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The Becker-Gottlieb transfer: a geometric description

Yi-Sheng Wang

Abstract

In this note, we examine geometric aspects of the Becker-Gottlieb transfer in terms of the Umkehr and index maps, and rework some classic index theorems, using the cohomological formulae of the Becker-Gottlieb transfer. The results are natural from the homotopy-theoretic point of view; they reveal subtle geometric information in the Umkehr map, and demonstrate the beauty of the Atiyah-Singer index theorem for families and its generalizations.

Introduction

The Becker-Gottlieb transfer is a homotopy-theoretically defined stable map associated to a compact smooth fiber bundle $E \xrightarrow{\pi} B$. Its construction involves no geometries and depends only on the smooth bundle structure π. For any spectrum $\mathbf{F}$, it induces a homomorphism [BG74]:

$$\mathrm{tr}^* : F^*(M) \to F^*(B),$$

where $F^*(-)$ is the associated cohomology theory.

Given a ring spectrum $\mathbf{E}$ and a module spectrum $\mathbf{F}$ over $\mathbf{E}$, if the tangent bundle along the fibers τM is $\mathbf{E}$-oriented, then one can define, using the Thom isomorphism induced from the orientation class, an Umkehr map or a Gysin homomorphism:

$$\pi^! : F^*(M) \to F^{*-k}(B),$$

where k is the dimension of X.

Now, suppose $\mathbf{F}$ is the topological K-theory spectrum $\mathbf{K}$. Then there is a topological index map [AS71], [LM89],

$$\mathrm{ind}_t : \mathbf{K}_{\mathrm{cpt}}(\tau M) \to \mathbf{K}(B),$$

where τM is the tangent bundle along the fiber, $K_{\mathrm{cpt}}(\tau M)$ is its topological K-group with compact support, which is equivalent to the reduced topological K-group $\tilde{K}(M^\tau)$ of its Thom space M^τ. This definition can be generalized to any module spectrum $\mathbf{F}$ over $\mathbf{K}$:

$$\mathrm{ind}_t : \tilde{F}^{*+k}(M^\tau) \to F^{*-k}(B),$$

where $\tilde{F}^*(-)$ is the reduced cohomology group.

The purpose of the note is to find, via the Umkehr map and index maps, concrete geometric expressions of the Becker-Gottlieb transfer. To this aim, we prove the following theorem, which is a corollary of Lemma 1.1.

Theorem 0.1. *Suppose* $\mathbf{F}$ *is a module spectrum over* $\mathbf{K}$ *and* τM *admits a spin structure. Then, for any element in* $F^*(M)$*, we have*

$$\mathrm{tr}^*(x) = \pi^!(s(M/B) \cdot x) = \mathrm{ind}_t[T_\tau(s(M/B) \cdot x)],$$ [1]

where $s(M/B)$ *is the Euler class induced from the complex spinor bundle* $\not{S}^{\pm}_{\mathbb{C}}(\tau M)$ *and* $T_\tau : F^*(M) \to \tilde{F}^{*+k}(M^\tau)$ *is the Thom isomorphism given by the conjugate complex spinor bundle* $\overline{\not{S}}^{\pm}_{\mathbb{C}}(\tau M)$*.*

Using the relation between tr^*, $\pi^!$, and ind_t, we reexamine some classic index theorems of families; we obtain a simple proof for the index theorem for flat vector bundles (Diagram 9) and extend Lott's index theorem to a $\mathbb{C}/\mathbb{Q}$-index theorem (Diagram 14).

The relation between the three maps as well as a cohomological formula for tr^* is established in Section 1. In Section 2, the Atiyah-Singer index theorem for families is used to to show an index theorem for flat vector bundles. The theorem is first proved in [BS82] with KR-theory; the proof presented here avoids KR-theory at the cost that τM need to have a spin structure. We investigate Lott's index theorem [Lot94] in Section 3, and apply it to the Becker-Gottlieb transfer through Theorem 0.1. In Section 4, we discuss the index theorems from homotopy-theoretic point of view. Some comments on approaches to a $\mathbb{C}/\mathbb{Z}$ index theorem for flat vector bundles (Diagram 17) are given in Section 5.

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[1]The first identity is an easy generalization of [BG74, Theorem 4.3] to module spectra.

1 Umkehr maps

Suppose $M \xrightarrow{\pi} B$ is a compact smooth fiber bundle with fiber X a k-dimensional manifold. Then there exists a fiber-wise embedding

$$\iota : M \to B \times \mathbb{R}^N,$$

where N is some large positive even integer, $B \times \mathbb{R}^N$ is considered as a trivial bundle over B, and $+$ is a base point. This embedding gives us a collapsing map

$$B_+ \wedge S^N \to M^\nu,$$

where M^ν is the Thom space of the normal bundle νM of ι. Let M^τ denote the Thom space of the tangent bundle along the fibers over M. Then the Beck-Gottlieb transfer [BG74] is a S-map given by the composition

$$\mathrm{tr} : B_+ \wedge S^N \to M^\nu \hookrightarrow M^{\nu \oplus \tau} \simeq M_+ \wedge S^N.$$

Given any spectrum $\mathbf{F}$, tr induces a homomorphism

$$\mathrm{tr}^* : F^*(M) \to F^*(B).$$

On the other hand, given a ring spectrum $\mathbf{E}$, if $\mathbf{F}$ is a module spectrum over $\mathbf{E}$ and M^ν is $\mathbf{E}$-oriented, then there exists an Umkehr map $\pi^!$ defined by the composition

$$\pi^! : F^*(M) \to \tilde{F}^{*+N-k}(M^\nu) \to \tilde{F}^{*+N-k}(B_+ \wedge S^N) \simeq F^{*-k}(B),$$

where the first homomorphism is the Thom isomorphism induced by the $\mathbf{E}$-orientation on M^ν.

The relation between the Umkehr map and the Becker-Gottlieb transfer can be summarized as follows [BG74, Theorem 4.3]: for any $x \in F^*(M)$,

$$\mathrm{tr}^*(x) = \pi^!(e(M/B) \cdot x),$$

where $e(M/B)$ is the Euler class of τM induced by the $\mathbf{E}$-orientation of M^ν and the trivialization $M^{\nu \oplus \tau} \simeq M_+ \wedge S^N$ given by pulling back the trivial bundle

$$B \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^N \to B \times \mathbb{R}^N$$

along the embedding ι.

Now, we recall the construction of the topological index map of $M \to B$ [AS71], [LM89, Chapter 15] and let $\mathbf{F}$ be the topological K-theory spectrum

K. The embedding ι induces a map between tangent bundles (along the fiber)

$$\tau\iota : \tau M \to B \times \mathbb{R}^{2N}$$

the normal bundle $\nu(\tau M)$ of $\tau\iota$ in $B \times \mathbb{R}^{2N}$ admits a complex structure and hence there is a Thom isomorphism

$$K_{\text{cpt}}(\tau M) \to K_{\text{cpt}}(\nu(\tau M)).$$

The topological index is then defined by the composition:

$$\text{ind}_t : K_{\text{cpt}}(\tau M) \to K_{\text{cpt}}(\nu(\tau M)) \to K_{\text{cpt}}(B \times \mathbb{R}^{2N}) \simeq K(B). \qquad (1)$$

Recall that an element in a complex topological K-theory with compact support can be represented as the formal difference of two vector bundles, each of which is trivialized outside a compact set [LM89, p.66]. Hence, we may rewrite the composition (1) in terms of Thom spaces and reduced cohomology groups [Rud08, Corollary 1.5]:

$$\text{ind}_t : \tilde{K}(M^{\tau}) \to \tilde{K}(M^{\tau\oplus\nu\oplus\nu}) \to \tilde{K}(B_+ \wedge S^{2N}) \simeq K(B). \qquad (2)$$

This expression allows us to generalize the composition (1) to any module spectrum **F** over **E** with **E** a ring spectrum equipped with a ring morphism $\mathbf{K} \to \mathbf{E}$.

$$\tilde{F}^{*+k}(M^{\tau}) \to \tilde{F}^{*+2N-k}(M^{\tau\oplus\nu\oplus\nu}) \to \tilde{F}^{*+2N-k}(B_+ \wedge S^{2N}) \simeq F^{*-k}(B). \qquad (3)$$

If, furthermore, the Thom space M^{τ} is **E**-oriented, then we can precompose (3) with the Thom isomorphism

$$T_{\tau} : F^*(M) \to \tilde{F}^{*+k}(M^{\tau})$$

to get a map

$$\text{ind}_{\pi} : F^*(M) \to F^{*-k}(B). \qquad (4)$$

For instance, if the τM is an even-dimensional *spin* vector bundle, then the complex spinor $\$^{\pm}_{\mathbb{C}}(\tau M)$ along with the Dirac operator

$$\not{D}^{+} : \Gamma(\$^{+}_{\mathbb{C}}(\tau M)) \to \Gamma(\$^{-}_{\mathbb{C}}(\tau M)),$$

gives a Thom class in $K_{\text{cpt}}(\tau M) \simeq \tilde{K}(M^{\tau})$ where $\Gamma(E)$ is the space of sections of a vector bundle $E \to B$ [LM89, Appendix C]. To compare with the Becker-Gottlieb transfer, we need to choose the Thom class induced from its conjugate bundle $\overline{\$}^{\pm}_{\mathbb{C}}(\tau M)$.

Lemma 1.1. *Suppose* $\mathbf{F}$ *is a module spectrum over* $\mathbf{K}$*, and let* T_τ *be the Thom isomorphism induced from the conjugate spinor bundle* $\overline{\slashed{S}}^{\pm}_{\mathbb{C}}(\tau M)$*. Then the Umkehr map* $\pi^!$ *is equal to* ind_π*.*

Proof. Observe that we have the isomorphism

$$\slashed{S}_{\mathbb{C}(\nu M)} \otimes \overline{\slashed{S}}_{\mathbb{C}}(\nu M) \simeq \mathrm{Cl}_{N-k}(\nu M) \otimes \mathbb{C},$$

where $\mathrm{Cl}_{N-k}(\nu M)$ is the Clifford bundle [LM89, IV.10.16]; taken into account the grading in $\slashed{S}_{\mathbb{C}}(\nu M)$ and $\overline{\slashed{S}}_{\mathbb{C}}(\nu M)$, their tensor recovers the complex spinor

$$\Lambda^{even/odd}_{\mathbb{C}}(\nu M \oplus \nu M) \simeq \slashed{S}^{\pm}_{C}(\nu M \oplus \nu M)$$

induced from the complex structure on $\nu M \oplus \nu M$; in addition, there is an identification [LM89, I.5.21]

$$\overline{\slashed{S}}_{\mathbb{C}}(\tau M) \otimes \overline{\slashed{S}}_{\mathbb{C}}(\nu M) \simeq \overline{\slashed{S}}_{\mathbb{C}}(\tau M \oplus \nu M) \simeq \overline{\slashed{S}}_{\mathbb{C}}(M \times \mathbb{R}^N).$$

Therefore, the diagram below commutes:

$$\begin{array}{ccccccc} & \overset{T_\tau}{\nearrow} & \tilde{F}^{*+k}(M^\tau) & \longrightarrow & \tilde{F}^{*+2N-k}(M^{\tau\oplus\nu\oplus\nu}) & \longrightarrow & \tilde{F}^{*+2N-k}(B_+ \wedge S^{2N}) \\ F^*(M) & & & \overset{T_o}{\nearrow} & & \nearrow & \downarrow \\ & \underset{T_\nu}{\searrow} & \tilde{F}^{*+N-k}(M^\nu) & \longrightarrow & \tilde{F}^{*+N-k}(B_+ \wedge S^N) & \longrightarrow & F^{*-k}(B) \end{array} \tag{5}$$

Note that T_v is induced from the spinor $\slashed{S}_{\mathbb{C}}(\nu M)$ without conjugation and T_o is induced from the spinor

$$\overline{\slashed{S}}^{\pm}_{\mathbb{C}}(M \times \mathbb{R}^N).$$

□

The lemma implies Theorem 0.1.

The index map $\mathrm{ind}_t : K_{\mathrm{cpt}}(\tau M) \to K(B)$ has a cohomological formula [AS71]

$$\mathrm{ch}[\mathrm{ind}_t(u)] = (-1)^k \tilde{\pi}^! [\mathrm{ch}(u) \cup \hat{A}(\tau M)^2], \tag{6}$$

for any $u \in K_{\mathrm{cpt}}(\tau M)$, where $\tilde{\pi}$ is the bundle map $\tau M \to B$, and $\tilde{\pi}^!$ is integration over the fiber, and $\cup$ is the cup product.

If τM is an even-dimensional *spin* vector bundle. Then the Thom isomorphism induced by the Thom class $\overline{\mathbf{s}}(\tau M)$ of the conjugate spinor bundle is given by the assignment:

$$K(M) \to K_{\text{cpt}}(\tau M)$$
$$y \to u = \overline{\mathbf{s}}(\tau M) \cdot \pi^* y,$$

where y is represented by the formal difference of complex vector bundle $[V] - [W]$. Now, recall that given a 2-dimensional spin bundle $V \to M$, its complexification has the decomposition

$$V \otimes \mathbb{C} \simeq L \oplus \overline{L}$$

and

$$\overline{\$^+_{\mathbb{C}}}(V) \oplus \overline{\$^-_{\mathbb{C}}}(V) \simeq L^{\frac{1}{2}} \oplus \overline{L}^{\frac{1}{2}}.$$

Since $L \simeq V$ as an oriented bundle [LM89, p.238]. Using the splitting principal, we obtain

$$\text{ch}(\overline{\mathbf{s}}(\tau M)) = \hat{A}^{-1}(\tau M)$$

(compare with [LM89, III.12.15])[2], the cohomological formula (6) can be rewritten as

$$\begin{aligned} \text{ch}[\text{ind}_t(u)] &= (-1)^k \tilde{\pi}^![\text{ch}(\overline{\mathbf{s}}(\tau M) \cdot \pi^* y) \cup \hat{A}(\tau M)^2] \\ &= (-1)^{\frac{k(k+1)}{2}} \pi^![\text{ch}(y) \cup \hat{A}(M/B)], \end{aligned} \tag{7}$$

where $\pi^!$ is integration over the fiber X.

Now, let y be the product of the Euler class $s(M/B) := s^*\mathbf{s}(\tau M)$ and another element x, where s is the zero section. Then one can compute the BG transfer:

$$\text{tr}^*(x) = \text{ind}_t[\mathbf{s}(\tau M) \cdot \pi^* y] = \text{ind}_t[\mathbf{s}(\tau M) \cdot \pi^*(s(M/B) \cdot x)].$$

Since $\text{ch}(s(M/B)) = (-1)^{\frac{k}{2}} e(M/B) \hat{A}(M/B)^{-1}$ [LM89, III. 11.24], we have the cohomological formula for the BG transfer:

$$\begin{aligned} \text{ch}[\text{tr}^*(x)] &= (-1)^{\frac{k(k+1)}{2}} \pi^![\text{ch}(u) \cup \hat{A}(M/B)] \\ &= (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{k}{2}} \pi^!(e(M/B) \cup \text{ch}(x)) = \pi^!(e(M/B) \cup \text{ch}(x)), \end{aligned}$$

which is equivalent to the commutative diagram below:

[2]The arguments in [LM89, III.11] actually require the non-triviality of the Euler class $e(M/B)$ of τM; however, Diagram 8 as well as Diagrams 13 and 14 holds trivially when $e(M/B)$ vanishes.

$$
\begin{array}{ccc}
K(M) & \xrightarrow{\ \mathrm{ch}\ } & H^{2*}(M,\mathbb{R}) \\
\Big\downarrow \mathrm{tr}^* & & \Big\downarrow \int_X e(M/B)\cup - \\
K(B) & \xrightarrow{\ \mathrm{ch}\ } & H^{2*}(B,\mathbb{R})
\end{array}
\tag{8}
$$

where $\int_X$ is integration over the fiber.

2 Analytic indices

We may think of the smooth fiber bundle $M \xrightarrow{\pi} B$ as a family of manifolds over the parameter space B with structure group $\mathrm{Diff}(X)$. A vector bundle over M can be viewed as a family of vector bundles parameterized by B with structure group of bundle diffeomorphisms $\mathrm{Diff}(E, X)$. Given two families V, W of vector bundles over M parameterized by B, a family of differential operators is a differential operator from $\Gamma(V) \to \Gamma(W)$ such that for each $\pi^{-1}(x)$, $x \in B$, it restricts to an elliptic operator

$$\Gamma(V\mid_{\pi^{-1}(x)}) \to \Gamma(W\mid_{\pi^{-1}(x)}).$$

The definition of families of elliptic operators can be extended to elliptic pseudo-differential operators. And the symbol $\sigma(P)$ of an elliptic pseudo-differential operator P gives rise to an element in $K_{\mathrm{cpt}}(\tau M)$; conversely, every element in $K_{\mathrm{cpt}}(\tau M)$ can be realized by some symbol [AS71]. The index of P is defined by

$$
\begin{aligned}
\mathrm{ind}_{\mathrm{a}} : K(\tau M) &\to K(B) \\
\sigma(P) &\longmapsto \mathrm{Ker}(\tilde{P}) - \mathrm{Coker}(\tilde{P}),
\end{aligned}
$$

where $\tilde{P}$ is the associated pseudo-differential elliptic operator which has locally constant kernel and cokernel [AS71, Proposition 2.2], [LM89, III 8.4]. The Atiyah-Singer theorem asserts that

Theorem 2.1 (Index theorem for families [AS71])**.**

$$\mathrm{ind}_t = \mathrm{ind}_a : K_{\mathrm{cpt}}(\tau M) \to K(B).$$

On the other hand, if τM is spin, any element $[V - W]$ in $K(M)$ induces a Dirac bundle $\overline{\not{S}}^{\pm}_{\mathbb{C}}(\tau M) \otimes V - \overline{\not{S}}^{\pm}_{\mathbb{C}}(\tau M) \otimes W$, and hence gives rise to an element in $K_{\mathrm{cpt}}(\tau M)$. This assignment is precisely the Thom isomorphism induced by $\bar{\mathbf{s}}(\tau M)$. Now take $s(M/B) = [\$^+_{\mathbb{C}}(\tau M) - \$^-_{\mathbb{C}}(\tau M)] \in K(M)$. Then the resulting Dirac bundle $\overline{\$}^{\pm}_{\mathbb{C}}(\tau M) \otimes s(M/B)$ corresponds to the de Rham complex $\Lambda^{even}(\tau M) \otimes \mathbb{C} - \Lambda^{odd}(\tau M) \otimes \mathbb{C}$ [BGV92, p.130;Prop. 3.40]. It implies that the analytic index for flat vector bundles given by taking fiberwise homology

$$\begin{aligned} [M, BU^{\delta}] &\xrightarrow{\mathrm{H}^*(\mathrm{M/B},-)} [M, BU^{\delta}] \\ V_{\alpha} &\longmapsto H^*(M/B, V_{\alpha}), \end{aligned}$$

can be realized by $\mathrm{ind}_a[\bar{\mathbf{s}}(\tau M)\pi^*(s(M/B) \cdot V_{\alpha})]$, where V_{α} is a flat vector bundle. Thus, we have the following commutative diagrams:

$$\begin{array}{ccc} [M, BU^{\delta}] & \xrightarrow{\quad\iota\quad} & K(M) \\ \Big\downarrow{\scriptstyle H^*(M/B,-)} & & \Big\downarrow{\scriptstyle \mathrm{tr}^*} \\ [B, BU^{\delta}] & \xrightarrow{\quad\iota\quad} & K(B) \end{array} \tag{9}$$

where ι is the canonical inclusion by forgetting flat connections.

3 Topological K-theory with coefficients

In this section, we discuss the Becker-Gottlieb transfer in Topological K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$. In [Lot94], Lott describes a geometric model for an $\mathbb{R}/\mathbb{Z}$-valued K-group of a compact manifold $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$. An element in $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ is a quadruple $\mathcal{E} = (V_{\pm}, h^{V_{\pm}}, \nabla^{V_{\pm}}, \omega)$, where $(V_{\pm}, h^{V_{\pm}})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded hermitian vector bundle on M, $\nabla^{V_{\pm}}$ is a hermitian connection on $V_{\pm}$, and $\omega \in \Omega^{odd}/\operatorname{Im}(d)$ satisfies $d\omega = \mathrm{ch}(\nabla^{V_+}) - \mathrm{ch}(\nabla^{V_-})$. The last condition implies that there exists a number k such that kV_+ and kV_- are isomorphic. Choosing an isomorphism $\phi : kV_+ \xrightarrow{\sim} kV_-$. we define a cohomology class

$$\begin{aligned} K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) &\to H^{odd}(M, \mathbb{R}) \\ \mathcal{E} = (V_{\pm}, h^V_{\pm}, \nabla^V_{\pm}, \omega) &\mapsto \frac{1}{k}\,\mathrm{cs}(k\nabla^V_+, \phi^* k\nabla^V_-) - \omega, \end{aligned}$$

where $\mathrm{cs}(-,-)$ is the Chern-Simons class. The image of the cohomology class under

$$H^{odd}(M, \mathbb{R}) \to H^{odd}(M, \mathbb{R}/\mathbb{Q})$$

is the $\mathbb{R}/\mathbb{Q}$ character of $\mathcal{E}$, and it is independent of the choice of an isomorphism ϕ. $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ is a $K(M)$-module, and given a cocycle $\mathcal{C} = (W, h^W, \nabla^W)$ in $K(M)$, $\mathcal{C} \cdot \mathcal{E}$ is defined by

$$(W \otimes V_\pm, h^W \otimes h^{V_\pm}, \nabla^W \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{V_\pm}, \mathrm{ch}(\nabla^W) \wedge \omega),$$

where (W, h^W) is a hermitian vector bundle and ∇^W a hermitian connection. The character $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ respects the module structures:

$$\mathrm{ch}(\mathcal{C} \cdot \mathcal{E}) = \mathrm{ch}(\mathcal{C}) \cdot \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}).$$

There is another $\mathbb{R}/\mathbb{Z}$-valued model, called Segal's model [APS76, p.88], which is closer to the homotopy-theoretic definition of topological K-theory with $\mathbb{R}/\mathbb{Z}$-coefficients. We want to show that Lott's model is equivalent to Segal's and hence also equivalent to the homotopy-theoretic definition using Moore spectrum.

Segal's model is generated by the quadruple $([\omega]; V, W, \phi)$, where ω is a closed odd form representing a class in $H^{odd}(M, \mathbb{R})$, and V, W are two vector bundles over M, and ϕ is an isomorphism between kV and kW. Two elements are equivalent

$$([\omega]; V, W, \phi) \quad \sim \quad ([\omega']; V', W', \phi')$$

if and only if (V, W, ϕ) and (V', W', ϕ') are the same in $K^{-1}_{\mathbb{Q}/\mathbb{Z}}(M)$, namely there exists bundle isomorphisms

$$\begin{array}{ccc} mV & \xrightarrow{\ \sim\ } & mV' \\ {\scriptstyle \tilde{\phi}}\downarrow\wr & & {\scriptstyle \tilde{\phi}'}\downarrow\wr \\ mW & \xrightarrow{\ \sim\ } & mW' \end{array} \tag{10}$$

such that $\omega - \omega'$ represents the Chern-Simon class of the commutative diagram, where $\tilde{\phi}$ and $\tilde{\phi}'$ are isomorphisms induced from ϕ and ϕ', and m is some positive integer.

Given an element in Segal's model $([\omega]; V, W, \phi)$, we let $V_+ = V$ and $V_- = W$ and choose a hermitian metric and a hermitian connection for each of $V_\pm$. This gives an assignment from Segal's model to Lott's model:

$$([\omega]; V_+, V_-, \phi) \longmapsto (V_\pm, h^{V_\pm}, \nabla^{V_\pm}, \frac{1}{k}\,\mathrm{cs}(k\nabla^{V_+}, \phi^* k\nabla^{V_-}) - \omega) \tag{11}$$

Conversely, given an element $(V_\pm, h^{V_\pm}, \nabla^{V_\pm}, \omega)$ in Lott's model, we choose an isomorphism $\phi : kV_+ \to kV_-$. Then the inverse to (11) is given by

$$(V_\pm, h^{V_\pm}, \nabla^{V_\pm}, \omega) \longmapsto ([\frac{1}{k}\operatorname{cs}(k\nabla^{V_+}, \phi^* k\nabla^{V_-}) - \omega]; V_+, V_-, \phi)]. \tag{12}$$

Segal's model is built from the cokernel of the homomorphism

$$K(M) \otimes \mathbb{Q} \xrightarrow{(-j,q)} K(M) \otimes \mathbb{R} \oplus K^{-1}_{\mathbb{Q}/\mathbb{Z}}(M),$$

where q and j are the natural projection and inclusion, respectively, and hence equivalent to the topological definition of topological K-theory with $\mathbb{R}/\mathbb{Z}$-coefficient using the Moore spectrum. In view of (11), (12), Lott's model is also equivalent to the homotopy theoretic definition of topological K-theory with $\mathbb{R}/\mathbb{Z}$-coefficient.

Now, suppose τM is an even dimensional spin vector bundle. Then we have the Umkehr map for $\pi^! : K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$, and using Diagram (5) and the fact that the $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ is a module morphism, we obtain a cohomological formula for $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\pi^!(\mathcal{E})) \in H^{odd}(M, \mathbb{R}/\mathbb{Q})$:

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\pi^!(\mathcal{E})) = (-1)^{\frac{k(k+1)}{2}} \pi^![\hat{A}(M/B) \cup \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})],$$

for any cocycle $\mathcal{E} \in K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ (Compare with [Lot94, p.295]).

On the other hand, we know that the Becker-Gottlieb transfer tr^* is related to $\pi^!$ via the formula

$$\mathrm{tr}^*(\mathcal{E}) = \pi^![(s(M/B) \cdot \mathcal{E}].$$

Since

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(s(M/B) \cdot \mathcal{E}) = \mathrm{ch}(s(M/B)) \cup \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$$

and

$$\mathrm{ch}(s(M/B)) = (-1)^{\frac{k}{2}} e(M/B) \cup \hat{A}(M/B)^{-1},$$

we obtain the cohomological formula for $\mathrm{tr}^*(\mathcal{E})$:

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}[\mathrm{tr}^*(\mathcal{E})] = \pi^![e(M/B) \cup \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})],$$

which is equivalent to the commutative diagram

$$\begin{array}{ccc} K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) & \xrightarrow{\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{odd}(M, \mathbb{R}/\mathbb{Q}) \\ \downarrow \mathrm{tr}^* & & \downarrow \int_X e(M/B) \cup - \\ K^{-1}_{\mathbb{R}/\mathbb{Z}}(B) & \xrightarrow{\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{odd}(B, \mathbb{R}/\mathbb{Q}) \end{array} \tag{13}$$

This index theorem can be easily extended to $K_{\mathbb{C}/\mathbb{Z}}$-theory. Since $K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ can be identified with the cokernel of the homomorphism:

$$K(M) \otimes \mathbb{R} \xrightarrow{(-j,q)} K(M) \otimes \mathbb{C} \oplus K^{-1}_{\mathbb{R}/\mathbb{Z}}(M),$$

every element in $K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$ can be expressed as a quadruple $\mathcal{E} = (V_\pm, h^{V_\pm}, \nabla^{V_\pm}, \omega)$ with ω is in $\Omega^{odd}(M) \otimes \mathbb{C}/\operatorname{Im}(d_\mathbb{C})$ and $d\omega = \operatorname{ch}(\nabla^{V_+}) - \operatorname{ch}(\nabla^{V_-})$. Next, we define $\operatorname{ch}_{\mathbb{C}/\mathbb{Q}}(\mathcal{E})$ to be the image of the form $\frac{1}{k}\operatorname{cs}(k\nabla^V_+, \phi^* k\nabla^V_-) - \omega$ under the homomorphism

$$H^{odd}(M, \mathbb{C}) \to H^{odd}(M, \mathbb{C}/\mathbb{Q});$$

note that $\operatorname{ch}_{\mathbb{C}/\mathbb{Q}}$ respects the module structures. We have the following $\mathbb{C}/\mathbb{Z}$ index theorem

$$\begin{array}{ccc} K^{-1}_{\mathbb{C}/\mathbb{Z}}(M) & \xrightarrow{\operatorname{ch}_{\mathbb{C}/\mathbb{Q}}} & H^{odd}(M, \mathbb{C}/\mathbb{Q}) \\ \Big\downarrow \operatorname{tr}^* & & \Big\downarrow \int_X e(M/B) \cup - \\ K^{-1}_{\mathbb{C}/\mathbb{Z}}(B) & \xrightarrow{\operatorname{ch}_{\mathbb{C}/\mathbb{Q}}} & H^{odd}(B, \mathbb{C}/\mathbb{Q}) \end{array} \tag{14}$$

and a cohomological formula for $\operatorname{tr}^*(\mathcal{E})$, $\mathcal{E} \in K^{-1}_{\mathbb{C}/\mathbb{Z}}(M)$:

$$\operatorname{ch}_{\mathbb{C}/\mathbb{Q}}(\operatorname{tr}^*)(\mathcal{E}) = \int_X e(M/B) \cup \operatorname{ch}_{\mathbb{C}/\mathbb{Q}}(\mathcal{E}).$$

In fact, Lott's index theorem can also be generalized to this context readily. Following [Lot94, Def. 13], we define the analytic index to be

$$\operatorname{ind}_a(\mathcal{E}) = (\operatorname{Ind}_\pm, h^{\operatorname{Ind}_\pm}, \nabla^{\operatorname{Ind}_\pm}, \int_X \hat{A}(M/B) \wedge w - \tilde{\eta}), \tag{15}$$

where the graded hermitian index bundle $(\operatorname{Ind}_\pm, h^{\operatorname{Ind}_\pm})$ with hermitian connection $\nabla^{\operatorname{Ind}_\pm}$ and the analytic-defined odd differential form $\tilde{\eta}$ are the same as in [Lot94][3]. Then the $\mathbb{C}/\mathbb{Z}$ index theorem says:

$$\operatorname{ind}_a(\mathcal{E}) = \pi^! : K^{-1}_{\mathbb{C}/\mathbb{Z}}(M) \to K^{-1}_{\mathbb{C}/\mathbb{Z}}(B).$$

[3] In case the index bundle is not well-defined, we may consult the construction in [Lot94, Def. 14]

4 Homotopy-theoretic aspects

In this section, we approach the discussed index theorems from homotopy-theoretic point of view. We have seen commutative diagrams (8), (13) and (14) are consequences of the cohomological formulae of the Umkehr map. Now, since the Becker-Gottlieb transfer is induced from an infinite loop map, the diagrams also follows, without employing Umkehr maps, if $\mathrm{ch}, \mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ and $\mathrm{ch}_{\mathbb{C}/\mathbb{Q}}$ are induced from infinite loop maps. To see this, we need to verify that they respect the Bott map [Hat, Prop. 4.3]. That is there are commutative diagrams

$$\begin{array}{ccccccc} \tilde{K}(X_+ \wedge S^2) & \xrightarrow{\mathrm{ch}} & \tilde{H}^{even}(X_+ \wedge S^2, \mathbb{R}) & \quad & \tilde{K}^{-1}_{\mathbb{F}/\mathbb{Z}}(X_+ \wedge S^2) & \xrightarrow{\mathrm{ch}_{\mathbb{F}/\mathbb{Q}}} & \tilde{H}^{odd}(X_+ \wedge S^2, \mathbb{F}/\mathbb{Q}) \\ \uparrow B & & \uparrow B & & \uparrow B & & \uparrow B \\ \tilde{K}(X_+) & \xrightarrow{\mathrm{ch}} & \tilde{H}^{even}(X_+, \mathbb{R}) & & \tilde{K}^{-1}_{\mathbb{F}/\mathbb{Z}}(X_+) & \xrightarrow{\mathrm{ch}_{\mathbb{F}/\mathbb{Q}}} & \tilde{H}^{odd}(X_+, \mathbb{F}/\mathbb{Q}) \end{array} \tag{16}$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and B is the Bott map, namely multiplying with the generator of $\tilde{K}(S^2)$ and $\tilde{H}^2(S^2, \mathbb{R})$, respectively. Note that the generator of $\tilde{K}(S^2)$ is $[H - 1]$, where H is the Hopf bundle, and hence $\mathrm{ch}(H - 1) = c_1(H) = e(H)$ is the generator of $\tilde{H}^2(S^2, \mathbb{R})$; in particular,

$$B \circ \mathrm{ch}(x) = e(H) \cup \mathrm{ch}(x) = \mathrm{ch}([H - 1] \cdot x) = \mathrm{ch} \circ B(x).$$

For the case of $\mathrm{ch}_{\mathbb{F}/\mathbb{Q}}$, the assertion follows from its being a module homomorphism.

5 A flat index theorem

In this section, we comment on some approaches to a conjectured index theorem for flat vector bundles, namely, the following commutative diagram

$$\begin{array}{ccc} \tilde{K}_a(M, \mathbb{C}) & \xrightarrow{\mathrm{e_{APS}}} & K^{-1}_{\mathbb{C}/\mathbb{Z}}(M) \\ \downarrow H^*(M/B, -) & & \downarrow \mathrm{tr}^* \\ \tilde{K}_a(B, \mathbb{C}) & \xrightarrow{\mathrm{e_{APS}}} & K^{-1}_{\mathbb{C}/\mathbb{Z}}(B) \end{array} \tag{17}$$

where e_{APS} is the map induced from the construction of the topological index for flat vector bundles in [APS76] (see also [JW95]), and $\tilde{K}_a(-,\mathbb{C})$ is the Grothendieck group of virtual flat vector bundles with zero dimension.

One way to approach the problem is to utilize the analytic index (15). For this, we need to analyze the η-form in $\mathrm{ind}_a(\mathcal{E})$, and compute the Chern-Simon class of the induced flat connection on $H_*(M/B,-) \to B$ and the associated unitary flat connection.

The conjectured index theorem refines the BL index theorem [BL95] and the MZ index theorem [MZ08]. On the other hand, from the homotopy-theoretic point of view, Diagram 17 is a quick consequence of the DWW index theorem [DWW03] if one can prove the associated map

$$e : K_a\mathbb{C} \to F_{t,\mathbb{C}/\mathbb{Z}}$$

is an infinite loop map.

The homomorphism e_{APS} is of interest in differential topology and geometry, given its relation with the Cheeger-Chern-Simons class [MZ08], [Ho18], the Borel regulator [JW95], and the $\tilde{\xi}$-invariant [APS76].

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