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Chirality of Real Non-Singular Cubic Fourfolds
and Their Pure Deformation Classification

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CHIRALITY OF REAL NON-SINGULAR CUBIC FOURFOLDS AND THEIR PURE DEFORMATION CLASSIFICATION

S. FINASHIN, V. KHARLAMOV

ABSTRACT. In our previous works we have classified real non-singular cubic hypersurfaces in the 5-dimensional projective space up to equivalence that includes both real projective transformations and continuous variations of coefficients preserving the hypersurface non-singular. Here, we perform a finer classification giving a full answer to the chirality problem: which of real non-singular cubic hypersurfaces can not be continuously deformed to their mirror reflection.

> ... I'll tell you all my ideas about Looking-glass House.
> First, there's the room you can see through the glass -
> that's just the same as our drawing room, only the things
> go the other way ... How would you like to live in
> Looking-glass House, Kitty? I wonder if they'd give you
> milk in there? Perhaps Looking-glass milk isn't good to
> drink...
>
> Lewis Carroll, *Through the Looking-Glass, and What
> Alice Found There.* (cf. note 6 on page 144 in [CTG])

1. INTRODUCTION

1.1. Chirality problem. There are two deformation equivalence relations emerging naturally in the study of real non-singular projective hypersurfaces in the framework of 16th Hilbert's problem. One of them is the *pure deformation equivalence* that assigns hypersurfaces to the same equivalence class if they can be joined by a continuous path (called a *real deformation*) in the space of real non-singular projective hypersurfaces of some fixed degree. Another one is the *coarse deformation equivalence*, in which real deformations are combined with real projective transformations.

If the dimension of the ambient projective space is even, then the group of real projective transformations is connected, and the above equivalence relations coincide. By contrary, if the dimension of the ambient projective space is odd, this group has two connected components, and some of coarse deformation classes may split into two pure deformation classes. The hypersurfaces in such a class are not pure deformation equivalent to their mirror images and are called *chiral*. The hypersurfaces in the other classes are called *achiral*, since each of them is pure deformation equivalent to its mirror image.

The first case where a discrepancy between pure and coarse deformation equivalences shows up is that of real non-singular quartic surfaces in 3-space (achirality

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of all real non-singular cubic surfaces is due to F. Klein [Kl]). In this case it was studied in [Kh1, Kh2], where it was used to upgrade the coarse deformation classification of real non-singular quartic surfaces obtained by V. Nikulin [N] to a pure deformation classification.

Real non-singular cubic fourfolds is a next by complexity case. Their deformation study was launched in [FK1], where we classified them up to coarse deformation equivalence. Then in [FK2] we began studying of the chirality phenomenon and gave complete answers for cubic fourfolds of maximal, and almost maximal, topological complexity. The approach, which we elaborated and applied in [FK2] relies on the surjectivity of the period map for cubic fourfolds established by R. Laza [La] and E. Looijenga [Lo].

Recall that according to [FK1] there exist precisely 75 coarse deformation classes of real non-singular fourfold cubic hypersurfaces $X \subset P^5$ (throughout the paper X stands both for the variety itself and for its complex point set, while $X_{\mathbb{R}} = X \cap P^5_{\mathbb{R}}$ denotes the real locus). These classes are determined by the isomorphism type of the pairs $(\text{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h \in \mathbb{M}(X))$ where $\mathbb{M}(X) = H^4(X; \mathbb{Z})$ is considered as a lattice, $h \in \mathbb{M}(X)$ is the *polarization class* that is induced from the standard generator of $H^4(P^5; \mathbb{Z})$, and conj^* is induced by complex conjugation $\text{conj} : X \to X$. This result can be simplified further and expressed in terms of a few simple numerical invariants. Namely, it is sufficient to consider the sublattice $\mathbb{M}^0_+(X) \subset \mathbb{M}(X)$, $\mathbb{M}^0_+(X) = \{x \in \mathbb{M}(X) : \text{conj}^* x = x, xh = 0\}$, and to retain only the following three invariants: the rank ρ of $\mathbb{M}^0_+$, the rank d of the 2-primary part $\text{discr}_2 \mathbb{M}^0_+$ of the discriminant $\text{discr} \mathbb{M}^0_+$, and the type, even or odd, of the discriminant form on $\text{discr}_2 \mathbb{M}^0_+$ (see Theorem 2.1.1 below).

Thus, to formulate the pure deformation classification of real non-singular cubic fourfolds, it is sufficient to list the triples of invariants (ρ, d, parity) which specify the coarse deformation classes and to indicate which of the coarse classes are chiral, and which ones are achiral.

1.2. **Main result.**

1.2.1. **Theorem.** *Among the 75 coarse deformation classes precisely 18 are chiral, and, thus, the number of pure deformation classes is 93. The chiral classes have pairs (ρ, d) satisfying $\rho + d \leqslant 12$. The only achiral classes with $\rho + d \leqslant 12$ are three classes with $4 \leqslant \rho = d \leqslant 6$ and one class with $(\rho, d) = (8, 4)$ and even pairity.*

A complete description of the pure deformation classes is presented in Table 1, where the coarse deformation classes are marked by letters c and a: by c, if the class is chiral, and by a, if it is achiral. We use ρ and d as Cartesian coordinates and employ bold letters to indicate even parity, while keeping normal letters for odd. For some pairs (ρ, d) there exist two coarse deformation classes, one with even discriminant form, and another with odd, and in this case, we put the even one in brackets.

In fact, the values of ρ and d determine the topology of the real locus of the cubic fourfold and are determined by it. Namely, for all pairs (ρ, d) except one the real locus of the fourfold is diffeomorphic to $\mathbb{RP}^4 \# a(S^2 \times S^2) \# b(S^1 \times S^3)$, where $a = \frac{1}{2}(\rho - d), b = \frac{1}{2}(22 - \rho - d)$. The exception is $(\rho, d, \text{parity}) = (12, 10, \text{even})$, in which case the real locus is diffeomorphic to $\mathbb{RP}^4 \sqcup S^4$ (see [FK3]). Comparing this with Table 1 we come to the following conclusion.

TABLE 1. Pure deformation classification via chirality

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
11											a										
10										a		a(**a**)									
9									a		a		a								
8								a(**a**)		a		a(**a**)		a							
7							a		a		a		a		a						
6					a		a(**a**)		a		a(**a**)		a		a						
5				a		c		a		a		a		a		a					
4			a		c		c(**a**)		a		a(**a**)		a		a(**a**)		a				
3		c		c		c		c		a		a		a		a		a			
2	c		c(**c**)		c		**c**		c		a(**a**)		a		**a**		a		a(**a**)		
1		c		c						c		a						a		a	
0		**c**										**c**								**a**	

ρ

1.2.2. **Corollary.** *Chirality of a cubic $X \subset P^4$ is determined by the topological type of its real locus $X_\mathbb{R}$ unless $X_\mathbb{R} = \mathbb{RP}^4 \# 2(S^2 \times S^2) \# 5(S^1 \times S^3)$, or equivalently, $(\rho, d) = (8, 4)$. If $(\rho, d) = (8, 4)$, then X is achiral in the case of even parity, and chiral in the case of odd.* ☐

The paper is organized as follows. In Section 2 we collect, and develop a bit further, some results from [FK1, FK2]. In particular, in Section 2.5 we elaborate some sufficient conditions for a sublattice or a superlattice to inherit achirality. The proof of the main results is divided into two parts respectively to the parity of the 2-primary part of the discriminant of the lattice $\mathbb{M}^0_+(X)$: in Section 3 we treat the even case, and in Section 4 the odd case. Our strategy there is, first, to use the results of Section 2.5 to reduce the study of chirality to some smaller generating set of lattices and, second, to apply the same approach as in [FK2] which is based on analysis of symmetries of fundamental polyhedra of the group generated by reflections determined by 2- and 6-roots of the lattice. In two cases considered in Section 4.4, we extend the reflection group by inclusion 4-root reflections, which simplifies control of symmetries of the fundamental polyhedra. In Section 5 we pose some related chirality questions and partially respond to them.

2. PRELIMINARIES

2.1. **Lattices under consideration.** Consider a non-singular cubic fourfold $X \subset P^5$. As is known, there exists a lattice isomorphism between $\mathbb{M}(X) = H^4(X)$ and $\mathbb{M} = 3I + 2U + 2E_8$ which sends the polarization class $h(X) \in \mathbb{M}(X)$ to $h = (1, 1, 1) \in 3I$. It follows then that the primitive sublattice $\mathbb{M}^0(X) = \{x \in \mathbb{M}(X) \mid xh = 0\}$ is isomorphic to $\mathbb{M}^0 = A_2 + 2U + 2E_8$.

For X defined over the reals, the complex conjugation $\mathrm{conj} : X \to X$ induces a lattice involution $\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X)$ such that $\mathrm{conj}^*(h) = h$ and, hence, induces also a lattice involution in $\mathbb{M}^0(X)$. We denote by $\mathbb{M}^0_\pm(X)$ and $\mathbb{M}_\pm(X)$ the eigen-sublattices $\{x \in \mathbb{M}^0(X) \,|\, \mathrm{conj}^*(x) = \pm x\}$ and $\{x \in \mathbb{M}(X) \,|\, \mathrm{conj}^*(x) = \pm x\}$, respectively. We have obviously $\mathbb{M}_- = \mathbb{M}^0_-$ and $\sigma_-(\mathbb{M}_+(X)) = \sigma_-(\mathbb{M}^0_+(X))$, where σ_- denotes the negative index of inertia (i.e., the number of negative squares in a diagonalization).

It was shown in [FK1, Theorem 4.8] that two non-singular real cubic fourfolds, X and Y, are coarse deformation equivalent if and only if the lattices $\mathbb{M}_-(X)$ and $\mathbb{M}_-(Y)$ are isomorphic. In terms of $\mathbb{M}^0_+$ this criterion can be translated as follows.

2.1.1. **Theorem.** *The coarse deformation class of a real non-singular cubic fourfold X is determined by the following three invariants: the rank ρ of the lattice $\mathbb{M}^0_+(X)$, the rank d of the 2-primary part $\mathrm{discr}_2 \mathbb{M}^0_+(X)$ of $\mathrm{discr}\, \mathbb{M}^0_+(X)$, and the type, even or odd, of $\mathrm{discr}_2 \mathbb{M}^0_+(X)$.*

Proof. As it follows from Nikulin's uniqueness theorem [N, Th. 3.6.2], the isomorphism class of the lattice $\mathbb{M}_-(X)$, as of any even 2-elementary hyperbolic lattice, is determined by its rank, the rank of its discriminant form, and the parity of the latter. Hence, there remain to notice that $\mathrm{discr}_2 \mathbb{M}_-(X) = -\mathrm{discr}_2 \mathbb{M}^0_+(X)$. $\square$

To determine the isomorphism class of $\mathbb{M}^0_+(X)$ using exclusively these three numerical invariants, we use the following uniqueness statement.

2.1.2. **Proposition.** *Let $\mathbb{L}$ and $\mathbb{L}'$ be even non-degenerate lattices which have only discriminant factors 2 and 3. If $\sigma_-(\mathbb{L}) = \sigma_-(\mathbb{L}') = 1$, $\mathrm{rk}\,\mathbb{L} = \mathrm{rk}\,\mathbb{L}'$, $\mathrm{discr}_3 \mathbb{L} = \mathrm{discr}_3 \mathbb{L}' = \mathrm{discr}_3 \langle 6 \rangle$, $\mathrm{rk}\,\mathrm{discr}_2 \mathbb{L} = \mathrm{rk}\,\mathrm{discr}_2 \mathbb{L}'$, and both $\mathrm{discr}_2 \mathbb{L}, \mathrm{discr}_2 \mathbb{L}'$ are of the same parity, then $\mathbb{L}$ and $\mathbb{L}'$ are isomorphic.*

Proof. It is trivial for lattices of rank 1. For lattices of rank 2, it follows from classification of binary integral quadratic forms by passing to a reduced form (using that $|\det \mathbb{L}| \leqslant 6$ if $\mathrm{rk}\,\mathrm{discr}_2 \mathbb{L} \leqslant 1$ and dividing the quadratic form of lattices by 2 if $\mathrm{rk}\,\mathrm{discr}_2 \mathbb{L} = 2$). For lattices of rank $\mathrm{rk}\,\mathbb{L} \geqslant 3$ with $\mathrm{rk}\,\mathrm{discr}_2 \mathbb{L} < \mathrm{rk}\,\mathbb{L}$, the claim of proposition follows from Nikulin's theorem [N, Theorem 1.14.2]. In the remaining case, $\mathrm{rk}\,\mathrm{discr}_2 \mathbb{L} = \mathrm{rk}\,\mathbb{L} \geqslant 3$ it is sufficient to divide the quadratic form of lattices by 2 and apply the same Nikulin's theorem to the obtained integral lattices. $\square$

Following [FK2], by $\mathrm{Aut}^+(\mathbb{M}^0)$ we denote the group of those automorphisms of $\mathbb{M}^0$ which preserve a simultaneous orientation of negative definite planes in $\mathbb{M}^0$, and put $\mathrm{Aut}^-(\mathbb{M}^0) = \mathrm{Aut}(\mathbb{M}^0) \setminus \mathrm{Aut}^+(\mathbb{M}^0)$.

We call a lattice involution $c : \mathbb{M} \to \mathbb{M}$ *geometric* if $c(h) = h$ and $\sigma_-(\mathbb{M}^0_\pm(c)) = 1$, where $\mathbb{M}^0_\pm(c)$ denotes the eigen-sublattices $\{x \in \mathbb{M}^0 \,|\, c(x) = \pm x\}$. Let us note that all geometric involutions preserve $\mathbb{M}^0$ and the involutions induced in $\mathbb{M}^0$ belong to $\mathrm{Aut}^-(\mathbb{M}^0)$. As was shown in [FK2, Lemma 3.1.1 and Theorem 3.1.2], a pair $(c : \mathbb{M} \to \mathbb{M}, h \in \mathbb{M})$ is isomorphic to a pair $(\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h(X) \in \mathbb{M}(X))$ for some non-singular real cubic fourfold X if and only if c is a geometric involution.

A pair $(c : \mathbb{M} \to \mathbb{M}, h \in \mathbb{M})$ isomorphic to $(\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h(X))$ is called the *homological type* of X.

2.2. **Lattice characterization of chirality.** In what follows $\mathbb{L}$ is an even lattice of signature $(n, 1)$, $n \geqslant 1$, whose discriminant $\mathrm{discr}(\mathbb{L})$ splits as $\mathrm{discr}_2(\mathbb{L}) + \mathrm{discr}_3(\mathbb{L})$, where $\mathrm{discr}_2(\mathbb{L})$ is a 2-periodic group, and $\mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$.

We let $\mathbb{L}_{\mathbb{R}} = \mathbb{L} \otimes \mathbb{R}$ and consider the cone $\Upsilon = \{p \in \mathbb{L}_{\mathbb{R}} \,|\, p^2 < 0\}$, together with the associated hyperbolic spaces $\Lambda = \Upsilon/\mathbb{R}^*$ and $\Lambda^{\#} = \Upsilon/\mathbb{R}_+$ where $\mathbb{R}^* = \mathbb{R} \smallsetminus \{0\}$ and $\mathbb{R}_+ = (0, \infty)$. In this context, given $v \in \mathbb{L}$ with $v^2 > 0$ we use notation H_v for the hyperplane $\{p \in \mathbb{L}_{\mathbb{R}} \,|\, vp = 0\}$ and $H_v^{\pm}$ for the half-spaces $\{p \in \mathbb{L}_{\mathbb{R}} \,|\, \pm vp \geqslant 0\}$. For $p \in \Upsilon$, H_v, etc., we use notation $[p] \in \Lambda$, $[H_v] \subset \Lambda$, $[p]^{\#} \in \Lambda^{\#}$, $[H_v]^{\#} \subset \Lambda^{\#}$, etc., for the corresponding object after projectivization.

We associate with each $v \in \mathbb{L}$, $v^2 > 0$, the reflection $R_v : \mathbb{L} \otimes \mathbb{Q} \to \mathbb{L} \otimes \mathbb{Q}, x \mapsto x - 2\frac{xv}{v^2}v$, across the hyperplane H_v. It preserves the lattice $\mathbb{L}$ invariant and belongs to the automorphism group $\mathrm{Aut}(\mathbb{L})$ if $v^2 = 2$, or if $v^2 = 6$ and xv is divisible by 3 for all $x \in \mathbb{L}$. We call such lattice elements *2-roots* and *6-roots*, denote their sets by V_2 and V_6, and let $\Phi = V_2 \cup V_6$.

Reflections R_v, $v \in \Phi$, generate the *reflection group* $W \subset \mathrm{Aut}(\mathbb{L})$ which, as known, acts discretely in both $\Lambda = \Upsilon/\mathbb{R}^*$ and $\Lambda^{\#} = \Upsilon/\mathbb{R}_+$. The hyperplanes $[H_v]$ (respectively $[H_v]^{\#}$), $v \in \Phi$, form a locally finite arrangement cutting Λ (respectively $\Lambda^{\#}$) into open polyhedra, whose closures are called the *cells*. The cells in Λ (respectively in $\Lambda^{\#}$) are the fundamental chambers of W.

A cell $P \subset \Lambda$ being fixed, the group $\mathrm{Aut}(\mathbb{L})$ splits into a semi-direct product $W_+ \rtimes \mathrm{Aut}(P)$, where $\mathrm{Aut}(P) = \{g \in \mathrm{Aut}(\mathbb{L}) \,|\, g(P) = P\}$ is the stabilizer of P.

The preimage of P in $\Lambda^{\#}$ is the union of a pair of cells, $P^{\#}$ and $-P^{\#}$. Each $g \in \mathrm{Aut}(P)$ either permutes $P^{\#}$ and $-P^{\#}$, and then we say that it is *P-reversing*, or it preserves both $P^{\#}$ and $-P^{\#}$, and then we call it *P-direct*. The subgroup of $\mathrm{Aut}(P)$ formed by P-direct elements will be denoted by $\mathrm{Aut}^+(P)$, while the coset of P-reversing elements will be denoted by $\mathrm{Aut}^-(P)$.

An additional characteristic of $g \in \mathrm{Aut}(\mathbb{L})$ is its 3-primary component, $\delta_3(g) \in \mathrm{Aut}(\mathbb{Z}/3)$, which may be trivial or not. We say that $g \in \mathrm{Aut}(\mathbb{L})$ is $\mathbb{Z}/3$-*direct* if $\delta_3(g) = \mathrm{id}$, and $\mathbb{Z}/3$-*reversing* if $\delta_3(g) \neq \mathrm{id}$ (that is $\delta_3(g) = -\mathrm{id}$).

The following theorem is an equivalent reformulation of Theorem 4.4.1 in [FK2].

2.2.1. Theorem. *A non-singular real cubic fourfold X is achiral if and only if the lattice $\mathbb{M}_+^0(X)$ admits a $\mathbb{Z}/3$-reversing automorphism $g \in \mathrm{Aut}^+(P)$ for some (or equivalently, for any) of the cells P of Λ.* $\qquad\square$

2.3. Chirality of a lattice. Let us pick a cell $P \subset \Lambda$ and fix a covering cell $P^{\#} \subset \Lambda^{\#}$. Choosing any vector $p \in \Upsilon$ so that $[p]^{\#}$ lies in the interior of $P^{\#}$, we let $\Phi^{\pm} = \{v \in \Phi \,|\, \pm vp > 0\}$. The minimal subset $\Phi^b \subset \Phi^-$ such that $P^{\#} = \bigcap_{v \in \Phi^b} [H_v^-]^{\#}$ is called the *basis of Φ defined by $P^{\#}$*. The hyperplanes $[H_v]$, $v \in \Phi^b$, support n-dimensional faces of P. Note that any $v \in \Phi^-$ is a linear combination of the roots in Φ^b with non-negative coefficients.

Theorem 2.2.1 motivates the following definitions. We call an automorphism of $\mathbb{L}$ *achiral* if it is $\mathbb{Z}/3$-reversing and P-direct for some cell P. Respectively, a lattice $\mathbb{L}$ is called *achiral* if it admits an achiral automorphism, and called *chiral* otherwise.

By definition, the *Coxeter graph* Γ of $\mathbb{L}$ has Φ^b as the vertex set. The vertices of Γ are colored: 2-roots are white and 6-roots are black. The edges are weighted: the weight of an edge connecting vertices $v, w \in \Phi^b$ is $m_{vw} = 4\frac{(vw)^2}{v^2 w^2}$, and $m_{vw} = 0$ means absence of an edge. These weights are non-negative integers, because $2\frac{vw}{v^2}, 2\frac{vw}{w^2} \in \mathbb{Z}$, and $v^2, w^2 > 0$ for any $v, w \in \Phi^b$. In the case of $m_{vw} = 1$, the angle between H_v and H_w is $\pi/3$, and $v^2 = w^2$; such edges are not labelled. The case of $m_{vw} = 2$ (which corresponds to angle $\pi/4$) cannot happen, since $v^2, w^2 \in \{2, 6\}$. An edge of weight $m_{vw} = 3$ connects always a 2-root with a 6-root; it corresponds

to angle $\pi/6$, and will be labelled by 6. The case of $m_{vw} = 4$ corresponds to parallel hyperplanes in Λ, and we sketch a thick edge between v and w. If $m_{vw} > 4$, then the corresponding hyperplanes in Λ are ultra-parallel (diverging), and we sketch a dotted edge.

For a subset $J \subset \Phi^b$ we may also consider the polyhedron $P^{\#}(J) = \cap_{v \in J} [H_v^-]^{\#}$ and the subgraph Γ_J of Γ spanned by J. We say that Γ_J is the *Coxeter's graph of J*. A permutation $\sigma : J \to J$ will be called a *symmetry of Γ_J* if it preserves the weight of edges and the length of the roots, i.e., $(\sigma(v))^2 = v^2$ and $m_{\sigma(v)\sigma(w)} = m_{vw}$ for all $v, w \in J$.

2.3.1. Theorem. [FK2] *Each P-direct automorphism of $\mathbb{L}$ permutes the elements of Φ_b and yields a symmetry of Γ. Conversely, if a subset $J \subset \Phi^b$ spans $\mathbb{L}$ over $\mathbb{Z}$, then any symmetry of Γ_J is induced by a P-direct automorphism of $\mathbb{L}$.* □

To recognize $\mathbb{Z}/3$-reversing symmetries of Γ, one can use the following observation. Considering some direct sum decomposition of $\mathbb{L}$, we observe that one of the direct summands, $\mathbb{L}_1$, has $\mathrm{discr}_3(\mathbb{L}_1) = \mathbb{Z}/3$, while the other direct summands have 2-periodic discriminants (because $\mathrm{discr}(\mathbb{L})$ gets an induced direct sum decomposition). For any vertex w of Γ viewed as a vector in $\mathbb{L}$, we can consider its $\mathbb{L}_1$-component. This leads to the following conclusion.

2.3.2. Proposition. *An automorphism of $\mathbb{L}$ induced by a symmetry σ of Γ_J as in Theorem 2.3.1 is $\mathbb{Z}/3$-direct if for some (equivalently for all) of the 6-roots vertices v of Γ_J the $\mathbb{L}_1$-components of v and $\sigma(v)$ are congruent modulo $3\mathbb{L}_1$. It is $\mathbb{Z}/3$-reversing if for some $v \in V_6$ we have $v - \sigma(v) \notin 3\mathbb{L}$.* □

2.4. Vinberg's criterion of termination. To calculate the root system Φ^b, and thus to determine its Coxeter's graph, we use Vinberg's algorithm which produces Φ^b in a certain natural order, see Section 5.3 in [FK2] for details. The following Vinberg's *finite volume criterion* (stated as Theorem 5.4.1 in [FK2]) assures that we found all the root vectors.

2.4.1. Theorem (Vinberg [V])**.** *A set of root vectors $J \subset \mathbb{L}$ in a hyperbolic lattice of signature $(n, 1)$, obtained at some step of Vinberg's algorithm is complete (admits no continuation) if and only if the hyperbolic volume of the polyhedron $P^{\#}(J)$ is finite.* □

We will use repeatedly the following sufficient criterion that guarantees finiteness of the volume (it is an advanced version of the criterion stated as Theorem 5.4.3 in [FK2]).

2.4.2. Proposition (McLeod [M])**.** *Under the assumptions of Theorem 2.4.1 the volume of the polyhedron $P^{\#}(J)$ is finite if the following two conditions are satisfied.*

- *Each connected parabolic subgraph Γ_I of Γ_J, $I \subset J$, should be a connected component of a parabolic subgraph of rank $n - 1$.*
- *For any Lannér's subgraph (for example, a dotted edge) Γ_S spanned by $S \subset J$ there should exist a set of vertices $T \subset J$ which includes neither vertices of S nor vertices adjacent to S, so that the graph Γ_T is elliptic and the sum of ranks of S and T is $n + 1$.* □

For the list of connected parabolic and Lannér's graphs see Vinberg's survey [V].

2.5. **Chirality under reductions and extensions.** Let Γ be the Coxeter graph of $\mathbb{L}$ and Γ_J be its elliptic subgraph whose vertex set, J, spans a sublattice $\mathbb{L}_J \subset \mathbb{L}$ with 2-periodic discriminant (this means that the connected components of Γ_J have types A_1, D_{2n} $(n \geq 2)$, E_7, and E_8). Then the orthogonal complement $\mathbb{L}^J \subset \mathbb{L}$ of $\mathbb{L}_J$ is, like $\mathbb{L}$, a hyperbolic lattice with the discriminant splitting into a direct sum of the component $\mathrm{discr}_3(\mathbb{L}^J) = \mathbb{Z}/3$ and a 2-periodic component $\mathrm{discr}_2(\mathbb{L}^J)$.

Below we put $H_J = \cap_{v \in J} H_v$.

2.5.1. **Lemma.** *Assume that the lattice $\mathbb{L}$ is achiral, and a P-direct $\mathbb{Z}/3$-reversing $f \in Aut(P)$ is induced by a symmetry of the graph Γ which preserves Γ_J invariant. Then $\mathbb{L}^J$ is also achiral.*

Proof. Since Γ_J is elliptic, the face $P^\# \cap H_J$ of $P^\#$ is of the same dimension as $\mathbb{L}^J$. Therefore, from $\mathrm{discr}_3(\mathbb{L}^J) = \mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$ it follows that $P^\# \cap H_J$ is contained in some cell of $\mathbb{L}^J$. Let us denote the latter by $P^{J\#}$. Since f preserves both $P^\#$ and H_J, it should preserve also $P^{J\#}$. The restriction $f_{|L^J}$ has the same, non-trivial, 3-primary component in $\mathrm{discr}_3(\mathbb{L}^J) = \mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$ as f. Thus, $f_{|L^J} \in Aut(P^J)$ is a P^J-direct $\mathbb{Z}/3$-reversing authomorphism, and hence $\mathbb{L}^J$ is also achiral. $\square$

2.5.2. **Lemma.** *Let $\mathbb{L}_h$ be an achiral lattice, and let $\mathbb{L}_e$ be an elliptic lattice which is generated by 2-roots and has 2-periodic discriminant. Then their direct sum, $\mathbb{L} = \mathbb{L}_e + \mathbb{L}_h$, is also an achiral lattice.*

Proof. Due to assumptions made, we may start Vinberg's root sequence Φ^b from a root basis of $\mathbb{L}_e$. Thus, we can identify $\mathbb{L}_e$ with $\mathbb{L}_J$ for a corresponding vertex set J of Coxeter's graph of $\mathbb{L}$, and $\mathbb{L}_h$ with the orthogonal complement $\mathbb{L}^J$ of $\mathbb{L}_J$.

Consider, now, a cell $P^J \subset \Lambda(\mathbb{L}^J)$ and the components of its preimage, $\pm P^{J\#} \subset \Lambda^\#(\mathbb{L}^J)$. By the same reason as in Lemma 2.5.1, P^J contains the face $P \cap H_J$ of some cell P in $\Lambda(\mathbb{L})$. However, here, the relation is stronger: $P \cap H_J = P^J$. Indeed, if a wall H_v of P is not disjoint from $P \cap H_J$, then either $v \in \mathbb{L}^J$ (and so $H_v \perp H_J$), or $v \in \mathbb{L}_J$ (and so $H_v \supset H_J$). To see it, let us consider a root $v = v_J + v^J \in \mathbb{L}_J + \mathbb{L}^J$ whose components $v_J + v^J$ are both non-zero. Since H_v intersects $\Lambda(\mathbb{L}^J)$, we have $(v^J)^2 > 0$. And since $\mathbb{L}_J$ is positive definite, we have $(v_J)^2 > 0$. The both squares are even. Hence, $v^2 > 2$. If $v^2 = 6$, then, in addition, $(v^J)^2$ and $(v_J)^2$ are divisible by 3, which also leads to a contradiction.

Choose a P^J-direct $\mathbb{Z}/3$-reversing automorphism $f^J \in Aut(P^J)$ and consider its extension $f : \mathbb{L} \to \mathbb{L}$ that acts as the identity map on $\mathbb{L}_J$. As it follows from above, it preserves all the walls of P which are not disjoint from $P \cap H_J$. Thus, it preserves P. Clearly, it is P-direct and $\mathbb{Z}/3$-reversing (it has the same 3-primary component as f^J). $\square$

3. Cases of Even Parity

3.1. **Lattices with even discriminant forms.** For the list of coarse deformation classes in terms of numerical invariants of $\mathbb{M}_-$, we address the reader to Fig. 1 in [FK3]. Since 2-primary parts of the discriminant forms of $\mathbb{M}_-$ are $\mathbb{M}_+^0$ are just opposite, we keep from this list only the classes of even parity (type I in terminology of [FK3]). Then, using the relation $\mathrm{rk}\,\mathbb{M}_+^0 = 22 - \mathrm{rk}\,\mathbb{M}_-$ we apply Proposition 2.1.2 and obtain Table 2 which shows for each of the classes of even parity the corresponding lattice $\mathbb{M}_+^0$. As in Table 1 the answers are arranged in columns and rows according to the values of d and ρ.

TABLE 2. Lattices with even discriminant form

d	4	8	12	16	20
10			$U(2) + A_2 + E_8(2) = U(2) + E_6(2) + D_4$		
8		$U(2) + E_6(2)$	$U + A_2 + E_8(2) = U + E_6(2) + D_4$		
6		$U + E_6(2)$	$U(2) + A_2 + 2D_4$		
4		$U(2) + A_2 + D_4$	$U + A_2 + 2D_4$	$U(2) + A_2 + D_4 + E_8$	
2	$U(2) + A_2$	$U + A_2 + D_4$	$U(2) + A_2 + E_8$	$U + A_2 + D_4 + E_8$	$U(2) + A_2 + 2E_8$
0	$U + A_2$		$U + A_2 + E_8$		$U + A_2 + 2E_8$

ρ

3.1.1. **Theorem.** *Table 2 contains 4 chiral lattices:* $U + A_2$, $U(2) + A_2$, $U + A_2 + D_4$, *and* $U + A_2 + E_8$. *All the others are achiral.*

The three lattices in the bottom row were already treated in [FK2]: $U + A_2$ and $U + A_2 + E_8$ were shown to be chiral, while $U + A_2 + 2E_8$ to be achiral. So, our task is to examine the remaining thirteen cases.
We start with one trivial example.

3.1.2. **Lemma.** *The lattice* $\mathbb{L} = U(2) + E_6(2)$ *is achiral.*

Proof. This lattice has neither 2-roots nor 6-roots. Hence, here $P = \Lambda_+(c)$, and therefore $g : U(2) + E_6(2) \to U(2) + E_6(2)$ that is equal to $-\operatorname{id}$ on $U(2)$ and id on $E_6(2)$ is $\mathbb{Z}/3$-direct and it belongs to $\operatorname{Aut}^-(P)$. Thus, $\mathbb{L}$ is achiral. $\square$

3.2. **A few more direct calculations.** Here we treat the cases $U(2) + A_2$, $U + A_2 + D_4$, $U(2) + A_2 + E_8$, and $U(2) + A_2 + D_4$ using the same approach (based on Vinberg's algorithm for finding Coxeter's graphs of the fundamental domains) as we applied in [FK2] for M- and (M-1)-lattices.

3.2.1. **Lemma.** (1) *The lattices* $U(2) + A_2$ *and* $U + A_2 + D_4$ *are chiral.*
 (2) *The lattices* $U(2) + A_2 + E_8$ *and* $U(2) + A_2 + D_4$ *are achiral.*

Proof. As in [FK2], we fix standard bases: u_1, u_2 for U and $U(2)$; a_1, a_2 for A_2; d_1, d_2, d_3, d_4 for D_4 along with its dual $d_1^*, d_2^*, d_3^*, d_4^*$ for $D_4^* \subset D_4 \otimes \mathbb{Q}$; and $e_1, \ldots, e_8$ for E_8 along with its dual $e_1^*, \ldots, e_8^*$ for $E_8^* = E_8$. We denote by d_1 the "central" root vector of D_4 and then $d_1^* = 2d_1 + d_2 + d_3 + d_4$, while $d_i^* = d_1 + \frac{d_2 + d_3 + d_4 + d_i}{2}$ for $i = 2, 3, 4$. For the expressions of e_i, see for instance [FK2, Fig.1].

Each time to lunch Vinberg's algorithm, we select as the initial vertex of the fundamental domain the point $p = u_1 - u_2$. The resulting Vinberg's sequences and their Coxeter graphs are shown below in Fig. 1, 2, 3, and 4, respectively. As in [FK2], we omit at level 0 of Vinberg's sequence the standard simple root vectors: e_i of E_8, e_i' of the second copy of E_8, $i = 1, \ldots, 8$, and d_i, $i = 1, \ldots 4$, of D_4 (when such summands appear).

(1) The completeness of Vinberg's sequences for $U(2) + A_2$ and $U + A_2 + D_4$ follows from Theorem 2.4.1 and Proposition 2.4.2. Indeed, in both cases the Coxeter graph does not contain Lannér's diagrams, and the only connected parabolic subgraphs are: $\widetilde{G}_2$ for $U(2) + A_2$; $\widetilde{G}_2$ and $\widetilde{D}_4$ for $U + A_2 + D_4$.

Thus, by Theorem 2.3.1 in the case of $U(2) + A_2$ the only P-direct symmetry is given by $v_1 \to v_1, v_2 \to v_2, v_3 \to v_4, v_4 \to v_3$. Since due to Proposition 2.3.2 this symmetry is $\mathbb{Z}/3$-direct (preserves v_2), we conclude that $U(2) + A_2$ is chiral.

Similarly, in the case of $U + A_2 + D_4$ each P-direct symmetry keeps fixed the vector v_3. Hence, all P-direct symmetries are $\mathbb{Z}/3$-direct, and we conclude that $U + A_2 + D_4$ is chiral.

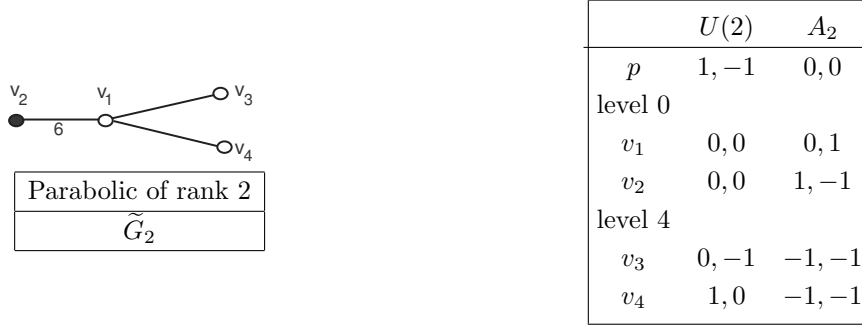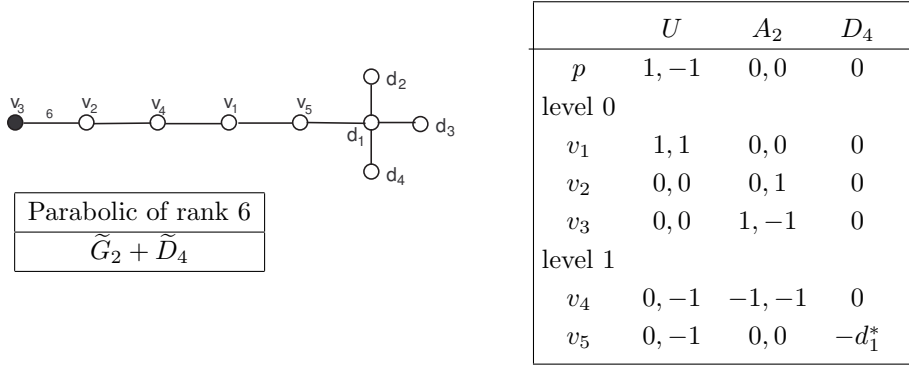FIGURE 1. Vinberg's vectors and their Coxeter's graph for $U(2) + A_2$

	$U(2)$	A_2
p	$1, -1$	$0, 0$
level 0		
v_1	$0, 0$	$0, 1$
v_2	$0, 0$	$1, -1$
level 4		
v_3	$0, -1$	$-1, -1$
v_4	$1, 0$	$-1, -1$

FIGURE 2. Vinberg's vectors and their Coxeter's graph for $U + A_2 + D_4$

	U	A_2	D_4
p	$1, -1$	$0, 0$	0
level 0			
v_1	$1, 1$	$0, 0$	0
v_2	$0, 0$	$0, 1$	0
v_3	$0, 0$	$1, -1$	0
level 1			
v_4	$0, -1$	$-1, -1$	0
v_5	$0, -1$	$0, 0$	$-d_1^*$

(2) When we treat $U(2) + A_2 + E_8$ and $U(2) + A_2 + D_4$, we perform Vinberg's algorithm only up to a step that provides a Coxeter subgraph which has a symmetry inducing a P-direct $\mathbb{Z}/3$-reversing involution, see Fig. 3 and 4. For $U(2) + A_2 + E_8$, this is the involution that permutes the vectors of Vinberg's sequence in accordance with the reflection in the vertical middle axs of the graph; by Theorem 2.3.1 it defines a P-direct automorphism which maps v_2 to $v_8 = -v_2 \mod 3M_+^0$ and by Proposition 2.3.2 is $\mathbb{Z}/3$-reversing. For $U(2) + A_2 + D_4$, this is again the involution permuting the vectors of Vinberg's system in accordance with the reflection in the vertical middle axis of the graph, and it also maps v_2 to $v_8 = -v_2 \mod 3M_+^0$, and by similar reason it is P-direct and $\mathbb{Z}/3$-reversing. $\square$

3.2.2. *Remark.* Coxeter's graphs on Fig. 3 and 4 are indeed complete.

3.3. Achirality of lattices via extension and reduction.

3.3.1. **Proposition.** *Lattices* $U(2) + A_2 + 2D_4$, $U(2) + A_2 + D_4 + E_8$, $U(2) + A_2 + E_8(2)$, *and* $U(2) + A_2 + E_8$ *are achiral.*

Proof. To prove achirality of lattices $U(2) + A_2 + 2D_4$, $U(2) + A_2 + D_4 + E_8$, and $U(2) + A_2 + E_8(2) = U(2) + E_6(2) + D_4$ we apply Lemma 2.5.2 to $U(2) + A_2 + D_4$, $U(2) + A_2 + E_8$, and $U(2) + E_6(2)$ respectively. The same lemma applied to $U(2) + A_2 + E_8$ shows achirality of $U(2) + A_2 + 2E_8$. $\square$

FIGURE 3. Vinberg's vectors and their Coxeter's subgraph for $U(2) + A_2 + E_8$

Parabolic of rank 10

$\widetilde{G}_2 + \widetilde{D}_8,\ 2\widetilde{G}_2 + \widetilde{E}_6,\ 2\widetilde{A}_1 + \widetilde{D}_8$
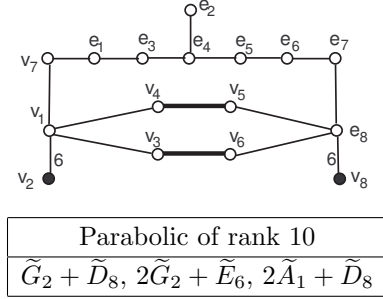
	$U(2)$	A_2	E_8
p	$1, -1$	$0, 0$	0
level 0			
v_1	$0, 0$	$0, 1$	0
v_2	$0, 0$	$1, -1$	0
level 4			
v_3	$0, -1$	$-1, -1$	0
v_4	$1, 0$	$-1, -1$	0
v_5	$0, -1$	$0, 0$	$-e_8^*$
v_6	$1, 0$	$0, 0$	$-e_8^*$
level 16			
v_7	$1, -1$	$-1, -1$	$-e_1^*$
level 48			
v_8	$3, -3$	$-4, -2$	$-3e_8^*$

FIGURE 4. Vinberg's vectors and their Coxeter's subgraph for $U(2) + A_2 + D_4$

Parabolic of rank 6

$6\widetilde{A}_1,\ \widetilde{G}_2 + \widetilde{D}_4$
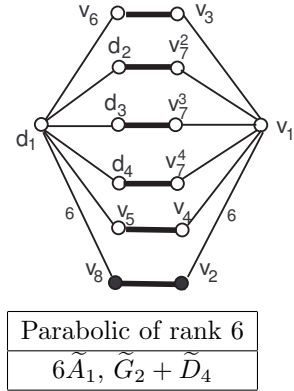
	$U(2)$	A_2	D_4
p	$1, -1$	$0, 0$	0
level 0			
v_1	$0, 0$	$0, 1$	0
v_2	$0, 0$	$1, -1$	0
level 4			
v_3	$0, -1$	$-1, -1$	0
v_4	$1, 0$	$-1, -1$	0
v_5	$0, -1$	$0, 0$	$-d_1^*$
v_6	$1, 0$	$0, 0$	$-d_1^*$
level 16			
v_7^2	$1, -1$	$-1, -1$	$-2d_2^*$
v_7^3	$1, -1$	$-1, -1$	$-2d_3^*$
v_7^4	$1, -1$	$-1, -1$	$-2d_4^*$
level 48			
v_8	$3, -3$	$-4, -2$	$-3d_1^*$

Figure 5 shows a hexagonal subgraph Γ of Coxeter's graph of the lattice $U + A_2 + 2E_8$ which we found in [FK2], where we proved also the achirality of the involution

FIGURE 5. Vinberg's vectors and their Coxeter's subgraph for $U + A_2 + 2E_8$
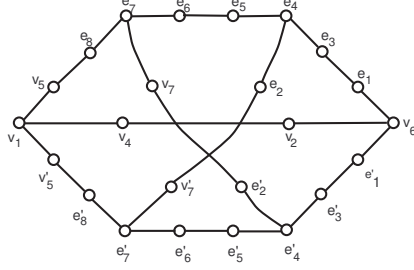
	U	A_2	E_8	E_8
p	$1, -1$	$0, 0$	0	0
level 0				
v_1	$1, 1$	$0, 0$	0	0
v_2	$0, 0$	$0, 1$	0	0
v_3	$0, 0$	$1, -1$	0	0
level 1				
v_4	$0, -1$	$-1, -1$	0	0
v_5	$0, -1$	$0, 0$	$-e_8^*$	0
v_5'	$0, -1$	$0, 0$	0	$-(e_8')^*$
level 16				
v_6	$2, -2$	$-1, -1$	$-e_1^*$	$-(e_1')^*$
level 36				
v_7	$3, -3$	$-2, -1$	$-e_7^*$	$-(e_2')^*$
v_7'	$3, -3$	$-2, -1$	$-e_2^*$	$-(e_7')^*$
level 48				
v_8	$6, -6$	$-4, -2$	$-3e_8^*$	$-3(e_1')^*$
v_8'	$6, -6$	$-4, -2$	$-3e_1^*$	$-3(e_8')^*$

$\Psi : \Gamma \to \Gamma$ that interchanges vertices v_1 and e_4 and keeps fixed the other trivalent vertices (at the corners of the hexagon).

3.3.2. **Lemma.** *For each $n = 1, 2, 3$ the vertex-set of the Coxeter graph Γ of the lattice $\mathbb{L} = U + A_2 + 2E_8$ admits a subset J with the following properties:*

(1) *J is invariant with respect to the achiral involution Ψ;*
(2) *the sublattice $\mathbb{L}_J$ spanned by J in $\mathbb{L}$ is primitive and isomorphic to nD_4;*
(3) *the orthogonal complement $\mathbb{L}^J$ of $\mathbb{L}_J$ in $\mathbb{L}$ is isomorphic to $U + A_2 + D_4 + E_8$, $U + A_2 + 2D_4$, and $U + E_6(2)$ for $n = 1, 2$, and 3, respectively.*

Proof. The set of vertices $J = \{v_7, e_6, e_7, e_8\} \cup \{v_7', e_6', e_7', e_8'\} \cup \{e_1, e_1', v_2, v_6\}$ of the graph Γ spans a sublattice $\mathbb{L}_J = 3D_4$, where each D_4-component is invariant under the involution Ψ. The embedding $\mathbb{L}_J \subset \mathbb{L}$ is primitive because for any subset S of J there exists a vertex of Γ adjacent only to one element of S. This proves (1) and (2) in the case $n = 3$. For the cases $n = 1$ and $n = 2$, we take subsets of J that span just one and, respectively, two of the above three summands D_4 and get (1) and (2) in the same way.

Part (3) follows from Proposition 2.1.2. Indeed, $\mathbb{L}^J$ is even and hyperbolic, $\operatorname{discr}_3 \mathbb{L}^J = \operatorname{discr} \langle 6 \rangle = \operatorname{discr}_3 E_6(2)$, $\operatorname{discr}_2 \mathbb{L}^J = -\operatorname{discr}_2 \mathbb{L}_J = \oplus_n \operatorname{discr}_2 D_4$, and $\operatorname{discr}_2 E_6(2) = \oplus_3 \operatorname{discr}_2 D_4$. $\square$

3.3.3. **Proposition.** *Lattices $U + A_2 + D_4 + E_8$, $U + A_2 + 2D_4$, and $U + E_6(2)$ are achiral.*

Proof. We deduce achirality of $U + A_2 + D_4 + E_8$, $U + A_2 + 2D_4$, and $U + E_6(2)$ from achirality of $U + A_2 + 2E_8$ by applying Lemma 2.5.1.   □

Proof of Theorem 3.1.1. The lattice $U + A_2 + E_8(2) = U + E_6(2) + D_4$ is obtained by adding D_4 to $U + E_6(2)$. Hence, by Lemma 2.5.2, $U + A_2 + E_8(2)$ is also achiral. All the other lattices from Table 2 are treated in Lemmas 3.1.2 and 3.2.1, Proposition 3.3.1, and in [FK2].   □

4. Cases of Odd Parity and Proof of Theorem 1.2.1

4.1. **Lattices with odd discriminant forms.** Similar to the even case, we select from the list given in [FK3] the classes of odd parity, translate them in terms of the invariants (d, ρ, parity), and apply Proposition 2.1.2 to obtain lattices shown in Table 3.

Table 3. Lattices $\mathbb{M}^0_+$ with odd discriminant form

$\rho - d$	d				
0	$t+2$	$-A_1$	$+\langle 6 \rangle$	$+tA_1,$	$0 \leqslant t \leqslant 9$
2	$t+1$	$-A_1$	$+A_2$	$+tA_1,$	$0 \leqslant t \leqslant 9$
4	t	U	$+A_2$	$+tA_1,$	$1 \leqslant t \leqslant 9$
6	$t+2$	U	$+A_2 + D_4$	$+tA_1,$	$1 \leqslant t \leqslant 6$
8	$t+2$	$-A_1$	$+\langle 6 \rangle + E_8$	$+tA_1,$	$0 \leqslant t \leqslant 5$
10	$t+1$	$-A_1$	$+A_2 + E_8$	$+tA_1,$	$0 \leqslant t \leqslant 5$
12	t	U	$+A_2 + E_8$	$+tA_1,$	$1 \leqslant t \leqslant 5$
14	$t+2$	U	$+A_2 + D_4 + E_8$	$+tA_1,$	$1 \leqslant t \leqslant 2$
16	$t+2$	$-A_1$	$+\langle 6 \rangle + 2E_8$	$+tA_1,$	$0 \leqslant t \leqslant 1$
18	$t+2$	$-A_1$	$+A_2 + 2E_8$	$+tA_1,$	$0 \leqslant t \leqslant 1$
20	t	U	$+A_2 + 2E_8$	$+tA_1,$	$t = 1$

4.2. **Achirality of lattices via extension and reduction.** Achirality of the lattices $U + A_2 + E_8 + A_1$ and $U + A_2 + 2E_8$ is already established in [FK2].

4.2.1. **Lemma.** *If a lattice $\mathbb{L}$ in Table 3 with some code (ρ, d) is achiral, then the lattice with the code $(\rho + 1, d + 1)$ (if belongs to this table) is also achiral.*

Proof. The lattice with the code $(\rho + 1, d + 1)$ must be in the same row of Table 3 as $\mathbb{L}$ but with the value of t increased by one, that is $\mathbb{L} + A_1$, and it is left to apply Lemma 2.5.2.   □

By Lemma 4.2.1, it is left to determine in each row of Table 3 an achiral lattice with the minimal value $t = t_0$, it follows then that all the lattices with $t > t_0$ are also achiral.

Our main result in this section can be stated as follows.

4.2.2. **Theorem.** *In Table 3 the first achiral lattice in the first row $\rho = d$ is defined by $t = 2$, in the rows $\rho = d + 2k$, $k = 1, \ldots, 6$, it is defined by $\rho + d = 14$, and in the last four rows all lattices are achiral.*

FIGURE 6. Vinberg's vectors and their partial Coxeter's subgraph for $U + A_2 + E_8 + A_1$
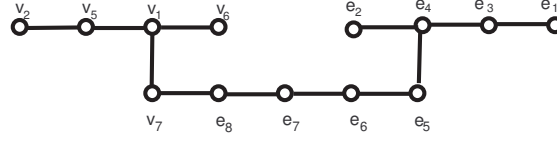
	U	A_2	A_1	E_8
p	$1, -1$	$0, 0$	0	0
level 0				
v_1	$1, 1$	$0, 0$	0	0
v_2	$0, 0$	$0, 1$	0	0
v_3	$0, 0$	$1, -1$	0	0
v_4	$0, 0$	$0, 0$	1	0
level 1				
v_5	$0, -1$	$-1, -1$	0	0
v_6	$0, -1$	$0, 0$	-1	0
v_7	$0, -1$	$0, 0$	0	$-e_8^*$
level 48				
v_8	$6, -6$	$-4, -2$	-3	$-3(e_1)^*$.

We complete its proof in section 4.5 after an analysis of several particular cases.

4.2.3. **Lemma.** *For any* $n = 1, \ldots, 4$, *Coxeter's graph* Γ *of the lattice* $\mathbb{L} = U + A_2 + E_8 + A_1$ *contains a set* J *of* n *pairwise non-incident vertices so that:*

    (1) *J is invariant under an achiral involution of $\mathbb{L}$;*
    (2) *J generates a sublattice $\mathbb{L}_J = nA_1$ primitively embedded into $\mathbb{L}$;*
    (3) *the orthogonal complement $\mathbb{L}^J$ of $\mathbb{L}_J$ has odd 2-discriminant form $\mathrm{discr}_2 \mathbb{L}^J$ whose rank is bigger by n than that of $\mathrm{discr}_2 \mathbb{L}$.*

Proof. In [FK2], Sect. 7.5, we determined the initial Vinberg's vectors of $\mathbb{L}$. These vectors and Coxeter's graph corresponding to the set of vectors $\{v_1, v_2, v_5, v_6, v_7, e_1, \ldots, e_8\}$ are reproduced on Fig. 6. This graph has an obvious reflection symmetry which induces on $\mathbb{L}$ an achiral involution as it was shown in [FK2].

As J we choose $\{e_7\}$, $\{e_6, e_8\}$, $\{v_7, e_7, e_5\}$, and $\{v_1, e_6, e_8, e_4\}$ for $n = 1, 2, 3, 4$ respectively. It is invariant under this achiral involution.

Property (2) is satisfied for the same reasons as in the proof of Proposition 3.3.2.

To prove (3) we consider $\det_2 \mathbb{L} = \det_2 A_1$ and its discriminant form $q = \langle \frac{1}{2} \rangle$ as a result of gluing of $(\det_2 \mathbb{L}_J, q_1)$ with $(\det_2 \mathbb{L}^J, q_2)$ (see [N, Proposition 1.3.1]) and notice that, since for any non-empty subset of J the sum of its elements has odd intersection with some element of $\mathbb{L}$, this gluing corresponds to a group-embedding $\det_2 \mathbb{L}_J \to \det_2 \mathbb{L}^J$ reversing the discriminant forms, so that the only nontrivial element in $\det_2 \mathbb{L}$ is an image of an element $e \in \det_2 \mathbb{L}^J$ with $q_2(e) = q(e) = \frac{1}{2}$. $\square$

4.2.4. **Lemma.** *For any* $n = 1, 2, 3$ *Coxeter's graph* Γ *of the lattice* $\mathbb{L} = U + A_2 + 2E_8$ *contains a set* J *of* n *pairwise non-incident vertices so that:*

FIGURE 7. Vinberg's vectors and their Coxeter's graph for $-A_1 + \langle 6 \rangle + A_1$
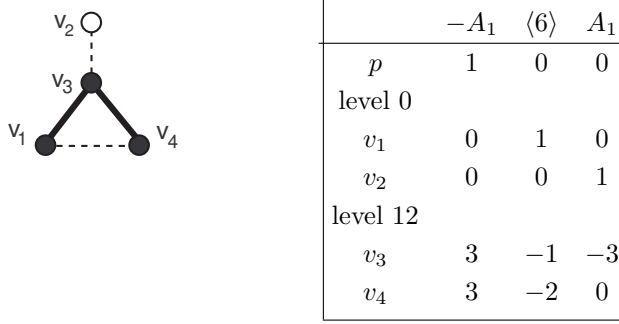
	$-A_1$	$\langle 6 \rangle$	A_1
p	1	0	0
level 0			
v_1	0	1	0
v_2	0	0	1
level 12			
v_3	3	-1	-3
v_4	3	-2	0

(1) J is invariant under an achiral involution of $\mathbb{L}$;
(2) J generates a sublattice $\mathbb{L}_J = nA_1$ primitively embedded into $\mathbb{L}$;
(3) the orthogonal complement $\mathbb{L}^J$ of $\mathbb{L}_J$ has odd 2-discriminant form $\mathrm{discr}_2 \mathbb{L}^J$ whose rank equals n.

Proof. For the cases $n = 1, 2, 3$ we choose as J the sets $\{e_7\}$, $\{e_7, e_7'\}$ and $\{e_7, e_7', v_6\}$ respectively (see Fig. 5), which are obviously invariant with respect to the achiral involution Ψ introduced in section 3.3. The proof of (2) is the same as in Lemma 4.2.3, while for (3) it is much easier, because $\mathrm{discr}_2 S = -\mathrm{discr}_2 S^\perp$ for primitive sublattices in a lattice with trivial discr_2. $\square$

4.2.5. **Proposition.** *All lattices in Table 3 for which $\rho + d \geqslant 14$ are achiral.*

Proof. Lemma 4.2.3 shows that all the lattices in Table 3 for which $\rho + d = 14$ are achiral. Lemma 4.2.4 treats similarly a fragment of line $\rho + d = 20$. From this, and achirality of $U + A_2 + 2E_8$, we obtain the required claim applying Lemma 4.2.1. $\square$

4.3. **A few more chiral lattices.** We start from analysis of lattices in the first row of Table 3.

4.3.1. **Proposition.** *Lattices $-A_1 + \langle 6 \rangle + tA_1$ are chiral for $t = 0, 1$ and achiral for $t \geqslant 2$.*

Proof. For $\mathbb{L} = -A_1 + \langle 6 \rangle + A_1$ (the case $t = 1$), Vinberg's algorithm gives easily four root vectors which together with the corresponding Coxeter's graph are shown on Figure 7. This graph describes a quadrilateral on the hyperbolic plane with consecutive sides defined by the roots v_1, v_3, v_4, v_2; two of its vertices (defined by the parabolic edges v_1, v_3 and v_3, v_4) lie at infinity and the other two vertices (defined by elliptic edges) are finite. The area of this quadrilateral is finite, which according to Vinberg's criterion 2.4.1 means that Coxter's graph that we found is complete. By Theorem 2.3.1 its only non-trivial P-direct symmetry keeps fixed v_2, v_3 and interchanges v_1 and v_4. It is chiral, because it does not change the $\langle 6 \rangle$-component modulo 3 and due to Proposition 2.3.2 induces a trivial automorphism of $\mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$.

Similar calculations in the case $t = 2$ give the result shown on Figure 8. The vertices of the graph obtained span our lattice, and by this reason the reflection over the vertical mirror line of the graph induces a P-direct automorphism of the

Figure 8. Vinberg's vectors and their Coxeter's subgraph for $-A_1 + \langle 6 \rangle + 2A_1$
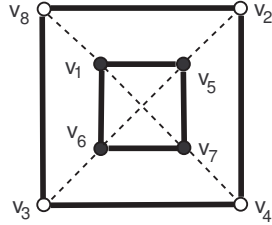
	$-A_1$	$\langle 6 \rangle$	A_1	A_1
p	1	0	0	0
level 0				
v_1	0	1	0	0
v_2	0	0	1	0
v_3	0	0	0	1
level 4				
v_4	1	0	-1	-1
level 12				
v_5	3	-1	-3	0
v_6	3	-1	0	-3
v_7	3	-2	0	0
level 16				
v_8	2	-1	-1	-1

lattice. This automorphism is $\mathbb{Z}/3$-reversing by Proposition 2.3.2, since it changes modulo 3 the $\langle 6 \rangle$-components of the 6-roots. Applying Theorem 2.3.1 we conclude that this lattice is achiral (note that this theorem does not require knowledge of the complete Coxeter's graph).

To deduce the result for other values of t, we use Lemma 4.2.1. $\square$

Analysis of chirality of lattices with $\rho + d \leqslant 12$ requires case-by-case study of the ones on the line $\rho + d = 12$.

4.3.2. **Proposition.** *Lattices $U + A_2 + D_4 + A_1$ and $-A_1 + \langle 6 \rangle + E_8$ (from the fourth and fifth rows of Table 3) are chiral.*

Proof. The result of calculation of Vinberg's vectors and Coxeter's graphs for these two lattices are shown on Fig. 9 and 10 respectively. Each of these graphs has an obvious symmetry:

- S_3-symmetry on Fig. 9 permuting 3 pairs of vertices: d_i, v_{6+i}, $i = 2, 3, 4$;
- S_2-symmetry on Fig. 10 permuting 2 pairs of vertices: v_5, v_1, and v_2, v_4.

We take into account this symmetry when we apply Proposition 2.4.2 to check that our calculation of Vinberg's vectors is complete. The details of this check (based on application of Theorem 2.4.1 and Proposition 2.4.2) are shown on the leftmost tables of Figures 9 and 10 (right below Coxeter's graphs). In the upper tables we list: in the first column, the sets of vertices I that generate a connected parabolic subgraph, Γ_I, one set in each symmetry class; in the second column, the types of these subgraphs; and, in the third column, the types of the parabolic subgraphs that complement Γ_I to a rank $(n-1)$ parabolic graph. In the bottom leftmost tables we list the sets (denoted by T) of vertices of elliptic complements to the dotted edges (whose endpoint pairs are denoted by S), taking again just one representative from each symmetry class.

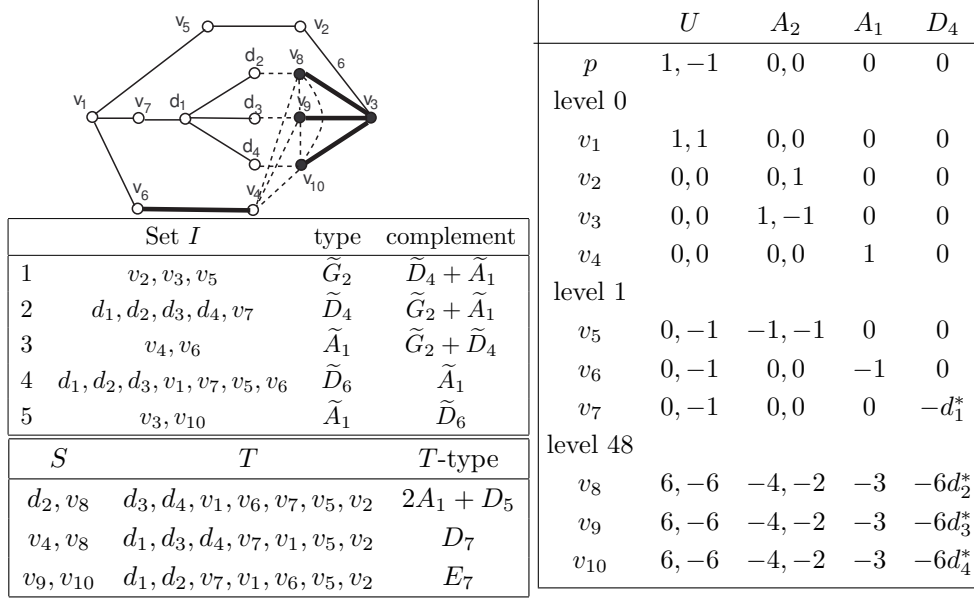FIGURE 9. Vinberg's vectors and their Coxeter's graph for $U + A_2 + A_1 + D_4$

			U	A_2	A_1	D_4
	p		$1,-1$	$0,0$	0	0
	level 0					
	v_1		$1,1$	$0,0$	0	0
	v_2		$0,0$	$0,1$	0	0
	v_3		$0,0$	$1,-1$	0	0
	v_4		$0,0$	$0,0$	1	0
	level 1					
	v_5		$0,-1$	$-1,-1$	0	0
	v_6		$0,-1$	$0,0$	-1	0
	v_7		$0,-1$	$0,0$	0	$-d_1^*$
	level 48					
	v_8		$6,-6$	$-4,-2$	-3	$-6d_2^*$
	v_9		$6,-6$	$-4,-2$	-3	$-6d_3^*$
	v_{10}		$6,-6$	$-4,-2$	-3	$-6d_4^*$

	Set I	type	complement
1	v_2, v_3, v_5	$\widetilde{G}_2$	$\widetilde{D}_4 + \widetilde{A}_1$
2	d_1, d_2, d_3, d_4, v_7	$\widetilde{D}_4$	$\widetilde{G}_2 + \widetilde{A}_1$
3	v_4, v_6	$\widetilde{A}_1$	$\widetilde{G}_2 + \widetilde{D}_4$
4	$d_1, d_2, d_3, v_1, v_7, v_5, v_6$	$\widetilde{D}_6$	$\widetilde{A}_1$
5	v_3, v_{10}	$\widetilde{A}_1$	$\widetilde{D}_6$

S	T	T-type
d_2, v_8	$d_3, d_4, v_1, v_6, v_7, v_5, v_2$	$2A_1 + D_5$
v_4, v_8	$d_1, d_3, d_4, v_7, v_1, v_5, v_2$	D_7
v_9, v_{10}	$d_1, d_2, v_7, v_1, v_6, v_5, v_2$	E_7

FIGURE 10. Vinberg's vectors and their Coxeter's graph for $-A_1 + \langle 6 \rangle + E_8$

			$-A_1$	$\langle 6 \rangle$	E_8
	p		1	0	0
	level 0				
	v_1		0	1	0
	level 4				
	v_2		1	0	$-e_8^*$
	level 12				
	v_3		3	-1	$-3e_8^*$
	v_4		3	-2	0
	level 16				
	v_5		2	-1	$-e_1^*$

	Set I	type	complement
1	$v_2, e_1, e_2, e_3, e_4, e_5, e_6, e_7$	$\widetilde{E}_7$	$\widetilde{A}_1$
2	v_1, v_3	$\widetilde{A}_1$	$\widetilde{E}_7$
3	$v_2, v_5, e_1, e_2, e_3, e_4, e_5$	$\widetilde{D}_6$	$\widetilde{G}_2$
4	v_3, v_7, e_8	$\widetilde{G}_2$	$\widetilde{D}_6$

S	T	T-type
v_1, v_4	$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$	E_8
v_1, v_5	$v_2, e_2, e_3, e_4, e_5, e_6, e_7, e_8$	$A_1 + D_7$
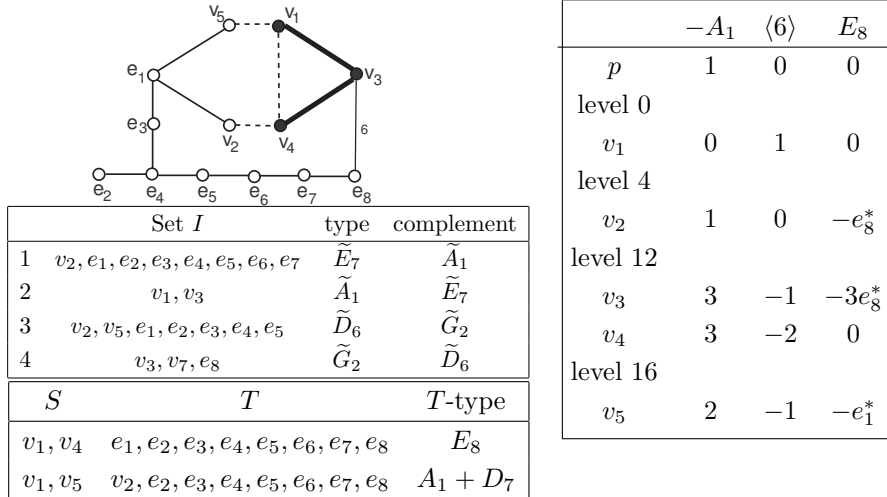
To conclude, we note that although Coxeter's graphs on Fig. 9 and 10 have symmetries that interchange 6-roots, such symmetries do not change the A_2-component (in Fig. 9), or the $\langle 6 \rangle$-component (in Fig. 10) modulo 3. Thus, by Proposition 2.3.2 these symmetries are $\mathbb{Z}/3$-direct and the corresponding lattices are chiral. $\square$

4.3.3. **Corollary.** *In Table 3 all lattices in the fourth row and below are chiral provided $\rho + d \leqslant 12$.*

Proof. Only the first seven rows of Table 3 contain some lattices with $\rho + d \leqslant 12$. The claim for the fourth and fifth rows follows from Proposition 4.3.2 and Lemma 4.2.1. For the sixth and seventh row, $\rho + d \leqslant 12$ holds only for $t = 0$, that is for lattices $-A_1 + A_2 + E_8$ and $U + A_2 + E_8$ which were analyzed in [FK2], and shown there to be achiral. $\square$

4.4. **Extension of the root system.** In this subsection, to prove chirality of lattices we apply a bit different method. Namely, we enlarge the root system by adding 4-roots to 2- and 6-roots. This extends the reflection group with new reflections (defined by 4-roots) and leads to subdivision of initial fundamental polyhedra (defined by 2- and 6-roots) into more simple fundamental polyhedra corresponding to the extended reflection group. The following lemma shows that we still keep control over the symmetries of the initial fundamental polyhedra.

4.4.1. **Lemma.** *Let W be a subgroup of a discrete group G generated by reflections in a hyperbolic space Λ. Then, for any fundamental polyhedron Π of G, there in only one fundamental polyhedron P of W containing Π, and the group of symmetries of P is generated by the group of symmetries of Π and those reflections in facets of Π that do not belong to G.*

Proof. Straightforward consequence of another simple, well known, lemma stating that for any discrete reflection group the group of symmetries of the system of its mirror hyperplanes is a semi-direct product of the group it-self with the group of symmetries of any of its fundamental polyhedra. $\square$

To find a sequence of roots defining a fundamental polyhedron of the extended reflection group we use Vinberg's algorithm like before. In corresponding Coxeter's graphs the 4-roots are marked by encircled black squares.

4.4.2. **Proposition.** *Lattices $-A_1 + A_2 + 4A_1$ and $-A_1 + A_2 + A_1 + D_4 = U + A_2 + 4A_1$ are chiral.*

Proof. We extend the group W generated by the reflections R_v, $x \mapsto x - 2\frac{ex}{e^2}v$, in the 2- and 6-roots, up to a group G generated, in addition, by the reflections $x \mapsto x - 2\frac{ex}{e^2}e$ in the 4-roots e, that is $e \in \Lambda$ with $e^2 = 4, e \cdot \Lambda = 0 \mod 2$. Applying Vinberg's algorithm to the group G we obtain the lists of roots shown in the rightmost tables of Figures 11 and 12, and derive from them the corresponding Coxeter's graphs. Completeness of these root-lists follows easily from Theorem 2.4.1 and Proposition 2.4.2: see the list of connected parabolic subgraphs supplied with the type of parabolic subgraphs that complement them to a rank $n - 1$ parabolic graph, and the list of the elliptic complements to Lannér's subgraphs (here, they are limited to dotted edges)

None of these graphs has a non-trivial symmetry. Hence, according to Lemma 4.4.1, the reflections defined by the 4-roots from our lists preserve a certain fundamental polyhedron P of W and generate all the symmetries of P. The action of each of these 4-reflections on the A_2-component is trivial modulo 3. Due to Proposition 2.3.2 this implies that all the P-direct symmetries are $\mathbb{Z}/3$-preserving. $\square$

4.4.3. **Corollary.** *Lattices in the second and third rows of table 3 with $\rho + d \leqslant 12$ are chiral.*

FIGURE 11. Vinberg's vectors and their Coxeter's graph for $-A_1 + A_2 + A_1 + D_4 = U + A_2 + 4A_1$

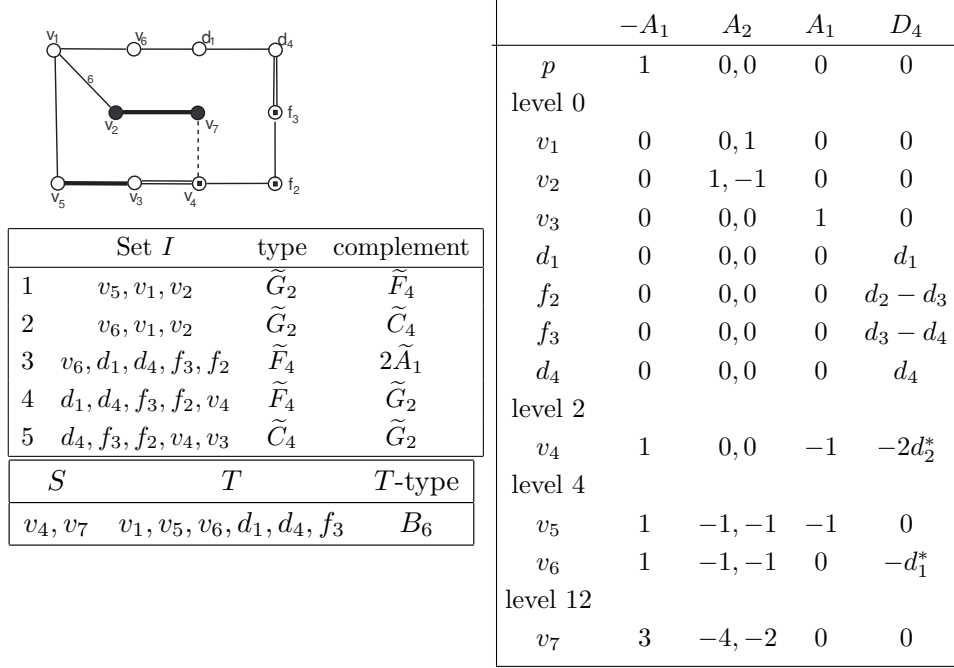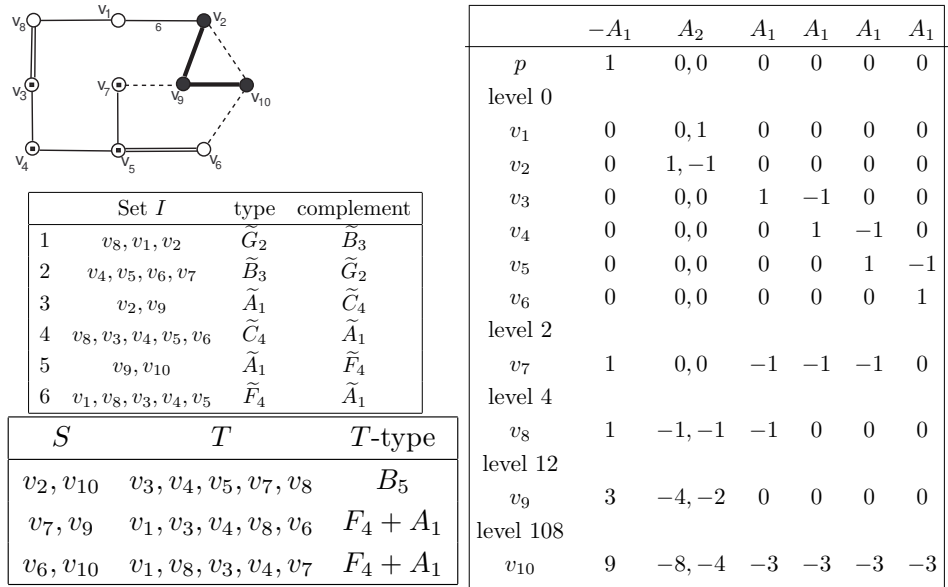|	Set I	type	complement
1	v_5, v_1, v_2	$\widetilde{G}_2$	$\widetilde{F}_4$
2	v_6, v_1, v_2	$\widetilde{G}_2$	$\widetilde{C}_4$
3	v_6, d_1, d_4, f_3, f_2	$\widetilde{F}_4$	$2\widetilde{A}_1$
4	d_1, d_4, f_3, f_2, v_4	$\widetilde{F}_4$	$\widetilde{G}_2$
5	d_4, f_3, f_2, v_4, v_3	$\widetilde{C}_4$	$\widetilde{G}_2$

S	T	T-type
v_4, v_7	$v_1, v_5, v_6, d_1, d_4, f_3$	B_6

	$-A_1$	A_2	A_1	D_4
p	1	$0,0$	0	0
level 0				
v_1	0	$0,1$	0	0
v_2	0	$1,-1$	0	0
v_3	0	$0,0$	1	0
d_1	0	$0,0$	0	d_1
f_2	0	$0,0$	0	$d_2 - d_3$
f_3	0	$0,0$	0	$d_3 - d_4$
d_4	0	$0,0$	0	d_4
level 2				
v_4	1	$0,0$	-1	$-2d_2^*$
level 4				
v_5	1	$-1,-1$	-1	0
v_6	1	$-1,-1$	0	$-d_1^*$
level 12				
v_7	3	$-4,-2$	0	0

FIGURE 12. Vinberg's vectors and their Coxeter's graph for $-A_1 + A_2 + 4A_1$

	Set I	type	complement
1	v_8, v_1, v_2	$\widetilde{G}_2$	$\widetilde{B}_3$
2	v_4, v_5, v_6, v_7	$\widetilde{B}_3$	$\widetilde{G}_2$
3	v_2, v_9	$\widetilde{A}_1$	$\widetilde{C}_4$
4	v_8, v_3, v_4, v_5, v_6	$\widetilde{C}_4$	$\widetilde{A}_1$
5	v_9, v_{10}	$\widetilde{A}_1$	$\widetilde{F}_4$
6	v_1, v_8, v_3, v_4, v_5	$\widetilde{F}_4$	$\widetilde{A}_1$

S	T	T-type
v_2, v_{10}	v_3, v_4, v_5, v_7, v_8	B_5
v_7, v_9	v_1, v_3, v_4, v_8, v_6	$F_4 + A_1$
v_6, v_{10}	v_1, v_8, v_3, v_4, v_7	$F_4 + A_1$

	$-A_1$	A_2	A_1	A_1	A_1	A_1
p	1	$0,0$	0	0	0	0
level 0						
v_1	0	$0,1$	0	0	0	0
v_2	0	$1,-1$	0	0	0	0
v_3	0	$0,0$	1	-1	0	0
v_4	0	$0,0$	0	1	-1	0
v_5	0	$0,0$	0	0	1	-1
v_6	0	$0,0$	0	0	0	1
level 2						
v_7	1	$0,0$	-1	-1	-1	0
level 4						
v_8	1	$-1,-1$	-1	0	0	0
level 12						
v_9	3	$-4,-2$	0	0	0	0
level 108						
v_{10}	9	$-8,-4$	-3	-3	-3	-3

Proof. It follows from Proposition 4.4.2 and Lemma 4.2.1.   □

4.5. **Proof of Theorem 4.2.2.** Lemma 4.2.1 shows existence of a border between the chiral and achiral lattices in each row of Table 3. For the first row the border is found in Proposition 4.3.1. For the other rows, the border is the line $\rho + d = 14$, since for $\rho + d \geqslant 14$ the lattices are achiral by Proposition 4.2.5, and for $\rho + d < 14$ are chiral by Corollaries 4.3.3 and 4.4.3.   □

4.6. **Proof of Theorem 1.2.1.** Theorem 2.2.1 reduces the question on chirality of non-singular real cubic fourfolds X to analysis of their lattices $\mathbb{M}^0_+(X)$. These lattices are listed in Tables 2 (even lattices) and 3 (odd lattices). The required analysis is performed in Theorems 3.1.1 and 4.2.2 and yields 18 chiral lattices described in Theorem 1.2.1.   □

5. CONCLUDING REMARKS

5.1. **Topological chirality.** It is natural to ask if it is possible for some chiral real non-singular cubic fourfold $X \subset P^5$ to distinguish it from its mirror image just by the topology of embedding of $X_\mathbb{R}$ into $P^5_\mathbb{R}$. More precisely, is it possible that $X_\mathbb{R}$ and its mirror image are not isotopic? The answer turns out to be in the negative. Indeed, all the coarse deformation classes shown on the upper-right side of Table 1 are achiral, and hence, for any X from these classes, $X_\mathbb{R}$ is isotopic to its mirror image. On the other hand, as is shown in [FK3], starting from these classes and performing surgeries through cuspidal-strata only, one can reach all the other classes except, in notation of [FK3], $\mathcal{C}^{10,1}$ and $\mathcal{C}^{2,1}_I$ (that is, in Table 1, the classes with even parity and (ρ, d) equal to $(20, 0)$ and $(12, 8)$), which are achiral by the results of the present paper. Hence, there remain to notice that a surgery through a cuspidal-stratum creates a handle embedded into $P^5_\mathbb{R}$ in a standard "non-knotted" way.

5.2. **Pointwise achirality.** If a real cubic $X \subset P^5$ is symmetric with respect to some hyperplane in $P^5_\mathbb{R}$, then X is obviously achiral. A naturally arising question is whether the converse is true: *if a coarse deformation class is achiral, does it contain a representative which is symmetric with respect to some mirror reflection?* The methods of this paper can be developed further to respond to this question as well, but requires an essential additional work. Already first inspection shows existence of such a representative (symmetric across a hyperplane) in each of the 3 achiral classes from the two bottom lines of Table 1 (that is all the achiral classes of M- and $(M-1)$-cubics).

Another approach is to look for explicit mirror-symmetric equations. For example, such an equation can be always written in an appropriate coordinate system in the form $f_3(x_0, \ldots, x_4) + x_5^2 f_1(x_0, \ldots, x_4) = 0$ where $x_5 = 0$ is the mirror-hyperplane and $f_3 = 0$ defines a real non-singular threefold cubic in P^4 transversal to a real hyperplane $f_1 = 0$ (compare, f.e., [LPZ]). This led us to a recurrent procedure that produces a sequence of maximal symmetric real non-singular cubic hypersurfaces in consecutive dimensions (the details are to be exposed in a separate publication).

Note finally that as was already pointed in [FK2], the equation $t(x_0^3 + \cdots + x_5^3) - (x_0 + \cdots + x_5)^3 = 0$ provides symmetric (across hyperplanes $x_i = x_j$) representatives for 4 achiral classes of cubics. Each of these classes is determined by topology of the real locus which is as follows: $\mathbb{RP}^4$ for $t < 0$ and $t > 36$, $\mathbb{RP}^4 \sqcup S^4$ for $16 < t < 36$, $\mathbb{RP}^4 \# 5(S^1 \times S^3)$ for $4 < t < 16$, and $\mathbb{RP}^4 \# 10(S^2 \times S^3)$ for $0 < t < 4$.

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