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The First Hochschild Cohomology as a Lie Algebra

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THE FIRST HOCHSCHILD COHOMOLOGY AS A LIE ALGEBRA

LLEONARD RUBIO Y DEGRASSI, SIBYLLE SCHROLL, AND ANDREA SOLOTAR

Abstract. In this paper we study sufficient conditions for the solvability of the first Hochschild cohomology of a finite dimensional algebra as a Lie algebra in terms of its Ext-quiver in arbitrary characteristic. In particular, we show that if the quiver has no parallel arrows and no loops then the first Hochschild cohomology is solvable. For quivers containing loops, we determine easily verifiable sufficient conditions for the solvability of the first Hochschild cohomology. We apply these criteria to show the solvabilty of the first Hochschild cohomology space for large families of algebras, namely, several families of self-injective tame algebras including all tame blocks of finite groups and some wild algebras including quantum complete intersections.

Contents

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1. Introduction

Let K be an algebraically closed field and let A be a finite dimensional K-algebra. The problem of describing the Hochschild cohomology of A as a Gerstenhaber algebra and in particular the first Hochschild cohomology space as a Lie algebra, and how this structure is related to A, has been studied in several recent articles, see for example [1, 4, 8, 17, 21]. The Gerstenhaber bracket in Hochschild cohomology has been defined more than fifty years ago. Recently, new methods to explicitly compute it have been introduced rendering the problem more tractable. An interesting question arising is which Lie algebras can actually appear in this way. The results in this article contribute to an answer to this question. Namely, we give sufficient criteria for the solvability as a Lie algebra of the first degree Hochschild cohomology of A, denoted by $\mathrm{HH}^1(A)$, in terms of its basic algebra A and, more precisely, in terms of its Ext-quiver and its relations. We prove that in many cases only solvable Lie algebras appear.

The Hochschild cohomology $\mathrm{HH}^\bullet(A) = \bigoplus_{n\geq 0} \mathrm{HH}^n(A)$ of A has a very rich structure. It is an associative, graded-commutative algebra with the cup product. It also has a graded Lie bracket of homological degree -1 and both structures are related by the graded Poisson identity. In particular, $\mathrm{HH}^\bullet(A)$ is a Gerstenhaber algebra, $\mathrm{HH}^1(A)$ is a Lie algebra with bracket induced by the usual commutator of derivations and $\mathrm{HH}^\bullet(A)$ is a Lie module for this Lie algebra. All these structures are invariant under derived equivalence [15].

In this paper we study the solvability of the Lie algebra given by the first Hochschild cohomology of a finite dimensional algebra. We develop sufficient conditions on the Ext-quiver with relations so that this Lie algebra is solvable and we give several applications of our methods. In particular, we show that for wild algebras the first Hochschild cohomology can both be solvable or semi-simple (or both). We show that it is solvable for quantum complete intersections for arbitrary parameter and semi-simple for a family of algebras related to Beilinson algebras.

The first condition for solvability is based on the Ext-quiver having no parallel arrows and no loops.

Theorem 1.1. *(see Theorem 3.1) Let A be a finite dimensional K-algebra with no loops in its Ext-quiver and such that $\dim_K(\mathrm{Ext}^1_A(S,T)) \leq 1$, for all nonisomorphic simple A-modules S and T. Then $\mathrm{HH}^1(A)$ is a strongly solvable Lie algebra.*

In particular, if the characteristic of K is zero then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.

The solvability of the first Hochschild cohomology has recently been extensively studied [4, 8, 16] and Theorem 1.1 also appears in [16] with a different proof. Our next main result concerns algebras whose Ext-quiver may have loops.

Theorem 1.2. *(see Theorems 3.4,3.5 and 3.11) Let A be a finite dimensional K-algebra such that the Ext-quiver has at most one loop at each vertex or two loops in the case of the algebra being local. Suppose that there are no derivations in $\mathrm{HH}^1(A)$ sending an arrow to a different parallel arrow.*

(1) *Suppose that $\mathrm{char}(K) = 2$. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*
(2) *Suppose that $\mathrm{char}(K) \neq 2$ and that there are no derivations sending a loop to its square. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

Furthermore, we also show some criteria for the solvability of the first Hochschild cohomology of graded algebras and we apply these to show that the first Hochschild cohomology of any quantum complete intersection is solvable in characteristic zero and supersolvable in characteristic p.

In [19] an extensive if not comprehensive classification of self-injective tame algebras up to derived equivalence has been given. We consider all families of algebras in this classification and we show that for almost all these algebras the first Hochschild cohomology is solvable in arbitrary characteristic. We also show that almost all tame blocks of finite groups have solvable first Hochschild cohomology.

Theorem 1.3. *(see Theorem 4.4) Let K be an algebraically closed field of arbitrary characteristic.*

(1) *Let A be a self-injective finite dimensional K-algebra of tame representation type appearing in the classification in* [19]*. Then $\mathrm{HH}^1(A)$ is solvable.*
(2) *Let A be a symmetric tame algebra of dihedral, semi-dihedral or quaternion type different from dihedral type in characteristic 2 with Klein defect. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

We note that there are two exceptions for the solvability of $\mathrm{HH}^1(A)$ in the first part of the Theorem. Namely, if $\mathrm{char}(K)|r$ and $r \geq 3$, the Lie algebra $\mathrm{HH}^1(K[X]/(X^r))$ is perfect. In particular, if $r = p = \mathrm{char(K)}$ then it is simple and it is the so-called Jacobson–Witt algebra. The second exception is if A is isomorphic to the trivial extension of the Kronecker algebra if $\mathrm{char(K)} \neq 2$, see [4].

The second part of the Theorem shows that almost all blocks of group algebras of finite groups of tame representation type have solvable first Hochschild cohomology. As part of the proof of Theorem 1.3 we also show that the first Hochschild cohomology of a Brauer graph algebra with any multiplicity function is solvable in arbitrary characteristic, that is with the exception of the trivial extension of the Kronecker quiver in characteristic different from 2.

In a recent preprint [8] the authors show the solvability of a Lie subalgebra of HH^1 for symmetric tame algebras. This Lie subalgebra is generated by the derivations that preserve the radical. In positive characteristic proving the solvability of this Lie subalgebra does not necessarily imply the solvability of $\mathrm{HH}^1(A)$. An example of this is given by the Lie algebra of derivations of $K[X]/(X^p)$ in characteristic p which is a simple Lie algebra, however the Lie subalgebra of the derivations preserving the radical is solvable.

The article is structured as follows. Section 2 contains background material on Lie algebras both in characteristic zero and in positive characteristic. It also contains a brief introduction of the Lie algebra structure of the first Hochschild cohomology of a finite dimensional algebra. Section 3 contains the main results of the paper, that is criteria for the solvability of the first Hochschild cohomology of a finite dimensional algebra in terms of its Ext-quiver, the relations on the quiver as well as some results for graded algebras. In Section 4, we apply the results of Section 3 to several families of algebras such self-injective algebras of tame representation type including Brauer graph algebras and blocks of groups algebras of finite groups as well as quantum complete intersections. We end the paper with an example of an infinite family of algebras for which the first Hochschild cohomology is semi-simple.

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Conventions

Let $A = KQ/I$ be a finite dimensional K-algebra, where $Q = (Q_1, Q_0)$ with Q_0 the vertices or Q and Q_1 the arrows in Q. Unless otherwise stated, all modules will be right modules. For an arrow a in Q, we write $s(a)$ for the source of a and $t(a)$ for the target of a. We write $A^e = A \otimes_K A^{op}$.

2. Background material

In this section we collect some background material on modular Lie algebras and the Lie bracket on the first Hochschild cohomology of a finite dimensional algebras (in any characteristic). For this let $\mathcal{L}$ be a Lie algebra and recall that the derived Lie algebra $\mathcal{L}^{(1)}$ is the Lie algebra defined by the linear span of all commutators $[x, y]$ for $x, y \in \mathcal{L}$. We denote by $\mathcal{L}^{(i)} = [\mathcal{L}^{(i-1)}, \mathcal{L}^{(i-1)}]$ the i-th term of the derived series of $\mathcal{L}$.

2.1. Modular Lie algebras.

We begin by collecting some well-known facts about modular Lie algebras. For this assume that the characteristic of K is $p \geq 0$.

Definition 2.1. A Lie algebra $\mathcal{L}$ is *strongly solvable* if its derived subalgebra is nilpotent, and $\mathcal{L}$ is *supersolvable* (also called *completely solvable*) if there exists a sequence of ideals $\mathcal{L} = \mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots \supset \mathcal{I}_n \supset \{0\}$ such that $\dim_k(\mathcal{I}_i) = n + 1 - i$.

Remark 2.2. If the characteristic of K is zero, the classes of strongly and supersolvable Lie algebras coincide with the class of solvable Lie algebras as a consequence of Lie's theorem (see, for example, [13]).

Over a field of positive characteristic Lie's theorem is false. However, a modular analogue holds for strongly solvable Lie algebras.

Theorem 2.3 ([7, Theorem 3])**.** *Let K be a field of characteristic $p > 0$. Then a Lie algebra $\mathcal{L}$ is strongly solvable if and only if $\mathcal{L}$ is supersolvable.*

Furthermore, we have the following.

Lemma 2.4. *Every subalgebra and every factor of a strongly solvable Lie algebra is strongly solvable.*

2.2. The Lie bracket on the first Hochschild cohomology.

For a finite dimensional K-algebra $A = KQ/I$, we briefly recall a construction of the Lie bracket on $HH^1(A)$ as defined in [6]. By the Wedderburn-Malcev theorem, we have $A = E \oplus J(A)$ where $E = KQ_0$ is a maximal semi-simple subalgebra and $J(A)$ the Jacobson radical of A. We denote by $\mathcal{B}$ the basis of A given by images of paths under the canonical map $KQ \to KQ/I$. We will freely refer to elements of $\mathcal{B}$ as paths.

Two paths ϵ and γ in Q are parallel if they have same source and target. If X and Y are sets of paths in Q, define

$$X||Y = \{(\epsilon, \gamma) \in X \times Y | \epsilon \text{ and } \gamma \text{ are parallel}\}.$$

The vector spaces $K(X||Y)$ and $\mathrm{Hom}_{E^e}(KX, KY)$ are isomorphic and we freely denote by $\alpha||\beta$ an element in $K(X||Y)$ and as well as the morphism in $\mathrm{Hom}_{E^e}(X, Y)$ sending α to β and any other basis element to zero.

Next we recall a construction of the Lie bracket on the $\mathrm{HH}^1(A)$ from [6]. Let $\mathcal{R} = (\mathcal{S}, \mathcal{F})$ be a reduction system. Let $\mathcal{R}$ be a minimal generating set of I. Then by [6, Theorems 4.1 and 4.2] the following is the start of an A-bimodule resolution of A

$$\cdots \longrightarrow A \otimes_E K\mathcal{R} \otimes_E A \xrightarrow{d_1} A \otimes_E KQ_1 \otimes_E A \xrightarrow{d_0} A \otimes_E A \longrightarrow 0,$$

with differentials $d_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v$ and, $d_1(1 \otimes r \otimes 1) = \sum_i \lambda_i \sum_{l=1}^{n_i} v_{j_1} \dots v_{j_{l-1}} \otimes v_{j_l} \otimes v_{j_{l+1}} \dots v_{j_{n_i}}$ where $r = \sum_i \lambda_i a_i \in \mathcal{R}$, with $a_i = v_{j_1} \dots v_{j_{n_i}}$ and $v_{j_k} \in Q_1$.

If we apply $\mathrm{Hom}_{E \otimes E^{op}}(-, A)$ to this chain complex, we obtain a cochain complex

$$0 \cdots \longrightarrow K(Q_0||A) \xrightarrow{\delta^0} K(Q_1||A) \xrightarrow{\delta^1} K(\mathcal{R}||A) \longrightarrow \dots,$$

with the induced differentials.

The Gerstenhaber structure is computed for example via comparison morphisms between the resolution described above and the E-reduced Bar resolution, introduced in [5]. The Lie bracket can then be expressed as in [1] by

$$[\alpha||h, \beta||b] = \beta||b^{\alpha||h} - \alpha||h^{\beta||b},$$

with $\alpha||h$, $\beta||b \in \mathrm{Hom}_{E^e}(KQ_1, A)$ and where $b^{\alpha||h}$ is the sum of all the nonzero paths obtained replacing every appearance of α in b by h. If b does not contain the arrow α or if when we replace α in b by h, the all the corresponding paths are zero, then we set $b^{\alpha||h} = 0$.

2.3. **Lie algebras of graded algebras.** Let $A = KQ/I$ with I an admissible ideal generated by homogeneous relations. Following [20], the Lie algebra $K(Q_1||\mathcal{B})$ admits a grading by considering the elements of $K(Q_1||\mathcal{B}_i)$ to have degree $i-1$ for all $i \in \mathbb{N}$. The Lie subalgebra $\mathrm{Ker}\delta^1$ preserves this grading and $\mathrm{Im}(\delta^0)$ is a graded ideal. We thus obtain an induced grading on $\mathrm{HH}^1(A)$. Set

$$\mathcal{L}_0 := K(Q_1||Q_0) \cap \mathrm{Ker}\delta^1$$
$$\mathcal{L}_1 := K(Q_1||Q_1) \cap \mathrm{Ker}\delta^1 / \mathcal{I}_1$$
$$\mathcal{L}_i := K(Q_1||\mathcal{B}_i) \cap \mathrm{Ker}\delta^1 / \mathcal{I}_i$$

where

$$\mathcal{I}_1 = \langle \sum_{a \in Q_1 e} a||a - \sum_{a \in eQ_1} a||a \mid e \in Q_0 \rangle$$

and for all $i > 1$

$$\mathcal{I}_i = \langle \sum_{a \in Q_1 e, \gamma a \in \mathcal{B}} a||\gamma a - \sum_{a \in eQ_1, a\gamma \in \mathcal{B}} a||a\gamma \mid e||\gamma \in Q_0||Q_i \rangle$$

With the above notation $\mathrm{HH}^1(A) = \oplus_{i \geq 0} \mathcal{L}_i$ and $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j-1}$ for all $i, j \geq 0$ where $\mathcal{L}_{-1} = 0$. Set $\mathcal{N} := \oplus_{i \geq 2} \mathcal{L}_i$. Then $\mathcal{N}$ is a nilpotent Lie algebra.

3. Criteria for solvability of $\mathrm{HH}^1(A)$

In this section we prove several criteria that if satisfied, induce the solvability of the first Hochschild cohomology of a finite dimensional K-algebra.

3.1. Algebras without self-extensions of simples.

Theorem 3.1. *Let K be an algebraically closed field and let $A = KQ/I$ be a finite dimensional K-algebra. Suppose that $\mathrm{Ext}^1_A(S,S) = \{0\}$ for every simple A-module S and that $\dim_K(\mathrm{Ext}^1_A(S,T)) \leq 1$ for all nonisomorphic simple A-modules S and T. Then $\mathrm{HH}^1(A)$ is a strongly solvable Lie algebra. Moreover, if the characteristic of K is zero then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

The following are direct consequences of the proof of Theorem 3.1.

Corollary 3.2. *Let $A = KQ/I$ be a finite dimensional algebra such that Q contains no loops and such that the Lie algebra $K(Q_1||\mathcal{B})$ is abelian. Then $\mathrm{HH}^1(A)$ is strongly solvable and if the characteristic of K is zero, then $\mathrm{HH}^1(A)$ is solvable.*

Set $K(Q_1||\mathcal{B}) = \oplus_{i\in\mathbb{N}} K(Q_1||\mathcal{B}_i)$, where $\mathcal{B}_i$ is the subset of $\mathcal{B}$ consisting of paths of length i.

Corollary 3.3. *The Lie algebra $\mathcal{L} := \oplus_{i\geq 2} K(Q_1||\mathcal{B}_i)$ is nilpotent.*

Proof of Theorem 3.1. If $\mathrm{Ext}^1_A(S,S) = \{0\}$ for every simple A-module S, then Q has no loops. Therefore $K(Q_1||\mathcal{B}_0) = \{0\}$. Since $\dim_K(\mathrm{Ext}^1_A(S,T)) \leq 1$ for every simple A-modules S and T, there are no parallel arrows in Q. Equation (2.2) then immediately gives that $K(Q_1||\mathcal{B}_1)$ is an abelian Lie algebra. Consequently the derived subalgebra of $K(Q_1||\mathcal{B})$ is contained in $\oplus_{i\geq 2} K(Q_1||\mathcal{B}_i)$. Furthermore, for $i \geq 2$ we have $\mathcal{B}_i \subseteq J(A)^2$. Let $\alpha||\beta \in K(Q_1||J(A)^i)$ and $\gamma||\delta \in K(Q_1||J(A)^j)$ where $i,j \in \mathbb{N}$. Suppose, $\delta = a_1a_2\dots a_j \in J(A)^j$ and $\beta = b_1b_2\dots b_i \in J(A)^i$. Then $\delta^{\alpha||\beta} = \sum_l a_1\dots a_{l-1}\beta a_{l+1}\dots a_k \in K(Q_1||J(A)^{i+j-1})$ and similarly, $\alpha||\beta^{\gamma||\delta} \in K(Q_1||J(A)^{i+j-1})$. Therefore $[\alpha||\beta, \gamma||\delta] \in K(Q_1||J(A)^{i+j-1})$. In particular, this implies that $\mathcal{L}^{'} \subseteq \oplus_{i\geq 2} K(Q_1||\mathcal{B}_i \cap J(A)^3)$. By iterating this process, since A is finite dimensional, it follows that $\oplus_{i\geq 2} K(Q_1||\mathcal{B}_i)$ is a nilpotent Lie algebra. Subalgebras of nilpotent Lie algebras are nilpotent, therefore $K(Q_1||\mathcal{B})$ is a strongly solvable Lie algebra. As a consequence of Lemma 2.4, the Lie algebras $\mathrm{Ker}(\delta^1)$, $\mathrm{Im}(\delta^0)$ and $\mathrm{HH}^1(A)$ are strongly solvable.

Now suppose that the characteristic of K is zero. Then combining the above, with Remark 2.2, it implies that $K(Q_1||\mathcal{B})$ is a solvable Lie algebra. Every subalgebra and every factor of a solvable Lie algebra is solvable and the result follows. □

We note that a different proof of Theorem 3.1 appears in [16].

3.2. Algebras with self-extensions of simples.

Let us now consider the case $\mathrm{Ext}^1_A(S,S) \neq \{0\}$, that is, when the quiver has loops.

For $i \in \mathbb{N}$, define the set

$$\Sigma_i := \{\alpha||\beta \in Q_1||\mathcal{B}_i \mid \alpha||\beta = \alpha_j||\beta_j \text{ for some } x = \sum \lambda_j\alpha_j||\beta_j \in \mathrm{Ker}(\delta^1) \text{ with } \lambda_j \in K^*\}.$$

We say that $\alpha||\beta, \gamma||\epsilon \in \Sigma_i$ are equivalent whenever $\alpha||\gamma$. This defines an equivalence relation on Σ_i. We denote by $\overline{\Sigma}_i$ its set of equivalence classes. By abuse of notation we denote the class of an element $\alpha||\beta$ by the same symbol. We call a class $\alpha||\beta \in \overline{\Sigma}_i$ nontrivial if it contains at least two elements.

We now state our first result on algebras with self-extensions of simples.

Theorem 3.4. *Let $A = KQ/I$ be a finite dimensional K-algebra such that $\mathrm{dim_K Ext^1_A(S,S)} \leq 1$ for every simple A-module S. Suppose that every class in $\overline{\Sigma}_1$ is trivial and that $\alpha||\alpha^2$ is not in Σ_2, for all $\alpha \in \mathcal{B}_0$. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

Before proving Theorem 3.4, we give some preliminary results and state a consequence in characteristic 2.

Corollary 3.5. *Let $A = KQ/I$ be a finite dimensional K-algebra over an algebraically closed field of characteristic 2 such that $\mathrm{dim_K Ext^1_A(S,S)} \leq 1$ for every simple A-module S. Suppose that every class in $\overline{\Sigma}_1$ is trivial. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

Proof. Since the characteristic of K is 2, we have for any loop α at vertex e that $[\alpha||e, \alpha||\alpha^2] = 0$. Therefore we can apply the same arguments as in the theorem and the result follows. □

Proposition 3.6. *Let A be a finite dimensional basic K-algebra. Suppose Σ_0 is empty and that every class in $\overline{\Sigma}_1$ is trivial. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

Proof. Using the description of the bracket, every element in the derived algebra of $\mathrm{HH}^1(A)$ has as summands elements in Σ_i, for $i \geq 2$. We denote by $\mathcal{N}$ the Lie algebra generated by these summands. By Corollary 3.3 we have that $\mathcal{N}$ is nilpotent. Note that the derived subalgebra of $\mathrm{Ker}(\delta^1)$ is a Lie subalgebra of $\mathcal{N}$, therefore it is solvable. Since quotients of solvable Lie algebras are solvable, it follows that $\mathrm{HH}^1(A)$ is solvable. □

Proof of Theorem 3.4 Let $\mathcal{L}$ be the Lie algebra generated by the elements in Σ_{i+1}, for all $i \in \{-1, \dots, n\}$ and for some for $n \in \mathbb{N}$. Note that since A is finite dimensional such n exists. Then $\mathcal{L}$ is a graded Lie algebra and can be written as $\mathcal{L} = \oplus_{i=-1}^{n} \mathcal{L}_i$ where $\mathcal{L}_i$ is generated by Σ_{i+1}. If $\mathcal{L}_{-1} = 0$, the statement follows from Proposition 3.6. So assume now that $\mathcal{L}_{-1} \neq 0$. Then $\alpha||\alpha$ is not in $\mathcal{L}^{(1)}$ since the only way to obtain it, is as $[\alpha||e, \alpha||\alpha^2]$. Then $\mathcal{L}^{(2)}$ does not contain $\alpha||e$ since the only way to obtain it is as $[\alpha||e, \alpha||\alpha]$. Thus the Lie algebra $\mathcal{L}^{(2)}$ satisfies the hypotheses of Proposition 3.6 and $\mathcal{L}^{(2)}$ is solvable. Consequently $(\mathcal{L}^{(2)})^{(s)} = \mathcal{L}^{(s+2)} = 0$, for all $s \geq 0$. Therefore $\mathcal{L}$ and hence $\mathrm{HH}^1(A)$ are solvable. □

The next result is similar but only applies to algebras which are not symmetric.

Proposition 3.7. *Let $A = KQ/I$ be a finite dimensional K-algebra such that $\mathrm{dim_K Ext^1_A(S,S)} \leq 1$ for every simple A-module S. Suppose that every class in $\overline{\Sigma}_1$ is trivial and that there are no paths parallel to any of the loops other than the loop itself. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

Proof. Let $\mathcal{L}$ be the Lie algebra generated by the elements in Σ_i for all $i \in \{0, \dots, n\}$. If $\mathrm{char}(K) = 0$, then every derivation preserves the radical. Thus $\alpha||\alpha^2 \notin \Sigma_2$ for all loops α and the result follows from Theorem 3.4. Let $\mathrm{char}(K) = p$ for some prime number p. If $p = 2$, the result follows from Theorem 3.5. Now suppose p is odd. Then since α is not parallel to any path in Q, the only potential nonzero derivations in Σ_i that send α to an element in $\mathcal{B}_i$ are $\alpha||\alpha^i$. Therefore the derivation $\alpha||\alpha^{p-1}$ is not in $\mathcal{L}^{(1)}$. Iterating this process it is easy to check that $\alpha||e$ is not in $\mathcal{L}^{(p+1)}$. So $\mathcal{L}^{(p)}$ is solvable and consequently $\mathcal{L}$ is solvable. The statement follows. □

Remark 3.8. The hypotheses in Proposition 3.7 imply that the Cartan matrix of A cannot be symmetric [9, I.6.1].

3.3. **Local algebras.**

We finish this section by considering the case when A is a local algebra.

Proposition 3.9. *Let $A = KQ/I$ be a local finite dimensional K-algebra such that $\Sigma_0 = \emptyset$. Suppose Q has two loops α_1 and α_2. If $\alpha_1||\alpha_2$ (or $\alpha_2||\alpha_1$) is not in Σ_1, then $\mathrm{HH}^1(A)$ is solvable.*

Proof. Let $\mathcal{L}$ be Lie algebra generated by the elements in Σ_i for all $i \in \{0, \dots, n\}$. Then $\mathcal{L}$ is a graded Lie algebra and can be written as $\oplus_{i=-1}^n \mathcal{L}_i$ where $\Sigma_i = \mathcal{L}_{i-1}$. Since $\Sigma_0 = \emptyset$, we have $\mathcal{L}_{-1} = 0$. Because of the grading, $\mathcal{L}_1^{(1)}$ is abelian (it only contains $\alpha_2||\alpha_1$ or $\alpha_1||\alpha_2$) and consequently $\mathcal{L}^{(2)}$ is solvable. The statement follows. □

Remark 3.10. Proposition 3.9 can be extended to n loops $\alpha_1, \dots, \alpha_n$ under the assumption that $\alpha_i||\alpha_j$ are not in Σ_1 for $1 \le i < j \le n$. Then an analogous argument shows that $\mathcal{L}_1^{(n-1)}$ is abelian.

The next result is for the more general case that Σ_0 is not empty.

Theorem 3.11. *Let $A = KQ/I$ be a local finite dimensional K-algebra with two loops α_1 and α_2 such that $\alpha_1||\alpha_2$ and $\alpha_2||\alpha_1$ are not in Σ_1. Then*

(1) *if $\mathrm{char}(K) = 2$, then $\mathrm{HH}^1(A)$ is solvable.*
(2) *if $\mathrm{char}(K) \neq 2$ and $\alpha_i||\alpha_i^2 \notin \Sigma_2$ for $i \in \{1, 2\}$, then $\mathrm{HH}^1(A)$ is solvable.*

Proof. Let $\mathcal{L}$ be the Lie algebra generated by the elements in Σ_i for all $i \in \{0, \dots, n\}$. Then $\mathcal{L}$ is a graded Lie algebra and can be written as $\oplus_{i=-1}^n \mathcal{L}_i$ where $\Sigma_i = \mathcal{L}_{i-1}$. If Σ_0 is empty then the statement follows from Proposition 3.9. So assume that Σ_0 is not empty.

(1) Firstly, we consider the case when both $\alpha_1||e$ and $\alpha_2||e$ are in Σ_0. Then the Lie algebra $\mathcal{L}$ does not contain $\alpha_i||\alpha_i\alpha_j$ and $\alpha_i||\alpha_j\alpha_i$ for $i, j \in \{1, 2\}$, $i \neq j$, because otherwise the derived subalgebra contains $[\alpha_i||e, \alpha_i||\alpha_i\alpha_j] = \alpha_i||\alpha_j$ and $[\alpha_j||e, \alpha_j||\alpha_i\alpha_j] = \alpha_j||\alpha_i$ which are not in $\mathcal{L}$ by hypothesis. Hence $\mathcal{L}^{(1)}$ does not contain $\alpha_i||\alpha_i$ for $i = \{1, 2\}$ since the only way to obtain them is through the bracket $[\alpha_i||\alpha_i^2, \alpha_i||e]$ for $i \in \{1, 2\}$ (note that the rest of the elements in $\mathcal{L}_2$ are zero). The Lie algebra $\mathcal{L}^{(2)}$ does not contain $\alpha_i||e$ for $i = \{1, 2\}$ since the only way to obtain them is through the bracket $[\alpha_i||\alpha_i, \alpha_i||e]$ for $i \in \{1, 2\}$ but $\alpha_i||\alpha_i$ are not in $\mathcal{L}^{(1)}$. Therefore the Lie algebra $\mathcal{L}^{(2)}$ is solvable and consequently $\mathcal{L}$ is solvable.

The only case left to consider is when $\alpha_1||e \in \Sigma_0$ and $\alpha_2||e \notin \Sigma_0$ (or viceversa). Using the same argument as the previous paragraph, $\alpha_1||\alpha_1\alpha_2$ and $\alpha_1||\alpha_2\alpha_1$ are not in $\mathcal{L}$ because otherwise the derived subalgebra has the element $[\alpha_1||e, \alpha_1||\alpha_1\alpha_2] = [\alpha_1||e, \alpha_1||\alpha_2\alpha_1] = \alpha_1||\alpha_2$ which is not in $\mathcal{L}$ by hypothesis. Therefore the derivation $\alpha_1||\alpha_1$ is not in the derived subalgebra $\mathcal{L}^{(1)}$ since the only way to obtain $\alpha_1||\alpha_1$ is through the bracket $[\alpha_1||\alpha_1^2, \alpha_1||e]$ and the rest of the elements in $\mathcal{L}_1$ are zero. The Lie algebra $\mathcal{L}^{(2)}$ does not contain $\alpha_1||e$ since the only way to obtain $\alpha_1||e$ is as $[\alpha_1||\alpha_1, \alpha_1||e]$ but we have seen that $\alpha_1||\alpha_1$ is not in $\mathcal{L}^{(1)}$. Therefore, by Proposition 3.9 the Lie algebra $\mathcal{L}^{(2)}$ is solvable and consequently $\mathcal{L}$ is solvable.

(2) The argument is analogous to that in (1). □

3.4. **Graded Algebras.** Let $A = KQ/I$ be a graded algebra. We adopt the notation of Section 2.3. Furthermore, from now on we suppose that $\mathcal{L}_0 = 0$. In the remaining part of the section we provide some sufficient conditions which lead to a method to verify the (super)solvability of $\mathrm{HH}^1(A)$. Since $\mathcal{L}_0 = 0$, in order to investigate the (super)solvability of $\mathrm{HH}^1(A)$ it is enough to look at $\mathcal{L}_1$.

First note that every element in $\mathcal{L}_1$ can be written as $\sum_{\alpha||\beta} \lambda_{\alpha,\beta}\alpha||\beta$ where $\alpha||\beta \in Q_1||Q_1$ and $\lambda_{\alpha,\beta} \in K$. In the monomial case the elements of the form $\alpha||\alpha$ are always in $\mathrm{Ker}(\delta^1)$ which is not true in general for graded algebras. Let $\sum_{\gamma||\epsilon} \mu_{\gamma,\epsilon}\gamma||\epsilon$ in $\mathcal{L}_1$ where $\gamma||\epsilon \in Q_1||Q_1$ and $\mu_{\gamma,\epsilon} \in K$. Then

$$[\sum_{\alpha||\beta} \lambda(\alpha||\beta), \sum_{\gamma||\epsilon} \mu(\gamma||\epsilon)] = \sum_{\alpha||\beta}\sum_{\gamma||\epsilon} \lambda\mu([\alpha||\beta, \gamma||\epsilon]) = \sum_{\alpha||\beta,\gamma||\epsilon} \lambda\mu\ (\delta_{\alpha,\epsilon}\gamma||\beta - \delta_{\beta,\gamma}\alpha||\epsilon)$$

where $\delta_{\alpha,\epsilon}$ and $\delta_{\beta,\gamma}$ are Kronecker coefficient and where for the sake of simplicity we have written the scalars without subscripts.

We use the same notation for the sets Σ_i and $\overline{\Sigma}_i$ as defined in the beginning of Section 3.2. For each equivalence class $\alpha||\beta \in \overline{\Sigma}_1$ we define $\overline{\mathcal{L}}_1^{\alpha||\beta}$ to be the Lie algebra generated by the elements in this class. We denote

$$\overline{\mathcal{L}}_1 := \prod_{\alpha||\beta\in\overline{\Sigma}_1} \overline{\mathcal{L}}_1^{\alpha||\beta}$$

the product of these Lie algebras.

Proposition 3.12. *If $\overline{\mathcal{L}}_1$ is a (super)solvable Lie algebra, then $\mathcal{L}_1$ is a (super)solvable Lie algebra.*

Proof. By construction $\mathcal{L}_1 \subset \overline{\mathcal{L}}_1$. Since $\overline{\mathcal{L}}_1$ is a (super)solvable then $\mathcal{L}_1$ is a (super)solvable Lie algebra. □

Remark 3.13.
(1) When we want to check the (super)solvability of $\mathcal{L}_1$, and consequently of $\mathrm{HH}^1(A)$, we just need to consider the nontrivial classes in $\overline{\Sigma}_1$. If every element $\alpha||\beta$, $\alpha \neq \beta$, in a nontrivial class satisfy $\beta||\alpha$ is not in $\overline{\Sigma}_1$, then $\overline{\mathcal{L}}_1$ is (super)solvable. This follows from [20, Lemma 4.9].
(2) If $\overline{\Sigma}_1$ contains only classes of the form $\alpha||\alpha$, then $\overline{\mathcal{L}}_1$, and consequently $\mathcal{L}_1$, are abelian Lie algebras.

4. Applications

In this section we prove the solvability of $\mathrm{HH}^1(A)$ for the self-injective tame algebras A classified in [19], as an application of the results previously stated. As usual, throughout this section K is an algebraically closed field.

4.1. **Dihedral, semi-dihedral and quaternion.** We first focus on symmetric tame algebras of dihedral, semi-dihedral and quaternion type, studied and classified up to Morita equivalence and up to scalars in [9]. This classification has been extended up to derived equivalences in [12] and more recently most of the algebras of dihedral, semi-dihedral and quaternion have been distinguished up to stable equivalence of Morita type [23]. For algebras of dihedral type the classification has been completed in [22].

Theorem 4.1. *Let K be an algebraically closed field of arbitrary characteristic. Let A be a symmetric tame algebra of dihedral, semi-dihedral or quaternion type different from dihedral type in characteristic* 2 *with Klein defect. Then* $\mathrm{HH}^1(A)$ *is a solvable Lie algebra.*

Proof. We organise the proof by the number of simple modules, and within each case we consider the dihedral, semi-dihedral and quaternion cases. The strategy is to show that we can use either Proposition 3.9, or, in characteristic 2, Theorem 3.11.

In the local algebra case the quiver Q has two loops X and Y and we denote the single vertex by e.

In dihedral type we have that A is either:

(1) $K[X,Y]/(XY, X^m - Y^n)$: for $m \geq n \geq 2, m+n > 4$,
(2) $D(1\mathcal{A})_1^1 = K[X,Y]/(X^2, Y^2)$,
(3) $D(1\mathcal{A})_1^k$, $k \geq 2$;
and when $\mathrm{char}(K) = 2$, there are two more cases:
(4) $K[X,Y]/(X^2, YX - Y^2)$,
(5) $D(1\mathcal{A})_2^k(d), k \geq 2, d \in \{0,1\}$.

In case (1), note that the derivations $X||e$, $Y||e$, $X||Y$ and $Y||X$ do not belong to Σ_i for $i = 0,1$, respectively, and the result follows from Theorem 3.11. In Case (2), we have that $\mathrm{HH}^1(A)$ is a Witt–Jacobson algebra which is a simple Lie algebra of Cartan type [14]. In case (3), $X||Y$ and $Y||X$ do not belong to Σ_1 and if $\mathrm{char}(K) \neq 2$, then $X||e$ and $Y||e$ do not belong to Σ_0 and we are done using Proposition 3.9. If $\mathrm{char}(K) = 2$, then we use Theorem 3.11. In case (4) it is enough to verify that $X||e$ and $Y||e$ c and that the Lie algebra generated by $Q_1\tilde{||}\mathcal{B}_1$ has K-basis $\{X||X + Y||Y, X||X + X||Y, Y||X\}$ and its derived algebra is one dimensional and therefore $\mathrm{HH}^1(K[X,Y]/(X^2, YX - Y^2))$ is solvable. Finally, case (5) is a consequence of the fact that $X||e$ and $Y||e$ are not in Σ_0, and $X||Y$ is not in Σ_1.

Next we consider local algebras of semi-dihedral type. In this situation, there are two possibilities, one of them only in characteristic 2.

(1) $SD(1\mathcal{A})_1^k, k \geq 2$. It is easy to verify that $X||Y$ and $Y||X$ do not belong to Σ_1 In addition, if $\mathrm{char}(K) \neq 2$, $X||e$ and $Y||e$ are neither in Σ_0. Hence the solvability of $\mathrm{HH}^1(A)$ follows from Proposition 3.9 if the characteristic of K is different from 2, otherwise it follows from Theorem 3.11.
(2) $SD(1\mathcal{A})_2^k(c,d)$, $\mathrm{char}(K) = 2$, $k \geq 2$, $(c,d) \neq (0,0)$. For $c, d \neq 0$, the derivations $X||Y$ and $Y||X$ are not in Σ_1, and solvability follows from Theorem 3.11. For $c = 0$ and for $d = 0$ the proof is analogous.

Finally, we consider local algebras of quaternion type. Again, there are two possibilities, one of them only in characteristic 2.

(1) $Q(1A)_1^k$, $k \geq 2$. Here, $X||Y$ and $Y||X$ are not in Σ_1. Moreover, if $\mathrm{char}(K) \neq 2$, we deduce that $X||e$ and $Y||e$ are not in Σ_0. The solvability of $\mathrm{HH}^1(A)$ follows from Proposition 3.9. For $\mathrm{char}(K) = 2$, we use Theorem 3.11.

(2) $Q(1\mathcal{A})_2^k(c,d)$, $\mathrm{char}(K) = 2$, $k \geq 2$, $(c,d) \neq (0,0)$. By length reasons, $X||e$ and $Y||e$ are not in Σ_0, and it is easy to see that $X||Y$ is not in Σ_1. For $c, d = 0$, the situation does not change. We apply Theorem 3.11.

Up to derived equivalence, there are four families of symmetric tame algebras with two simple modules and they all have the same quiver. We denote the two loops by α and η having source e_0 and e_1, respectively.

In characteristic 2 the solvability of $\mathrm{HH}^1(A)$ follows from Cor. 3.5, so it is enough to consider the case $\mathrm{char}(K) \neq 2$.

(1) $D(2\mathcal{B})^{k,s}(c)$. Again, by length reasons, $\eta||e_1$ and $\alpha||e_0$ are not in Σ_0, and the statement follows from Proposition 3.6.

(2) $SD(2\mathcal{B})_1^{k,t}(c)$. Analogously to case $D(2B)^{k,s}(c)$, we deduce that $\eta||e_1$ is not in Σ_0, and neither does $\alpha||e_0$. We conclude as before.

(3) $SD(2\mathcal{B})_2^{k,t}(c)$. The elements $\eta||e_0$ and $\alpha||e_0$ are not in in Σ_0. Again we use Proposition 3.6.

(4) $Q(2\mathcal{B})_1^{k,s}(a,c)$. The reasoning is similar to the previous one.

For symmetric tame algebras with 3 simple modules, the solvability of HH^1 for algebras of type $3\mathcal{K}$ and of type $3\mathcal{A}$ follows from Theorem 3.1. For the algebra $D(3\mathcal{R})^{k,s,t,u}$ the results holds in case $\mathrm{char}(K) = 2$ by Corollary 3.5 and in any other characteristic, from Proposition 3.9. □

Corollary 4.2. *Suppose* $\mathrm{char}(K) = 2$ *and let* A *be a tame nonlocal symmetric algebra. If* $\dim_K(\mathrm{Ext}_A^1(S,T)) \leq 1$ *for all simple* A*-modules* $S \neq T$, *then* $\mathrm{HH}^1(A)$ *is solvable.*

Proof. Since A is tame symmetric $\dim_K(\mathrm{Ext}_A^1(S,S)) \leq 1$. The statement follows from Corollary 3.5. □

4.2. **Brauer graph algebras.** In this short subsection we prove the solvability of $\mathrm{HH}^1(A)$ for A a Brauer graph algebra different from the trivial extension of the Kronecker algebra. In the latter case, $\mathrm{HH}^1(A)$ is isomorphic to $\mathfrak{gl}_2(K)$ which is not solvable except in characteristic 2.

The solvability of the first Hochschild cohomology of Brauer graph algebras with multiplicity function identically equal to one and different from the trivial extension of the Kronecker algebra has been shown in [4]. We now prove that the first Hochschild cohomology of any Brauer graph algebra with any multiplicity function (apart from the trivial extension of the Kronecker algebra) is solvable in any characteristic.

Theorem 4.3. *Let* B *be Brauer graph algebra different from the trivial extension of the Kronecker algebra if the characteristic of* K *is not 2. Then* $HH^1(B)$ *is a solvable Lie algebra.*

Proof. The first step is to show that if B is a Brauer graph algebra different from the trivial extension of the Kronecker algebra, then $\overline{\Sigma}_1$ is trivial. Since B is a Brauer graph algebra, there are at most two arrows starting and ending at each vertex of the quiver Q. Given two parallel arrows α_1 and α_2 in Q, there exist β_i and γ_i such that $\alpha_i\beta_i \notin I$ and $\gamma_i\alpha_i \notin I$ for $i \in \{1,2\}$. Since B is special biserial, we infer that $\overline{\Sigma}_1$ is trivial. The only exception is when the Brauer graph has exactly two vertices and two edges. If the valencies at each vertex are 1, then B is the trivial extension of the Kronecker algebra. Let the valency at each vertex be m, n respectively, such that $m+n > 2$ and let α_1, α_2,β_1, β_2 be respectively the two pairs of parallel arrows. It is easy to show that two Type III relations are given by $\beta_2\alpha_1$ and $\alpha_2\beta_1$. In addition $\alpha_2\alpha_1$, $\beta_2\beta_1 \notin I$. Therefore the derivations $\alpha_1||\alpha_2$ and $\beta_1||\beta_2$ are not in Σ_1.

The next step is to apply the results from Section 3. More precisely, if there are no loops in the quiver, then the statement follows from Proposition 3.6. If there are loops and if $\mathrm{char}(K) = 2$, then from Theorem 3.11 we deduce that $\mathrm{HH}^1(A)$ is solvable. In any other characteristic, we proceed as follows. We note that there are two types of loops in Ext quiver of a Brauer graph algebra, say ϵ and α, with vertices e_1 and e_2, respectively. They are induced, respectively, from vertices with multiplicities greater than 1 and from loops of the Brauer graph. The former appears in Type III relations in the form $\alpha\epsilon$ or $\epsilon\alpha$ for some arrow α. Therefore the derivation $\epsilon||e_1$ is not in Σ_1. The latter also occurs in the Type III relations as $\alpha^2 = 0$. Since $\mathrm{char}(K) \neq 2$, the derivation $\alpha||e_2$ does not belong to Σ_0. The statement follows from Proposition 3.6. □

4.3. **Self-injective tame algebras.** In this subsection A is a self-injective tame algebra which is in the classification of Skowroński's survey paper in [19]. Our aim is to prove the following.

Theorem 4.4. *Let A be a self-injective tame algebra that appears in the classification in* [19] *different from $K[X]/(X^r)$ when $\mathrm{char}(K) \mid r$ and different from the trivial extension of the Kronecker algebra if $\mathrm{char(K)} \neq 2$. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

We start by recalling the following result from [19] (using the notation in that paper), which provides the derived equivalent classes of algebras of nonsimple connected symmetric algebras of finite type.

Theorem 4.5 ([19])**.** *The algebras $N_e^{em}, m \geq 2, e \geq 1$, $D(m), m \geq 2$, $T(K\Delta(A_n))$ $n \geq 1$, $T(K\Delta(D_n))$ $n \geq 4$, $T(K\Delta(E_n))$, $6 \leq n \leq 8$ and $D^{'}(m)$, $m \geq 2$ and $\mathrm{char}(K) = 2$, form a complete family of representatives of the derived classes of the nonsimple connected symmetric algebras of finite type.*

Proposition 4.6. *Let A be as in Theorem 4.5, excluding Nakayama algebras N_e^{em} where $e = 1$ and $\mathrm{char}(K)$ divides $m+1$. The Lie algebra $\mathrm{HH}^1(A)$ is solvable.*

Proof. Let $N_e^{em}, m \geq 2, e \geq 1$ be a Nakayama algebra with e vertices and such that all the compositions of $em+1$ consecutive arrows generate the admissible ideal. If $e = 1$, then $N_e^{em} \cong K[x]/(x^{m+1})$. If p divides $m+1$, then $\mathrm{HH}^1(K[x]/(x^{m+1}))$ is a perfect Lie algebra, therefore not solvable. If p does not divide $m+1$, then the derivation $x||e$ is not in Σ_0 and so by Proposition 3.6 we have that $\mathrm{HH}^1(N_e^{em})$ is a solvable Lie algebra. If $e > 1$ the statement follows from Theorem 3.1.

The trivial extension algebra $T(K\Delta(A_n))$ for $n \geq 1$ is the Nakayama algebra N_n^n therefore the solvability of $\mathrm{HH}^1(T(K\Delta(A_n))$ follows from the previous paragraph. For the other self-injective algebras of Dynkin type, we proceed as follows. The algebras $D(m)$ and $D^{'}(m)$ have the same Ext-quiver which has only one loop, say α, at a vertex e, and $\dim_K(\mathrm{Ext}_A^1(S,T)) \leq 1$ for every two simple modules S and T. Since $D^{'}(m)$ for $m \geq 2$ is defined only in characteristic 2, the solvability of $\mathrm{HH}^1(D^{'}(m))$ follows from Theorem 3.11. For $D(m)$, it is enough to verify that the derivation $\alpha||e$ is not in Σ^0. The solvability follows then from Proposition 3.6. From Theorem 3.1 we obtain the solvability of $\mathrm{HH}^1(T(K\Delta(D_n)))$ and of $\mathrm{HH}^1(T(K\Delta(E_n)))$. □

The description of self-injective algebra of Euclidean type given in [19] is as follows.

Theorem 4.7 ([19]). *Let A be a self-injective algebra of Euclidean type. The following are equivalent*

- *A is symmetric and has nonsingular Cartan matrix.*
- *A is derived equivalent to an algebra of the form $A(p,q)$, $\Lambda(n)$ or $\Gamma(n)$.*

Proposition 4.8. *The first Hochschild cohomology space of any symmetric algebra A of Euclidean type with nonsingular Cartan matrix is a solvable Lie algebra.*

Proof. Such an algebra is a Brauer graph algebra, therefore the statement follows from Theorem 4.3. □

In order to describe what happens when A is a self-injective symmetric algebra of Euclidean type with singular Cartan matrix we need a preliminary result.

Proposition 4.9 ([19]). *Let A be a self-injective symmetric algebra of Euclidean type with singular Cartan matrix. There exists an Euclidean canonical algebra C such that A is isomorphic to the trivial extension $T(C)$.*

Proposition 4.10. *The first Hochschild cohomology space of any self-injective symmetric algebra A of Euclidean type with singular Cartan matrix is a solvable Lie algebra.*

Proof. We already know that A is isomorphic to the trivial extension $T(C)$ of an Euclidean canonical algebra C. Let $C(2,3,3)$ be the canonical algebra with three parallel branches (α_1,α_2), $(\beta_1,\beta_2,\beta_3)$ and $(\gamma_1,\gamma_2,\gamma_3)$. Its trivial extension has two more arrows m_β, m_γ such that $s(m_\beta) = s(m_\gamma) = t(\alpha_2)$ and $t(m_\beta) = t(m_\gamma) = s(\alpha_1)$. Note that $\beta_3 m_\gamma$ and $\gamma_3 m_\beta$ belong to the admissible ideal I, while $\beta_3 m_\beta$ and $\gamma_3 m_\gamma$ are not in I. Therefore the derivations $m_\beta||m_\gamma$ and $m_\gamma||m_\beta$ are not in Σ_1. From Proposition 3.6, $\mathrm{HH}^1(T(C(2,3,3)))$ is solvable. The proof is analogous for the algebras $C(2,3,4)$ and $C(2,3,5)$ and $C(2,2,r)$ with $r \geq 2$. □

Next we consider self-injective algebras of tubular type. We first recall the following two results.

Theorem 4.11 ([19]). *Let A be a standard weakly symmetric algebra of tubular type with nonsingular Cartan matrix. Then A is derived equivalent to an algebra of the form $A_1(\lambda)$, $A_2(\lambda)$ with $\lambda \in K\ \{0,1\}$, A_3,A_4, A_5 or A_{12}.*

Theorem 4.12 ([19])**.** *Let A be a nonstandard nondomestic weakly symmetric algebra of polynomial growth with nonsingular Cartan matrix. Then A is derived equivalent to an algebra of the form Λ_1, $\Lambda_3(\lambda)$ where $\lambda \in K\ \{0,1\}$, Λ_4 or Λ_9.*

Proposition 4.13. *Let A be an algebra either as in Theorem 4.11 or as in Theorem 4.12. The Lie algebra $\mathrm{HH}^1(A)$ is solvable.*

Proof. For the algebras $A_1(\lambda)$,A_3,A_4, A_{12}, Λ_4 and Λ_9 the solvability of the first Hochschild cohomology space follows from Theorem 3.1, while for $A_2(\lambda)$, A_5, $\Lambda_1(\lambda)$, $\Lambda_3(\lambda)$ in $\mathrm{char}(K) = 2$ it is a consequence of Cor. 3.5. In any other characteristic, we need to do some checks. We start with $A_2(\lambda)$ by observing that the derivations $\alpha||e_1$ and $\beta||e_2$ are not in Σ_0. Similarly, we deduce the solvability of $\mathrm{HH}^1(\Lambda_3(\lambda))$. For A_5, we deduce that $\alpha||e$ is not in Σ_0 and we apply the same argument to Λ_1. □

The case where A has singular Cartan matrix is covered by the following results. Let us recall the following theorem.

Theorem 4.14 ([19])**.** *Let A be a self-injective algebra. The following statements are equivalent:*

- *A is symmetric of tubular type and has singular Cartan matrix.*
- *A is derived equivalent to the trivial extension of a canonical tubular algebra.*

Proposition 4.15. *Let A be derived equivalent to the trivial extension of a canonical tubular algebra. Then $\mathrm{HH}^1(A)$ is solvable.*

Proof. Let A be derived equivalent to the trivial extension $T(C) = KQ/I$ of a canonical tubular algebra C. Let $C(2,4,4)$ be the canonical algebra with three parallel branches (α_1, α_2), $(\beta_1, \beta_2, \beta_3, \beta_4)$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Its trivial extension has two more arrows m_β, m_γ such that $s(m_\beta) = s(m_\gamma) = t(\alpha_2)$ and $t(m_\beta) = t(m_\gamma) = s(\alpha_1)$. Note that $\beta_4 m_\gamma$ and $\gamma_4 m_\beta$ belong to the admissible ideal I, while $\beta_4 m_\beta$ and $\gamma_4 m_\gamma$ are not in I. As a consequence, the derivations $m_\beta||m_\gamma$ and $m_\gamma||m_\beta$ are not in Σ_1. Using Proposition 3.6, we conclude that $\mathrm{HH}^1(T(C(2,4,4)))$ is solvable. An analogous proofs holds for the algebras $C(3,3,3)$, $C(2,3,6)$ and $C(2,2,2,2,\lambda)$ for $\lambda \in K\ \{0,1\}$. □

The structure of arbitrary standard self-injective algebras of polynomial growth is described by the following theorem.

Theorem 4.16 ([19])**.** *Let A be a nonsimple basic connected self-injective algebra. The algebra A is standard of polynomial growth if and only if A is isomorphic to a self-injective algebra of Dynkin type, Euclidean type or tubular type.*

Another class of tame self-injective algebras that is in the classification in [19] are quaternion type. For these the solvability of HH^1 follows from Theorem 4.1.

4.4. **Quantum complete intersections.** The Hochschild cohomology of quantum complete intersections has been extensively studied, see for example [2, 3, 10, 18]. In this section we apply Proposition 3.12 together with Remark 3.13 to calculate the Lie algebra structure of the first Hochschild cohomology of arbitrary quantum complete intersection, that is of rank r with nonzero arbitrary parameter. In particular, it follows from our result that to a certain degree this Lie algebra structure is independent of the rank and the parameters.

Recall that KQ/I is a quantum complete intersection of rank r, if

$$KQ/I = K\langle X_1,\dots,X_r\rangle/\langle X_iX_j - q_{ij}X_jX_i, X_i^{n_i}, \text{ for } 1\le i,j\le r,\rangle$$

where if the characteristic of K is zero we set $n_i \in \mathbb{N}_{\ge 2}$ and if the characteristic of K is p, we set $n_i = p^{m_i}$ for $m_i \in \mathbb{N}_{\ge 2}$.

Proposition 4.17. *Let A be a quantum complete intersection of rank r.*

(1) *Suppose the characteristic of K is zero. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*
(2) *Suppose the characteristic of K is odd. Then $\mathrm{HH}^1(A)$ is a supersolvable Lie algebra.*

Proof. We write the relations $\rho_{ij} = (X_iX_j, q_{ij}X_jX_i)$ and $\rho_{ii} = (X_i^{p^{n_i}}, 0)$, for $1 \le i < j \le r$, where $n_i \ge 1$ are integers. The first step in order to apply Proposition 3.12 is to show that $\mathcal{L}_0 = 0$. The term of the differential of $X_i||e_0$ given by $\rho_{ii+1}||X_{i+1} - q_{ii+1}\rho_{ii+1}||X_{i+1}$ is never in the kernel. The only element that would have a nonzero term for that relation is $X_{i+1}||e_0$ and is given by $\rho_{ii+1}||X_i - q_{ii+1}\rho_{ii+1}||X_i$. Since $X_i \ne X_{i+1}$ the previous statement follows. By Remark 3.13 it is enough to check that $X_i||X_j$ is not in $\overline{\Sigma}_1$ for $i \ne j$. If we apply δ^1 to $X_i||X_j$, then one of the nonzero elements in the image is given by $\rho_{ij}||X_j^2 - q_{ij}\rho_{ij}||X_j^2$. The only element that has a nonzero image for these relations is $X_j||X_i$ which gives $\rho_{ij}||X_i^2 - q_{ij}\rho_{ij}||X_i^2$. Since $X_i \ne X_j$, it follows that $\mathrm{HH}^1(A)$ is a supersolvable Lie algebra and both (1) and (2) follow. Note that in the characteristic zero case, if $\rho_{ii} = (X_i^2, 0)$ for some i, then it is enough to notice that in the image of $X_i||X_j$, we have $\rho_{ii}||X_jX_i + X_iX_j$.

□

4.5. **Examples of algebras with nonsolvable first Hochschild cohomology.** In this section we give two examples of families of algebras for which the Lie algebra given by the first Hochschild cohomology is not solvable but rather semi-simple. These examples generalise the example of the Kronecker algebra given in [4], see also [8].

The first family consists of monomial algebras with radical square zero over a field of arbitrary characteristic. Let $n, m \in \mathbb{N}_{\ge 1}$. We denote by $Q_{n,m}$ the quiver with n vertices $1,\dots,n$ and for each consecutive pair of vertices $(i, i+1)$ there are m parallel arrows denoted by $\alpha_{i,1},\dots\alpha_{i,m}$ from i to $i+1$. For $n = 2$, the quiver $Q_{2,m}$ is the m-Kronecker quiver. Set $A_{n,m} = KQ_{n,m}/\mathrm{rad}^2 KQ_{n,m}$.

Proposition 4.18. *There is an isomorphism of Lie algebras*

$$\mathrm{HH}^1(A_{n,m}) \cong \prod_{i=1}^{n} \mathfrak{sl}_m(K),$$

for $n, m \ge 2$. In particular, $\mathrm{HH}^1(A_{n,m})$ is a solvable Lie algebra if and only if $\mathrm{char}(K) = 2$ and $m = 2$.

Proof. A short calculation shows that, for $j \ne l$ where $j, l \in \{1,\dots m\}$, $h \in \{1,\dots m-1\}$ and $i \in \{1,\dots n\}$, the set $\{\alpha_{i,h}||\alpha_{i,h} - \alpha_{i,h+1}||\alpha_{i,h+1}, \alpha_{i,j}||\alpha_{i,l}\}$ is a basis of $\mathrm{HH}^1(A_{n,m})$. Since the Lie bracket of different classes of parallel arrows is zero, the Lie algebra structure of $\mathrm{HH}^1(A_{n,m})$

decomposes as a product of Lie algebras. The next step is to show that for a fixed i each of these Lie algebras is isomorphic to $\mathfrak{sl}_m(k)$. We consider the basis for $\mathfrak{sl}_m(K)$ given by the set of elementary matrices $\{e_{st}\}_{s,t}$ for $s \neq t$, together with $h_s = e_{ss} - e_{s+1,s+1}$. Given i, we denote $\alpha_{i,j}$ by α_j. Then if we write $e_{st} := \alpha_s || \alpha_t$ and $h_s := \alpha_s || \alpha_s - \alpha_{s+1} || \alpha_{s+1}$ it is easy to show that the above statement follows. Therefore $\mathrm{HH}^1(A)$ is isomorphic to $\prod_{i=1}^n \mathfrak{sl}_m(k)$. If $\mathrm{char}(K) = 2$ and $m = 2$ this Lie algebra is not solvable. □

An analogous proof to the above shows the following generalisation of Proposition 4.18. Let $n, \mathbf{m}$ be such that $\mathbf{m} = (m_1, \ldots, m_k)$ and $n, m_1, \ldots, m_k \in \mathbb{N}_{\geq 1}$. We denote by $Q_{n,\mathbf{m}}$ the quiver with n vertices $1, \ldots, n$ and for each consecutive pair of vertices $(i, i+1)$ there are m_i parallel arrows denoted by $\alpha_{i,1}, \ldots \alpha_{i,m_i}$ from i to $i+1$. For $n = 2$ and $\mathbf{m} = m$, the quiver $Q_{2,m}$ is the m-Kronecker quiver. Set $A_{n,\mathbf{m}} = KQ_{n,\mathbf{m}}/\mathrm{rad}^2 KQ_{n,\mathbf{m}}$.

Corollary 4.19. *For $A_{n,\mathbf{m}}$ as above, we have*

$$\mathrm{HH}^1(A_{n,\mathbf{m}}) \cong \prod_{i=1}^n \mathfrak{sl}_{m_i}(K),$$

for $n, m_1, \ldots, m_k \geq 2$.

It can be shown in a similar way that $\mathrm{HH}^1(KQ_{n,\mathbf{m}}) \cong \prod_{i=1}^n \mathfrak{sl}_{m_i}(K)$, for $n, m_1, \ldots, m_k \geq 2$. In fact, this result also follows from the fact that $A_{n,\mathbf{m}}$ is the Koszul dual of $KQ_{n,\mathbf{m}}$ so their respective Hochschild cohomologies –that are isomorphic as graded algebras– are also isomorphic as Gerstenhaber algebras [11].

Corollary 4.20. *Let $A = KQ/I$ be a finite dimensional algebra such that Q contains $Q_{n,m}$ as a subquiver and the other arrows of Q form a simple directed graph. Suppose that I contains all or none of the relations of degree 2 involving the arrows of $Q_{n,m}$ and that for any arrow α in $Q_{n,m}$ and any arrow β not in $Q_{n,m}$, either $\alpha\beta \notin I$ or $\beta\alpha \notin I$. Then*

$$\mathrm{HH}^1(A)/\mathrm{rad}(\mathrm{HH}^1(A)) \cong \prod_{i=1}^n \mathfrak{sl}_m(K).$$

Proof. By construction, $\mathrm{HH}^1(A) = \prod_{i=1}^n \mathfrak{sl}_m(K) \oplus \mathcal{S}$, where $\mathcal{S}$ is a solvable Lie algebra. The statement follows. □

It follows from Corollary 4.20, see also [8] in characteristic zero, that the first Hochschild cohomology of a special biserial algebra A (beyond the Kronecker algebra) is not necessarily solvable. This is the case, for example, when the quiver contains $Q_{2,m}$ for some m as subquiver with all relations of length two for all arrows in $Q_{2,m}$. In that case $\mathfrak{sl}_2(k)$ is a Lie subalgebra of $\mathrm{HH}^1(A)$.

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