uniformly, while the spaces C and RBV_p (for $p > 1$) uniformly separate points, but not strongly. With this terminology, the following necessary and sufficient conditions may be proved.

Proposition 2.2.2 [KRw]. *The following statements are true for function spaces X and Y over* $[0, 1]$.

(a) *Suppose that X strongly separates points. Then the operator $M_\mu : X \to Y$ is injective if and only if* $\text{supp}(\mu) = [0, 1]$.

(b) *Suppose that X uniformly separates points. Then the operator $M_\mu : X \to Y$ is injective if and only if* $\text{supp}(\mu)$ *is dense in* $[0, 1]$.

(c) *Suppose that Y separates points. Then the operator $M_\mu : X \to Y$ is surjective if and only if* $\text{supp}(\mu) = [0, 1]$ *and* $1/\mu \in X/Y$.

From Proposition 2.2.2 (c) it follows, in particular, that *surjectivity of $M_\mu : X \to Y$ implies injectivity, whenever the target space Y separates points.*

Moreover, taking into account that all spaces $Y \in \{B, C, BV, WBV_p, \Lambda BV, RBV_p\}$ separate points, we get a simple surjectivity criterion. If $M_\mu : X \to Y$ is surjective, we obtain from (c) that $\text{supp}(\mu) = [0, 1]$ and $1/\mu \in X/Y$. Since the constant function $z(t) \equiv 1$ belongs to all indicated spaces, we conclude that $X/Y \subseteq X \subseteq B$ and $1/\mu$ is bounded, which means that $|\mu(t)| \geq \delta$ on $[0, 1]$ for some $\delta > 0$. But the latter condition is also sufficient for surjectivity, because it implies both $\text{supp}(\mu) = [0, 1]$ and $1/\mu \in Y \subseteq X/Y$. So we arrive at the following

Corollary 2.2.3 [KRw]. *The following statements are true for function spaces X and Y over $[0,1]$.*

(a) *For $X \in \{BV, WBV_p, \Lambda BV\}$, the operator $M_\mu : X \to Y$ is injective if and only if* $\operatorname{supp}(\mu) = [0,1]$.
(b) *For $X \in \{C, RBV_p\}$ with $p > 1$, the operator $M_\mu : X \to Y$ is injective if and only if* $\overline{\operatorname{supp}(\mu)} = [0,1]$.
(c) *For $Y \subseteq X \subseteq B$ and $Y \in \{B, C, BV, WBV_p, \Lambda BV, RBV_p\}$, the operator $M_\mu : X \to Y$ is surjective if and only if* $\operatorname{supp}_\delta(\mu) = [0,1]$ *for some $\delta > 0$. In this case M_μ is also injective.*

We remark that Corollary 2.2.3 was proved for the special case $X = BV$ in [AsRa], for $X = WBV_p$ in [AsCRa].
The following Examples 2.2.4 and 2.2.5 show that injectivity of the operator $M_\mu : BV \to BV$ does not imply its surjectivity, and that M_μ may be injective in RBV_p for $p > 1$ without being injective in $BV = RBV_1$.

Example 2.2.4. Define $\mu : [0,1] \to \mathbb{R}$ by

$$\mu(t) := \begin{cases} 1 & \text{for} \quad t = 0, \\ t & \text{for} \quad 0 < t \leq 1. \end{cases}$$

Clearly, $\mu \in BV$ with $Var(\mu) = 2$, and so M_μ maps BV into itself. The operator M_μ is injective, by Corollary 2.2.3 (a). However, M_μ is not surjective, since the function $y(t) \equiv 1$ belongs to BV, but not to the range of M_μ. In fact, any function x satisfying $M_\mu(x) = y$ would be unbounded near zero. □

The explanation of Example 2.2.4 is of course that the function μ is different from zero, but not bounded away from zero.

Example 2.2.5. Define $\mu : [0,1] \to \mathbb{R}$ by $\mu(t) := t$. Since $\mu \in RBV_p$ for every $p \geq 1$ with $RVar_p(\mu) = 1$, the operator M_μ maps RBV_p into itself and is injective in case $p > 1$, by Corollary 2.2.3 (b). However, M_μ is not injective in BV, since the function $x = \chi_0$ is not identically zero, but the function $y = M_\mu(x)$ is. □

The explanation of Example 2.2.5 is of course that the support of μ is dense in $[0,1]$, but does not coincide with $[0,1]$.
Combining the surjectivity and injectivity results from Corollary 2.2.3 with the fact that the inverse of the operator M_μ, if it exists, is the operator $M_{1/\mu}$, we obtain a necessary and sufficient condition for M_μ to be an isomorphism:

Theorem 2.2.6. *For $X \in \{BV, WBV_p, RBV_p, \Lambda BV\}$, the following two conditions are equivalent.*
(a) *The function μ belongs to X and* $\operatorname{supp}_\delta(\mu) = [0,1]$ *for some $\delta > 0$.*
(b) *The operator $M_\mu : X \to X$ is an isomorphism.*

We remark that Theorem 2.2.6 was proved for the special case $X = BV$ in [AsRa], for $X = WBV_p$ in [AsCRa].
Apart from boundedness (equivalent: continuity), an important property of linear operators is compactness. For multiplication operators, this often leads to a strong degeneracy. For instance, it is well-known that the operator (2.0.2) maps the space C [resp. the space L_p] into itself if and only if $\mu \in C$ [resp. $\mu \in L_\infty$], and it is compact if and only if $\mu(t) = 0$ for all [resp. almost all] $t \in [0,1]$.
Surprisingly enough, in the space BV many multiplication operators may be compact for nonconstant multipliers. To describe the corresponding class, we recall that X_c denotes the set of all functions $\mu \in X$ with countable support.

Theorem 2.2.7. *For $X \in \{BV, WBV_p, RBV_p, \Lambda BV\}$, the following two conditions are equivalent.*
(a) *The function μ belongs to X_c.*
(b) *The operator $M_\mu : X \to X$ is compact.*

This result was proved in [AsRa] for $X = BV$ and in [AsCRa] for $X = WBV_p$. Note that condition (a) may be fulfilled for continuous μ only in case $\mu(t) \equiv 0$, which shows that the zero operator is the only compact multiplication operator in $X = RBV_p$ for $p > 1$.
Recall that the *essential norm* of a bounded linear operator A between two Banach spaces X and Y is defined by

(2.2.4) $$|||A|||_{X\to Y} := \inf\{||A - K||_{X\to Y} : K : X \to Y \text{ linear and compact}\}.$$

In other words, the essential norm measures the distance of an operator from the (closed) ideal of compact operators; in particular, $|||A||| = 0$ if and only if A itself is compact. Our previous discussion suggests that

(2.2.5) $$|||M_\mu|||_{BV\to BV} = \inf\{\delta > 0 : \text{supp}_\delta(\mu) \text{ finite}\}.$$

We were not able to prove this conjecture in the general case; however, some partial results are possible. Recall that the *right regularization* $\mu^\#$ of a BV-function $\mu : [0,1] \to \mathbb{R}$ is defined by

(2.2.6) $$\mu^\#(t) := \begin{cases} \lim\limits_{s\to t+} \mu(s) & \text{for} \quad 0 \le t < 1, \\ \mu(1) & \text{for} \quad t = 1 \end{cases}$$

Then $Var(\mu^\#) \le Var(\mu)$, and $\lambda := \mu - \mu^\#$ is in BV and has countable support, because $\mu^\#$ differs from μ only in the (at most countably many) points of discontinuity of μ. Thus, λ generates a compact multiplication operator $M_\lambda : BV \to BV$, by Theorem 2.2.7, which implies that

$$|||M_{\mu^\#}|||_{BV\to BV} = |||M_\mu|||_{BV\to BV}.$$

We claim that M_μ satisfies the upper estimate

(2.2.7) $$|||M_\mu|||_{BV\to BV} \le \|\mu^\#\|_{BV}$$

and, if μ is bounded away from zero, also the lower estimate

(2.2.8) $$|||M_\mu|||_{BV\to BV} \ge \|1/\mu^\#\|_{BV}^{-1}.$$

In order to prove (2.2.7) we fix $\mu \in BV$ and obtain

$$|||M_\mu|||_{BV\to BV} \le \|M_\mu - M_\lambda\|_{BV\to BV} = \|M_{\mu-\lambda}\|_{BV\to BV} = \|\mu - \lambda\|_{BV} = \|\mu^\#\|_{BV},$$

where $\lambda = \mu - \mu^\#$ as before. For the proof of (2.2.8) we assume that μ is bounded away from zero, and hence generates an invertible operator $M_\mu : BV \to BV$ with $M_\mu^{-1} = M_{1/\mu}$. For any compact operator $K : BV \to BV$ we must have

$$\|M_{\mu^\#} - K\|_{BV\to BV} \ge \frac{1}{\|M_{\mu^\#}^{-1}\|_{BV\to BV}} = \frac{1}{\|1/\mu^\#\|_{BV}},$$

because a compact operator is never invertible in an infinite dimensional space. In particular, (2.2.8) follows after passing to the infimum over all compact operators K. We illustrate our discussion with two examples.

Example 2.2.8. This example shows that the estimates (2.2.7) and (2.2.8) become worse the closer μ comes to zero. For $\alpha > 0$, consider the function $\mu_\alpha(t) := t + \alpha$. Then μ_α is continuous and bounded away from zero, and has bounded variation. Since $\mu_\alpha^\# = \mu_\alpha$, from (2.2.7) and (2.2.8) we get

$$\frac{\alpha^2 + \alpha}{\alpha + 2} = \frac{1}{\|1/\mu_\alpha^\#\|_{BV}} \le |||M_{\mu_\alpha}|||_{BV\to BV} \le \|\mu_\alpha^\#\|_{BV} = 1 + \alpha.$$

Consequently,

$$\lim_{\alpha\to 0+} \frac{1}{\|1/\mu_\alpha^\#\|_{BV}} = 0, \qquad \lim_{\alpha\to 0+} \|\mu_\alpha^\#\|_{BV} = 1,$$

and a direct calculation shows that

$$\inf\{\delta > 0 : \operatorname{supp}_\delta(\mu_\alpha) \text{ finite}\} = 1 + \alpha.$$

This illustrates the "gap"between the lower and upper estimate for $|||M_{\mu_\alpha}|||_{BV\to BV}$. □

Example 2.2.9. In contrast to Example 2.2.8, in this example the functions μ and $\mu^\#$ are quite different. Let $\{r_0, r_1, r_2, \ldots\}$ be an enumeration of all rational numbers in $[0, 1]$, and define $\mu : [0, 1] \to \mathbb{R}$ by

$$\mu(t) := \begin{cases} 2^{-k} & \text{if } t = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \in BV$, and μ has countable support, so M_μ is compact, by Theorem 2.2.7. Moreover, since

$$\{r_k \in \mathbb{Q} \cap [0, 1] : 2^{-k} > \delta\,(k \in \mathbb{N})\} \subseteq \{2^{-k} : k < \log_2(1/\delta)\,(k \in \mathbb{N})\},$$

and the last set is finite for each $\delta > 0$, the equality (2.2.5) is true here. Alternatively, we could have used the estimate (2.2.7) and the fact that $\mu^\#(t) \equiv 0$ in this example. □

2.3. Integral operators. The purpose of this section is to study the linear integral operator (2.0.3), as well as its variant

$$V(x)(t) = \int_0^t k(t, s)x(s)\, ds, \tag{2.3.1}$$

in spaces of functions of bounded variation. There is a vast literature on the behaviour of the operators (2.0.3) and (2.3.1) in spaces of continuous or measurable functions, but considerably less is known in the space BV and its various generalizations. Let us point out that integral operators are more important than substitution or multiplication operators, because the operator (2.0.3) is closely related to boundary value problems, and the operator (2.3.1) is closely related to initial value problems for (second order) differential equations.

To begin with, we state two conditions on the kernel function k which will be used over and over in this section. Here the symbol $\forall' s$ means that the following property holds for almost all s.

$$\forall t \in [0, 1] : \; k(t, \cdot) \in L_1; \tag{A}$$

$$\exists m \in L_1, \; \forall' s \in [0, 1] : \; Var(k(\cdot, s)) \le m(s). \tag{B}$$

Condition (A) is indispensable for the integral in (2.0.3) to make sense. Condition (B) in turn is what we need to generate an integral operator in BV:

Theorem 2.3.1 [Bi]. *Under the conditions* (A) *and* (B), *the operator K maps the space BV into itself and is bounded.*

We make some comments on Theorem 2.3.1. First of all, if (B) holds, condition (A) may be weakened to

$$\forall t \in [0, 1] : \; k(t, \cdot) \text{ is measurable and } k(0, \cdot) \in L_1. \tag{A′}$$

In fact, one may show that (A') and (B) together imply (A). On the other hand, if we weaken (B) by requiring

$$\exists m \in L_1, \; \forall' s \in [0, 1] : \; |k(\cdot, s)| \le m(s), \tag{B′}$$

i.e., replacing the majorization for the variation by a pointwise majorization. In this case we cannot guarantee that K maps BV into itself; here is a very simple example.

Example 2.3.2. Let $k(t,s) := \chi_{\mathbb{Q}}(t)$. Then (A) and (B') hold (with $m(s) \equiv 1$), but (B) does not. The corresponding operator K maps the function $x(t) \equiv 1$ into the function $K(x) = \chi_{\mathbb{Q}}$ which does not belong to BV. □

We point out that Theorem 2.3.1 gives only a sufficient condition on k for $K(BV) \subseteq BV$; later we will show (Example 2.3.4) that this condition is not necessary. In fact, conditions (A) and (B) are so strong that they are even sufficient for the inclusions $K(L_\infty) \subseteq BV$ or $K(WBV_p) \subseteq BV$.

In order get a milder condition which is both necessary and sufficient, we introduce yet another requirement for k:

$$\text{(C)} \qquad \exists M > 0 \ \forall \xi \in [0,1] : \ Var\left(\int_0^\xi k(\cdot,s)\,ds\right) \leq M.$$

Observe that (B) implies (C), because for any partition $\{t_0, t_1, \ldots, t_m\}$ from (B) it follows that

$$\sum_{j=1}^m \left|\int_0^\xi k(t_j,s)\,ds - \int_0^\xi k(t_{j-1},s)\,ds\right| \leq \int_0^1 m(s)\,ds,$$

and so (C) is true with $M = \|m\|_{L_1}$. It turns out that, combining (C) with (A) (which must be our general hypothesis), we precisely get what we want.

Theorem 2.3.3 [BiGK]. *Suppose that k satisfies condition* (A). *Then the following two conditions are equivalent.*

(a) *The kernel function k satifies condition* (C).

(b) *The operator K maps the space BV into itself and is bounded.*

It is not hard to see that, under the hypotheses of Theorem 2.3.3, the norm of the operator K in BV may be estimated by

$$\text{(2.3.2)} \qquad \|K\|_{BV \to BV} \leq 2M + \|k(0,\cdot)\|_{L_1}.$$

Theorem 2.3.3 illustrates the fact that both condition (A) and condition (C) for k contribute to the boundedness of K. We remark that the following slightly more general fact was proved in [BiGK]: *The operator K maps the space BV into the Wiener space WBV_p and is bounded if and only if condition (C) holds with Var replaced by $WVar_p$.*

The following example shows that condition (C) does not imply condition (B).

Example 2.3.4 [BuGK]. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$k(t,s) := \begin{cases} \dfrac{t}{t^2+s^2} & \text{for} \quad (t,s) \neq (0,0), \\ 0 & \text{for} \quad (t,s) = (0,0). \end{cases}$$

Let us show that k satisfies (C). For every $\xi \in [0,1]$ and $t \in [0,1]$ we have

$$\int_0^\xi k(t,s)\,ds = \begin{cases} \arctan \dfrac{\xi}{t} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0. \end{cases}$$

Since the function $t \mapsto \arctan(\xi/t)$ is strictly decreasing for fixed $\xi > 0$, we can estimate the variation occurring in condition (C) by the maximal difference of the arctan function, i.e., by $M = \pi$.

To see that k does not satisfy (B), note that, for fixed $s \in (0,1)$, the function $k(\cdot, s)$ is increasing on $[0, s]$ and decreasing on $[s, 1]$, attaining its maximum $1/2s$ for $t = s$. So we have

$$Var(k(\cdot, s)) = k(s,s) - k(0,s) - k(1,s) + k(s,s) = \frac{1}{s} - \frac{1}{1+s^2},$$

and the function on the right-hand side does not belong to L_1. □

Example 2.3.4 is interesting also from another viewpoint, since the kernel function in this example satisfies condition (B'). To see this, observe that

$$\int_0^1 k(t,s)\,dt = \int_0^1 \frac{t}{t^2+s^2}\,dt = \int_s^{\sqrt{1+s^2}} \frac{du}{u} = \log \frac{\sqrt{1+s^2}}{s},$$

and the last expression is finite for $0 < s \leq 1$. This shows that the requirement $|k(\cdot, s)| \leq m(s)$ in (B') to hold only for *almost* all s is important.

There are two special cases for the integral operator (2.3.1) which are particularly important in applications, namely *separated kernels* and *Volterra kernels*. While we will treat Volterra kernels in the next section, we deal now with separated kernels, which means that k has the form

$$k(t,s) = k_1(t)k_2(s). \tag{2.3.3}$$

In this situation it is easy to express the condition (A), (B), and (C) in terms of conditions on k_1 and k_2. For instance, (A) and (B) hold if $k_1 \in BV$ and $k_2 \in L_1$; in this case we may choose $m(s) := Var(k_1)|k_2(s)|$. Similarly, (C) holds under the same assumption; in this case we may choose $M := Var(k_1)\|k_2\|_{L_1}$. However, neither (B) nor (C) for (2.3.3) implies that $k_1 \in BV$ and $k_2 \in L_1$.

Example 2.3.5. Let $k_1 = k_2 = \chi_{\mathbb{Q}}$. Then (B) is true, since

$$Var(k(\cdot, s)) = Var(k_2(s)k_1) = 0 \qquad (s \in [0,1] \setminus \mathbb{Q}).$$

However, we have $k_1(t) = k(t, 0) = \chi_{\mathbb{Q}}(t)$, and this function does not belong to BV. □

In view of Example 2.3.5 we remark that condition (C) (and so also (B)) for (2.3.3) implies $k_1 \in BV$ and $k_2 \in L_1$ if

$$\int_0^\xi k_2(s)\,ds \neq 0$$

for some $\xi \in (0, 1]$.

From (2.3.2) it follows that for separated kernels we have

$$\|K\|_{BV \to BV} \leq 2M + |k_1(0)|\,\|k_2\|_{L_1} = 2Var(k_1)\|k_2\|_{L_1} + |k_1(0)|\,\|k_2\|_{L_1} \leq 2\|k_1\|_{BV}\|k_2\|_{L_1}.$$

The study of solutions to integral equations, both linear and nonlinear, in BV spaces is motivated by numerous applications to real world problems. Sometimes it is useful, or even necessary, to look for solutions in the space $BV \cap C$, i.e., to add continuity. So there is some interest to find conditions which guarantee, or are even equivalent to, the inclusion $K(BV \cap C) \subseteq BV \cap C$. To this end, we still introduce another two conditions on the kernel function k:

(D) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall t, \tau \in [0,1]\ \forall' s \in [0,1]:\ |t - \tau| \leq \delta\ \Rightarrow\ |k(t,s) - k(\tau,s)| \leq \varepsilon;$

(E) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall t, \tau, \xi \in [0,1]:\ |t - \tau| \leq \delta\ \Rightarrow\ \left|\int_0^\xi [k(t,s) - k(\tau,s)]\,ds\right| \leq \varepsilon.$

Let us briefly check these two conditions by means of the kernel function k from Example 2.3.4. Choosing $t_n := 1/n$ and $\tau = 0$ in condition (E) we get

$$\left|\int_0^\xi [k(t_n,s) - k(\tau,s)]\,ds\right| = |\arctan(n\xi)| \to \frac{\pi}{2} \qquad (n \to \infty)$$

for $\xi > 0$ and so (E) fails. Similarly, (D) fails as well, because for the same choice of t_n and τ we have

$$|k(1/n,s) - k(0,s)| = \frac{n}{1+n^2s^2} \to \infty \qquad (n \to \infty)$$

for $0 < s \leq 1$.

Trivially, condition (D) implies condition (E); the converse, however, is not true, as we will see later (Example 2.3.6).

Let us check a moment what conditions (D) and (E) mean for separated kernel function (2.3.3). Obviously, the assumptions $k_1 \in C$ and $k_2 \in B$ are sufficient for (D) (take $\delta := \varepsilon/\|k_2\|_\infty$), while the assumptions $k_1 \in C$ and $k_2 \in L_\infty$ are sufficient for (E) (take $\delta := \varepsilon/\|k_2\|_{L_\infty}$). However, here we have to require that $k_2(s) \not\equiv 0$. In general these assumptions are far from being necessary: indeed, in case $k_2(s) \equiv 0$ we may choose k_1 arbitrary, and in case $k_1(t) = \text{const}$ we may choose k_2 arbitrary.

One could ask whether or not we get more implications for the kernel function (2.3.3) than those already established. The following example shows that even for separated kernel functions (D) is stronger than (E).

Example 2.3.6. Let k be of the form (2.3.3) with $k_1(t) := t$ and $k_2(s) := 1/\sqrt{s}$. Then (D) fails, because

$$|k(t,s) - k(\tau,s)| = \frac{|t-\tau|}{\sqrt{s}}$$

is unbounded for s near zero. However, the equality

$$\left|\int_0^\xi [k(t,s) - k(\tau,s)]\, ds\right| = |t-\tau| \int_0^\xi \frac{ds}{\sqrt{s}} = 2|t-\tau|\sqrt{\xi}$$

shows that (E) is fulfilled for any $\xi \in [0,1]$ with $\delta := \varepsilon/2$. □

For the reader's ease, let us recall which of the conditions (A) – (E) stated at the beginning are fulfilled by the examples of this section.

Example	(A)	(B)	(A')	(B')	(C)	(D)	(E)
2.3.2	yes	no	yes	yes	no	no	no
2.3.4	yes	no	yes	yes	yes	no	no
2.3.6	yes	yes	yes	yes	yes	no	yes

Table 2.1: Properties of k in the above examples

The importance of the new condition (E) is illustrated by the following

Theorem 2.3.7 [BiGK]. *Suppose that k satisfies condition* (A). *Then the following two conditions are equivalent.*

(a) *The kernel function k satifies conditions* (C) *and* (E).

(b) *The operator K maps the space $BV \cap C$ into itself and is bounded, and the set $\{K(x) : x \in BV \cap C, \|x\|_{BV} \leq 1\}$ is equicontinuous.*

In Chapter 4 we will apply our theoretical results to nonlinear integral equations involving linear integral operators like (2.0.3) and nonlinear composition or superposition operators. A useful tool to prove existence of solutions to such equations is fixed point theory. In most cases, it suffices to apply Banach's fixed point theorem for contractions or Schauder's fixed point theorem for compact maps. Since a bounded linear operator is always Lipschitz continuous, for applying Banach's theorem we have to ensure only a Lipschitz condition for the nonlinear operator; this is a delicate problem, as we shall see in the next chapter. On the other hand, for applying Schauder's theorem we have to ensure that one of the operators involved is compact, while for the other it suffices to be bounded. So there is some interest in having (at least sufficient) conditions on the kernel function k which imply the compactness of the corresponding operator K.

We consider this problem in the Wiener spaces WBV_p which reduces to BV for the special choice $p = 1$. To this end, we need condition (B) which we introduced at the beginning of this section.

Theorem 2.3.8 [BiGK1]. *Under the conditions* (A) *and* (B), *the integral operator K maps WBV_p into WBV_q and is compact for any $p, q \geq 1$.*

A comparison of Theorems 2.3.1 and 2.3.8 shows that the hypotheses for the boundedness and compactness of K are the same. Of course, these hypotheses are only sufficient, but not necessary. Let us show by means of an example that not every bounded integral operator in BV is compact.

Example 2.3.9. Since the kernel function k from Example 2.3.4 satisfies (A) and (C), but not (D), the corresponding operator maps the space BV into itself and is bounded, but we may suspect that it not compact. In fact, it is not. Consider the sequence $(x_n)_n$ of characteristic functions $x_n(t) := \chi_{[0,2^{-n}]}(t)$. This sequence is bounded, since $\|x_n\|_{BV} = 2$. Moreover,

$$K(x_m - x_n)(t) = \int_{2^{-n}}^{2^{-m}} \frac{t}{t^2 + s^2}\, ds = \arctan\left(\frac{1}{2^m t}\right) - \arctan\left(\frac{1}{2^n t}\right) \qquad (m < n,\ 0 < t \leq 1).$$

Since the function $t \mapsto K(x_m - x_n)(t)$ is nonnegative and attains its maximum at $t = 2^{-(m+n)/2}$, we conclude that

$$Var(K(x_m) - K(x_n)) \geq \arctan(2^{(n-m)/2}) - \arctan(2^{-(n-m)/2}) \geq \arctan\sqrt{2} - \arctan\frac{1}{\sqrt{2}},$$

which shows that $(K(x_n))_n$ cannot contain a convergent subsequence. □

Now we are going to study the linear integral operator (2.0.3) in spaces of bounded Riesz variation or Waterman variation. Denoting by RBV_p the space of all functions of bounded Riesz p-variation, see Section 1.3, we introduce a new condition (F). Here we suppose that the partial derivative $k_t(\cdot, s)$ of k with respect to the argument t is continuous for almost all s, and denote by κ the function $\kappa(t) := \|k_t(\cdot, s)\|_C$. The new condition reads then

$$\text{(F)} \qquad \forall' s \in [0,1] : \ k_t(\cdot, s) \in C \text{ and } \kappa \in L_p.$$

Note that condition (A) implies that $k(\cdot, s) \in RBV_p$ for almost all $s \in [0,1]$, so by Riesz's theorem (Theorem 1.3.5) we know that $k(\cdot, s) \in AC$, provided that $p > 1$. Then $k_t(\cdot, s) \in L_p$ for almost all $s \in [0,1]$ which shows that condition (F) is not as restrictive as it may seem.

Theorem 2.3.10 [AD]. *Under the conditions* (A), (B), *and* (F), *the integral operator K maps RBV_p into itself and is bounded.*

Now let Λ be a Waterman sequence, and denote by Var_Λ the corresponding Waterman variation introduced in Section 1.4. In analogy to conditions (B) and (C), we consider now the conditions

$$(\mathrm{B}_\Lambda) \qquad \exists m \in L_1,\ \forall' s \in [0,1] : \ Var_\Lambda(k(\cdot, s) \leq m(s);$$

$$(\mathrm{C}_\Lambda) \qquad \exists M > 0\ \forall \xi \in [0,1] : \ Var_\Lambda\left(\int_0^\xi k(\cdot, s)\, ds\right) \leq M.$$

With this notation, we have then the following analogues to Theorem 2.3.3 and Theorem 2.3.8:

Theorem 2.3.11 [BiCGS]. *Suppose that k satisfies condition* (A). *Then the following two conditions are equivalent.*

(a) *The kernel function k satifies condition* (C_Λ).

(b) *The operator K maps the space BV into the space ΛBV and is bounded.*

Theorem 2.3.12 [BiCGS]. *Under the conditions* (A) *and* (B_Λ), *the integral operator K maps ΛBV into itself and is compact.*

Observe that there is an asymmetry in Theorem 2.3.10: although condition (C_Λ) explicitly refers to the Waterman sequence Λ, we do not need the inclusion $K(\Lambda BV) \subseteq \Lambda BV$ in (b) to ensure (a), but only the (apparently weaker) inclusion $K(BV) \subseteq \Lambda BV$. This can be explaned as follows. Since the characteristic function $x = \chi_{[0,\xi]}$ belongs to BV, Theorem 2.3.7 (b) implies that $K(x) \in \Lambda BV$. But

$$K(x)(t) = \int_0^1 k(t,s)x(s)\,ds = \int_0^\xi k(t,s)\,ds,$$

and taking Λ-variations on both sides yields precisely condition (C_Λ).

2.4. Singular operators. A particularly interesting special case of linear integral operators is that of *Volterra kernels*, which means that $k(t,s) \equiv 0$ for $s \geq t$. The corresponding operator (2.0.3) has then the form

$$V(x)(t) = \int_0^t k(t,s)x(s)\,ds \tag{2.4.1}$$

and is called *Volterra operator*. Thus, we may write the kernel function in (2.4.1) in the form

$$v(t,s) := \begin{cases} k(t,s) & \text{for} \quad 0 \leq s < t \leq 1, \\ 0 & \text{for} \quad 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4.2}$$

Volterra operators have in general much nicer properties than just integral operators like (2.0.3). For example, the operator (2.4.1) has, in contrast to the operator (2.0.3), always spectral radius zero, which is useful in the search for invariant balls for nonlinear operators of Volterra type, see Section 4.2.

Let us see how our conditions (A), (B), (C), (B_Λ), and (C_Λ) translate for the Volterra operator (2.4.1).

As before, (A) requires that the function $v(t,\cdot)$ is in L_1, but here only on the interval $[0,t]$. condition (B) becomes

$$\forall' s \in [0,1]: \ |v(s,s)| + Var(v(\cdot,s);[s,1]) \leq m(s), \tag{2.4.3}$$

where m is some nonnegative L_1-function. Observe that, if we require the majorant m in (2.4.3) to belong not only to L_1, but to L_p for some $p > 1$, then also $v(t,\cdot) \in L_p[0,t]$ for all t. This follows from the estimate

$$|v(t,s)| \leq |v(s,s) - v(t,s)| + |v(s,s)| \leq Var(v(\cdot,s);[s,t]) + |v(s,s)| \leq m(s)$$

which holds for almost all $s \in [0,t]$.

Condition (C) may be replaced by

$$\exists M > 0 \ \forall \xi \in [0,1]: \ Var\left(\int_0^{\min\{\xi,\cdot\}} k(\cdot,s))\,ds\right) \leq M.$$

The following theorem shows that, due to the special structure (2.4.2) of the kernel, a Volterra operator maps a very large space into BV.

Theorem 2.4.1 [BiGK1]. *Suppose that k satisfies condition* (A) *and* (2.4.3), *where $m \in L_p$ for some $p \in [1, \infty)$. Then the operator* (2.4.1) *maps the space $L_{p/(p-1)}$ into the space BV and is bounded. In case $p = 1$ the operator $V : BV \to BV$ is compact.*

So Theorem 2.4.1 shows that, the milder the condition on the majorant m in (2.4.3) (i.e., the smaller p), the smaller we may choose the space of departure $L_{p/(p-1)}$ for V. Theorem 2.4.1 is proved by showing that the estimate

$$\|V(x)\|_{BV} \le \|m\|_{L_p} \|x\|_{L_{p/(p-1)}} \tag{2.4.4}$$

holds for every $x \in L_{p/(p-1)}$.

From Theorem 2.4.1 it follows that, whenever $(x_n)_n$ is a bounded sequence which converges in the $L_{p/(p-1)}$-norm to zero for some $p \in [1, \infty)$, the sequence $(V(x_n))_n$ converges in the BV-norm to zero. One may show that this is also true if $(x_n)_n$ converges only a.e. on $[0, 1]$ to zero. However, it is false for $p = \infty$, i.e., $L_{p/(p-1)} = L_1$, as the following example shows.

Example 2.4.2 [BiGK1]. For $0 \le t \le 1$, let $x_n(t) := n\chi_{[0,1/n]}(t)$. Clearly, the sequence $(x_n)_n$ is bounded in L_1 and converges a.e. on $[0, 1]$, but not in the L_1-norm, to zero. Taking $k(t, s) \equiv 1$ for $0 \le s < t \le 1$ we obtain

$$V(x_n)(t) = \int_0^t x_n(s)\, ds = n \min\{t, 1/n\} \qquad (0 \le t \le 1)$$

which shows that $Var(V(x_n)) \equiv 1$ for all $n \in \mathbb{N}$. □

In the setting of Waterman spaces, condition (B_Λ) for Volterra operators reads

$$|v(s, s)| + Var_\Lambda(v(\cdot, s); [s, 1]) \le m(s), \tag{2.4.5}$$

while condition (C_Λ) becomes

$$Var_\Lambda \left(\int_0^{\min\{\xi,\cdot\}} k(\cdot, s))\, ds \right) \le M. \tag{2.4.6}$$

We get then the following result which is parallel to Theorem 2.4.1.

Theorem 2.4.3 [BiCGS]. *Suppose that v satisfies condition* (A) *and* (2.4.5), *where $m \in L_p$ for some $p \in [1, \infty)$. Then the operator* (2.4.1) *maps the space $L_{p/(p-1)}$ into the space ΛBV and is bounded.*

Similarly as in the proof of Theorem 2.4.1, one gets here the estimate

$$\|V(x)\|_{\Lambda BV} \le \max\{\lambda_1, 1\} \|m\|_{L_p} \|x\|_{L_{p/(p-1)}}$$

for every $x \in L_{p/(p-1)}$, where m is the L_p function from (2.4.5), and λ_1 is the first element of the Waterman sequence $\Lambda = (\lambda_n)_n$.

In many applications, Volterra integral operators of type (2.4.1) with kernel function

$$v_\alpha(t, s) := \begin{cases} \dfrac{1}{(t-s)^\alpha} & \text{for} \quad 0 \le s < t \le 1, \\ 0 & \text{for} \quad 0 \le t \le s \le 1 \end{cases} \tag{2.4.7}$$

are particularly important. In case $0 < \alpha < 1$ such operators are called *weakly singular*, in case $\alpha = 1$ strongly singular. We consider here the weakly singular Volterra operator

$$V_\alpha(x)(t) = \int_0^t \frac{x(s)}{(t-s)^\alpha}\, ds \qquad (0 \le t \le 1). \tag{2.4.8}$$

It is obvious that the operator V_α satisfies condition (A) (see Section 2.3) for $0 < \alpha < 1$. We claim that V_α also satisfies condition (C), but not condition (B), and so it exhibits the same behaviour as the operator from Example 2.3.4.

To see this, observe that, for fixed $\xi \in [0,1]$ we have

$$\int_0^\xi v_\alpha(0,s)\,ds = 0$$

and

$$\int_0^\xi v_\alpha(t,s)\,ds = \int_0^{\min\{\xi,t\}} \frac{ds}{(t-s)^\alpha} = \frac{t^{1-\alpha}}{1-\alpha} - \frac{(t-\min\{\xi,t\})^{1-\alpha}}{1-\alpha} \qquad (0 < t \le 1). \tag{2.4.9}$$

Since the function on the right-hand side of this equality is increasing in t on $[0,\xi]$, and decreasing in t on $[\xi,1]$, we obtain, similarly as in Example 2.3.4,

$$\begin{aligned} Var\left(\int_0^\xi v_\alpha(\cdot,s)\,ds\right) &= 2\int_0^\xi v_\alpha(\xi,s)\,ds - \int_0^\xi v_\alpha(0,s)\,ds - \int_0^\xi v_\alpha(1,s)\,ds \\ &= \frac{2}{1-\alpha}\xi^{1-\alpha} - \frac{1}{1-\alpha} - \frac{1}{1-\alpha}(1-\xi)^{1-\alpha} \le \frac{2}{1-\alpha} =: M. \end{aligned}$$

However, the kernel function v_α cannot satisfy condition (B), because the function $t \mapsto v_\alpha(t,s)$ is not bounded, let alone of bounded variation, for any $s \in (0,1)$. For the same reason, v_α does not satisfy (D). To check condition (E), observe that (2.4.9) implies

$$\begin{aligned} \left|\int_0^\xi [v_\alpha(t,s) - v_\alpha(\tau,s)]\,ds\right| &= \left|\frac{t^{1-\alpha}}{1-\alpha} - \frac{(t-\min\{\xi,t\})^{1-\alpha}}{1-\alpha} - \frac{\tau^{1-\alpha}}{1-\alpha} + \frac{(\tau-\min\{\xi,\tau\})^{1-\alpha}}{1-\alpha}\right| \\ &\le \frac{1}{1-\alpha}\left(|t^{1-\alpha} - \tau^{1-\alpha}| + |t - \min\{\xi,t\} - \tau + \min\{\xi,\tau\}|^{1-\alpha}\right) \\ &\le \frac{1}{1-\alpha}\left(|t-\tau|^{1-\alpha} + |t - \min\{\xi,t\} - \min\{\xi,\tau\}|^{1-\alpha}\right) \\ &\le \frac{1}{1-\alpha}\left(|t-\tau|^{1-\alpha} + (|t-\tau| + |\min\{\xi,t\} - \tau + \min\{\xi,\tau\}|)^{1-\alpha}\right) \\ &\le \frac{1}{1-\alpha}\left(|t-\tau|^{1-\alpha} + 2^{1-\alpha}|t-\tau|^{1-\alpha}\right) = \frac{1+2^{1-\alpha}}{1-\alpha}|t-\tau|^{1-\alpha}. \end{aligned}$$

So for $\varepsilon > 0$ it suffices to choose

$$0 < \delta \le \left(\frac{1-\alpha}{1+2^{1-\alpha}}\right)^{1/(1-\alpha)} \varepsilon^{1/(1-\alpha)}$$

to verify condition (E). To conclude, let us collect the various (mostly only sufficient) conditions on k under which the corresponding integral operator K has certain analytical properties.

(A) & (B)	$\Rightarrow$	$K(BV) \subseteq BV$ bounded
(A) & (C)	$\Leftrightarrow$	$K(BV) \subseteq BV$ bounded
(A) & (B)	$\Rightarrow$	$K(WBV_p) \subseteq WBV_q$ compact
(C) & (E)	$\Rightarrow$	$K(BV \cap C) \subseteq BV \cap C$ compact
(A) & (B) & (F)	$\Rightarrow$	$K(RBV_p) \subseteq RBV_p$ bounded
(A) & (C_Λ)	$\Leftrightarrow$	$K(BV) \subseteq \Lambda BV$ bounded
(A) & (2.4.3)	$\Rightarrow$	$V(L_\infty) \subseteq BV$ bounded
(A) & (2.4.5)	$\Rightarrow$	$V(L_\infty) \subseteq \Lambda BV$ bounded

Table 2.2: The operators K and V in BV spaces

Chapter 3. Nonlinear Operators in BV Spaces

In this chapter we study composition operators and superposition operators in BV and related more general spaces.

Recall that the *composition operator* C_f generated by some function $f : \mathbb{R} \to \mathbb{R}$ acting on functions $x : [0,1] \to \mathbb{R}$ is defined by

$$C_f(x)(t) := f(x(t)) \qquad (0 \leq t \leq 1). \tag{3.0.1}$$

More generally, the *superposition operator* (or *Nemytskij operator*) S_f generated by some function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ acting on functions $x : [0,1] \to \mathbb{R}$ is defined by

$$S_f(x)(t) := f(t, x(t)) \qquad (0 \leq t \leq 1). \tag{3.0.2}$$

In the study of these operators in spaces of functions of bounded variation, the following 10 crucial questions naturally arise.

1. Under what conditions on f we have $C_f(BV) \subseteq BV$?
2. Under what conditions on f is the operator C_f bounded on BV?
3. Under what conditions on f is the operator C_f continuous on BV?
4. Under what conditions on f is the operator C_f uniformly continuous on bounded subsets?
5. Under what conditions on f is the operator C_f Lipschitz continuous on bounded subsets?
6. Under what conditions on f we have $S_f(BV) \subseteq BV$?
7. Under what conditions on f is the operator S_f bounded on BV?
8. Under what conditions on f is the operator S_f continuous on BV?
9. Under what conditions on f is the operator S_f uniformly continuous on bounded subsets?
10. Under what conditions on f is the operator S_f Lipschitz continuous on bounded subsets?

Of course, the term "conditions" means here that we are interested in criteria which are both necessary and sufficient. Conditions which are only sufficient, or only necessary, are often easily found. Recall that a (linear or nonlinear) operator is called *bounded* if it maps bounded sets into bounded sets. In contrast to linear operators, a nonlinear operator may be bounded and discontinuous, or continuous and unbounded.

While the answer to almost all questions for the operator (3.0.1) is now known, but many of these questions for the operator (3.0.2) are open. This is not surprising, because the "interaction" of t and u in the function $f(t,u)$ in (3.0.2) makes the theory much more complicated (and, in fact, sometimes leads to completely unexpected phenomena).

3.1. Composition operators in BV. We start this section with the solution of the first and second question for the composition operator (3.0.1).

Theorem 3.1.1 [Jo]. *The following two conditions are equivalent:*
(a) *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the local Lipschitz condition*

$$\forall r > 0\, \exists L_r > 0\, \forall u, v \in [-r, r] : \ |f(u) - f(v)| \leq L_r |u - v|. \tag{3.1.1}$$

(b) *The operator* C_f *defined in* (3.0.1) *maps* BV *into itself.*

Theorem 3.1.1 is rather subtle, inasmuch it is not possible to replace (local) Lipschitz continuity of f in (a) by (global) Hölder continuity:

Example 3.1.2. Let C_f be the composition operator generated by the "seagull function"

$$f(u) := \min\left\{\sqrt{|u|}, 1\right\}.$$

Then f is absolutely continuous and Hölder continuous (with Hölder exponent 1/2). However, C_f maps the function

$$x(t) := \begin{cases} t^2 \sin^2 \dfrac{1}{t} & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

which belongs to BV, into the function

$$f(x(t)) := \begin{cases} t \left|\sin \dfrac{1}{t}\right| & \text{for} \quad 0 < t \leq 1, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

which does not. □

Observe that Example 3.1.2 also shows that the composition of two functions of bounded variation need not have bounded variation.

Before studying analytical properties of the composition operator (3.0.1), let us study some mapping properties like injectivity, surjectivity, and bijectivity. It is rather obvious that *the operator* C_f *is bijective in* BV *if and only if the function* f *is bijective and both* f *and* f^{-1} *satisfy the local Lipschitz condition* (3.1.1) on $\mathbb{R}$. Indeed, this is a direct consequence of Theorem 3.1.1 and the fact that C_f^{-1} (if it exists!) is the composition operator $C_{f^{-1}}$. Moreover, the following simple example shows that we really need the condition $f^{-1} \in Lip_{loc}(\mathbb{R})$ to ensure the bijectivity of $C_f : BV \to BV$.

Example 3.1.3. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(u) := u^3$ is bijective with $f^{-1} \notin Lip_{loc}(\mathbb{R})$. Clearly, the corresponding composition operator C_f is injective in BV. However, C_f is not surjective. To see this, observe that the function

$$y(t) := \begin{cases} \dfrac{1}{n^3} & \text{for } t = \dfrac{1}{n}, \\ 0 & \text{otherwise} \end{cases}$$

belongs to BV. The only possible preimage x of y is

$$x(t) = \begin{cases} \dfrac{1}{n} & \text{for } t = \dfrac{1}{n}, \\ 0 & \text{otherwise,} \end{cases}$$

which does not belong to BV. □

Surprisingly enough, the symmetry between f and C_f holds for injectivity, but not for surjectivity, as the following theorem shows.

Theorem 3.1.4 [KRw]. (a) *The injectivity of the operator* C_f *on* BV *implies the injectivity of the function* f *on* $\mathbb{R}$*, and vice versa.*
(b) *The surjectivity of the operator* C_f *on* BV *implies the surjectivity of the function* f *on* $\mathbb{R}$.

Note the similarity between the mapping behaviour of the inner composition operator Σ_φ and the outer composition operator C_f: for the injectivity of Σ_φ we have a necessary and sufficient condition in terms of φ, while for its surjectivity we only have a necessary condition; for the injectivity of C_f we have a necessary and sufficient condition in terms of f, while for its surjectivity we only have a necessary condition.

The operator C_f in Example 3.1.3 is injective, but not surjective in BV. Conversely, in contrast to multiplication operators in BV, where surjectivity implies injectivity (see Corollary 2.2.3 (c)), there are composition operators in BV which are surjective, but not injective.

Example 3.1.5 [KRw]. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(u) := \min\{u+2, |u|\} = \begin{cases} 2+u & \text{for} \quad u < -1, \\ -u & \text{for} \quad -1 \leq u \leq 0, \\ u & \text{for} \quad u > 0. \end{cases}$$

Geometrically, the graph of f consists of three linear pieces with corner points at $(-1, 1)$ and $(0, 0)$. Obviously, f is locally (even globally) Lipschitz on $\mathbb{R}$, so the operator C_f maps BV into itself, by Theorem 3.1.1. Moreover, C_f is not injective, which follows from Theorem 3.1.4 (a) or may be checked directly. However, $C_f : BV \to BV$ is surjective, which may be seen as follows.

For fixed $v \in \mathbb{R}$ we have

$$f^{-1}(\{v\}) = \begin{cases} \{v-2\} & \text{for} \quad v < 0, \\ \{v-2, -v, v\} & \text{for} \quad 0 \leq v \leq 1, \\ \{v\} & \text{for} \quad v > 1. \end{cases}$$

Being bounded, every $y \in BV$ maps $[0, 1]$ into an interval $[a, b]$ for some $a < b$. A somewhat cumbersome reasoning [KRw, Lemma 6] shows that one can find a set $A \subseteq [0, 1]$ with only finitely many connected components such that

$$y^{-1}([a, 1/3]) \subseteq A \subseteq y^{-1}([a, 2/3)).$$

With the function y we associate a function x defined by

$$x(t) := (y(t) - 2)\chi_A(t) + y(t)\chi_{[0,1]\setminus A}(t) = \begin{cases} y(t) - 2 & \text{for} \quad t \in A, \\ y(t) & \text{for} \quad t \notin A. \end{cases}$$

Since A (and so also $[0, 1] \setminus A$) has only finitely many connected components, both functions χ_A and $\chi_{[0,1]\setminus A}$ belong to BV, hence $x \in BV$. Moreover, by construction we have $y = C_f(x)$, which proves surjectivity. □

We may summarize our comparison between mapping properties of $f \in Lip_{loc}(\mathbb{R})$ and $C_f : BV \to BV$ as follows:

- Injectivity of C_f implies injectivity of f, and vice versa.
- Surjectivity of C_f implies surjectivity of f, but not vice versa.
- There are injective composition operators which are not surjective.
- There are surjective composition operators which are not injective.
- Bijectivity of C_f implies bijectivity of f with $f^{-1} \in Lip_{loc}$, and vice versa.

Let us take a closer look at the last assertion. The crucial condition $f^{-1} \in Lip_{loc}$ was reformulated in [KRw] in the following form. Let us call a function $f : \mathbb{R} \to \mathbb{R}$ *non-flat at* $u \in \mathbb{R}$ if there are compact intervals $I, J \subset \mathbb{R}$ such that u is an interior point of I, $f|_I : I \to J$ is bijective, and $(f|_I)^{-1} \in Lip(J)$. This name is motivated by the fact that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is *non-flat at* $u \in \mathbb{R}$ if and only if there are numbers $m, \delta > 0$ such that $|f(u_1) - f(u_2)| \geq m|u_1 - u_2|$ for all $u_1, u_2 \in [u-\delta, u+\delta]$, which means that f has locally a Lipschitz continuous inverse.

The surjective function f from Example 3.1.3, which does not generate a surjective operator C_f in BV, is non-flat at every point $u \in \mathbb{R} \setminus \{0\}$, but not at $u = 0$, the only preimage of $v = 0$. Similarly, the surjective function f from Example 3.1.5, which generates a surjective operator C_f in BV, is non-flat at every point $u \in \mathbb{R} \setminus \{-1, 0\}$, where $f(-1) = 1$ and $f(0) = 0$. However, in contrast to Example 3.1.3 we have more choices for preimages: for $v = 1$ we may take $u = 1 \in f^{-1}(1)$, while for $v = 0$ we may take $u = -2 \in f^{-1}(0)$, and f is non-flat at both points u. This suggests the following conjecture: *The operator C_f defined by some $f \in Lip_{loc}(\mathbb{R})$ is surjective in BV if and only if for each $v \in \mathbb{R}$ there is some $u \in f^{-1}(v)$ such that f is non-flat at u.* We were able to prove the "if" part (which is not easy at all) [KRw], but not the "only if" part of this conjecture.

In the following Table 3.1 we summarize what we know about the mapping properties of the operators we considered so far: the substitution operator (2.0.1), the multiplication operator (2.0.2), and the composition operator (3.0.1).

φ surjective	φ injective	$\mathrm{supp}(\mu) = [0,1]$	$\mathrm{supp}_\delta(\mu) = [0,1]$	f injective	f surjective
$\Updownarrow$	$\Uparrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Uparrow$
Σ_φ injective	Σ_φ surjective	M_μ injective	M_μ surjective	C_f injective	C_f surjective

Table 3.1: Mapping properties of some operators in BV

Example 3.1.3 shows that the implication for C_f in the fifth box in Table 3.1 cannot be inverted. On the other hand, we do not know whether or not the implication for Σ_φ in the second box can be inverted.

Now we are going to analyze the analytical properties of the operator C_f, like boundedness, continuity, or Lipschitz continuity. Throughout this survey, we denote by

$$B_r = B_r(X) := \{x \in X : \|x\| \leq r\} \tag{3.1.2}$$

the closed ball centered at zero with radius $r > 0$ in a normed space $(X, \|\cdot\|)$. Recall that an operator between two normed spaces is called *bounded* if it maps bounded sets into bounded sets.

Theorem 3.1.6. *Under the hypothesis* (3.1.1), *the operator C_f is automatically bounded in the norm* (1.1.14).

The proof of the following result is straightforward: indeed, condition (3.1.1) immediately implies that $C_f(B_r(BV)) \subseteq B_{2L_r}(BV)$.

To decide whether or not the operator C_f is also continuous in the norm (1.1.14) if $C_f(BV) \subseteq BV$ was a surprisingly difficult open problem for more than 50 years. Some sufficient conditions have been given in the literature. Thus, in [BaBiKMc,BiGK1] it is shown that C_f is continuous if f is a sum of power series centered at 0 with infinite radius of convergence, or if f is of class C^1. In [DN, Corollary 6.64] the authors prove that the condition $f \in C^1$ even implies the uniform continuity of C_f on bounded subsets of BV. On the other hand, C_f need not be uniformly continuous on bounded subsets if f is locally Lipschitz [DN, Proposition 6.66], which by Josephy's result is necessary and sufficient for $C_f(BV) \subseteq BV$.

The problem of deciding whether or not the operator C_f is always continuous in the norm (1.1.14) whenever it maps BV into itself is now positively solved:

Theorem 3.1.7. *Under the hypothesis* (3.1.1), *the operator C_f is automatically continuous in BV.*

Theorem 3.1.7 has an interesting history. In the paper [Mo] from 1937, Morse claims to prove that the local Lipschitz continuity of f guarantees the continuity of C_f in the BV norm. However, the proof of this claim is about 30 pages long, it uses cryptic tools from other fields, and its correctness is at least doubtful. In the recent paper [Mc1], the author gives a more straightforward and elegant proof of this fact.

An even shorter (and almost elementary) proof of Theorem 3.1.7 was given quite recently by Reinwand in [Rw]. Since this proof gives some insight into general continuity properties of operators, we briefly sketch the idea.

It is taught in every first year calculus course that the continuity of a sequence of functions between two metric spaces carries over to the limit function under uniform convergence, but not necessarily under pointwise convergence. In [Rw] the author introduces another type of convergence, called *semi-uniform convergence*, and proves the following result: *if $(C_n)_n$ is a sequence of continuous functions $C_n : X \to Y$, where X and Y are metric spaces, which converges semi-uniformly to some function $C : X \to Y$, then C is also continuous.*

Now, the point is that this may be made more explicit if C_n and C are composition operators. Given a sequence $(f_n)_n$ in $Lip_{loc}(\mathbb{R})$ and a function $f \in Lip_{loc}(\mathbb{R})$, the author proves the following

Proposition 3.1.8 [Rw]. *The following two statements are equivalent.*

(a) *The operator sequence $(C_{f_n})_n$ converges semi-uniformly in BV to the operator C_f.*

(b) *The relations*

$$\lim_{n\to\infty} \|f_n - f\|_{BV[-R,R]} = 0, \qquad \sup_n \, lip(f_n; [-R, R]) < \infty$$

hold for each $R > 0$.

Observe that (b) is much weaker than convergence in $Lip_{loc}(\mathbb{R})$ (i.e., in the norm $\|\cdot\|_{Lip[-R,R]}$), because we only impose convergence in the BV norm (1.1.14).

Now Theorem 3.1.7 follows readily from Proposition 3.1.8. In fact, $f \in Lip_{loc}$ implies that (and is even equivalent to) $f' \in L_\infty[-R, R]$ for each $R > 0$. We can therefore choose a sequence of polynomials $(p_n)_n$ such that $\|p_n - f'\|_{L_1[-R,R]} \to 0$ as $n \to \infty$, $\|p_n\|_{L_\infty[-R,R]} \le c_R < \infty$ for each $R > 0$, and the sequence $(p_n)_n$ does not depend on R. The functions $f_n : [-R, R] \to \mathbb{R}$ defined by

$$f_n(u) := f(0) + \int_0^u p_n(v)\, dv \qquad (|u| \le R)$$

satisfy then

$$f_n \in Lip_{loc}, \qquad \|f_n - f\|_{BV[-R,R]} = \|p_n - f'\|_{L_1[-R,R]} \to 0, \qquad lip(f_n; [-R, R]) \le c_R$$

for each $R > 0$. By Proposition 3.1.8, the operator sequence $(C_{f_n})_n$ converges semi-uniformly in BV to the operator C_f, and since each C_{f_n} is continuous in BV, the operator C_f is continuous as well.

We already mentioned a sufficient condition for the uniform continuity of C_f on bounded subsets of BV; let us restate this for further reference as

Theorem 3.1.9 [DN]. *If the function $f : \mathbb{R} \to \mathbb{R}$ is C^1, the operator C_f defined in* (3.0.1) *is uniformly continuous on bounded subsets of BV.*

Now we turn to the problem of characterizing the Lipschitz continuity of C_f on bounded subsets of BV. One might ask why we did not consider the *global* Lipschitz continuity of C_f in BV in our list of questions, i.e., conditions for

$$\|C_f(x) - C_f(\tilde{x})\|_{BV} \le L\|x - \tilde{x}\|_{BV} \quad (x, \tilde{x} \in BV) \tag{3.1.3}$$

with some constant $L > 0$. The reason is simple: this would lead to a strong degeneracy for the generating function f:

Theorem 3.1.10 [MM]. *The following two conditions are equivalent:*
(a) *The function $f : \mathbb{R} \to \mathbb{R}$ is affine, i.e.,*

$$f(u) = \alpha u + \beta \qquad (u \in \mathbb{R}) \tag{3.1.4}$$

for some constants $\alpha, \beta \in \mathbb{R}$.
(b) *The operator C_f defined in* (3.0.1) *satisfies the global Lipschitz condition* (3.1.3).

Theorem 3.1.10 is quite surprising: it shows that, if we want to apply the Banach fixed point principle to a problem involving composition operators in the space BV, we can do this only if the problem is actually *linear*. On the other hand, in most applications it suffices to impose a *local* Lipschitz condition of the form

$$\|C_f(x) - C_f(\tilde{x})\|_{BV} \leq L_r \|x - \tilde{x}\|_{BV} \quad (x, \tilde{x} \in B_r(BV)) \tag{3.1.5}$$

with some constant $L_r > 0$ which may depend on the "size" of the bounded set containing x and $\tilde{x}$. Here we arrive at a condition which is milder than being affine:

Theorem 3.1.11 [AMS]. *The following two conditions are equivalent:*
(a) *The function $f : \mathbb{R} \to \mathbb{R}$ is C^1, and its derivative satisfies the local Lipschitz condition*

$$\forall r > 0\, \exists L_r > 0\, \forall u, v \in [-r, r] : \ |f'(u) - f'(v)| \leq L_r |u - v|. \tag{3.1.6}$$

(b) *The operator C_f defined in* (3.0.1) *satisfies the local Lipschitz condition* (3.1.5).

The following example illustrates the difference between the preceding two theorems.

Example 3.1.12. Let C_f be the composition operator generated by the function $f(u) := u^2$. By Theorems 3.1.1 and 3.1.8, the corresponding operator C_f maps BV into itself and is both bounded and continuous. We claim that C_f satisfies a local, but not a global Lipschitz condition in the BV-norm (1.1.14).
To see this, we first recall that BV is an algebra, as we have seen in Section 1.1, which is imbedded into the algebra of all bounded functions with the sup norm $\|\cdot\|_\infty$. So for $x, \tilde{x} \in BV$ with $\|x\|_{BV}, \|\tilde{x}\|_{BV} \leq r$ we have

$$\|C_f(x) - C_f(\tilde{x})\|_{BV} = \|(x + \tilde{x})(x - \tilde{x})\|_{BV} \leq \|x + \tilde{x}\|_\infty \|x - \tilde{x}\|_{BV} + \|x + \tilde{x}\|_{BV} \|x - \tilde{x}\|_\infty.$$

Since $\|x\|_\infty \leq \|x\|_{BV}$, we conclude that (3.1.5) holds with $L_r := 4r$.
On the other hand, Theorem 3.1.10 implies that C_f cannot be globally Lipschitz continuous on BV. Indeed, if we suppose that (3.1.3) is true, we could take $x(t) \equiv 2L$ and $\tilde{x}(t) \equiv 0$ and get a contradiction. □

Our final result in this section is a characterization of compact composition operators in BV. It is well-known that the operator (3.0.1) (and even the operator (3.0.2)) can be compact in many function spaces, like C or L_p, only if the generating function is constant. This shows, loosely speaking, that it does not make sense to apply the Schauder fixed point theorem to a problem which contains a composition operator, but no other operator which is compact. The same is true in the space BV:

Theorem 3.1.13. *The following two conditions are equivalent:*
(a) *The function $f : \mathbb{R} \to \mathbb{R}$ is constant on $\mathbb{R}$.*
(b) *The operator C_f defined in* (3.0.1) *is compact in BV.*

Proof. The implication (a) $\Rightarrow$ (b) is trivial. So assume that $C_f : BV \to BV$ is compact, but the generating function is not constant. This means that we can find real numbers a and b with $a < b$ and (without loss of generality) $f(a) < f(b)$. Here we may assume that $a = 0$, $f(a) = 0$, $b = 1$, and $f(b) = 1$, because otherwise we may pass from f to the function

$$g(t) := \frac{f((b-a)t + a) - f(a)}{f(b) - f(a)}$$

which generates a composition operator $C_g : BV \to BV$ that is compact if and only if C_f is compact.

The sequence $(x_n)_n$ defined by $x_n(t) := \chi_{(0,1/n)}(t)$ is bounded in BV, so by our compactness assumption the sequence $(y_n)_n$ defined by $y_n(t) := f(x_n(t))$ has a convergent subsequence $(y_{n_k})_k$. Denote its limit by y, so

$$\|y_{n_k} - y\|_{BV} \to 0 \qquad (k \to \infty).$$

But

$$y_{n_k}(t) = (f \circ x_{n_k})(t) = f(\chi_{(0,1/n_k)}(t)) = \chi_{(0,1/n_k)}(t),$$

since $f(0) = 0$ and $f(1) = 1$. Moreover, it is clear that the sequence $(y_{n_k})_k$ converges pointwise on $[0, 1]$ to zero, so $y(t) \equiv 0$. On the other hand

$$\|y_{n_k}\|_{BV} = \|\chi_{(0,1/n_k)}\|_{BV} \equiv 2,$$

a contradiction. It follows that our assumption was false, and so f has to be constant as claimed. ■

To conclude, we summarize our results on the autonomous composition operator (3.0.1) in the following synoptic table.

$f \in Lip_{loc}(\mathbb{R})$	$\Leftrightarrow$	$C_f(BV) \subseteq BV$
$f \in Lip_{loc}(\mathbb{R})$	$\Leftrightarrow$	C_f bounded
$f \in Lip_{loc}(\mathbb{R})$	$\Leftrightarrow$	C_f continuous
$f \in C^1(\mathbb{R})$	$\Rightarrow$	C_f uniformly continuous on bounded sets
$f' \in Lip_{loc}(\mathbb{R})$	$\Leftrightarrow$	C_f Lipschitz continuous on bounded sets
f affine	$\Leftrightarrow$	C_f globally Lipschitz continuous
f constant	$\Leftrightarrow$	C_f compact

Table 3.2: The operator C_f in BV

Table 3.2 shows that the problem of characterizing boundedness, continuity, and (global or local) Lipschitz continuity of the composition operator C_f are completely solved. As we shall see in the following section, this is far from being true for the superposition operator S_f.

3.2. Superposition operators in BV**.** As pointed out above, the behavior of the superposition operator (3.0.2) in BV is much more complicated than in the autonomous case (3.0.1). For making the presentation more coherent and for not overburdening the formulation of the theorems which follow, we collect right from the beginning of this section 7 technical conditions (A) – (G) on the function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$.

(A) $$\exists L > 0\, \forall u, v \in \mathbb{R} : \sup_{0 \le t \le 1} |f(t, u) - f(t, v)| \le L|u - v|.$$

(B) $$\forall r > 0\, \exists L_r > 0\, \forall u, v \in [-r, r] : \sup_{0 \le t \le 1} |f(t, u) - f(t, v)| \le L_r|u - v|.$$

(C) $$\exists M>0\,\forall u\in\mathbb{R}:\ Var(f(\cdot,u);[0,1])\le M.$$

(D) $$\forall r>0\,\exists M_r>0\,\forall u\in[-r,r]:\ \mathrm{Var}(f(\cdot,u);[0,1])\le M_r.$$

(E) $$\exists M>0\,\forall\{t_0,\dots,t_m\}\in\mathcal{P}[0,1]\,\forall u_0,\dots,u_{m-1}\in\mathbb{R}:\ \sum_{j=1}^{m}|f(t_j,u_{j-1})-f(t_{j-1},u_{j-1})|\le M.$$

(F) $$\begin{cases}\forall r>0\,\exists M_r>0\,\forall\{t_0,\dots,t_m\}\in\mathcal{P}[0,1]\,\forall u_0,\dots,u_{m-1}\in[-r,r]:\\[2mm] \displaystyle\sum_{j=1}^{m-1}|u_j-u_{j-1}|\le r\ \Rightarrow\ \sum_{j=1}^{m}|f(t_j,u_{j-1})-f(t_{j-1},u_{j-1})|\le M_r.\end{cases}$$

(G) $$\begin{cases}\forall r>0\,\exists M_r>0\,\forall\{t_0,\dots,t_m\}\in\mathcal{P}[0,1]\,\forall u_0,\dots,u_m\in[-r,r]:\ \displaystyle\sum_{j=1}^{m}|u_j-u_{j-1}|\le r\\[2mm] \Rightarrow\ \displaystyle\sum_{j=1}^{m}|f(t_j,u_j)-f(t_{j-1},u_j)|\le M_r \text{ and } \sum_{j=1}^{m}|f(t_{j-1},u_j)-f(t_{j-1},u_{j-1})|\le M_r.\end{cases}$$

One could call (A) a *Lipschitz condition* for $f(t,\cdot)$, uniformly in t, (B) a *local Lipschitz condition* for $f(t,\cdot)$, uniformly in t, (C) a *variation condition* for $f(\cdot,u)$, uniformly in u, (D) a *variation condition* for $f(\cdot,u)$, locally uniformly in u, (E) a *mixed condition* for f, (F) a *local mixed condition* for f, and (G) a (local) *crossed mixed condition* for f.

Observe that (B) implies (G) with $M_r := L_r \max\{r,1\}$; we will use this fact later in Chapteer 4.

There are some obvious interconnections between the conditions (A) – (G) which we collect in the following Table 3.3.

(A)	$\Rightarrow$	(B)	(C)	$\Rightarrow$	(D)		
			(E)	$\Rightarrow$	(F)	$\Leftarrow$	(G)

Table 3.3: Interconnection between (A) – (G) (general f)

None of these conditions are equivalent. Clearly, (B) $\not\Rightarrow$ (A) (take $f(t,u)=u^2$), and (D) $\not\Rightarrow$ (C) (take $f(t,u)=tu$). Moreover, we will show in Example 3.2.2 below that the implication (F) $\Rightarrow$ (E) is not true.

Keeping in mind the interconnections between the conditions (A) – (G), we start now a series of results which connect these conditions to the properties of the superposition operator S_f generated by $f:[0,1]\times\mathbb{R}\to\mathbb{R}$. Here functions of the special type

(3.2.1) $$f(t,u) := \begin{cases}\phi(u) & \text{for}\quad t=0,\\ 0 & \text{for}\quad 0<t\le 1\end{cases}$$

will play a prominent role, where $\phi:\mathbb{R}\to\mathbb{R}$ is given. Several examples and counterexamples may be constructed by adjusting the properties of ϕ in order to fulfill (or not to fulfill) a prescribed part of the conditions (A) – (G). This fact is based on the observation that, by definition of f, the equality

$$S_f(x)(t)-S_f(y)(t)=[\phi(x(0))-\phi(y(0))]\chi_{\{0\}}(t)\qquad(0\le t\le 1)$$

holds for all functions $x,y\in BV$, which implies the following

Lemma 3.2.1. *Let S_f be the superposition operator genated by the function* (3.2.1). *Then the following assertions are true.*
(a) *The function $\phi : \mathbb{R} \to \mathbb{R}$ is bounded if and only if the operator $S_f : BV \to BV$ is bounded.*
(b) *The function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the operator $S_f : BV \to BV$ is continuous.*
(c) *The function $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if and only if the operator $S_f : BV \to BV$ is Lipschitz continuous.*
(d) *The function $\phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous iff the operator $S_f : BV \to BV$ is locally Lipschitz continuous.*

It is illuminating to see how the technical conditions (A) – (G) look like for the special function (3.2.1). We summarize with the following scheme.

- Condition (A) for (3.2.1) means $\phi \in Lip(\mathbb{R})$.
- Condition (B) for (3.2.1) means $\phi \in Lip([-r, r])$ and $Var(f(\cdot, u); [0, 1]) = |\phi(u)|$.
- Condition (C) for (3.2.1) means $\phi \in B(\mathbb{R})$.
- Condition (D) for (3.2.1) means $\phi \in B([-r, r])$.
- Condition (E) for (3.2.1) means $\phi \in B(\mathbb{R})$.
- Condition (F) for (3.2.1) means $\phi \in B([-r, r])$.
- Condition (G) for (3.2.1) means $\phi \in BV([-r, r])$.

This scheme shows that we obtain additional implications in Table 3.3 and get the stronger Table 3.4.

(A)	$\Rightarrow$	(B)	(C)	$\Rightarrow$	(D)			
			$\Updownarrow$		$\Updownarrow$			
			(E)	$\Rightarrow$	(F)	$\Leftarrow$	(G)	

Table 3.4: Interconnection between (A) – (G) (f from (3.2.1))

Our first example in this section shows that (F) does not imply (E).

Example 3.2.2 [BiGK1]. Choosing $\phi(u) := u$ in (3.2.1), the corresponding function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ does not satisfy (E), because for the partition $\mathcal{P}[0, 1] := \{0, 1\}$ and an arbitrary point $u_0 \in \mathbb{R}$ we have $|f(1, u_0) - f(0, u_0)| = |u_0|$, contradicting (E) if $|u_0| > M$.
On the other hand, choosing $\{t_0, \ldots, t_m\} \in \mathcal{P}[0, 1]$ and $u_0, \ldots, u_{m-1} \in [-r, r]$ with

$$\sum_{j=1}^{m-1} |u_j - u_{j-1}| \leq r,$$

we get

$$\sum_{j=1}^{m} |f(t_j, u_{j-1}) - f(t_{j-1}, u_{j-1})| = |f(0, u_0)| = |u_0| \leq r,$$

so (F) holds with $M_r := r$. Observe that f in this example satisfies (A) (and so also (B)), as well as (D), since

$$Var(f(\cdot; [0, 1]) = |f(0, u)| = |u|.$$

For the same reason, however, f does not satisfy (C). □

We remark that the first example of a function f with the same properties as the function in Example 3.2.2 was contructed in the paper [BaBiKMc].

Let us now check the sufficiency (or necessity) of the conditions (A) – (G) for the acting condition $S_f(BV) \subseteq BV$ and the analytical properties of S_f. To begin with, we remark that Lyamin [L] claimed that conditions (B) and (D) together imply $S_f(BV) \subseteq BV$. This seems quite natural, but unfortunately, it is false, as is shown by the following counterexample due to Maćkowiak.

Example 3.2.3 [Mc]. For $n = 2, 3, 4, \ldots$, let

$$c_n := 1 - \frac{1}{n}, \; w_n := \frac{1}{2n}, \; I_n := (c_n - w_n, c_n + w_n),$$

and define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,u) := \begin{cases} \frac{1}{n}\left(1 - \frac{|u - c_n|}{w_n}\right) & \text{for } t = c_n \text{ and } u \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then f satisfies (B) (even (A)) with $L = 2$, as well as (D) (even (C)) with $M = 22$. However, for $x(t) := t$ we have

$$S_f(x)(t) = f(t,t) = \begin{cases} \frac{1}{n} & \text{for } t = 1 - \frac{1}{n}, \\ 0 & \text{otherwise,} \end{cases}$$

and this function clearly does not belong to BV. □

The first correct sufficient conditions have been obtained by Bugajewska in 2010 and Bugajewska et al. in 2015 and read as follows.

Theorem 3.2.4 [Ba]. *Suppose that the conditions* (A) *and* (E) *are fulfilled. Then $S_f(BV) \subseteq BV$, and S_f is bounded.*

Theorem 3.2.5 [BaBiKMc]. *Suppose that the conditions* (B) *and* (F) *are fulfilled. Then $S_f(BV) \subseteq BV$, and S_f is bounded.*

Since (A) ⇒ (B) and (E) ⇒ (F), but neither (B) ⇒ (A) nor (F) ⇒ (E), Theorem 3.2.5 is actually stronger than Theorem 3.2.4. In fact, the function f in Example 3.2.3 satisfies (B) (even (A)) with $L = 1$, but not (E), as we have seen there. Since $S_f(BV) \not\subseteq BV$, the function f cannot satisfy (F), by Theorem 3.2.5.

The hypotheses given in Theorems 3.2.4 and 3.2.5 are only sufficient. Closer scrutiny reveals that condition (G) is exactly what we need to characterize bounded superposition operators in BV:

Theorem 3.2.6 [BaBiKMc]. *The following two conditions are equivalent:*

(a) *The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies condition* (G).

(b) *The operator S_f maps BV into itself and is bounded.*

As pointed out above, the condition (B) and (F) in Theorem 3.2.5 are only sufficient for the boundedness of the operator S_f. We illustrate this for condition (B) by the following simple

Example 3.2.7. Let $\phi(u) := \min\left\{\sqrt{|u|}, 1\right\}$ be the seagull function from Example 3.1.2, and define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by (3.2.1). A straightforward calculation shows that $x \in BV$ with $\|x\|_{BV} \leq r$ implies $\|S_f(x)\|_{BV} \leq 2\sqrt{r}$, so S_f maps BV into itself and is bounded. On the other hand, condition (B) is certainly not fulfilled for $t = 0$.

However, by Theorem 3.2.6 the function f satisfies condition (G). Indeed, if a partition $\{t_0, t_1, \ldots, t_m\}$ of $[0,1]$ and a collection $\{u_0, u_1, \ldots, u_m\}$ of real numbers meet the hypotheses of (G), then

$$\sum_{j=1}^{m} |f(t_j, u_j) - f(t_{j-1}, u_j)| = |f(0, u_1)| = \min\left\{\sqrt{|u_1|}, 1\right\} \leq 1$$

and

$$\sum_{j=1}^{m} |f(t_{j-1}, u_j) - f(t_{j-1}, u_{j-1})| = |f(0, u_1) - f(0, u_0)|$$
$$= \left|\min\left\{\sqrt{|u_1|}, 1\right\} - \min\left\{\sqrt{|u_0|}, 1\right\}\right| \leq \sqrt{|u_1 - u_0|} \leq \sqrt{2r}.$$

Consequently, we may choose $M_r := \max\{\sqrt{2r}, 1\}$ in condition (G). □

The fact that boundedness is included in Theorem 3.2.6 (b) is somewhat annoying: One could ask whether or not condition (G) is also necessary for the mere inclusion $S_f(BV) \subseteq BV$ without the boundedness requirement on S_f. The following example shows that this is not true because, in contrast to the composition operator C_f, the superposition operator S_f need not be bounded, nor continuous, if it maps BV into itself:

Example 3.2.8 [BaBiKMc]. Let $\phi(u) := 1/u$ for $u \neq 0$ and $\phi(0) := 0$, and define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by (3.2.1). Clearly, we have $S_f(BV) \subseteq BV$. However, the sequence $(x_n)_n$ defined by

$$x_n(t) := \begin{cases} 1/n & \text{for } t = 0, \\ 0 & \text{for } 0 < t \leq 1 \end{cases}$$

is bounded in BV, and even converges to zero, since $\|x_n\|_{BV} = \frac{2}{n}$. However, S_f maps this sequence into the sequence

$$S_f(x_n)(t) = f(t, x_n(t)) = \begin{cases} n & \text{for } t = 0, \\ 0 & \text{for } 0 < t \leq 1 \end{cases}$$

which is unbounded, since $\|S_f(x_n)\|_{BV} = 2n$. Consequently, S_f is not bounded in BV. □

Note that we might have used also Lemma 3.2.1 in Example 3.2.8, because that function ϕ is neither bounded nor continuous on $\mathbb{R}$.

As far as we know, a criterion, both necessary and sufficient, for the acting condition $S_f(BV) \subseteq BV$ is not known. Such a criterion should be weaker than condition (G), but cover the function f from the preceding Example 3.2.8. From Theorems 3.2.4, 3.2.5, and 3.2.6 it follows that the function f in Example 3.2.8 cannot satisfy any of the conditions (E), (F), or (G); this can also be proved directly.

Concerning the boundedness of S_f, the following result seems to be of independent interest. It shows that the boundedness of f is reflected in the boundedness of S_f:

Theorem 3.2.9 [BaBiKMc]. (a) *If condition* (B) *holds for f, then S_f is bounded if and only if f is locally bounded.*
(b) *If $S_f(BV) \subseteq BV$, then the set*

$$T_r := \left\{t : 0 \leq t \leq 1, \sup_{|u| \leq r} |f(t, u)| = \infty\right\}$$

is finite for every $r > 0$.

Part (b) in Theorem 3.2.9 shows, loosely speaking, that $f(\cdot, u)$ cannot become unbounded in the neighbourhood of "too many points" if S_f maps BV into itself. For instance, in Example 3.2.8 we have $T_r = \{0\}$ for any $r > 0$.

3.3. Continuity properties. Now we consider the problem of finding conditions for the continuity (or stronger properties) of the operator (3.0.2) in BV. Example 3.2.8 shows that, in contrast to the composition operator C_f, the superposition operator S_f is not automatically continuous if it maps BV into itself. In fact, the sequence $(x_n)_n$ constructed there satisfies $\|x_n\|_{BV} \to 0$, but $\|S_f(x_n)\|_{BV} \to \infty$ as $n \to \infty$, so S_f cannot be continuous at the zero function $x_0(t) \equiv 0$.

A sufficient condition for the continuity of S_f is that f be continuously differentiable on $[0,1] \times \mathbb{R}$. Under this hypothesis, one can say even more:

Theorem 3.3.1 [Mc1]. *Suppose that $f \in C^1$. Then S_f is uniformly continuous on bounded subsets of BV.*

Interestingly, the converse of Theorem 3.3.1 is far from being true, for two reasons. First of all, one may show that S_f need not be continuous if f is Lipschitz in both variables, instead of being C^1. It is even more striking that even a *discontinuous* function f may generate a superposition operator S_f which is continuous in BV.

Example 3.3.2 [Mc1]. First we define an auxiliary function $\phi : [0,1] \times [1/2,1] \to \mathbb{R}$ by

$$\phi(t,u) := \begin{cases} t & \text{for } \ 0 \le t \le 1/2 \text{ and } u \ge 2t, \\ u-t & \text{for } \ 0 \le t \le 1/2 \text{ and } u \le 2t, \\ u+t-1 & \text{for } \ 1/2 \le t \le 1 \text{ and } u \le 2-2t, \\ 1-t & \text{for } \ 1/2 \le t \le 1 \text{ and } u \ge 2-2t. \end{cases}$$

Afterwards we define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ for $n = 0,1,2,\ldots$ by

$$f(t,u) := \begin{cases} \frac{1}{2} - \left| t - \frac{1}{2} \right| & \text{for } \ u \ge 1, \\ 0 & \text{for } \ u \le 0, \\ 2^{-n}\phi\left(2^n[t - 2^{-n}\mathrm{ent}(t2^n)], 2^n u\right) & \text{for } \ 2^{-(n+1)} \le u \le 2^{-n}, \end{cases}$$

where $\mathrm{ent}(r)$ denotes the integer part of $r \in \mathbb{R}$. It is not hard to check that f is well-defined and continuous. A more cumbersome calculation shows that f is globally Lipschitz (with Lipschitz constant 2) on $[0,1] \times \mathbb{R}$, which implies that $S_f(BV) \subseteq BV$. However, f is not C^1, since it is not differentiable at $(t,u) = (1/2, 1)$.

We claim that S_f is not continuous at the zero function $x_0(t) \equiv 0$. From the definition of f it follows that $S_f(x_0) = x_0$. For $n = 0,1,2,\ldots$, consider the sequence $(x_n)_n$ of constant functions $x_n(t) \equiv 2^{-n}$. Clearly, $\|x_n\|_{BV} = 2^{-n}$, so $(x_n)_n$ converges to x_0 as $n \to \infty$. On the other hand,

$$Var(S_f(x_n); [0,1]) = Var(f(\cdot, 2^{-n}); [0,1]) \equiv 1 \qquad (n = 1,2,3,\ldots),$$

and so the sequence $(S_f(x_n))_n$ cannot converge to x_0. □

Example 3.3.3 [BiGK1]. The function f from Example 3.2.2 is certainly discontinuous on the positive vertical axis. Nevertheless, the corresponding operator S_f not only maps BV into itself, but is everywhere continuous in the BV-norm. □

For the reader's ease, let us recall which of the conditions (A) – (G) stated at the beginning are fulfilled by the examples of this section.

Example	(A)	(B)	(C)	(D)	(E)	(F)	(G)
3.2.2	yes	yes	no	yes	no	yes	yes
3.2.3	yes	yes	yes	yes	no	no	no
3.2.7	no	no	yes	yes	yes	yes	yes
3.2.8	no	no	no	no	no	no	no
3.3.2	yes	yes	yes	yes	yes	yes	yes

Table 3.5: Properties of f in the above examples

Sometimes one is not interested in the global continuity on the whole space, but in the continuity at just a single point. In this connection, we fix $x \in BV$ and impose the following new condition.

$$\text{(H)} \qquad \begin{cases} \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall \{t_0, \dots, t_m\} \in \mathcal{P}[0,1] \, \forall u_0, \dots, u_m \in [-\delta, \delta] : \sum_{j=1}^{m} |u_j - u_{j-1}| \leq \delta \\ \Rightarrow \sum_{j=1}^{m} |[f(t_j, u_j + x(t_j)) - f(t_{j-1}, u_j + x(t_{j-1})] - [f(t_j, x(t_j)) - f(t_{j-1}, x(t_{j-1}))]| \leq \varepsilon \\ \text{and } \sum_{j=1}^{m} |f(t_{j-1}, u_j + x(t_{j-1})) - f(t_{j-1}, u_{j-1} + x(t_{j-1}))| \leq \varepsilon. \end{cases}$$

Observe that (H) implies (G) if $f(t,0) \equiv 0$ (i.e., $S_f(x_0) = x_0$) and we take $x_0(t) \equiv 0$. The following result gives a necessary and sufficient condition for continuity at a point.

Theorem 3.3.4 [Mc1]. *The following two conditions are equivalent:*
(a) *The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies condition* (H).
(b) *The operator S_f maps BV into itself and is continuous at x.*

Clearly, if f is C^1 on $[0,1] \times \mathbb{R}$, then f is absolutely continuous on each compact subset of $[0,1] \times \mathbb{R}$, and so condition (H) holds for every $x \in BV$. The functions f in the Examples 3.2.8 and 3.3.2 have been constructed in such a way that the corresponding operator S_f is discontinuous at $x = \theta$. By Theorem 3.3.4, neither of the two functions can satisfy condition (H) at $x = \theta$. Indeed, this follows from the fact that they do not satisfy (G), and that $f(t,0) \equiv 0$ in both cases.
Now we consider global Lipschitz continuity of S_f in BV, i.e., the condition

$$\|S_f(x) - S_f(\tilde{x})\|_{BV} \leq L \|x - \tilde{x}\|_{BV} \quad (x, \tilde{x} \in BV) \tag{3.3.1}$$

with some constant $L > 0$. The following is parallel to Theorem 3.1.10 and shows that (3.3.1) leads to a strong degeneracy for the regularization (2.2.6) of the generating function f in the first argument:

Theorem 3.3.5 [MM]. *The following two conditions are equivalent:*
(a) *The right regularization of the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f^{\#}(t,u) := \begin{cases} \lim_{s \to t+} f(s,u) & \text{for } 0 \leq t < 1, \\ f(1,u) & \text{for } t = 1 \end{cases} \tag{3.3.2}$$

is affine, i.e.,

$$f^{\#}(t,u) = \alpha(t)u + \beta(t) \qquad (t \in [0,1], u \in \mathbb{R})$$

for some functions $\alpha, \beta \in BV$.
(b) *The operator S_f defined in* (3.0.2) *satisfies the global Lipschitz condition* (3.3.1).

It is clear that in the autonomous case $f : \mathbb{R} \to \mathbb{R}$ there is no difference between f and $f^{\#}$, since the regularization refers only to the variable t. One might ask whether or not, under the hypothesis (3.3.1), the function f itself is also affine, and not only its right regularization $f^{\#}$. The following example shows that the answer is negative:

Example 3.3.6 [MM]. Here we go back to Example 2.2.9. Let $\{r_0, r_1, r_2, \ldots\}$ be an enumeration of all rational numbers in $[0, 1]$ ($r_0 := 0$), and let $\phi : \mathbb{R} \to \mathbb{R}$ be any function satisfying $\phi(0) = 0$ and $|\phi(u) - \phi(v)| \leq \ell|u - v|$. We define $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t, u) := \begin{cases} 2^{-k}\phi(u) & \text{if } t = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

For any partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}[0, 1]$ and $x \in BV$ we have

$$\sum_{j=1}^{m} |S_f(x)(t_j) - S_f(x)(t_{j-1})| \leq 2\sum_{k=0}^{\infty} |f(r_k, x(r_k))| = 2\sum_{k=0}^{\infty} 2^{-k}|\phi(x(r_k))| \leq 4\ell\|x\|_\infty,$$

which shows that S_f maps the space BV into itself and is bounded. Furthermore, for $x, y \in BV$ and $P = \{t_0, t_1, \ldots, t_m\}$ as above we obtain the estimate

$$\begin{aligned} &\sum_{j=1}^{m} |S_f(x)(t_j) - S_f(y)(t_j) - S_f(x)(t_{j-1}) + S_f(y)(t_{j-1})| \\ &\leq 2\sum_{j=0}^{m} |f(t_j, x(t_j)) - f(t_j, y(t_j))| \leq 2\sum_{k=0}^{\infty} |f(r_k, x(r_k)) - f(r_k, y(r_k))| \\ &\leq 2\sum_{k=0}^{\infty} 2^{-k}|\phi(x(r_k)) - \phi(y(r_k))| \leq 2\ell\sum_{k=0}^{\infty} 2^{-k}|x(r_k) - y(r_k)| \\ &= 2\ell|x(0) - y(0)| + 2\ell\sum_{k=1}^{\infty} 2^{-k}|x(r_k) - y(r_k)| \leq 2\ell\|x - y\|_{BV}. \end{aligned}$$

This together with the trivial estimate $|S_f(x)(0) - S_f(y)(0)| \leq \ell|x(0) - y(0)|$ shows that S_f satisfies the global Lipschitz condition (3.3.1) with $L = 2\ell$, although f is not affine. □

It is not hard to see that $f^{\#}(t, u) \equiv 0$ on $[0, 1)$ for the function f in Example 3.3.6, in accordance with Theorem 3.3.5.

Note that also the function f from Example 3.2.2 can serve as an example of a non-affine function which generates a Lipschitz continuous operator in BV. Indeed, this follows from the global Lipschitz continuity of $\phi(u) = u$ on $\mathbb{R}$ and Lemma 3.2.1 (c). Also in this case it is easy to see that $f^{\#}(t, u) \equiv 0$, in accordance with Theorem 3.3.5.

Let us now take a closer look at compactness. As Theorem 3.1.13 shows, the composition operator C_f is never compact in BV, except for the trivial case when f is constant. To see that the situation is different for superposition operators, let us go back for a moment to the multiplication operator M_μ which we studied in Section 2.2.

Clearly, the multiplication operator M_μ in (2.2.1) may be viewed as a special superposition operator S_f which is generated by the function $f(t, u) = \mu(t)u$, with $\mu \in BV$. We have seen in Section 2.2 that in case of a countable support $\mathrm{supp}(\mu)$, the operator M_μ is compact, so there are compact superposition operators in BV which are not constant. Let us briefly consider an example.

Example 3.3.7. The function f from Example 3.2.2 is discontinuous, but generates a continuous superposition operator S_f in BV, as we have observed in Example 3.3.3. Since S_f is a multiplication operator M_μ with $\mu = \chi_{\{0\}}$, the operator S_f is also compact in BV. □

One could conjecture that the result for multiplication operators carries over to general superposition operators, requiring that $\mathrm{supp}(f(\cdot, u))$ is countable for each $u \in \mathbb{R}$. However, the following examples show that this condition is not necessary for the compactness of S_f, and it even does not guarantee the acting condition $S_f(BV) \subseteq BV$.

Example 3.3.8. Define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $f(t,u) \equiv 1$. Then the corresponding operator S_f is clearly compact in BV, but $\mathrm{supp}(f(\cdot,u)) = [0,1]$ for each u. □

Example 3.3.9. Define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,u) := \begin{cases} 1 & \text{for } t = u = 1/n \ (n \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{supp}(f(\cdot,u))$ is finite (or empty) for each $u \in \mathbb{R}$. On the other hand, the operator S_f does not map BV into itself, as can be seen by the choice $x(t) := t$. □

In view of these examples, the question arises to find nonconstant compact superposition operators which are not multiplication operators. The following theorem shows that this is not possible, *under the additional assumption that $f(\cdot,u)$ be continuous for all u.* Observe that this assumptions excludes multiplication operators, because in this case $\mathrm{supp}(f(\cdot,u))$ cannot be countable, by the intermediate value theorem.

Theorem 3.3.10. *Assume that $f(\cdot,u)$ is continuous for all $u \in \mathbb{R}$. Moreover, suppose that the corresponding operator S_f maps BV into itself and is both continuous and compact. Then f does not depend on u, and so S_f is constant.*

Proof. Fix $u \in \mathbb{R}\backslash\{0\}$ and $a \in [0,1]$, and let $a_n := a+1/n$ (or $a_n := a-1/n$ if $a = 1$). For n sufficiently large, the sequence $(x_n)_n$ defined by $x_n := u\chi_{a_n}$ is then bounded in BV, since $\|x_n\|_{BV} \leq 2|u|$. This sequence is mapped by S_f into the sequence $(y_n)_n$ with

$$y_n(t) = f(t, x_n(t)) = \begin{cases} f(t,u) & \text{for} \quad t = a_n, \\ f(t,0) & \text{for} \quad t \neq a_n. \end{cases}$$

Since S_f is compact, there exists some subsequence $(y_{n_k})_k$ of $(y_n)_n$ converging in the BV-norm (and so also pointwise) to some $y \in BV$. We claim that $y(t) = f(t,0)$ for all $t \in [0,1]$. To see this, we distinguish two cases.

<u>1st case</u>: $t = a$. Then $y_{n_k}(t) = y_{n_k}(a) = f(a, x_{n_k}(a)) = f(a,0)$, since $a \neq a_n$ for all n. Consequently, $y(a) = f(a,0)$ as well.

<u>2nd case</u>: $t \neq a$. Then also $t \neq a_n$ for n large enough, and so again $y_{n_k}(t) = f(t, x_{n_k}(t)) = f(t,0)$, hence $y(t) = f(t,0)$ as claimed.

Now, considering the special partition $\{0, a, a_{n_k}, 1\}$ yields

$$\begin{aligned} Var(y_{n_k} - y; [0,1]) &\geq |y_{n_k}(a) - y(a) - y_{n_k}(a_{n_k}) + y(a_{n_k})| \\ &= |f(a,0) - f(a,0) - f(a_{n_k}, u) + f(a_{n_k}, 0)| \\ &= |f(a_{n_k}, u) - f(a_{n_k}, 0)| \to |f(a,u) - f(a,0)| \end{aligned}$$

as $k \to \infty$. But $\|y_{n_k} - y\|_{BV} \to 0$, as $k \to \infty$, since y is the limit of the sequence $(y_{n_k})_k$. We conclude that $f(a,u) = f(a,0)$, hence $f(t,u) = f(t)$, since a and u were arbitrary. ■

The following Table 3.6 shows that the description of analytical properties in terms of f is much more difficult for the superposition operator S_f than it was for the composition operator C_f.

(B) & (F)	$\Rightarrow$	$S_f : BV \to BV$ bounded
(G)	$\Leftrightarrow$	$S_f : BV \to BV$ bounded
(B) & f locally bounded	$\Leftrightarrow$	$S_f : BV \to BV$ bounded
$f \in C^1$	$\Rightarrow$	$S_f : BV \to BV$ uniformly continuous
$f^\#$ affine	$\Leftrightarrow$	$S_f : BV \to BV$ globally Lipschitz
$f = f(t)$	$\Leftrightarrow$	$S_f : BV \to BV$ continuous and compact

Table 3.6: The operator S_f in BV

The results of this section show that most conditions on f are only sufficient to imply a certain property of the corresponding superposition operator S_f. There is an interesting condition which shows that conditions (B) and (D) are "almost" necessary for the boundedness of S_f in BV. The precise formulation is as follows.

Theorem 3.3.11 [DN]. *Suppose that the operator S_f generated by some function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ maps BV into itself and is bounded. Then f may be represented as a sum*

$$f(t,u) = g(t,u) + h(t,u),$$

where the functions g and h have the following properties.
(a) *The function g satisfies the conditions* (B) *and* (D).
(b) *The function h is zero on* $([0,1] \setminus C) \times \mathbb{R}$*, where $C \subset [0,1]$ is some countable subset.*

At this time let us take a deep breath and see what we have learned so far. Our results show that the composition operator (3.0.1) and the superposition operator (3.0.2) have a completely different behavior in the space BV. To conclude this section, let us briefly summarize these differences:

- Whenever the operator C_f maps BV into itself, it is automatically bounded; this is not true for the operator S_f.
- Whenever the operator C_f maps BV into itself, it is automatically continuous; this is not true for the operator S_f.
- The condition $C_f(BV) \subseteq BV$ holds precisely for locally Lipschitz functions f; the condition $S_f(BV) \subseteq BV$ may hold even for discontinuous functions f.
- Local Lipschitz continuity of $u \mapsto f(u)$ guarantees the continuity of C_f in the BV-norm; however, even global Lipschitz continuity of $(t,u) \mapsto f(t,u)$ does not guarantee this for the operator S_f.
- Only affine functions f generate globally Lipschitz continuous operators C_f in the BV-norm; this is not true for the operator S_f.
- Only constant functions f generate compact operators C_f in the BV-norm; this is not true for the operator S_f.

In the following section we are concerned with the problem which of our results carry over from BV to other spaces of functions of bounded (Wiener, Riesz, Waterman) variation. This will be important in applications to the integral equations and boundary value problems discussed in the last chapters.

3.4. Operators in generalized BV spaces. Let us now study the behavior of the composition operator (3.0.1) and the superposition operator (3.0.2) in the more general spaces WBV_p, RBV_p, and ΛBV introduced in the first chapter. We start with a necessary and sufficient acting condition for the composition operator C_f which is parallel to Theorem 3.1.1.

Theorem 3.4.1 [AGV]. *The following four conditions are equivalent:*
(a) *The function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the local Lipschitz condition* (3.1.1).
(b) *The operator* C_f *defined in* (3.0.1) *maps the Wiener space* WBV_p *into itself.*
(c) *The operator* C_f *defined in* (3.0.1) *maps the Riesz space* RBV_p *into itself.*
(d) *The operator* C_f *defined in* (3.0.1) *maps the Waterman space* ΛBV *into itself.*
Moreover, in this case the operator C_f *is automatically bounded.*

Sometimes it is interesting to consider composition operator (3.0.1) between WBV_p and WBV_q, or RBV_p and RBV_q, for $p \neq q$. Here one has to replace the local Lipschitz condition (3.1.1) by the local Hölder condition

(3.4.1) $$\forall r > 0\, \exists L_r > 0\, \forall u, v \in [-r, r] : \ |f(u) - f(v)| \leq L_r |u - v|^{p/q}.$$

However, this is a reasonable condition only for $p \leq q$, since a function which satisfies a Hölder condition with exponent $\alpha > 1$ is constant. So the problem for Wiener spaces is different from that for Riesz spaces. In fact, if (3.4.1) holds, then for $x \in WBV_p$ with $\|x\|_{WBV_p} \leq r$ and any partition $\{t_0, t_1, \ldots, t_m\}$ of $[0, 1]$ we obtain

$$\sum_{j=1}^{m} |f(x(t_j)) - f(x(t_{j-1}))|^q \leq L_r^q \sum_{j=1}^{m} |x(t_j) - x(t_{j-1})|^p \leq L_r^q WVar_p(x; [0, 1]),$$

which shows that $C_f(WBV_p) \subseteq WBV_q$. On the other hand, for $x \in RBV_p$ with $\|x\|_{RBV_p} \leq r$ and any partition $\{t_0, t_1, \ldots, t_m\}$ of $[0, 1]$ we obtain

$$\sum_{j=1}^{m} \frac{|f(x(t_j)) - f(x(t_{j-1}))|^q}{(t_j - t_{j-1})^{q-1}} \leq L_r^q \sum_{j=1}^{m} \frac{|x(t_j) - x(t_{j-1})|^p}{(t_j - t_{j-1})^{q-1}},$$

and we can replace $(t_j - t_{j-1})^{q-1}$ in the denominator by $(t_j - t_{j-1})^{p-1}$ only if $p \geq q$. So the only possibility which does not lead to a degeneracy is $p = q$, and this case is already covered by Theorem 3.4.1 (c). This is not surprising, since RBV_q is strictly contained in RBV_p for $p < q$.

Theorem 3.4.1 shows that in all spaces of functions of bounded variation we consider in this survey, we get boundedness of the composition operator for free. The continuity problem is much more delicate, as we have seen in Section 3.1. The proof of the fact that the acting condition $C_f(BV) \subseteq BV$ implies the continuity of C_f is by no means trivial, and it is even not clear whether or not this also holds in the Wiener space WBV_p for $p > 1$.

In the Riesz space, however, we may use a trick to get continuity. As was shown by Marcus and Mizel [MaMi], the composition operator C_f is automatically continuous in the Sobolev space $W^{1,p}[0, 1]$ for $p > 1$, provided that C_f maps this space into itself. But as we have seen in Section 1.3, the space $W^{1,p}[0, 1]$ is isomorphic to $RBV_p[0, 1]$, since both spaces basically contain for $p > 1$ all absolutely continuous functions with first L_p-derivative, and their norms also coincide (up to equivalence). So the following is true:

Theorem 3.4.2. *Under the hypothesis* (3.1.1), *the operator* C_f *is automatically continuous in* RBV_p *for* $p > 1$.

We point out that sufficient conditions for the continuity of C_f are known if C_f acts from BV into the larger space WBV_p for some $p > 1$. Let us denote by $AC_{loc}(\mathbb{R})$ the space of all functions $f : \mathbb{R} \to \mathbb{R}$ which are absolutely continuous on each compact interval.

Theorem 3.4.3 [BiGK1]. *Suppose that* $f \in AC_{loc}(\mathbb{R})$, *and* $f' \in L_{p/(p-1)}[-a, a]$ *for some* $p > 1$ *and every* $a > 0$. *Then* C_f *maps* BV *into* WBV_p *and is both bounded and continuous.*

Theorem 3.4.3 has an interesting consequence. Suppose that $C_f(BV) \subseteq BV$. Then we know that $C_f(BV) \subseteq WBV_p$ for every $p > 1$, and $f \in Lip_{loc}(\mathbb{R})$, by Theorem 3.1.1. But then $f' \in L_\infty[-a,a] \subset L_{p/(p-1)}[-a,a]$ for every $a > 0$, and so C_f is continuous, viewed as an operator from BV into WBV_p.

The following theorem provides a sufficient condition on f which implies the continuity of C_f in a Waterman space.

Theorem 3.4.4 [BiCGS]. *In case $f \in C^1(\mathbb{R})$, the composition operator C_f maps ΛBV into itself and is both bounded and continuous.*

Recall that we introduced the relation $\Lambda \preceq \Gamma$ in Section 1.4; this relation is equivalent to the inclucion $\Gamma BV \subseteq \Lambda BV$. We also considered a stronger realtion denoted by $\Lambda \prec \Gamma$; this relation implies (but is not equivalent to) the strict inclucion $\Gamma BV \subset \Lambda BV$. Now, the next theorem shows that the composition operator C_f never maps the space ΛBV into the (smaller) space ΓBV for $\Lambda \prec \Gamma$, unless C_f is trivial.

Theorem 3.4.5 [BiCGS]. *Suppose that $C_f(\Lambda BV) \subseteq \Gamma BV$, where $\Lambda \prec \Gamma$. Then the function f is differentiable with $f'(u) \equiv 0$, so f is constant.*

Let us now pass to the nonautonomous superposition operator (3.0.2). As one could expect, here the situation becomes more complicated. First of all, we have to replace condition (F) from Section 3.2 by a stronger condition; here by $\mathfrak{S}_m$ we denote the set of all permutations of $\{1, 2, \ldots, m\}$.

$$(\mathrm{F}_\Lambda) \qquad \begin{cases} \forall r > 0\, \exists M_r > 0\, \forall \{t_0, \ldots, t_m\} \in \mathcal{P}[0,1]\, \forall u_0, \ldots, u_m \in [-r, r] : \\ \sup\limits_{\sigma \in \mathfrak{S}_m} \sum\limits_{j=1}^{m} \lambda_{\sigma(j)} |u_j - u_{j-1}| \leq r \;\Rightarrow\; \sup\limits_{\sigma \in \mathfrak{S}_m} \sum\limits_{j=1}^{m} \lambda_{\sigma(j)} |f(t_j, u_{j-1}) - f(t_{j-1}, u_{j-1})| \leq M_r. \end{cases}$$

Theorem 3.4.6 [BiCGS]. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions* (B) *and* (F_Λ). *Then the superposition operator S_f maps ΛBV into itself and is bounded.*

In Theorem 3.2.6 we have seen a certain converse of Theorem 3.4.6 for BV: if the superposition operator S_f maps BV into itself and is bounded, then condition (G) holds, and so also condition (F). The following somewhat surprising example shows that an analogous result for ΛBV (with (F) replaced by (F_Λ)) is not true.

Example 3.4.7 [BiCGS]. Let $\phi(u) := u$, and define $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ by (3.2.1). Then the superposition operator S_f generated by f maps any space ΛBV into any space ΓBV and is bounded. However, the function f does not satisfy condition (F_Λ).
Indeed, take $\lambda_n := 1/n$ and $a_n := 1/\sqrt{n}$; then

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \qquad \sum_{n=1}^{\infty} a_n = \infty \qquad \sum_{n=1}^{\infty} \lambda_n a_n =: r < \infty. \tag{3.4.2}$$

Suppose that f satisfies (F_Λ), fix $M_r > 0$ in correspondence to r from (3.4.2), and choose $m \in \mathbb{N}$ so large that

$$\sum_{j=1}^{m+1} a_j > M_r.$$

For $j = 0, 1, \ldots, m$, let $u_j := a_{j+1} + a_{j+2} + \ldots + a_{m+1}$. Then

$$\sup_{\sigma \in \mathfrak{S}_m} \sum_{j=1}^{m} \lambda_{\sigma(j)} |u_j - u_{j-1}| = \sup_{\sigma \in \mathfrak{S}_m} \sum_{j=1}^{m} \lambda_{\sigma(j)} a_j \leq r.$$

On the other hand, the last term in (F_Λ) is then

$$\sup_{\sigma\in\mathfrak{S}_m}\sum_{j=1}^{m}\lambda_{\sigma(j)}|f(t_j,u_{j-1})-f(t_{j-1},u_{j-1})| = \sup_{\sigma\in\mathfrak{S}_m}\lambda_{\sigma(1)}|f(0,u_0)| = \sum_{j=1}^{m+1}a_j > M_r,$$

and so condition (F_Λ) fails. □

The following Table 3.7 summarizes what we know about composition and superposition operators in generalized spaces of functions of bounded variation, where X stands for WBV_p, RBV_p, or ΛBV.

$f \in Lip_{loc}(\mathbb{R})$	$\Leftrightarrow$	$C_f(X) \subseteq X$
$f \in Lip_{loc}(\mathbb{R})$	$\Leftrightarrow$	$C_f : X \to X$ bounded
f constant	$\Leftrightarrow$	$C_f(\Lambda BV) \subseteq \Gamma BV$ for $\Gamma \prec \Lambda$
$f \in AC_{loc}(\mathbb{R}), f' \in L_{p/(p-1)}$	$\Rightarrow$	$C_f : BV \to WBV_p$ bounded and continuous
$f \in C^1(\mathbb{R})$	$\Rightarrow$	$C_f : \Lambda BV \to \Lambda BV$ continuous
$f \in C^1$	$\Rightarrow$	$S_f : X \to X$ uniformly continuous
$f^\#$ affine	$\Leftrightarrow$	$S_f : X \to X$ globally Lipschitz
(B) & (F_Λ)	$\Rightarrow$	$S_f : \Lambda BV \to \Lambda BV$ bounded

Table 3.7: The operators C_f and S_f in other spaces

Chapter 4. Applications to Integral Equations

In this chapter we will focus on applications of results from previous chapters to nonlinear integral equations considered in spaces of functions of bounded variation of various types. We will prove existence, or existence and uniqueness, of solutions for either nonlinear Hammerstein or nonlinear Volterra-Hammerstein integral equations. Our main tool is fixed point theory, so we will impose suitable conditions on the data which make it possible to apply well-known fixed point theorems. Although those conditions have been already considered in the preceding two chapters, we will repeat them here to make the presentation self-contained.

In the first two sections we are going to concentrate on the space BV of functions of bounded variation in the sense of Jordan. In the third section some results will be placed in other spaces of functions of bounded variation, namely, in ΛBV, WBV_p and RBV_p spaces.

There are several motivations to consider BV-solutions to integral equations. In particular, let us point out the fact that solutions to many equations which describe specific physical phenomena are often functions of bounded variation in the sense of Jordan. For example, the integral equation

$$x(t) = \omega^2 \int_0^1 G(t,s)\rho(s)x(s)\,ds + \int_0^1 G(t,s)q(s)\,ds \qquad (0 \le t \le 1), \tag{4.0.1}$$

where

$$G(t,s) = \begin{cases} t(1-s) & \text{for} \quad 0 \le t \le s \le 1, \\ s(1-t) & \text{for} \quad 0 \le s \le t \le 1, \end{cases} \tag{4.0.2}$$

is the classical Green's function of the second derivative, describes the amplitude of forced vibrations of a string (see [Pi]). Under suitable assumptions on the data ρ, q, and ω, (4.0.1) possesses a unique solution which is a function of bounded variation in the sense of Jordan on the interval $[0,1]$. This fact follows from the theorems proved in the paper [Bi].

4.1. Hammerstein integral equations. Consider the nonlinear Hammerstein integral equation

$$x(t) = g(t) + \lambda \int_0^1 k(t,s) f(x(s))\,ds \qquad (0 \le t \le 1), \tag{4.1.1}$$

with $\lambda \in \mathbb{R}$, where $g : [0,1] \to \mathbb{R}$, $k : [0,1] \times [0,1] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given functions, and the function $x : [0,1] \to \mathbb{R}$ is unknown. We point out that the role of λ in (4.1.1) is very important. For example, in the problems concerning calculation of either free pulsation of harmonic vibrations of a string or critical speed of a shaft is reduced to calculating such values of λ for which the corresponding integral equations, being special cases of (4.1.1), have a nontrivial solution.

As mentioned above, in this section we are going to look for solutions of equation (4.1.1) of bounded variation in the sense of Jordan. Using the notation from Chapter 2 and Chapter 3 we may write (4.1.1) equivalently as operator equation

$$x = g + \lambda K(C_f(x)), \tag{4.1.2}$$

where K denotes the integral operator (2.0.3) generated by k, and C_f denotes the composition operator (3.0.1) generated by f. Firstly, we will prove two existence and uniqueness type theorems for equation (4.1.2) by using Banach's fixed point theorem for contractions, which is sufficient in this situation. For convenience of the reader we will repeat here some assumptions which appear in previous chapters and will be needed in the study of this equation; we denote them by (H1), (H2), ... without referring to identical conditions in Chapter 2 or 3. The following conditions will be used throughout the sequel; as before, the symbol $\forall' s$ means that the indicated property holds only for almost all s.

$$\forall t \in [0,1] : \ k(t,\cdot) \in L_1; \tag{H1}$$

(H2) $$\exists m \in L_1,\ \forall' s \in [0,1]:\ Var(k(\cdot,s)) \le m(s);$$

(H3) $$f \in Lip_{loc}(\mathbb{R}).$$

Note that condition (H1) coincides with condition (A), and condition (H2) with condition (B') in Section 3.2, while condition (H3) is the crucial hypothesis (3.1.1) in Theorem 3.1.1. The above conditions suffice to obtain our first existence and uniqueness result.

Theorem 4.1.1 [Bi]. *Under the assumptions* (H1) – (H3), *there exists a number* $\rho > 0$ *such that equation* (4.1.1) *has, for fixed* $g \in BV$*, a unique solution* $x \in BV$ *for every* λ *satisfying* $|\lambda| \le \rho$.

Proof. As usual, the proof builds on the Banach-Caccioppoli fixed point theorem, applied to the operator $G : BV \to BV$ defined by $G(x) := g + \lambda K(C_f(x))$. To this end, choose $r > 0$ with $\|g\|_{BV} < r$, and $\rho > 0$ in such a way that both

$$\|g\|_{BV} + \rho \sup_{|u|\le r} |f(u)|\, \|m + |k(0,\cdot)|\|_{L_1} \le r \tag{4.1.3}$$

and

$$\rho L_r\, \|m + |k(0,\cdot)|\|_{L_1} < 1, \tag{4.1.4}$$

where L_r is the local Lipschitz constant from Theorem 3.1.1 (a). Since the L_1-norm of $m + |k(0,\cdot)|$ occurring on the right-hand side of (4.1.3) and (4.1.4) will be used several times in the sequel, we introduce the special abbreviation

$$\mu(k) := \|m + |k(0,\cdot)|\|_{L_1}. \tag{4.1.5}$$

By Theorem 2.3.1 and Theorem 3.1.1, we have $K(C_f(x)) \in BV$, and so the operator G maps the space BV into itself. Denoting as in (3.1.2) by $B_r = B_r(BV)$ the closed ball of radius r in BV centered at zero, for any $x \in B_r$ and every partition $\{t_0, t_1, \ldots, t_m\}$ of $[0,1]$ we have

$$\begin{aligned}\sum_{j=1}^{m} \left| \int_0^1 [k(t_j,s) - k(t_{j-1},s)] f(x(s))\, ds \right| &\le \sup_{0\le s\le 1} |f(x(s))| \sum_{j=1}^{m} \int_0^1 |k(t_j,s) - k(t_{j-1},s)|\, ds \\ = \sup_{0\le s\le 1} |f(x(s))| \int_0^1 \sum_{j=1}^{m} |k(t_j,s) - k(t_{j-1},s)|\, ds &\le \sup_{0\le s\le 1} |f(x(s))| \int_0^1 m(s)\, ds.\end{aligned}$$

Consequently, for $|\lambda| \le \rho$ we get

$$\|G(x)\|_{BV} \le \|g\|_{BV} + |\lambda| \sup_{|u|\le r} |f(u)|\, \mu(k),$$

which shows that $G(B_r) \subseteq B_r$, by (4.1.3). Now we have to show that G is a contraction on B_r. For any $x, y \in B_r$ and every partition $\{t_0, t_1, \ldots, t_m\}$ of $[0,1]$ we obtain

$$\begin{aligned}&\sum_{j=1}^{m} \left| \int_0^1 k(t_j,s)[f(x(s)) - f(y(s))]\, ds - \int_0^1 k(t_{j-1},s)[f(x(s)) - f(y(s))]\, ds \right| \\ &\le \sup_{0\le s\le 1} |f(x(s)) - f(y(s))| \int_0^1 \sum_{j=1}^{m} |k(t_j,s) - k(t_{j-1},s)|\, ds \\ &\le L_r \sup_{0\le s\le 1} |x(s) - y(s)| \int_0^1 m(s)\, ds \le L_r \|m\|_{L_1} \|x - y\|_{BV},\end{aligned} \tag{4.1.6}$$

hence

$$Var(K(C_f(x)) - K(C_f(y))) \le L_r \|m\|_{L_1} \|x - y\|_{BV}.$$

Moreover,

$$(4.1.7)\qquad \begin{aligned}|K(C_f(x))(0) - K(C_f(y))(0)| &\leq \sup_{0\leq s\leq 1} |f(x(s)) - f(y(s))| \int_0^1 |k(0,s)|\, ds \\ &\leq L_r \|k(0,\cdot)\|_{L_1} \|x-y\|_{BV}.\end{aligned}$$

Combining (4.1.6) and (4.1.7) we arrive at

$$\begin{aligned}\|G(x)-G(y)\|_{BV} &= |G(x)(0) - G(y)(0)| + |\lambda| Var(K(C_f(x)) - K(C_f(y))) \\ &\leq |\lambda| L_r\, \mu(k) \|x-y\|_{BV},\end{aligned}$$

with $\mu(k)$ as in (4.1.5), and the statement follows from (4.1.4). ■

We illustrate Theorem 4.1.1 for the case of separated kernels $k(t,s) = k_1(t)k_2(s)$, where $k_1 \in BV$ and $k_2 \in L_1$. As we have seen in Section 2.3, in this case (H1) holds trivially, while (H2) holds with $m(s) = Var(k_1)k_2(s)$. The L_1-norm of $k(0,\cdot)$ occurring in (4.1.3) and (4.1.4) is here

$$\|k(0,\cdot)\|_{L_1} = \int_0^1 |k_1(0)|\, |k_2(s)|\, ds = |k_1(0)|\, \|k_2\|_{L_1}.$$

So for $\|g\|_{BV} < r$ condition (4.1.3) reads

$$\rho \leq \frac{r - \|g\|_{BV}}{\|k_1\|_{BV} \|k_2\|_{L_1} \sup_{|u|\leq r} |f(u)|},$$

while condition (4.1.4) becomes

$$\rho < \frac{1}{L_r \|k_1\|_{BV} \|k_2\|_{L_1}}.$$

Observe that both (4.1.3) and (4.1.4) show that, the "larger" k_1, k_2 and f are, the smaller is the set of admissible parameters λ. □

Let us consider now the more general nonlinear Hammerstein integral equation

$$(4.1.8)\qquad x(t) = g(t) + \lambda \int_0^1 k(t,s) f(s, x(s))\, ds \qquad (0 \leq t \leq 1),$$

in which the composition operator C_f is replaced by the superposition operator S_f. So the corresponding operator equation reads

$$(4.1.9)\qquad x = g + \lambda K(S_f(x)),$$

where S_f denotes the operator (3.0.2) generated by $f : [0,1] \times \mathbb{R} \to \mathbb{R}$. Now we impose the following hypotheses:

$$(\text{H4})\qquad \sup_{0\leq t\leq 1} \|k(t,\cdot)\|_{L_1} < \infty;$$

$$(\text{H5})\qquad \exists M > 0\ \forall \xi \in [0,1] :\ Var\left(\int_0^\xi k(\cdot,s)\, ds\right) \leq M;$$

$$(\text{H6})\qquad f \in Lip_{loc}([0,1] \times \mathbb{R}).$$

Note that condition (H4) implies condition (A), while condition (H5) coincides with condition (C) in Section 2.3. Condition (H6) is satisfied, for example, in case $f \in C^1$. Also, observe that (H6) implies condition (G) which plays a crucial role in Theorem 3.2.6, because

$$\sum_{j=1}^m |u_j - u_{j-1}| \leq r$$

for points $u_0, \ldots, u_m \in [-r, r]$ implies both

$$\sum_{j=1}^{m} |f(t_j, u_j) - f(t_{j-1}, u_j)| \leq L_r \sum_{j=1}^{m} |t_j - t_{j-1}| = L_r$$

and

$$\sum_{j=1}^{m} |f(t_j, u_j) - f(t_{j-1}, u_{j-1})| \leq L_r \sum_{j=1}^{m} |u_j - u_{j-1}| \leq r L_r,$$

where L_r denotes the Lipschitz constant of f on $[0,1] \times [-r, r]$. Combining this with the estimate (2.3.2) for the integral operator (2.3.1) we obtain the following

Theorem 4.1.2 [BiGK]. *Under the assumptions* (H4) – (H6), *there exists a number* $\rho > 0$ *such that equation* (4.1.8) *has, for fixed* $g \in BV$, *a unique solution* $x \in BV$ *for every* λ *satisfying* $|\lambda| \leq \rho$.

Proof. The first part of the proof is very similar to that of Theorem 4.1.1. Again, we apply the Banach-Caccioppoli fixed point theorem, but now to the operator $G : B_r(BV) \to BV$ defined by $G(x) := g + \lambda K(S_f(x))$. To this end, suppose that $\|g\|_{BV} < r$, let L_r denote the Lipschitz constant of f on $[0,1] \times [-r, r]$, and choose $\rho > 0$ in such a way that both

$$\|g\|_{BV} + \rho(2M + \|k(0, \cdot)\|_{L_1})(L_r + rL_r + |f(0,0)|) \leq r \tag{4.1.10}$$

and

$$\rho L_r \sup_{0 \leq t \leq 1} \|k(t, \cdot)\|_{L_1} < 1, \tag{4.1.11}$$

where M is the constant appearing in condition (H5). By Theorem 2.3.3 and Theorem 3.2.6, we have $K(S_f(x)) \in BV$, and so the operator G maps the space BV into itself. We claim that the operator G maps the ball $B_r = B_r(BV)$ into itself. For any $x \in B_r$ and every partition $\{t_0, t_1, \ldots, t_m\}$ of $[0,1]$ we have

$$\begin{aligned}
&\sum_{j=1}^{m} |f(t_j, x(t_j)) - f(t_{j-1}, x(t_{j-1}))| \\
&\leq \sum_{j=1}^{m} |f(t_j, x(t_j)) - f(t_j, x(t_{j-1}))| + \sum_{j=1}^{m} |f(t_j, x(t_{j-1})) - f(t_{j-1}, x(t_{j-1}))| \\
&\leq L_r \sum_{j=1}^{m} |x(t_j) - x(t_{j-1})| + L_r \sum_{j=1}^{m} |t_j - t_{j-1}| \leq L_r(Var(x) + 1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|S_f(x)\|_{BV} &\leq |f(0, x(0)) - f(0,0)| + |f(0,0)| + Var(S_f(x)) \\
&\leq L_r|x(0)| + |f(0,0)| + L_r(Var(x) + 1) \leq L_r + rL_r + |f(0,0)|.
\end{aligned}$$

Consequently, using (2.3.2) we get

$$\|G(x)\|_{BV} \leq \|g\|_{BV} + |\lambda| \, \|K(S_f(x))\|_{BV} \leq \|g\|_{BV} + \rho(2M + \|k(0, \cdot)\|_{L_1})(L_r + rL_r + |f(0,0)|),$$

and the inclusion $G(B_r) \subseteq B_r$ follows from (4.1.10).

To show that G is a contraction is slightly more tricky than in Theorem 4.1.1. First of all, denoting by $\| \cdot \|_\infty$ the supremum norm (1.1.11) on BV, we get

$$\begin{aligned}
\|G(x) - G(y)\|_\infty &= |\lambda| \sup_{0 \leq t \leq 1} \left| \int_0^1 k(t,s)[f(s, x(s)) - f(s, y(s))] \, ds \right| \\
&\leq \rho L_r \sup_{0 \leq t \leq 1} \int_0^1 |k(t,s)| \, |x(s) - y(s)| \, ds \leq \rho L_r \sup_{0 \leq t \leq 1} \|k(t, \cdot)\|_{L_1} \|x - y\|_\infty.
\end{aligned} \tag{4.1.12}$$

Consider the set $G(B_r(BV))$, equipped with the metric $d(x,y) := \|x - y\|_\infty$. We claim that this metric space is complete. Suppose that $(y_n)_n$, where $y_n = G(x_n)$ and $\|x_n\|_{BV} \leq r$, is a Cauchy sequence with respect to the supremum norm. It is clear that there exists a bounded function $y : [0,1] \to [-r,r]$ which is the uniform limit of the sequence $(y_n)_n$. On the other hand, by Helly's selection principle (Theorem 1.4.14 (b)), there exist a function $x \in B_r(BV)$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \to x$ pointwise on $[0,1]$ as $k \to \infty$. But then, by the Lebesgue dominated convergence theorem, we have $G(x_{n_k}) \to G(x)$ pointwise on $[0,1]$. This shows that $y = G(x)$ and proves the completeness of the considered metric space.

Now we consider the restriction of the operator G to the set $G(B_r(BV))$. The estimate (4.1.12) shows, together with (4.1.11), that G is a contraction on $(G(B_r(BV)), d)$, and we already know that G maps this space into itself. So from the Banach-Caccioppoli fixed point theorem we conclude that equation (4.1.8) has a unique solution $x \in BV$ for every λ satisfying $|\lambda| \leq \rho$. ■

We point out that Theorem 4.1.2 has a wider range of applications than Theorem 4.1.1, for two reasons. First, superposition operators are of course more general than composition operators. Second, there are kernel functions which satisfy hypothesis (H5), but not hypothesis (H2).

Example 4.1.3. A prominent example is the weakly singular kernel function $k(t,s) = v_\alpha(t,s)$, see (2.4.6). As we have seen in Section 2.4, this kernel function does not satisfy condition (H2), and so Theorem 4.1.1 does not apply. On the other hand, it satisfies (H4), since

$$\sup_{0 \leq t \leq 1} \int_0^1 v_\alpha(t,s)\, ds = \sup_{0 \leq t \leq 1} \int_0^t \frac{ds}{(t-s)^\alpha} = \sup_{0 \leq t \leq 1} \frac{t^{1-\alpha}}{1-\alpha} = \frac{1}{1-\alpha},$$

as well as (H5) with $M := 2/(1-\alpha)$. □

Let us assume for simplicity that $f(0,0) = 0$ in Example 4.1.3. Then for $\|g\|_{BV} < r$ condition (4.1.10) reads

$$\rho \leq \frac{(1-\alpha)(r - \|g\|_{BV})}{4(1+r)L_r},$$

while condition (4.1.11) becomes

$$\rho < \frac{1-\alpha}{L_r}.$$

Both restrictions for ρ are quite reasonable: the larger $\alpha < 1$, or the larger $L_r > 0$, or the closer g to the boundary of B_r, the smaller we must choose ρ.

Since seeking solutions to integral equations like (4.1.1) and (4.1.8) is stimulated and motivated by physical phenomena, sometimes it is necessary to consider BV-solutions which are in addition continuous. Here we impose the hypothesis

(H7) $\quad \forall \varepsilon > 0\, \exists \delta > 0\, \forall t, \tau \in [0,1]\, \forall' s \in [0,1] : \ |t - \tau| \leq \delta \implies |k(t,s) - k(\tau,s)| \leq \varepsilon$

which is nothing else but the condition (D) from Section 2.3.

Theorem 4.1.4 [Bi]. *Under the assumptions* (H1), (H2), (H3) *and* (H7), *there exists a number* $\rho > 0$ *such that equation* (4.1.1) *has, for fixed* $g \in BV \cap C$, *a unique solution* $x \in BV \cap C$ *for every* λ *satisfying* $|\lambda| \leq \rho$.

Proof. We only sketch the differences to the proof of Theorem 4.1.1. Let $(x_n)_n$ be a sequence in $BV \cap C$ such that $\|x_n - x\|_{BV} \to 0$, as $n \to \infty$, for some $x \in BV$. From $\|x_n - x\|_C \to 0$ it follows then that $x \in C$, which shows that $(BV \cap C, \|\cdot\|_{BV})$ is a Banach space. Considering again the operator $G(x) := g + \lambda K(C_f(x))$, but now on the ball $B_r(BV \cap C)$, we get the estimate

$$|G(x)(t) - G(x)(\tau)| \leq |g(t) - g(\tau)| + |\lambda| \sup_{0 \leq s \leq 1} |f(x(s))| \int_0^1 |k(t,s) - k(\tau,s)|\, ds$$

for every $x \in B_r(BV \cap C)$ and $t, \tau \in [0,1]$. From (H7) and $g \in BV \cap C$ it follows that $G(x)$ is a continuous function. The remaining part of the proof goes exactly as in Theorem 4.1.1. ■

The preceding existence results all build on the contraction mapping principle, which explains that we also obtained uniqueness of solutions. Now we prove an existence theorem based on the Schauder fixed point principle. The advantage is that we need not impose a Lipschitz condition on the nonlinearity f, and we get existence for *every* $\lambda \in \mathbb{R}$. Of course, we have to pay a price for this: we loose uniqueness of solutions.

Let us assume for simplicity that $g(t) \equiv 0$ in (4.1.1), i.e., we consider solutions of the equation

$$x(t) = \lambda \int_0^1 k(t,s) f(x(s))\, ds \qquad (0 \le t \le 1), \tag{4.1.13}$$

or, equivalently, fixed points of the operator $G := \lambda K C_f$. Suppose that the map $f : \mathbb{R} \to \mathbb{R}$ satisfies the two hypotheses

$$\exists q > 1\, \forall r > 0 : \; f \in RBV_q[-r,r] \tag{H8}$$

and

$$\lim_{|u|\to\infty} \frac{|f(u)|}{|u|} = 0. \tag{H9}$$

Here RBV_q in (H8) denotes the Riesz space introduced in Section 1.3. Condition (H9) means that f has strictly sublinear growth for large values of the argument.

Theorem 4.1.5 [BiGK1]. *Under the assumptions* (H1), (H2), (H8) *and* (H9), *equation* (4.1.13) *has, for every $\lambda \in \mathbb{R}$, a solution $x \in BV$.*

Proof. Without loss of generality, we may assume that $\lambda \neq 0$. For $p := q/(q-1)$, with q as in (H8), we consider the composition operator C_f from BV into WBV_p, and the integral operator K from WBV_p into BV; here WBV_p is the Wiener space introduced in Section 1.2. From Theorem 2.3.8 we already know that $K : WBV_p \to BV$ is compact. Moreover, the estimate $|x(t) - x(0)|^p \le WVar_p(x)$ implies that $\|x\|_\infty \le \|x\|_{WBV_p}$, see (1.2.4), and so

$$\|K(x)\|_{BV} \le \mu(k) \|x\|_{WBV_p}, \tag{4.1.14}$$

where $\mu(k)$ is given by (4.1.5).

Theorem 1.3.5 and (H9) imply that $f' \in L_q[-r,r]$ for every $r > 0$. So for every $x \in B_r(BV)$ we have

$$WVar_p(C_f(x)) \le \|f'\|^p_{L_q[-r,r]} Var(x) \qquad (q = p/(p-1)),$$

which shows that $C_f : BV \to WBV_q$ is both continuous and bounded. We claim that there exists $R > 0$ such that

$$\sup_{|u|\le R} |f(u)| \le \frac{R}{|\lambda|\mu(k)}. \tag{4.1.15}$$

In fact, otherwise we find an unbounded real sequence $(u_n)_n$ such that

$$\frac{|f(u_n)|}{|u_n|} \ge \frac{1}{|\lambda|\mu(k)},$$

contradicting our assumption (H9). Combining (4.1.14) and (4.1.15) yields

$$\|G(x)\|_{BV} = |\lambda|\, |K(C_f(x))(0)| + |\lambda|\, Var(K(C_f(x))) \le |\lambda|\, \mu(k) \sup_{|u|\le R} |f(u)| \le R,$$

and the assertion follows from Schauder's fixed point theorem. ■

We remark that Theorem 4.1.5 was proved in [BiGK], even for the nonautonomous case $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ and for $g(x) \not\equiv 0$, by degree theoretic methods. More precisely, the authors show that

$$H(\tau, x)(t) := \tau g(t) + \tau\lambda \int_0^1 k(t,s)f(s,x(s))\,ds \qquad (0 \leq \tau \leq 1)$$

defines an admissible compact homotopy joining the operator $G(x) := g + \lambda K(S_f(x))$ with the zero operator on a suitable ball in BV, and then apply the Leray-Schauder degree on that ball.

Let us give an example which is extremely simple, but illustrates quite well the difference between our various assumptions.

Example 4.1.6. For $\lambda > 0$, consider the equation

$$x(t) = \lambda t \int_0^1 x(s)^\tau\,ds \qquad (0 \leq t \leq 1). \tag{4.1.16}$$

We distinguish the special values $\tau = 2$, $\tau = 1$ and $\tau = 1/2$ and are interested in nonnegative BV solutions of (4.1.16). To begin with, let us see which of our hypotheses are satisfied for $k(t,s) = t$ and $f(u) = u^\tau$ for these values of τ.

	(H1)	(H2)	(H3)	(H4)	(H5)	(H6)	(H7)	(H8)	(H9)
$\tau = 2$	yes	yes	yes	yes	yes	yes	yes	yes	no
$\tau = 1$	yes	yes	yes	yes	yes	yes	yes	yes	no
$\tau = 1/2$	yes	yes	no	yes	yes	no	yes	yes	yes

Table 4.1: Properties of k and f in Example 4.1.6

Of course, $x(t) \equiv 0$ is always a solution of (4.1.16), so we are interested in nontrivial solutions (which may occur in balls, where the operator G defined by the right-hand side of (4.1.16) is not contracting). The structure of the equation shows that all solutions are of the form $x(t) = At$ for some $A \in \mathbb{R}$. Putting this function into $G(x)(t)$ and calculating the integral gives a condition for A.

1st case: $\tau = 2$. Here we get $A = 3/\lambda$, so

$$x_\lambda(t) = \frac{3}{\lambda}t$$

is the unique nontrivial solution. Since $\|x_\lambda\|_{BV} = 3/\lambda$, the operator G cannot be a contraction on the ball B_r for $r = 3/\lambda$. Indeed, the estimate

$$|x(t)^2 - y(t)^2| = |x(t) + y(t)|\,|x(t) - y(t)| \leq 2r|x(t) - y(t)| \qquad (x, y \in B_r)$$

shows that G is a contraction on B_r only for $2\lambda r < 1$.

Observe that $m(s) \equiv 1$, $k(0,s) \equiv 0$, hence $\mu(k) = 1$, and $L_r = 2r$ in this case, so the conditions (4.1.3) and (4.1.4) become

$$\rho \leq \frac{1}{r}, \qquad \rho < \frac{1}{2r},$$

where the second condition is more restrictive. Since $f(u) = u^2$ does not satisfy the growth condition (H9), Theorem 4.1.1 applies here, but Theorem 4.1.5 does not.

2nd case: $\tau = 1$. Here A may be arbitrary, but (4.1.16) is solvable only for $\lambda = 2$. In this case

$$x(t) = At \qquad (A \in \mathbb{R})$$

are infinitely many nontrivial solutions. Since the operator G here is not a contraction, and $f(u) = u$ does not satisfy the growth condition (H9), neither Theorem 4.1.1 nor Theorem 4.1.5 applies.

3rd case: $\tau = 1/2$. Here we get $A = 4\lambda^2/9$, so

$$x_\lambda(t) = \frac{4\lambda^2}{9}t$$

is the unique nontrivial solution. Of course, the structure of the nonlinear part shows that the operator G here is not a contraction on any ball B_r, and so Theorem 4.1.1 does not apply. However, $f(u) = \sqrt{|u|}$ satisfies the growth condition (H9), and it also belongs to $RBV_q[-r, r]$ for $1 < q < 2$ and every $r > 0$.

An easy calculation shows that condition (4.1.15) becomes here $R \geq \lambda^2$, so for these values of R we may apply Theorem 4.1.5. This example shows again that we cannot expect uniqueness of solutions when applying the Schauder fixed point theorem. It also illustrates the fact that, in case of nonlinearities of strictly sublinear growth at ∞, only balls of large radius are invariant. □

We remark that the various condition we have given in this section to ensure the existence of invariant balls for the fixed point operator G have been generalized in the literature. We cite a sample result which uses milder a priori estimates. For example, in [BiR] the authors impose the following conditions:

- There exists a function $\Psi : [0, \infty) \to [0, \infty)$ such that $\Psi(r) > 0$ for $r > 0$ and

$$\sup\{|f(x(t))| : 0 \leq t \leq 1\} \leq \Psi(\|x\|_{BV})$$

  for all $x \in BV$.

- There exists a continuous increasing function $\Phi_r : [0, \infty) \to [0, \infty)$ such that

$$\|C_f(x) - C_f(y)\|_{BV} \leq \Phi_r(\|x - y\|_{BV}) \qquad (x, y \in B_r).$$

It is then shown that, under additional appropriate hypotheses, equation (4.1.1) has a solution in $B_r(BV)$ for fixed $g \in BV$ and all $|\lambda| \leq \rho$, provided that

$$\|g\|_{BV} + \rho\Psi(r)\mu(k) \leq r$$

and

$$\Phi_r(\rho) < \frac{\rho}{\mu(k)} \qquad (\rho > 0),$$

where $\mu(k)$ is again given by (4.1.5). Since $\|x\|_\infty \leq \|x\|_{BV}$, we may choose $\Psi(r) := \sup\{|f(u)| : |u| \leq r\}$ to recover Theorem 4.1.1. Similarly, if f satisfies the local Lipschitz condition (3.1.1), we may choose $\Phi_r(\rho) := L_r\rho$ to recover Theorem 4.1.1. More general choices, however, are also possible (e.g., $\Phi_r(\rho) := \arctan\rho$ or $\Phi_r(\rho) := \log(1 + \rho)$) which enlarge the applicability of the result in [BiR].

4.2. Hammerstein-Volterra integral equations. Now we pass to the nonlinear Hammerstein-Volterra integral equations

$$x(t) = g(t) + \lambda\int_0^t k(t, s)f(x(s))\,ds \qquad (t \geq 0), \tag{4.2.1}$$

and

$$x(t) = g(t) + \lambda\int_0^t k(t, s)f(s, x(s))\,ds \qquad (t \geq 0). \tag{4.2.2}$$

Note that we can make the right-hand side of (4.2.1) or (4.2.2) small by either choosing the factor λ or the upper limit t of the integral sufficiently small. Our choice depends on what we need in a specific problem: choosing λ small enlarges the existence interval $[0, T]$ of solutions, while choosing t small gives us a larger set of admissible values for λ.

Since most results are parallel to those obtained in the previous section, we will be brief. We restrict ourselves to the equation (4.2.1) which is simpler than (4.2.2). Adapting the hypotheses on f, the extension to (4.2.2) is straightforward. Denoting by

$$V(x)(t) = \int_0^t k(t,s)x(s)\,ds = \int_0^1 v(t,s)x(s)\,ds, \tag{4.2.3}$$

with $v(t,s)$ given by (2.4.2), the linear integral operator (2.4.1), we may rewrite (4.2.1) as fixed point equation

$$x = g + \lambda V(C_f(x)), \tag{4.2.4}$$

where C_f is the composition operator (3.0.1). As in Section 2.4, we assume that

$$\forall t > 0: \;\; v(t,\cdot) \in L_1[0,t] \tag{H10}$$

and

$$|v(s,s)| + Var(v(\cdot,s);[s,1]) \leq m(s), \tag{H11}$$

for some L_1-function m. Observe that Hypothesis (H11) is nothing else but (2.4.3) from Section 2.4. The following result is then parallel to Theorem 4.1.1.

Theorem 4.2.1 [Bi]. *Under the assumptions* (H3), (H10), *and* (H11), *there exist numbers* $\rho > 0$ *and* $T > 0$ *such that equation* (4.2.1) *has, for fixed* $g \in BV$, *a unique solution* $x \in BV[0,T]$ *for every* λ *satisfying* $|\lambda| \leq \rho$.

Proof. As before, the proof builds on the Banach-Caccioppoli fixed point theorem, applied to the operator $G : B_r(BV) \to BV$ defined by $G(x) := g + \lambda V(C_f(x))$. Let $r > 0$ and $\rho > 0$ be the same as in the proof of Theorem 4.1.1. In addition, we choose $T > 0$ in such a way that both

$$\|g\|_{BV} + \rho \sup_{|u|\leq r} |f(u)| \int_0^T m(s)\,ds \leq r \tag{4.2.5}$$

and

$$\rho L_r \int_0^T m(s)\,ds < 1, \tag{4.2.6}$$

where L_r is the local Lipschitz constant from Theorem 3.1.1 (a). In rather the same way as in the proof of Theorem 4.1.1, for any $x \in B_r$ and every partition $\{t_0, t_1, \ldots, t_m\}$ of $[0,T]$ we have then

$$\sum_{j=1}^m \left| \int_0^{t_j} k(t_j,s)f(x(s))\,ds - \int_0^{t_{j-1}} k(t_{j-1},s)f(x(s))\,ds \right|$$
$$\leq \sup_{0\leq s\leq 1} |f(x(s))| \sum_{j=1}^m \int_0^T |v(t_j,s) - v(t_{j-1},s)|\,ds$$
$$= \sup_{0\leq s\leq 1} |f(x(s))| \int_0^T \sum_{j=1}^m |v(t_j,s) - v(t_{j-1},s)|\,ds \leq \sup_{0\leq s\leq 1} |f(x(s))| \int_0^T m(s)\,ds.$$

Consequently, for $|\lambda| \leq \rho$ we get

$$\|G(x)\|_{BV} \leq \|g\|_{BV} + |\lambda| \sup_{|u|\leq r} |f(u)|\, \|m\|_{L_1[0,T]}. \tag{4.2.7}$$

This shows that $G(B_r) \subseteq B_r$, by (4.2.5). Now we have to show that G is a contraction on B_r. For any $x, y \in B_r$ and every partition $\{t_0, t_1, \dots, t_m\}$ of $[0,T]$ we obtain

$$
\begin{aligned}
&\sum_{j=1}^{m} \left| \int_0^T v(t_j,s)[f(x(s)) - f(y(s))]\,ds - \int_0^T v(t_{j-1},s)[f(x(s)) - f(y(s))]\,ds \right| \\
&\qquad \le \sup_{0\le s\le 1} |f(x(s)) - f(y(s))| \int_0^T \sum_{j=1}^{m} |v(t_j,s) - v(t_{j-1},s)|\,ds \qquad (4.2.8)\\
&\qquad \le L_r \sup_{0\le s\le 1} |x(s) - y(s)| \int_0^T m(s)\,ds \le L_r \|m\|_{L_1[0,T]} \|x-y\|_{BV},
\end{aligned}
$$

hence

$$Var(V(C_f(x)) - V(C_f(y))) \le L_r \|m\|_{L_1[0,T]} \|x-y\|_{BV}.$$

Moreover,

$$
\begin{aligned}
|V(C_f(x))(0) - V(C_f(y))(0)| &\le \sup_{0\le s\le 1} |f(x(s)) - f(y(s))| \int_0^T |v(0,s)|\,ds \qquad (4.2.9)\\
&\le L_r \|v(0,\cdot)\|_{L_1[0,T]} \|x-y\|_{BV}.
\end{aligned}
$$

Combining (4.2.8) and (4.2.9) we arrive at

$$
\begin{aligned}
\|G(x) - G(y)\|_{BV} &= |G(x)(0) - G(y)(0)| + |\lambda| Var(V(C_f(x)) - V(C_f(y))) \\
&\le |\lambda| L_r \, \|m\|_{L_1[0,T]} \|x-y\|_{BV},
\end{aligned}
$$

and the statement follows from (4.2.6). ■

Of course, if we suppose that $g \in BV \cap C$, we may also formulate and prove an analogue to Theorem 4.1.4. We confine ourselves to the formulation of such a result without proof.

Theorem 4.2.2 [Bi]. *Under the assumptions* (H3) *and* (H7), *there exist numbers* $\rho > 0$ *and* $T > 0$ *such that equation* (4.2.1) *has, for fixed* $g \in BV \cap C$, *a unique solution* $x \in BV[0,T] \cap C[0,T]$ *for every* λ *satisfying* $|\lambda| \le \rho$.

We also give a theorem which is parallel to Theorem 4.1.5, inasmuch as it builds on Schauder's theorem, rather than Banach's theorem.

Theorem 4.2.3. *Under the assumptions* (H8), (H9), (H10) *and* (H11), *equation* (4.2.1) *has, for every* $\lambda \in \mathbb{R}$, *a solution* $x \in BV[0,T]$ *for sufficiently small* $T > 0$.

Proof. The proof of the continuity and boundedness of $C_f : BV \to WBV_p$ and compactness of $V : WBV_p \to BV$ goes as in Theorem 4.1.5. However, due to the special structure of a Volterra operator, existence of invariant balls is more easily established. In fact, (4.1.14) may now be replaced by

$$\|V(x)\|_{BV[0,T]} \le \|m\|_{L_1[0,T]} \|x\|_{WBV_p}, \qquad (4.2.10)$$

and Lebegue's dominated convergence theorem shows that $\|m\|_{L_1[0,T]}$ may be made arbitrarily small by choosing $T > 0$ sufficiently small. ■

The following example illustrates the applicability of Theorem 4.2.3 in the case when the preceding Theorem 4.2.1 and 4.2.2 do not apply for lack of Lipschitz continuity of the nonlinearity f.

Example 4.2.4. Consider again the third case of Example 4.1.6, i.e., the equation

$$x(t) = \lambda t \int_0^t \sqrt{x(s)}\,ds \qquad (t \ge 0) \qquad (4.2.11)$$

which is the Volterra analogue to (4.1.16).

Hypothesis (H10) holds trivially. Since $v(s,s) = s$ and $Var(v(\cdot,s);[s,1]) = 1-s$ for $v(t,s) = t$, we may choose $m(s) \equiv 1$ in Hypothesis (H11). Consequently, we have

$$\|V(x)\|_{BV[0,T]} \leq \|m\|_{L_1[0,T]} = T.$$

Theorem 4.2.3 guarantees the existence of a solution of (4.2.11) for every $\lambda \in \mathbb{R}$ on a sufficiently small intervall $[0,T]$. Of course, $x(t) \equiv 0$ is such a solution, and it is even defined on the whole semiaxis $[0,\infty)$.

On the other hand, a direct calculation shows that $x(t) = \lambda^2 t^4/9$ is another solution of (4.2.11). So Theorem 4.2.3 again does not guarantee uniqueness of solutions, as one could expect.

We point out that equations like (4.2.11) may have many more solutions even if the integral in (4.2.11) becomes singular. Thus, if we consider the equation

$$x(t) = \lambda t \int_0^t x(s)^\tau \, ds \qquad (t \geq 0) \tag{4.2.12}$$

and make the *Ansatz* $x(t) := At^\alpha$, the integral exists not only for positive values of $\tau\alpha$, but also for $-1 < \tau\alpha < 0$, say. Comparing exponents leads then to the condition $\tau\alpha = \alpha - 2$. Now, the function $\tau = \varphi(\alpha) := (\alpha-2)/\alpha$ is a homeomorphism between $(1,2)$ and $(-1,0)$, with inverse $\alpha = \psi(\tau) = 2/(1-\tau)$, and the conditions $1 < \alpha < 2$ and $-1 < \tau < 0$ precisely correspond to our requirement $-1 < \tau\alpha < 0$. Adjusting the constant $A = A(\alpha,\lambda)$ we obtain a nontrivial solution of bounded variation also in this weakly singular case. □

The reason why Volterra-Hammerstein equations have better properties than Hammerstein equations is of course that the operator V given in (4.2.3) has, in contrast to the operator (2.0.3), spectral radius zero. This makes the problem of finding invariant balls, and so of proving existence, quite easy. A well-known consequence is that an initial value problem for an ordinary differential equation is very often (uniquely) solvable, while the existence and uniqueness of solutions of a boundary value problem may be a subtle problem. In Section 4.4 below we will derive some properties of the solution set of equations like (4.1.1) and (4.2.1) if these equations are not uniquely solvable.

4.3. Solvability in generalized BV spaces. Let us now see to what extent our results carry over to more general spaces of functions of bounded variation. Here our focus will be put on the Waterman space ΛBV and the Young space YBV_ϕ which have been introduced in Section 1.4. In the preceding Sections 4.1 and 4.2 we have given existence and uniqueness results

- for the Hammerstein equation (4.1.1) in the space BV,
- for the Hammerstein-Voltarre equation (4.2.1) in the space BV,
- for the Hammerstein equation (4.1.1) in the space $BV \cap C$, and
- for the Hammerstein-Voltarre equation (4.2.1) in the space $BV \cap C$.

At the risk of being redundant, we will do the same in this section for the spaces ΛBV and YBV_ϕ. However, some proofs will be skipped if they require only minor technical modifications of proofs which have been given before.

We start with the Waterman space ΛBV. Of course, we have to adapt our hypotheses by replacing, in particular, Hypothesis (H2) by

$$\exists m \in L_1,\ \forall' s \in [0,1]:\ Var_\Lambda(k(\cdot,s)) \leq m(s); \tag{H2$_\Lambda$}$$

which coincides with condition (B_Λ) in Section 2.3. The following existence and uniqueness result is parallel to Theorem 4.1.1.

Theorem 4.3.1 [BaR]. *Under the assumptions* (H1), (H2), *and* (H3), *there exists a number* $\rho > 0$ *such that equation* (4.1.1) *has, for fixed* $g \in \Lambda BV$, *a unique solution* $x \in \Lambda BV$ *for every* λ *satisfying* $|\lambda| \leq \rho$.

Proof. Consider again the fixed point problem for the operator $G(x) := g + \lambda K(C_f(x))$. To this end, choose $r > 0$ with $\|g\|_{\Lambda BV} < r$, and $\rho > 0$ in such a way that both

$$\|g\|_{\Lambda BV} + \rho \sup_{|u|\leq r} |f(u)|\, \mu(k) \leq r \tag{4.3.1}$$

and

$$\rho L_r \max\{1, 1/\lambda_1\}\mu(k) < 1, \tag{4.3.2}$$

where $\mu(k)$ is given by (4.1.5) and L_r is the local Lipschitz constant from Theorem 3.1.1 (a).

By Theorem 2.3.12, the operator K maps ΛBV into itself. Moreover, by Theorem 3.4.1 the operator C_f maps ΛBV into itself and is bounded. Fix $x \in B_r = B_r(\Lambda BV)$, and let $\{[a_n, b_n] : n \in \mathbb{N}\}$ be any collection of non-overlapping intervals $[a_n, b_n] \subset [0,1]$. For fixed $N \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{n=1}^{N} \lambda_n\, |K(C_f(x))(b_n) - K(C_f(x))(a_n)| &= \sum_{n=1}^{N} \lambda_n \left| \int_0^1 [k(b_n, s) - k(a_n, s)] f(x(s))\, ds \right| ds \\
&\leq \sup_{0\leq s\leq 1} |f(x(s))| \int_0^1 \sum_{n=1}^{N} \lambda_n\, |k(b_n, s) - k(a_n, s)|\; ds \\
&\leq \sup_{0\leq s\leq 1} |f(x(s))| \int_0^1 Var_\Lambda(k(\cdot, s)\, ds \leq \sup_{0\leq s\leq 1} |f(x(s))| \int_0^1 m(s)\, ds.
\end{aligned}$$

Consequently, for $|\lambda| \leq \rho$ we get

$$\|G(x)\|_{\Lambda BV} \leq \|g\|_{\Lambda BV} + |\lambda| \sup_{|u|\leq r} |f(u)|\, \mu(k),$$

which shows that $G(B_r) \subseteq B_r$, by (4.3.1). The proof of the contraction property of G on B_r is similar. For $x, y \in B_r$ we get

$$\begin{aligned}
&\sum_{n=1}^{N} \lambda_n\, |K(C_f(x))(b_n) - K(C_f(x))(a_n) - K(C_f(y))(b_n) + K(C_f(y))(a_n)| \\
&= \sum_{n=1}^{N} \lambda_n \left| \int_0^1 k(b_n, s)[f(x(s)) - f(y(s))]\, ds - \int_0^1 k(a_n, s)[f(x(s)) - f(y(s))]\, ds \right| \\
&\leq \int_0^1 \sum_{n=1}^{N} \lambda_n |k(b_n, s) - k(a_n, s)|\, |f(x(s)) - f(y(s))|\, ds \\
&\leq L_r \sup_{0\leq s\leq 1} |x(s) - y(s)| \int_0^1 \sum_{n=1}^{N} \lambda_n |k(b_n, s) - k(a_n, s)|\, ds \\
&\leq L_r \max\{1, 1/\lambda_1\} \|x - y\|_{\Lambda BV} \int_0^1 Var_\Lambda(k(\cdot, s)\, ds \\
&\leq L_r \max\{1, 1/\lambda_1\} \|m\|_{L_1} \|x - y\|_{\Lambda BV}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|G(x) - G(y)\|_{\Lambda BV} &= |G(x)(0) - G(y)(0)| + |\lambda| Var(K(C_f(x)) - K(C_f(y))) \\
&\leq |\lambda| L_r \max\{1, 1/\lambda_1\} \|k(0, \cdot)\|_{L_1} \|x - y\|_{\Lambda BV} + |\lambda| L_r \max\{1, 1/\lambda_1\} \|m\|_{L_1} \|x - y\|_{\Lambda BV} \\
&= |\lambda| L_r \max\{1, 1/\lambda_1\} \mu(k) \|x - y\|_{\Lambda BV}.
\end{aligned}$$

The statement follows now from (4.3.2) and the Banach-Caccioppoli fixed point theorem. ■

Of course, the term max $\{1, 1/\lambda_1\}$ which appears here comes from the imbedding constant $c(\Lambda BV, B)$ in Table 1.4. The next example illustrates the estimates (4.3.1) and (4.3.2) in the special Waterman space $\Lambda_q BV$.

Example 4.3.2. Let $0 < q < 1$, $k(t,s) := \zeta_{2-q}(t)k_2(s)$, where ζ_θ is the zigzag function (1.1.7), $k_2 \in L_1$, and $f(u) := |u|^\tau$ for some $\tau > 1$. For these data, we consider equation (4.1.1) in the space $\Lambda_q BV$. From (1.4.12) and $\zeta_{2-q}(0) = 0$ it follows that

$$\mu(k) = \|m\|_{L_1} + \|k(0,\cdot)\|_{L_1} = \|k_2\|_{L_1} Var_{\Lambda_q}(\zeta_{2-q}) = \|k_2\|_{L_1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \|k_2\|_{L_1} \frac{\pi^2}{6}.$$

So for given $g \in \Lambda_q BV$ with $\|g\|_{\Lambda_q BV} < r$, condition (4.3.1) reads

$$\rho \leq \frac{6(r - \|g\|_{\Lambda_q BV})}{\pi^2 r^\tau \|k_2\|_{L_1}},$$

while condition (4.3.2) becomes

$$\rho < \frac{6}{\pi^2 \tau r^{\tau-1} \|k_2\|_{L_1}}.$$

For these values of $\rho > 0$, the equation

$$x(t) = g(t) + \lambda \zeta_{2-q}(t) \int_0^1 k_2(s) x(s)^\tau \, ds \qquad (0 \leq t \leq 1)$$

has a unique solution $x \in \Lambda_q BV$ for $|\lambda| \leq \rho$. □

Similarly, for proving an existence and uniqueness result for the Hammerstein-Volterra equation (4.2.1) we have to replace Hypothesis (H11) by

$$(\mathrm{H11}_\Lambda) \qquad \exists m \in L_1: \ \lambda_1 |v(s,s)| + Var_\Lambda(v(\cdot,s);[s,1]) \leq m(s).$$

Theorem 4.3.3 [BaR]. *Under the assumptions* (H3), (H10), *and* ($\mathrm{H11}_\Lambda$), *there exist numbers* $\rho > 0$ *and* $T > 0$ *such that equation* (4.2.1) *has, for fixed* $g \in \Lambda BV$, *a unique solution* $x \in \Lambda BV[0,T]$ *for every* λ *satisfying* $|\lambda| \leq \rho$.

Proof. Again, let $g \in \Lambda BV$, $r > 0$ and $\rho > 0$ be the same as in the proof of Theorem 4.3.1. In addition, we choose $T > 0$ in such a way that both

$$\|g\|_{BV} + \rho \sup_{|u| \leq r} |f(u)| \int_0^T m(s)\, ds \leq r \tag{4.3.3}$$

and

$$\rho L_r \max\{1, 1/\lambda_1\} \int_0^T m(s)\, ds < 1, \tag{4.3.4}$$

where L_r is the local Lipschitz constant from Theorem 3.1.1 (a). By the definition (2.4.2) of the triangular kernel function v, we have

$$Var_\Lambda(v(\cdot,s);[0,s]) \leq \lambda_1 |k(s,s)|.$$

Consequently, for $x \in B_r = B_r(\Lambda BV)$ and any collection of non-overlapping intervals $[a_n, b_n] \subset [0,T]$ we obtain

$$\sum_{n=1}^{N} \lambda_n |V(C_f(x))(b_n) - V(C_f(x))(a_n)| = \sum_{n=1}^{N} \lambda_n \left| \int_0^1 [v(b_n,s) - v(a_n,s)] f(x(s))\, ds \right| ds$$

$$\leq \sup_{0 \leq s \leq 1} |f(x(s))| \int_0^T \sum_{n=1}^{N} \lambda_n |k(b_n,s) - k(a_n,s)|\, ds$$

$$\leq \sup_{0 \leq s \leq 1} |f(x(s))| \int_0^T Var_\Lambda(k(\cdot,s)\, ds \leq \sup_{0 \leq s \leq 1} |f(x(s))| \int_0^T [\lambda_1 |k(s,s)| + Var_\Lambda(k(\cdot,s);[s,T])]\, ds$$

$$\leq \sup_{0 \leq s \leq 1} |f(x(s))| \int_0^T m(s)\, ds.$$

So for $|\lambda| \le \rho$ we get

$$\|G(x)\|_{\Lambda BV} \le \|g\|_{\Lambda BV} + |\lambda| \sup_{|u|\le r} |f(u)| \, \|m\|_{L_1[0,T]},$$

and $G(x) = g + \lambda V(C_f(x))$ as before. This shows that $G(B_r) \subseteq B_r$, by (4.3.3). To show that G is a contraction on B_r, we proceed as in the proof of Theorem 4.2.1. For $x, y \in B_r$ we get

$$\sum_{n=1}^{N} \lambda_n \left| V(C_f(x))(b_n) - V(C_f(x))(a_n) - V(C_f(y))(b_n) + V(C_f(y))(a_n) \right|$$

$$= \sum_{n=1}^{N} \lambda_n \left| \int_0^T k(b_n, s)[f(x(s)) - f(y(s))] \, ds - \int_0^T k(a_n, s)[f(x(s)) - f(y(s))] \, ds \right|$$

$$\le \int_0^T \sum_{n=1}^{N} \lambda_n |v(b_n, s) - v(a_n, s)| \, |f(x(s)) - f(y(s))| \, ds$$

$$\le L_r \sup_{0\le s\le 1} |x(s) - y(s)| \int_0^T \sum_{n=1}^{N} \lambda_n |k(b_n, s) - k(a_n, s)| \, ds$$

$$\le L_r \max\{1, 1/\lambda_1\} \|x - y\|_{\Lambda BV} \int_0^T Var_\Lambda(k(\cdot, s) \, ds$$

$$\le L_r \max\{1, 1/\lambda_1\} \|m\|_{L_1} \|x - y\|_{\Lambda BV}.$$

Consequently,

$$\|G(x) - G(y)\|_{\Lambda BV} = |G(x)(0) - G(y)(0)| + |\lambda| Var(K(C_f(x)) - K(C_f(y)))$$

$$\le |\lambda| L_r \max\{1, 1/\lambda_1\} \|k(0, \cdot)\|_{L_1[0,T]} \|x - y\|_{\Lambda BV} + |\lambda| L_r \max\{1, 1/\lambda_1\} \|m\|_{L_1[0,T]} \|x - y\|_{\Lambda BV}$$

$$= |\lambda| L_r \max\{1, 1/\lambda_1\} \|m\|_{L_1[0,T]} \|x - y\|_{\Lambda BV}.$$

The statement follows now from (4.3.4) and the Banach-Caccioppoli fixed point theorem. ∎

Of course, if we suppose that $g \in \Lambda BV \cap C$, we may also formulate and prove an analogue to Theorem 4.3.1 and Theorem 4.3.3. We confine ourselves to the formulation of such results without proof.

Theorem 4.3.4 [BaR]. *Under the assumptions* (H1), (H3), (H7), *and* ($H2_\Lambda$), *there exists a number* $\rho > 0$ *such that equation* (4.1.1) *has, for fixed* $g \in \Lambda BV \cap C$, *a unique solution* $x \in \Lambda BV \cap C$ *for every* λ *satisfying* $|\lambda| \le \rho$.

Theorem 4.3.5 [BaR]. *Under the assumptions* (H3) *and* ($H11_\Lambda$), *there exist numbers* $\rho > 0$ *and* $T > 0$ *such that equation* (4.2.1) *has, for fixed* $g \in \Lambda BV \cap C$, *a unique solution* $x \in \Lambda BV[0,T] \cap C[0,T]$ *for every* λ *satisfying* $|\lambda| \le \rho$.

Let us now turn to the space YBV_ϕ generated to some Young function $\phi : [0, \infty) \to [0, \infty)$. Here we have to replace hypothesis (H2) by the following condition:

($H2_\phi$) $\qquad \exists m \in L_1, \ \exists \alpha > 0, \ \forall' s \in [0,1] : \ YVar_\phi(k(\cdot, s)/\alpha) \le m(s);$

The following existence and uniqueness result is then similar to Theorem 4.1.1 and Theorem 4.3.1.

Theorem 4.3.6 [BaBiH]. *Under the assumptions* (H1), ($H2_\phi$), *and* (H3), *there exists a number* $\rho > 0$ *such that equation* (4.1.1) *has, for fixed* $g \in YBV_\phi$, *a unique solution* $x \in YBV_\phi$ *for every* λ *satisfying* $|\lambda| \le \rho$.

Proof. First we observe that ($H2_\phi$) implies that

$$\int_0^1 YVar_\phi(k(\cdot, s)/\alpha) \, ds \le \|m\|_{L_1} < \infty.$$

So for $\gamma := \alpha \max\{\|m\|_{L_1}, 1\}$ we get

$$\int_0^1 YVar_\phi(K(\cdot,s)/\gamma)\, ds \le \frac{1}{\max\{\|m\|_{L_1},1\}} \int_0^1 YVar_\phi(k(\cdot,s)/\alpha)\, ds = \frac{\|m\|_{L_1}}{\max\{\|m\|_{L_1},1\}} \le 1.$$

Consequently,

$$\|k(\cdot,s)\|_{YBV_\phi} \le |k(0,s)| + \gamma,$$

by the definition (1.4.16) of the norm in YBV_ϕ. Choose $r > 0$ with $\|g\|_{YBV_\phi} < r$, and $\rho > 0$ in such a way that both

$$\|g\|_{YBV_\phi} + \rho\mu(k) \sup_{|u|\le r} |f(u)| \le r \tag{4.3.5}$$

and

$$\rho L_r\, \mu(k)\phi(1) < 1, \tag{4.3.6}$$

where L_r is the local Lipschitz constant from Theorem 3.1.1 (a), and $\mu(k)$ is given by (4.1.5). We claim that the operator $G(x) := g + \lambda K(C_f(x))$ maps the ball $B_r = B_r(YBV_\phi)$ into itself.
In fact, given $x \in B_r$ and any partition $\{t_0, t_1, \dots, t_m\}$ of $[0,1]$, by Jensen's inequality we have

$$\begin{aligned}
&\sum_{j=1}^m \phi\left(\frac{|\lambda|}{\gamma} |K(C_f(x))(t_j) - K(C_f(x))(t_{j-1})|\right) \\
&\le \sum_{j=1}^m \phi\left(\int_0^1 \frac{|\lambda|}{\gamma} \sup_{|u|\le r} |f(u)|\, |k(t_j,s) - k(t_{j-1},s)|\, ds\right) \\
&\le \int_0^1 YVar_\phi\left(|\lambda| \sup_{|u|\le r} |f(u)| \frac{k(\cdot,s)}{\gamma}\right) ds.
\end{aligned}$$

Taking the supremum over all partitions we deduce that

$$\begin{aligned}
&\inf\left\{\mu > 0 : YVar_\phi\left(\frac{\lambda K(C_f(x))}{\mu}\right) \le 1\right\} \\
&\le \inf\left\{\mu > 0 : \int_0^1 YVar_\phi\left(|\lambda| \sup_{|u|\le r} |f(u)| \frac{k(\cdot,s)}{\mu}\right) ds \le 1\right\} \\
&\le |\lambda| \sup_{|u|\le r} |f(u)| \int_0^1 YVar_\phi\left(\frac{k(\cdot,s)}{\gamma}\right) ds \le |\lambda| \sup_{|u|\le r} |f(u)|.
\end{aligned}$$

Moreover,

$$|G(x)(0)| \le \|g\|_{YBV_\phi} + \rho \|k(0,\cdot)\|_{L_1} \sup_{|u|\le r} |f(u)|.$$

Consequently, for $|\lambda| \le \rho$ we have

$$\|G(x)\|_{YBV_\phi} \le \|g\|_{YBV_\phi} + |\lambda| \sup_{|u|\le r} |f(u)|\, \mu(k),$$

which shows that $G(B_r) \subseteq B_r$, by (4.3.5). Now we have to show that, as before, G is a contraction on B_r. For any $x, y \in B_r$ and every partition $\{t_0, t_1, \dots, t_m\}$ of $[0,1]$ we obtain

$$\begin{aligned}
&\sum_{j=1}^m \phi\left(\int_0^1 \frac{|\lambda|}{\gamma} |k(t_j,s) - k(t_{j-1},s)|\, |f(x(s)) - f(y(s))|\right) ds \\
&\le \sum_{j=1}^m \int_0^1 \phi\left(\frac{|\lambda|}{\gamma} \sup_{0\le s\le 1} |f(x(s)) - f(y(s))|\, |k(t_j,s) - k(t_{j-1},s)|\right) ds \\
&\le \int_0^1 YVar_\phi\left(|\lambda| L_r \sup_{0\le s\le 1} |x(s) - y(s)| \frac{k(\cdot,s)}{\gamma}\right) ds,
\end{aligned}$$

hence

$$(4.3.7)\qquad YVar_\phi(K(C_f(x))-K(C_f(y)))\le L_r\gamma\|x-y\|_\infty\le L_r\gamma\phi(1)\|x-y\|_{YBV_\phi}.$$

The statement follows now from (4.3.6). ■

Let us check what our hypotheses mean for separated kernel functions of the form (2.3.3).

Example 4.3.7. Suppose that $k(t,s)=k_1(t)k_2(s)$, where $k_1\in YBV_\phi$ and $k_2\in L_\infty$. Obviously,

$$YVar_\phi\left(\frac{k(\cdot,s)}{\|k_2\|_{L_\infty}}\right)=YVar_\phi\left(\frac{k_1k_2(s)}{\|k_2\|_{L_\infty}}\right)\le\frac{k_2(s)}{\|k_2(\cdot)\|_{L_\infty}}YVar_\phi(k_1),$$

so hypothesis (H2$_\phi$) holds with

$$\alpha=\|k_2\|_{L_\infty},\qquad m(s)=\frac{YVar_\phi(k_1)}{\|k_2\|_{L_\infty}}k_2(s).$$

So

$$\mu(k)=YVar_\phi(k_1)\frac{\|k_2\|_{L_1}}{\|k_2\|_{L_\infty}}+|k_1(0)|\,\|k_2\|_{L_1}\le\max\{1/\|k_2\|_{L_\infty},1\}\,\|k_1\|_{YBV_\phi}\,\|k_2\|_{L_1}$$

in this case. To be specific, let $\phi(u):=u^p$ for some $p\in(1,\infty)$, i.e., we consider equation (4.1.1) in the Wiener space WBV_p. Similarly as in Example 4.3.2, we take $k(t,s):=\zeta_{2/p}(t)k_2(s)$, where ζ_θ is the zigzag function (1.1.7), $k_2\in L_\infty$, and $f(u):=|u|^\tau$ for some $\tau>1$. From (1.2.7) it follows that

$$\mu(k)=\|m\|_{L_1}+\|k(0,\cdot)\|_{L_1}=\|k_2\|_{L_1}WVar_p(\zeta_{2/p})=\|k_2\|_{L_1}\sum_{n=1}^\infty\frac{1}{n^2}=\|k_2\|_{L_1}\frac{\pi^2}{6}.$$

So for given $g\in WBV_p$ with $\|g\|_{WBV_p}<r$, condition (4.3.5) reads

$$\rho\le\frac{6(r-\|g\|_{WBV_p})}{\pi^2r^\tau},$$

while condition (4.3.2) becomes

$$\rho<\frac{6}{\pi^2\tau r^{\tau-1}}.$$

For these values of $\rho>0$, the equation

$$x(t)=g(t)+\lambda\zeta_{2/p}(t)\int_0^1k_2(s)x(s)^\tau\,ds\qquad(0\le t\le1)$$

has a unique solution $x\in WBV_p$ for $|\lambda|\le\rho$. □

If we want to prove unique solvability in YBV_ϕ for the Hammerstein-Volterra equation (4.2.1), we have to replace (H11) by the condition

$$(\mathrm{H}11_\phi)\qquad\exists m\in L_1\ \exists\alpha>0:\ \phi(|v(s,s)|/\alpha)+YVar_\phi(v(\cdot,s)/\alpha;[s,1])\le m(s).$$

The next theorem gives an existence and uniqueness result for solutions $x\in YBV_\phi$ of the Hammerstein-Volterra equation (4.2.1). Since the argument is here slightly more tricky than for BV and ΛBV, we also give the proof. Now we have to suppose, in addition, that the Young function ϕ satisfies a Δ_2 *condition* (for small u) which means that there exist numbers $u_0>0$ and $k>0$ such that

$$(4.3.8)\qquad\phi(2u)\le k\phi(u)\qquad(0\le u\le u_0).$$

Obviously, the Young function $\phi(u)=u^p$ with $1<p<\infty$ satisfies a Δ_2 condition, so the following theorem applies, in particular, to the Wiener space WBV_p.

Theorem 4.3.8 [BaBiH]. *Under the assumptions* (H3), (H10), *and* (H11$_\phi$), *there exist numbers* $\rho > 0$ *and* $T > 0$ *such that equation* (4.2.1) *has, for fixed* $g \in YBV_\phi$, *a unique solution* $x \in YBV_\phi[0, T]$ *for every* λ *satisfying* $|\lambda| \le \rho$.

Proof. Let $r > 0$ and L_r be the same as in the proof of Theorem 4.3.6. Choose $N \in \mathbb{N}$ so large that both

$$\rho \frac{\alpha}{2^N} \sup_{|u| \le r} |f(u)| + \|g\|_{YBV_\phi} \le r \tag{4.3.9}$$

and

$$\rho \frac{\alpha L_r \phi(1)}{2^N} < 1. \tag{4.3.10}$$

Let $0 < T \le \min\{u_0, 1\}$, where u_0 is the upper bound for the Δ_2 condition, be so small that

$$\int_0^T \phi\left(\frac{2^N |k(s,s)|}{\alpha}\right) ds + \int_0^T YVar_\phi\left(\frac{2^N |k(\cdot, s)|}{\alpha}; [s, T]\right) ds \le k^N \int_0^T m(s)\, ds \le 1, \tag{4.3.11}$$

where α and m are from condition (H11$_\phi$), and $k > 0$ is the constant occurring in the Δ_2 condition. Such a $T > 0$ exists in view of the absolute continuity of the Lebesgue integral and the estimate

$$\int_0^T \phi\left(\frac{2|k(s,s)|}{\alpha}\right) ds + \int_0^T YVar_\phi\left(\frac{2|k(\cdot, s)|}{\alpha}; [s, T]\right) ds \le k \int_0^T m(s)\, ds,$$

so it suffices to choose $T > 0$ such that $\|m\|_{L_1} \le 1/k$. From (4.3.11) we get

$$\inf\left\{\mu > 0 : \int_0^T \phi\left(\frac{|k(s,s)|}{\mu}\right) ds + \int_0^T YVar_\phi\left(\frac{|k(\cdot, s)|}{\mu}; [s, T]\right) ds \le 1\right\} \le \frac{\alpha}{2^N}. \tag{4.3.12}$$

Given $x \in B_r$ and any partition $\{t_0, t_1, \dots, t_m\}$ of $[0, 1]$, and denoting $v(t, s)$ as in (2.4.2), we have

$$\begin{aligned}
&\sum_{j=1}^m \phi\left(\frac{|K(C_f(x))(t_j) - K(C_f(x))(t_{j-1})|}{\mu}\right) \\
&= \sum_{j=1}^m \phi\left(\left|\int_0^{t_j} \frac{k(t_j, s) f(x(s))}{\mu}\, ds - \int_0^{t_{j-1}} \frac{k(t_{j-1}, s) f(x(s))}{\mu}\, ds\right|\right) \\
&= \sum_{j=1}^m \phi\left(\left|\int_0^T \frac{[v(t_j, s) - v(t_{j-1}, s)] f(x(s))}{\mu}\right|\right) \\
&\le \sum_{j=1}^m \phi\left(\frac{1}{T} \int_0^T \frac{T}{\mu} \sup_{|u| \le r} |f(u)|\, |v(t_j, s) - v(t_{j-1}, s)|\, ds\right) \\
&\le \sum_{j=1}^m \frac{1}{T} \int_0^T \phi\left(T \sup_{|u| \le r} |f(u)| \frac{|v(t_j, s) - v(t_{j-1}, s)|}{\mu}\right) ds \\
&\le \int_0^T YVar_\phi\left(\sup_{|u| \le r} |f(u)| \frac{v(\cdot, s)}{\mu}\right) ds.
\end{aligned}$$

Taking the supremum over all partitions we conclude that

$$\begin{aligned}
&\inf\left\{\mu > 0 : YVar_\phi\left(\frac{K(C_f(x))}{\mu}\right) \le 1\right\} \\
&\le \inf\left\{\mu > 0 : \int_0^T YVar_\phi\left(\sup_{|u| \le r} |f(u)| \frac{v(\cdot, s)}{\mu}\right) ds \le 1\right\} \\
&\le \sup_{|u| \le r} |f(u)| \inf\left\{\mu > 0 : \int_0^T YVar_\phi\left(\frac{v(\cdot, s)}{\mu}\right) ds \le 1\right\}.
\end{aligned}$$

Combining this with (4.3.11) and the obvious equality

$$\int_0^T YVar_\phi\left(\frac{v(\cdot,s)}{\mu}\right)ds = \int_0^T \phi\left(\frac{v(s,s)}{\mu}\right)ds + \int_0^T YVar_\phi\left(\frac{v(\cdot,s)}{\mu};[s,T]\right)ds$$

we finally obtain

$$\inf\left\{\mu>0 : YVar_\phi\left(\frac{K(C_f(x))}{\mu}\right)\le 1\right\} \le \sup_{|u|\le r}|f(u)|\,\frac{\alpha}{2^N}.$$

This shows that the operator G maps, by our choice (4.3.9) of $r>0$, the ball B_r into itself. A similar reasoning, building on (4.3.10) shows that G is a contraction on this ball. ■

To conclude, we give two existence and uniqueness results for continuous solutions $x \in YBV_\phi$ of (4.1.1) resp. (4.2.1). The proof is the same as for Theorem 4.1.1 resp. Theorem 4.2.1, with obvious modifications.

Theorem 4.3.9 [BaBiH]. *Under the assumptions* (H1), (H3) *and* (H7), *there exists a number* $\rho>0$ *such that equation* (4.1.1) *has, for fixed* $g \in YBV_\phi \cap C$*, a unique solution* $x \in YBV_\phi \cap C$ *for every* λ *satisfying* $|\lambda|\le\rho$.

Theorem 4.3.10 [BaBiH]. *Under the assumptions* (H3) *and* (H7), *there exist numbers* $\rho>0$ *and* $T>0$ *such that equation* (4.2.1) *has, for fixed* $g\in YBV_\phi\cap C$*, a unique solution* $x\in YBV_\phi[0,T]\cap C[0,T]$ *for every* λ *satisfying* $|\lambda|\le\rho$.

We collect all existence and uniqueness results of Sections 4.1 – 4.3 in the following synoptic Table 4.2. Here we restrict ourselves to the autonomous case of a nonlinearity $f:\mathbb{R}\to\mathbb{R}$.

Unique solvability of	*Equation* (4.1.1)	*Equation* (4.2.1)
in BV	Theorem 4.1.1 [Bi]	Theorem 4.2.1 [Bi]
in $BV\cap C$	Theorem 4.1.4 [BiGK]	Theorem 4.2.2 [Bi]
in ΛBV	Theorem 4.3.1 [BaR]	Theorem 4.3.3 [BaR]
in $\Lambda BV\cap C$	Theorem 4.3.4 [BaR]	Theorem 4.3.5 [BaR]
in YBV_ϕ	Theorem 4.3.6 [BaBiH]	Theorem 4.3.8 [BaBiH]
in $YBV_\phi\cap C$	Theorem 4.3.9 [BaBiH]	Theorem 4.3.10 [BaBiH]

Table 4.2: Existence and uniqueness theorems

All existence and uniqueness results of Table 4.2 refer to the autonomous case of functions $f:\mathbb{R}\to\mathbb{R}$, i.e., to the equations (4.1.1) and (4.2.1). Building on the acting, boundedness, and continuity conditions given in Section 3.2 for the superposition operator S_f, one may obtain analogous results for the non-autonomous case of functions $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, i.e., for the equations (4.1.8) and (4.2.2).

4.4. Structure of the solution set. If the solution of a nonlinear equation, like those considered in this chapter, is not unique, it is of some interest to have information on the topological structure of the solution set. One prominent example is the R_δ*-property* which means that the set of solutions is homeomorphic to the intersection of a decreasing sequence of absolute retracts.

In this section we give a sample result of this type for solutions $x\in\Lambda BV\cap C$ of the general Hammerstein-Volterra integral equation (4.2.2). Since we reformulated this equation as fixed point problem (4.2.4), our discussion will rely upon the following structural result on fixed point sets of continuous maps in $C=C[0,1]$ which we recall without proof.

Proposition 4.4.1 [Sz]. *Let $G : C \to C$ be a continuous map which satisfies the following four condition.*

(a) *The set $G(C) \subset C$ is equicontinuous.*

(b) *There exist $t_0 \in [0,1]$ and $y_0 \in \mathbb{R}$ such that $G(x)(t_0) = y_0$ for all $x \in C$.*

(c) *For any $\varepsilon > 0$ and every $x, y \in C$ from $x|_{I_\varepsilon} = y|_{I_\varepsilon}$ it follows that $G(x)|_{I_\varepsilon} = G(y)|_{I_\varepsilon}$, where $I_\varepsilon := [t_0 - \varepsilon, t_0 + \varepsilon]$.*

(d) *Every sequence $(x_n)_n$ in C which satisfies*

$$\lim_{n\to\infty} (x_n - G(x_n)) = 0 \tag{4.4.1}$$

has an accumulation point.

Then the fixed point set of G is a compact R_δ-set.

Let us make some comments on Proposition 4.4.1. This proposition has been proved in [Sz] in the much more general setting of maps $G : K \to E$, where K is a bounded convex subset of a normed space, and E is a Banach space.

Condition (a) in Proposition 4.4.1 suggests to use some Arzelà-Ascoli type result, while condition (c) means, loosely speaking, that G is an *operator with memory*; Volterra integral operators provide an important example. Finally, condition (d) is usually called a *Palais-Smale condition*; this is an important ingredient of topological and variational methods in nonlinear analysis.

We are going to apply Proposition 4.4.1 to the operator G in (4.2.4). For the precise formulation, recall that $f : [0,1] \times \mathbb{R}$ is said to *satisfy an L_p-Carathéodory condition* if $t \mapsto f(t,u)$ is Lebesgue measurable for each $u \in \mathbb{R}$, $u \mapsto f(t,u)$ is continuous for (almost) each $t \in [0,1]$, and $|f(t,u)| \leq m_p(t)$ for some function $m_p \in L_p$. Moreover, we need the following technical hypothesis

$$\text{(H12)} \qquad \forall \varepsilon > 0\, \exists \delta > 0\, \forall t, \tau \in [0,1] : \ 0 \leq \tau - t \leq \delta \implies \int_0^t |k(t,s) - k(\tau,s)| m_p(s)\, ds \leq \varepsilon,$$

which is some integral-type modification of (H7) and involves the function m_p from the Carathéodory condition.

Theorem 4.4.2 [BiCGS]. *Suppose that $f : [0,1] \times \mathbb{R}$ satisfies an L_p-Carathéodory condition for some $p \in (1,\infty]$, $g \in BV \cap C$, and $\lambda \in \mathbb{R}$. Assume that the hypotheses* (H1), (H10), (H11$_\Lambda$) (H12) *and* (F$_\Lambda$) *from Section 3.4 hold, where Λ is some Waterman sequence and $m \in L_{p/(p-1)}$ in hypothesis* (H11$_\Lambda$)*. Then the solution set*

$$\Sigma := \{x \in BV \cap C : x = g + \lambda V(S_f(x))\} \tag{4.4.2}$$

of equation (4.2.2) *is a compact R_δ-set in the Banach space $\Lambda BV \cap C$ endowed with the norm* (1.1.14).

Proof. We apply Proposition 4.4.1 to the operator

$$G(x)(t) = g(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds \qquad (0 \leq t \leq 1) \tag{4.4.3}$$

in the space C of continuous functions and divide the proof into four steps.

<u>1st step</u>: To verify condition (a), let $\varepsilon > 0$ be given. Our hypotheses (H11$_\Lambda$) and (H12) imply that we can find $\delta > 0$ such that

$$|g(t) - g(\tau)| \leq \varepsilon \qquad (0 \leq t, \tau \leq 1,\ |t - \tau| \leq \delta),$$

$$\int_0^t |k(t,s) - k(\tau,s)| m_p(s)\, ds \leq \varepsilon \qquad (0 \leq t \leq \tau \leq 1,\ |t - \tau| \leq \delta),$$

and

$$\max\{\lambda_1, 1\} \int_A m_p(s)m(s)\,ds \le \varepsilon$$

for any Lebesgue meaasurable set $A \subset [0,1]$ of measure $\le \delta$, where $m_p \in L_p$ is the Carathéodory bound of f and $m \in L_{p/(p-1)}$ is the function from (H11$_\Lambda$). It follows that, for $0 \le t \le \tau \le 1$ with $|t-\tau| \le \delta$, we have

$$|G(x)(t) - G(x)(\tau)| \le |g(t) - g(\tau)| + |\lambda| \left| \int_0^t k(t,s)f(s,x(s))\,ds - \int_0^\tau k(\tau,s)f(s,x(s))\,ds \right|$$

$$\le |g(t) - g(\tau)| + |\lambda| \int_0^t |k(t,s) - k(\tau,s)|\,|f(s,x(s))|\,ds + |\lambda| \int_t^\tau |k(\tau,s)|\,|f(s,x(s))|\,ds$$

$$\le |g(t) - g(\tau)| + |\lambda| \int_0^t |k(t,s) - k(\tau,s)|\,m_p(s)\,ds + \max\{\lambda_1, 1\}\,|\lambda| \int_t^\tau m_p(s)m(s)\,ds \le 3\lambda\varepsilon.$$

Since δ depends only on ε, but not on $x \in C$, we have shown that the set $G(C)$ is equicontinuous.

2nd step: The conditions (b) and (c) are trivially satisfied for $t_0 := 0$ and $y_0 := g(0)$.

3rd step: We have to show that the operator (4.4.3) is continuous. But this follows for the integral operator (2.4.1) from Theorem 3.4.3 and the estimate

$$\|K(S_f(x))\|_{\Lambda BV} \le \max\{\lambda_1, 1\}\,\|m_p\|_{L_p}\|x\|_{L_{p/(p-1)}}, \tag{4.4.4}$$

where m_p is the majorizing function for f.

4th step: We claim that the operator (4.4.3) satisfies the Palais-Smale condition (d). So let $(x_n)_n$ be a sequence of continuous functions which satisfies (4.4.1). Using (4.4.4) and Helly's Theorem 1.1.14 (b), there exists a subsequence $(G(x_{n_k}))_k$ of $(G(x_n))_n$ which converges pointwise to some $y \in \Lambda BV$. The structure of the fixed point equation (4.2.2) shows that then $(x_{n_k})_k$ also converges pointwise to y, and so the sequence $(S_f(x_{n_k}))_k$ is bounded in $L_{p/(p-1)}$ and converges a.e. on $[0,1]$ to $S_f(y)$.

Now we use a ΛBV variant of the remark after Theorem 2.4.1 and conclude that the sequence $(G(x_{n_k}))_k$ converges in the ΛBV-norm to $G(y) = y$. Since the ΛBV-norm (1.4.6) is stronger than the supremum norm (1.1.11), it follows that the sequence $(x_n)_n$ has an accumulation point in the space C.

By Proposition 4.4.1, the set (4.4.2) is a compact R_δ in C. To end the proof we show that this set, endowed with the norm (1.1.11), is homeomorphic to the set

$$\Sigma_\Lambda := \{x \in \Lambda BV \cap C : x = g + \lambda V(S_f(x))\}, \tag{4.4.5}$$

endowed with the norm (1.4.6). Of course, we have equality $\Sigma = \Sigma_\Lambda$ as sets, and the identity map $id : (\Sigma, \|\cdot\|_\infty) \to (\Sigma_\Lambda, \|\cdot\|_{\Lambda BV})$ is continuous, since the norm (1.4.6) is stronger than the norm (1.1.11). Now, if $(x_n)_n$ is a sequence in Σ which converges to some $x \in \Sigma$ in the norm (1.1.11), we conclude as above that the sequence $(G(x_n))_n$ converges to $G(x)$ in the norm (1.4.6). But $G(x_n) = x_n$ and $G(x) = x$, hence $\|x_n - x\|_{\Lambda BV} \to 0$ as $n \to \infty$. We conclude that the identity map $id : (\Sigma_\Lambda, \|\cdot\|_{\Lambda BV}) \to (\Sigma, \|\cdot\|_\infty)$ is continuous as well, and so we are done. ■

It is not hard to find examples of functions k and f which satisfy the hypotheses of Theorem 4.4.2. For example, let $k(t,s) = k_1(t)k_2(s)$ with $k_1 \in Lip$ and $k_2 \in L_2$, and $f(t,u) = g(t)h(u)$, where $g \in L_2$ and $h : \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Then (H11$_\Lambda$) holds with

$$m(s) := (\lambda_1\|k_1\|_\infty + Var_\Lambda(k_1))\,\|k_2\|_{L_2},$$

and (H12) holds with

$$\delta := \frac{\varepsilon}{2lip(k_1)\,\max\{\lambda_1, 1\}\|k_1\|_{\Lambda BV}}.$$

The other hypotheses of Theorem 4.4.2 are easily verified.

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