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EXPERIMENTING WITH SYMPLECTIC HYPERGEOMETRIC MONODROMY GROUPS

A. S. DETINKO, D. L. FLANNERY, AND A. HULPKE

ABSTRACT. We present new computational results for symplectic monodromy groups of hypergeometric differential equations. In particular, we compute the arithmetic closure of each group, sometimes justifying arithmeticity. The results are obtained by extending our previous algorithms for Zariski dense groups, based on the strong approximation and congruence subgroup properties.

1. INTRODUCTION

This paper continues work in [6], which treated symplectic monodromy groups of hypergeometric differential equations as a test case. Deciding arithmeticity of such a group in its Zariski closure is a basic problem (see [16, Section 3.5] and [2, p. 326]). More generally, one asks whether the group is arithmetic, or whether it is *thin*, i.e., Zariski dense but not arithmetic in the ambient algebraic group. This problem has received considerable attention. It was solved completely for monodromy groups associated with Calabi-Yau manifolds [3, 17, 18], which are 4-dimensional symplectic linear groups over $\mathbb{Q}$. Note also the results of [9], that demonstrate thinness of certain orthogonal hypergeometric monodromy groups.

Our approach to all questions emphasizes computer-aided experimentation. We compute the arithmetic closure $\mathrm{cl}(H)$ of a dense group H, the 'closest' arithmetic overgroup of H. Then $\mathrm{cl}(H)$ is used to investigate H. Sometimes we are able to prove that H is arithmetic. Moreover, we process large amounts of data by computer, producing information about all symplectic hypergeometric monodromy groups of a specified degree.

Our methods are based on the strong approximation and congruence subgroup properties for the symplectic group. In Section 2, we extend algorithms developed in [6] for dense subgroups of $\mathrm{Sp}(n, \mathbb{Z})$ to accept dense subgroups of $\mathrm{Sp}(n, \mathbb{Q})$. Section 3 provides relevant background on hypergeometric groups, and details of

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our experimental strategy. Output for all dense hypergeometric monodromy subgroups of the symplectic group of degree 4 is tabulated in the Appendix. As further illustration, sample output for groups of degree 6 is also given.

We set down some notation. Let $S = \{g_1, \ldots, g_r\}$ be a generating set of $H \leq \mathrm{GL}(n, \mathbb{Q})$. The subring of $\mathbb{Q}$ generated by the entries of the g_i and g_i^{-1} will be denoted R. Thus $R = \frac{1}{\mu}\mathbb{Z}$ for a positive integer μ. If m is coprime to μ then the congruence homomorphism φ_m induced by natural surjection $\mathbb{Z} \to \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ maps $\mathrm{GL}(n, R)$ into $\mathrm{GL}(n, \mathbb{Z}_m)$.

Throughout, $\mathbb{F}$ is a field and 1_m is the $m \times m$ identity matrix. Let V be the $\mathbb{F}$-vector space of dimension $n = 2s > 2$, and let Φ be the matrix of a non-degenerate skew-symmetric bilinear form on V with respect to a basis of V. The full symplectic group in $\mathrm{GL}(n, \mathbb{F})$ preserving Φ is denoted $\mathrm{Sp}(\Phi, \mathbb{F})$. If $D \subseteq \mathbb{F}$ is a unital subring then $\mathrm{Sp}(\Phi, D) := \mathrm{Sp}(\Phi, \mathbb{F}) \cap \mathrm{GL}(n, D)$. We write $\mathrm{Sp}(n, D)$ instead of $\mathrm{Sp}(\Phi, D)$ if

$$\Phi = J_n := \begin{pmatrix} 0_s & 1_s \\ -1_s & 0_s \end{pmatrix}.$$

Since $\mathrm{Sp}(\Phi, \mathbb{F})$ and $\mathrm{Sp}(n, \mathbb{F})$ are $\mathrm{GL}(n, \mathbb{F})$-conjugate, often it suffices to deal with the latter rather than the former group. The shorthand Sp_n denotes the symplectic group when field and matrix of the form do not matter.

2. Computing with dense subgroups of symplectic groups

In this section we establish the theoretical foundation for our algorithms.

2.1. **Strong approximation and computing.** Let $H \leq \mathrm{Sp}(n, \mathbb{Q})$ be dense. The strong approximation theorem guarantees that H surjects onto $\mathrm{Sp}(n, p)$ for almost all primes $p \in \mathbb{Z}$ [13, Window 9]. Let $\Pi(H)$ be the (finite) set of primes p such that $\varphi_p(H) \neq \mathrm{Sp}(n, p)$. Below we discuss how to compute $\Pi(H)$.

In [8] we developed a method to compute the set of primes p such that $\varphi_p(H) \neq \mathrm{SL}(n, p)$ for dense $H \leq \mathrm{SL}(n, \mathbb{Q})$. This relies on irreducibility testing of the adjoint module of H, and uses the classification of maximal subgroups of $\mathrm{SL}(n, p)$. Something similar could be done for dense subgroups of $\mathrm{Sp}(n, \mathbb{Q})$.

For a dense subgroup H of $\mathrm{SL}(n, \mathbb{Z})$ or $\mathrm{Sp}(n, \mathbb{Z})$, another way to compute $\Pi(H)$ is described in [6, Section 3] (see also [7, Section 2.5]). Here we must know a transvection in H (recall: a transvection $\tau \in \mathrm{GL}(n, \mathbb{F})$ is a unipotent element such that $1_n - \tau$ has rank 1). Although an arbitrary dense subgroup of Sp_n may not contain a transvection, the groups in our experiments do.

Proposition 2.1. *Suppose that $H \leq \mathrm{Sp}(n, \mathbb{Q})$ contains a transvection τ. Then H is dense if and only if $\langle \tau \rangle^H$ is absolutely irreducible.*

Proof. The proof of [6, Proposition 3.7] for $H \leq \mathrm{Sp}(n,\mathbb{Z})$ remains valid for $H \leq \mathrm{Sp}(n,\mathbb{Q})$. □

Proposition 2.1 allows us to apply the procedure $\texttt{IsDense}(H,\tau)$ from [6, Section 3.2] to test density of $H \leq \mathrm{Sp}(n,\mathbb{Q})$ knowing a transvection $\tau \in H$. Given dense H, we compute $\Pi(H)$ using $\texttt{PrimesForDense}(H,\tau)$ from [6, Section 3.2] as follows. Let $\{A_1,\ldots,A_{n^2}\}$ be a basis of the enveloping algebra $\langle N\rangle_{\mathbb{Q}}$, where $N = \langle\tau\rangle^H$ and the A_i are words in S. We can find a finite set Π_1 of primes such that the $\varphi_p(A_i)$ are linearly independent and $\varphi_p(1_n - \tau) \neq 0$ for any prime $p \notin \Pi_1$. That is, if $p \notin \Pi_1$ then $\varphi_p(N)$ is absolutely irreducible and contains the transvection $\varphi_p(\tau)$; so $\varphi_p(H) = \mathrm{Sp}(n,p)$ by [6, Theorem 3.2]. Thus $\Pi(H) \subseteq \Pi_1$. We obtain $\Pi(H)$ after checking whether $\varphi_p(H) = \mathrm{Sp}(n,p)$ for each $p \in \Pi_1$. This last step uses recognition algorithms for matrix groups over finite fields [14].

2.2. **Integrality and computing the $\mathbb{Z}$-intercept.** Some of our algorithms require us to compute the '$\mathbb{Z}$-points' $H_{\mathbb{Z}} := H \cap \mathrm{GL}(n,\mathbb{Z})$ of input $H \leq \mathrm{GL}(n,\mathbb{Q})$. This is possible by the next result.

Lemma 2.2 ([5, Lemma 5.1])**.** *For a finitely generated subgroup H of $\mathrm{GL}(n,\mathbb{Q})$, the following are equivalent:*

- *H is integral, i.e., H is conjugate to a subgroup of $\mathrm{GL}(n,\mathbb{Z})$,*
- *$|H : H_{\mathbb{Z}}|$ is finite,*
- *there exists a positive integer d such that dH consists of $\mathbb{Z}$-matrices.*

In [5, Section 5] we explain how to find d if $|H : H_{\mathbb{Z}}|$ is finite. The procedure $\texttt{IntegralIntercept}(S,d)$ from [4] then computes a generating set of $H_{\mathbb{Z}}$. However, its practicality is limited. In our experiments, we calculated a transversal of $H_{\mathbb{Z}}$ in H using an orbit algorithm for the multiplication action by H, starting with 1_n. Suppose that g is an image so obtained. We test whether $gh^{-1} \in \mathrm{GL}(n,\mathbb{Z})$ for each known orbit element h. If this happens for some h then g lies in the same coset of $H_{\mathbb{Z}}$ (and will yield a Schreier generator of $H_{\mathbb{Z}}$). If no such h exists then g is a representative of a new coset.

We avail of the following reduction when $|H : H_{\mathbb{Z}}|$ is large. Let $\sigma \in \mathbb{Z}$ be divisible by a set of primes dividing the denominators of entries in elements of H; so $\frac{1}{\sigma}\mathbb{Z} \subseteq R$. Then $H_{\mathbb{Z}} \leq K \leq H$ for $K = H \cap \mathrm{Sp}(n, \frac{1}{\sigma}\mathbb{Z})$. Membership in K is tested by inspection of matrix denominators. We thus divide the transversal length into two factors, first calculating a transversal of K in H, and then a transversal of $H_{\mathbb{Z}}$ in K.

A potential complication is too many Schreier generators for $H_{\mathbb{Z}}$. Rather than keeping them all, we randomly select about 300 subproducts of Schreier generators for each transversal step (cf. [1]). Conceivably we may not then compute all of $H_{\mathbb{Z}}$, but merely a proper subgroup. At the end we therefore verify, by a calculation in the congruence image modulo the level of $H_{\mathbb{Z}}$ (see Section 2.3), that all Schreier generators indeed lie in the subgroup generated by the chosen set.

2.3. **The congruence subgroup property and computing.** Suppose that $H \leq \mathrm{Sp}(n, \mathbb{Q})$ is arithmetic, i.e., commensurable with $\mathrm{Sp}(n, \mathbb{Z})$. Since the congruence subgroup property holds for $\mathrm{Sp}(n, \mathbb{Z})$, it follows that $H_{\mathbb{Z}}$ contains a principal congruence subgroup, i.e., the kernel of φ_r on $\mathrm{Sp}(n, \mathbb{Z})$ for some integer $r > 1$. The *level* of H, denoted $M(H)$, is the modulus m of the unique maximal principal congruence subgroup $\ker \varphi_m \cap \mathrm{Sp}(n, \mathbb{Z})$ in $H_{\mathbb{Z}}$.

Now suppose that $H \leq \mathrm{Sp}(n, \mathbb{Z})$ is dense. The *arithmetic closure* $\mathrm{cl}(H)$ of H in $\mathrm{Sp}(n, \mathbb{Z})$ is the intersection of all arithmetic subgroups of $\mathrm{Sp}(n, \mathbb{Z})$ containing H (see [6, Section 3.3]). If $H \leq \mathrm{Sp}(n, \mathbb{Q})$ is not necessarily arithmetic, but $H_{\mathbb{Z}}$ is dense, then we set $M(H) = M(\mathrm{cl}(H_{\mathbb{Z}}))$.

The level of a dense subgroup H of $\mathrm{Sp}(n, \mathbb{Z})$ is a key component of various algorithms for computing with the group [6]. If $|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(H)|$ is not too large, then we may test arithmeticity by coset enumeration [11, Chapter 5].

The algorithm `LevelMaxPCS` from [6] returns $M(H)$ for an input dense subgroup H of $\mathrm{Sp}(n, \mathbb{Z})$ and $\Pi(H)$. Here we will use `LevelMaxPCS`$(H_{\mathbb{Z}}, \Pi(H_{\mathbb{Z}}))$, which by definition returns $M(H)$. Note that while $\Pi(H) \subseteq \Pi(H_{\mathbb{Z}})$, these sets need not coincide. For example, it could be that p divides μ and hence $p \notin \Pi(H)$, whereas $p \in \Pi(H_{\mathbb{Z}})$, i.e., $p \mid M(H)$.

3. Hypergeometric groups

3.1. **Background.** We adhere mainly to the notation and definitions in [2]. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, where $a_j,\, b_k \in \mathbb{C}^{\times}$ and $a_j \neq b_k$ for $1 \leq j, k \leq n$. A subgroup of $\mathrm{GL}(n, \mathbb{C})$ generated by elements h_∞, h_0 such that $\det(t1_n - h_\infty) = \prod_{j=1}^{n}(t - a_j)$ and $\det(t1_n - h_0^{-1}) = \prod_{j=1}^{n}(t - b_j)$ is called a *hypergeometric group*, and denoted $H(a, b)$. It is absolutely irreducible by [2, Proposition 3.3]. The element $h_1 := (h_0 h_\infty)^{-1}$ of $H(a, b)$ is a reflection, i.e., $h_1 - 1_n$ has rank 1.

If $a_j = \exp(2\pi \mathrm{i} \alpha_j)$ and $b_j = \exp(2\pi \mathrm{i} \beta_j)$ for $\alpha_j, \beta_j \in \mathbb{C}$ then $H(a, b)$ is the monodromy group of a hypergeometric differential equation [2, Proposition 3.2].

Theorem 3.1 ([2, Theorem 3.5])**.** *For a_j, b_k as above, let*

$$f(t) = \textstyle\prod_{j=1}^{n}(t - a_j) = t^n + A_1 t^{n-1} + \cdots + A_n$$

and

$$g(t) = \textstyle\prod_{j=1}^{n}(t - b_j) = t^n + B_1 t^{n-1} + \cdots + B_n.$$

Further, let

$$A = \begin{pmatrix} 0 & \cdots & 0 & -A_n \\ 1 & \cdots & 0 & -A_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -A_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \cdots & 0 & -B_n \\ 1 & \cdots & 0 & -B_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -B_1 \end{pmatrix}.$$

Then $h_\infty = A$*,* $h_0 = B^{-1}$ *generate a hypergeometric group* $H(a,b)$ *for* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$*. Any hypergeometric group with the same* a*,* b *is* $\mathrm{GL}(n, \mathbb{C})$*-conjugate to this one.*

We are concerned with $H(a,b)$ that are

(i) symplectic,
(ii) dense in Sp_n,
(iii) integral.

There are only finitely many $\mathrm{GL}(n, \mathbb{Q})$-conjugacy classes of such $H(a,b)$. By [2, Proposition 6.1], $H(a,b)$ is symplectic if and only if $\{a_1, \ldots, a_n\} = \{a_1^{-1}, \ldots, a_n^{-1}\}$, $\{b_1, \ldots, b_n\} = \{b_1^{-1}, \ldots, b_n^{-1}\}$, and $\delta := \det(h_1) = 1$ (whence h_1 is a transvection). We remark that $H(a,b)$ need not be dense in Sp_n (by, e.g., [2, Theorem 6.5]). Additionally, $H(a,b)$ is integral if and only if the a_j and b_k are roots of unity. Hence the characteristic polynomials $f(t)$, $g(t)$ of A, B should be products of coprime cyclotomic polynomials. Under these conditions, $H(a,b) \leq \mathrm{Sp}(\Phi, \mathbb{Z})$ for some Φ. Since $H(a,b)$ is absolutely irreducible, Φ is unique up to a scalar multiple.

3.2. **Experiments.** Assuming that $H(a,b) \leq \mathrm{GL}(n, \mathbb{Q})$ satisfy the requirements (i), (ii), (iii) of Section 3.1, we proceed as follows.

(I) We list all pairs $f(t)$, $g(t)$ of polynomials of degree n where each is the product of coprime cyclotomic polynomials, and such that $\delta = 1$ (for h_∞, h_0 as in Theorem 3.1). Non-dense $H(a,b)$ are excluded by running $\texttt{IsDense}(H(a,b), h_1)$.

(II) The matrix Φ of a symplectic form fixed by $H(a,b)$ may be interpreted as a homomorphism between the natural module of $H(a,b)$ and its dual. We use MeatAxe methods [11, Section 7.5.2] to compute Φ. Next, $g \in \mathrm{GL}(n, \mathbb{Q})$ such that $gJ_ng^\top = \Phi$ is found by simple linear algebra. Then $L = L(a,b) := g^{-1}H(a,b)g \leq \mathrm{Sp}(n, \mathbb{Q})$. (We seek a copy of $H(a,b)$ that preserves the standard form because it is more convenient for computing; e.g., we have a presentation of $\mathrm{Sp}(n, \mathbb{Z})$ but not of $\mathrm{Sp}(\Phi, \mathbb{Z})$.)

Since Φ is not strictly unique, and g can vary by factors stabilizing the form, L depends on the choices made. These might also impact $|L : L_{\mathbb{Z}}|$, which we want to keep small for reasons of efficiency and to reduce the number of Schreier generators arising (remember that $|L : L_{\mathbb{Z}}| < \infty$ by Lemma 2.2). Our code therefore uses heuristics to determine candidates for Φ and g. It then calculates L and the least common multiple $\overline{k}$ of $\{k \mid l^k \in L_{\mathbb{Z}}$ for all generators $l \in L\}$ (as a stand-in for $|L : L_{\mathbb{Z}}|$), and chooses g such that $\overline{k}$ is minimal.

(III) We compute $L_{\mathbb{Z}}$ (see Section 2.2). The group L contains the transvection $h = g^{-1}h_1 g$. Although perhaps $h \notin L_{\mathbb{Z}}$, we can always find a transvection $\lambda = h^k \in L_{\mathbb{Z}}$ for some k.

(IV) We compute

$\Pi(L_{\mathbb{Z}}) =$ `PrimesForDense`$(L_{\mathbb{Z}}, \lambda)$,
`LevelMaxPCS`$(L_{\mathbb{Z}}, \Pi(L_{\mathbb{Z}}))$,
$|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(L_{\mathbb{Z}})|$.

(V) When $|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(L_{\mathbb{Z}})|$ is sufficiently small, we express the generators of $L_{\mathbb{Z}}$ as words in generators of $\mathrm{Sp}(n, \mathbb{Z})$ [12], and try to find $|\mathrm{Sp}(n, \mathbb{Z}) : L_{\mathbb{Z}}|$ by coset enumeration. If this succeeds, i.e., confirms that the index of $L_{\mathbb{Z}}$ is equal to that of $\mathrm{cl}(L_{\mathbb{Z}})$, then we have proved arithmeticity of $L_{\mathbb{Z}}$ and thereby also of $H(a, b)$ (cf. [15, Theorem 4.1, p. 204]).

As its cost is bounded below by the index, we restricted our attempts at coset enumeration to groups with (presumed) indices less than 10^7. If the index was expected to be in the range $10^7, \ldots, 10^{14}$, then we tried to find an intermediate subgroup $L_{\mathbb{Z}} < U < \mathrm{Sp}(n, \mathbb{Z})$ such that $|U : L_{\mathbb{Z}}| \leq 10^7$. Enumeration was then undertaken with a presentation for U found by Reidemeister-Schreier rewriting [11, Chapter 5]. Suitable U are generated by $L_{\mathbb{Z}}$ together with congruence subgroups in $\mathrm{Sp}(n, \mathbb{Z})$ of level dividing the level of $\mathrm{cl}(L_{\mathbb{Z}})$.

By [18, Theorem 1.1], if the leading coefficient of $f(t) - g(t)$ has absolute value at most 2 then $H(a, b)$ is arithmetic in $\mathrm{Sp}(\Phi, \mathbb{Z})$. At least in degree 4, we proved arithmeticity (and computed the level and index) whenever the criterion from [18] applies, and occasionally when it does not. Unfortunately, we lack a method for proving non-arithmeticity if coset enumeration fails.

APPENDIX

Our algorithms have been implemented in GAP [10]. In this appendix we present the complete experimental results for $n = 4$ (Table 1), and a sample for $n = 6$ (Table 2). The experimental results for all 916 groups of degree 6 are available at `https://www.math.colostate.edu/~hulpke/paper/hypergeom6.pdf`.

A group with entry ≤ 60 in column 'Nr' of Table 1 has the same number in [18, Table 1], and Nr $= m \geq 100$ here matches number $m - 100$ in [18, Table 2]. The column 'Polynomials' lists $f(t)$, $g(t)$ satisfying (I) in Section 3.2 (the Nr entries in Table 2 stem from the listing of these polynomials). All primes divisors of μ are given in column 'Mu'. 'Int' is $|L : L_{\mathbb{Z}}|$. 'iLevel' and 'iIndex' are level and index of $\mathrm{cl}(L_{\mathbb{Z}})$ in $\mathrm{Sp}(n, \mathbb{Z})$. 'Coeff' is the absolute value of the leading coefficient of $f(t) - g(t)$. The column 'Enum' records whether coset enumeration was able to calculate $|\mathrm{Sp}(n, \mathbb{Z}) : L_{\mathbb{Z}}|$. We reiterate that if an enumeration succeeds (indicated by ✓), then the input group is arithmetic. A dash means that coset enumeration did not terminate (of course, this does not prove that the group is not arithmetic). In some cases, indicated by $\times$, large $|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(L_{\mathbb{Z}})|$ implies that coset enumeration is unlikely to succeed.

Indices in degree 4 were small enough to attempt coset enumeration, bar one example (Nr $= 101$). The size of indices for many groups of degree 6 dissuades any attempt at coset enumeration.

We comment on some test groups of interest.

Table 1 shows that the groups Nr $= 104, 109$ are arithmetic; the question is open in [18, Table 3]. We could not decide arithmeticity for the groups Nr $= 102$ and 106, just as in [18, Table 3].

Let H_n, G_n be $H(a, b)$ with $f(t) = (t - 1)^n$ and $g(t) = (t^{n+1} - 1)/(t - 1)$, $(t^{n+1} + 1)/(t + 1)$, respectively. The arithmeticity problem for H_n and G_n was posed at the 'Workshop on Thin Groups and Super Approximation', Institute for Advanced Study, Princeton, March 2016. If $n = 4$ then H_n is thin [3]; see row 107 in Table 1, or row 8 in [6, Table 3] for $\mathrm{cl}(H_4)$. The group G_4 is arithmetic [18, Corollary 1.4] (row 1 in [6, Table 3] and row 112 in Table 1). Note that indices stated here might differ from those in [6], due to different conjugating matrices (see (II), Section 3.2).

In degree 6, when [18, Theorem 1.1] does not apply we proved arithmeticity for (much) fewer groups; two notable exceptions are rows 468 and 534 of Table 2. We have not yet solved the arithmeticity problem for G_6 or H_6; however, the level and index of their arithmetic closures are given in rows 774 and 838 of Table 2.

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
1	$t^4-4t^3+6t^2-4t+1$ $t^4-2t^3+3t^2-2t+1$	1	1	2	$2{\cdot}5$	2	✓
2	$(t-1)^2(t+1)^2$ $t^4+2t^3+3t^2+2t+1$	3	2^2	$2{\cdot}3^2$	$2^7 3^4 5^2$	2	✓
3	$(t-1)^2(t+1)^2$ $(t^2+1)(t^2+t+1)$	$2, 3$	3^2	$2^4 3^2$	$2^{10} 3^3 5^2$	1	✓
4	$(t-1)^2(t+1)^2$ $t^4+t^3+t^2+t+1$	5	5	$2{\cdot}5^2$	$2^5 3^2 5{\cdot}13$	1	✓
5	$t^4-2t^3+3t^2-2t+1$ $(t-1)^2(t+1)^2$	3	2^2	$2{\cdot}3^2$	$2^7 3^4 5^2$	2	✓
6	$(t^2-t+1)(t^2+1)$ $(t-1)^2(t+1)^2$	$2, 3$	3^2	$2^4 3^2$	$2^{10} 3^3 5^2$	1	✓
7	$t^4-t^3+t^2-t+1$ $(t-1)^2(t+1)^2$	5	5	$2{\cdot}5^2$	$2^4 3^2 5{\cdot}13$	1	✓
8	$t^4+2t^3+3t^2+2t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	2	$2{\cdot}5$	2	✓
9	$(t-1)^2(t^2+t+1)$ t^4+2t^2+1	2	$2{\cdot}3$	2^4	$2^6 3^2 5$	1	✓
10	$(t-1)^2(t^2+t+1)$ $t^4+t^3+t^2+t+1$	$2, 5$	$2^3 3{\cdot}5^2$	$2^3 5^2$	$2^8 3^3 5^2 13$	2	✓
11	$t^4-2t^3+3t^2-2t+1$ $(t-1)^2(t^2+t+1)$	2	3^2	2^4	$2^8 3^2 5$	1	✓
12	$(t-1)^2(t^2+t+1)$ $(t+1)^2(t^2-t+1)$	2	2	2^4	$2^{11} 3{\cdot}5$	2	✓
13	$(t-1)^2(t^2+t+1)$ $(t^2-t+1)(t^2+1)$	1	1	2^2	$2^6 3{\cdot}5$	2	✓
14	$(t-1)^2(t^2+t+1)$ t^4+1	2	2	2^3	$2^2 3{\cdot}5$	1	✓
15	$(t-1)^2(t^2+t+1)$ $t^4-t^3+t^2-t+1$	1	1	2	$2{\cdot}3$	1	✓
16	$(t-1)^2(t^2+t+1)$ t^4-t^2+1	2	3	2^3	$2^3 3{\cdot}5$	1	✓
17	t^4+2t^2+1 $t^4+2t^3+3t^2+2t+1$	1	1	2	$2{\cdot}5$	2	✓
18	$(t+1)^2(t^2+1)$ $t^4+2t^3+3t^2+2t+1$	1	1	2	$2{\cdot}5$	1	✓
19	$t^4+t^3+t^2+t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	1	1	1	✓

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
20	$(t{+}1)^2(t^2{-}t{+}1)$ $t^4{+}2t^3{+}3t^2{+}2t{+}1$	2	3^2	2^4	$2^7 3^2 5$	1	✓
21	$t^4{+}1$ $t^4{+}2t^3{+}3t^2{+}2t{+}1$	1	1	2	$2{\cdot}5$	2	✓
22	$t^4{-}t^2{+}1$ $t^4{+}2t^3{+}3t^2{+}2t{+}1$	1	1	2^2	$2^6 3{\cdot}5$	2	✓
23	$t^4{+}t^3{+}t^2{+}t{+}1$ $(t{+}1)^2(t^2{+}t{+}1)$	1	1	2	$2{\cdot}3$	2	✓
24	$(t{-}1)^2(t^2{+}1)$ $t^4{-}2t^3{+}3t^2{-}2t{+}1$	1	1	2	$2{\cdot}5$	1	✓
25	$(t{-}1)^2(t^2{+}1)$ $(t^2{-}t{+}1)(t^2{+}t{+}1)$	3	3	$2{\cdot}3^2$	$2^5 3{\cdot}5^2$	2	✓
26	$(t{-}1)^2(t^2{+}1)$ $t^4{+}1$	1	1	2^2	$2^7 3^2 5$	2	✓
27	$(t{-}1)^2(t^2{+}1)$ $t^4{-}t^3{+}t^2{-}t{+}1$	1	1	2	$2{\cdot}3$	1	✓
28	$(t{-}1)^2(t^2{+}1)$ $t^4{-}t^2{+}1$	3	2	$2{\cdot}3^2$	$2^5 3{\cdot}5^2$	2	✓
29	$t^4{+}2t^2{+}1$ $t^4{+}t^3{+}t^2{+}t{+}1$	1	1	2	$2{\cdot}3$	1	✓
30	$t^4{-}2t^3{+}3t^2{-}2t{+}1$ $t^4{+}2t^2{+}1$	1	1	2	$2{\cdot}5$	2	✓
31	$t^4{+}2t^2{+}1$ $(t{+}1)^2(t^2{-}t{+}1)$	2	$2{\cdot}3$	2^4	$2^6 3^2 5$	1	✓
32	$t^4{-}t^3{+}t^2{-}t{+}1$ $t^4{+}2t^2{+}1$	1	1	2	$2{\cdot}3$	1	✓
33	$t^4{+}t^3{+}t^2{+}t{+}1$ $(t{+}1)^2(t^2{+}1)$	1	1	2	$2{\cdot}3$	1	✓
34	$(t^2{-}t{+}1)(t^2{+}t{+}1)$ $(t{+}1)^2(t^2{+}1)$	3	3	$2{\cdot}3^2$	$2^5 3{\cdot}5^2$	2	✓
35	$t^4{+}1$ $(t{+}1)^2(t^2{+}1)$	1	1	2^2	$2^7 3^2 5$	2	✓
36	$t^4{-}t^2{+}1$ $(t{+}1)^2(t^2{+}1)$	3	2	$2{\cdot}3^2$	$2^5 3{\cdot}5^2$	2	✓
37	$t^4{+}t^3{+}t^2{+}t{+}1$ $(t^2{+}1)(t^2{+}t{+}1)$	1	1	2	$2{\cdot}3$	1	✓
38	$(t{+}1)^2(t^2{-}t{+}1)$ $(t^2{+}1)(t^2{+}t{+}1)$	1	1	2^2	$2^6 3{\cdot}5$	2	✓

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
39	t^4+1 $(t^2+1)(t^2+t+1)$	1	1	2^3	$2^2 3{\cdot}5$	1	✓
40	$t^4-t^3+t^2-t+1$ $(t^2+1)(t^2+t+1)$	1	1	2	$2{\cdot}3$	2	✓
41	t^4-t^2+1 $(t^2+1)(t^2+t+1)$	$2,3$	$2{\cdot}3$	$2^3 3^2$	$2^6 3^2 5^2$	1	✓
42	$(t+1)^2(t^2-t+1)$ $t^4+t^3+t^2+t+1$	1	1	2	$2{\cdot}3$	1	✓
43	$(t^2-t+1)(t^2+t+1)$ $t^4+t^3+t^2+t+1$	1	1	1	1	1	✓
44	$(t^2-t+1)(t^2+1)$ $t^4+t^3+t^2+t+1$	1	1	2	$2{\cdot}3$	2	✓
45	t^4+1 $t^4+t^3+t^2+t+1$	1	1	2	$2{\cdot}3$	1	✓
46	$t^4-t^3+t^2-t+1$ $t^4+t^3+t^2+t+1$	1	1	2^2	$2^9 3^2$	2	✓
47	t^4-t^2+1 $t^4+t^3+t^2+t+1$	1	1	1	1	1	✓
48	$(t-1)^2(t^2-t+1)$ $t^4-t^3+t^2-t+1$	1	1	2	$2{\cdot}3$	2	✓
49	$t^4-2t^3+3t^2-2t+1$ t^4+1	1	1	2	$2{\cdot}5$	2	✓
50	$t^4-2t^3+3t^2-2t+1$ $t^4-t^3+t^2-t+1$	1	1	1	1	1	✓
51	$t^4-2t^3+3t^2-2t+1$ t^4-t^2+1	1	1	2^2	$2^6 3{\cdot}5$	2	✓
52	t^4+1 $(t+1)^2(t^2-t+1)$	2	2	2^3	$2^2 3{\cdot}5$	1	✓
53	$t^4-t^3+t^2-t+1$ $(t+1)^2(t^2-t+1)$	5	5	$2{\cdot}5^2$	$2^4 3^2 5{\cdot}13$	2	✓
54	t^4-t^2+1 $(t+1)^2(t^2-t+1)$	2	3	2^3	$2^3 3{\cdot}5$	1	✓
55	$t^4-t^3+t^2-t+1$ $(t^2-t+1)(t^2+t+1)$	1	1	1	1	1	✓
56	$(t^2-t+1)(t^2+1)$ t^4+1	1	1	2^3	$2^2 3{\cdot}5$	1	✓
57	$t^4-t^3+t^2-t+1$ $(t^2-t+1)(t^2+1)$	1	1	2	$2{\cdot}3$	1	✓

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
58	$(t^2{-}t{+}1)(t^2{+}1)$ $t^4{-}t^2{+}1$	$2, 3$	$2{\cdot}3$	$2^3 3^2$	$2^6 3^2 5^2$	1	✓
59	$t^4{-}t^3{+}t^2{-}t{+}1$ $t^4{+}1$	1	1	2	$2{\cdot}3$	1	✓
60	$t^4{-}t^3{+}t^2{-}t{+}1$ $t^4{-}t^2{+}1$	1	1	1	1	1	✓
101	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{+}4t^3{+}6t^2{+}4t{+}1$	2	2	2^8	$2^{41} 3^2 5$	8	×
102	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{+}2t^3{+}3t^2{+}2t{+}1$	1	1	$2{\cdot}3^4$	$2^8 3^{14} 5^2$	6	—
103	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $(t{+}1)^2(t^2{+}t{+}1)$	3	$2^2 3$	$2^4 3^3$	$2^{19} 3^7 5^2$	7	—
104	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{+}2t^2{+}1$	1	1	2^4	$2^{20} 3^2 5$	4	✓
105	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $(t{+}1)^2(t^2{+}1)$	2	2^3	2^8	$2^{27} 3^2 5$	6	—
106	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $(t^2{+}1)(t^2{+}t{+}1)$	$2, 3$	$2^3 3^2$	$2^4 3^3$	$2^{13} 3^8 5^2$	5	—
107	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{+}t^3{+}t^2{+}t{+}1$	1	1	$2{\cdot}5^2$	$2^8 3^3 5^8 13$	5	—
108	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $(t{+}1)^2(t^2{-}t{+}1)$	2	$2^3 3$	2^7	$2^{16} 3^2 5$	5	—
109	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $(t^2{-}t{+}1)(t^2{+}t{+}1)$	3	$2^2 3$	$2^2 3^3$	$2^{11} 3^6 5^2$	4	✓
110	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $(t^2{-}t{+}1)(t^2{+}1)$	2	$2{\cdot}3$	2^4	$2^7 3^2 5$	3	✓
111	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{+}1$	1	1	2^3	$2^{11} 3^2 5$	4	—
112	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{-}t^3{+}t^2{-}t{+}1$	1	1	2	$2{\cdot}3$	3	✓
113	$t^4{-}4t^3{+}6t^2{-}4t{+}1$ $t^4{-}t^2{+}1$	1	1	2^2	$2^5 5$	4	—
114	$(t{-}1)^2(t^2{+}t{+}1)$ $t^4{+}4t^3{+}6t^2{+}4t{+}1$	1	1	2^5	$2^{13} 3{\cdot}5$	5	—
115	$(t{-}1)^2(t^2{+}1)$ $t^4{+}4t^3{+}6t^2{+}4t{+}1$	2	2^3	2^8	$2^{27} 3^2 5$	6	—
116	$t^4{+}2t^2{+}1$ $t^4{+}4t^3{+}6t^2{+}4t{+}1$	1	1	2^4	$2^{20} 3^2 5$	4	✓

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
117	$(t^2+1)(t^2+t+1)$ $t^4+4t^3+6t^2+4t+1$	2	$2{\cdot}3$	2^4	$2^7 3^2 5$	3	✓
118	$t^4+t^3+t^2+t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	2	$2{\cdot}3$	3	✓
119	$(t-1)^2(t^2-t+1)$ $t^4+4t^3+6t^2+4t+1$	3	2^2	$2^5 3^3$	$2^{19} 3^6 5^2$	7	—
120	$t^4-2t^3+3t^2-2t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	$2{\cdot}3^4$	$2^8 3^{14} 5^2$	6	—
121	$(t^2-t+1)(t^2+t+1)$ $t^4+4t^3+6t^2+4t+1$	3	$2^2 3$	$2^2 3^3$	$2^{11} 3^6 5^2$	4	✓
122	$(t^2-t+1)(t^2+1)$ $t^4+4t^3+6t^2+4t+1$	$2, 3$	$2^3 3$	$2^4 3^3$	$2^{13} 3^7 5^2$	5	—
123	t^4+1 $t^4+4t^3+6t^2+4t+1$	1	1	2^3	$2^{11} 3^2 5$	4	—
124	$t^4-t^3+t^2-t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	$2{\cdot}5^2$	$2^7 3^3 5^8 13$	5	—
125	t^4-t^2+1 $t^4+4t^3+6t^2+4t+1$	1	1	2^2	$2^5 5$	4	—
126	$(t-1)^2(t^2+t+1)$ $(t+1)^2(t^2+1)$	2	$2^2 3$	2^6	$2^{11} 3^2 5$	3	✓
127	$(t-1)^2(t^2+1)$ $t^4+2t^3+3t^2+2t+1$	3	2^2	$2{\cdot}3^2$	$2^7 3^4 5^2$	4	✓
128	$(t-1)^2(t^2-t+1)$ $t^4+2t^3+3t^2+2t+1$	$2, 3$	$2^2 3^2$	$2^4 3^3$	$2^{13} 3^7 5^2$	5	—
129	$t^4-2t^3+3t^2-2t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	2^4	$2^{16} 3{\cdot}5$	4	✓
130	$(t^2-t+1)(t^2+1)$ $t^4+2t^3+3t^2+2t+1$	2	3^2	2^4	$2^7 3^2 5$	3	✓
131	$t^4-t^3+t^2-t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	1	1	3	✓
132	$(t-1)^2(t^2+1)$ $(t+1)^2(t^2+t+1)$	$2, 3$	$2{\cdot}3$	$2^5 3^2$	$2^{14} 3^2 5^2$	5	—
133	t^4+2t^2+1 $(t+1)^2(t^2+t+1)$	2	$2{\cdot}3$	$2^4 3$	$2^7 3^4 5^2$	3	✓
134	$(t-1)^2(t^2-t+1)$ $(t+1)^2(t^2+t+1)$	2	2	$2^4 3^2$	$2^{18} 3^8 5^2$	6	—
135	$t^4-2t^3+3t^2-2t+1$ $(t+1)^2(t^2+t+1)$	$2, 3$	$2^2 3^2$	$2^4 3^2$	$2^{13} 3^7 5^2$	5	—

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
136	$(t^2-t+1)(t^2+1)$ $(t+1)^2(t^2+t+1)$	3	3	$2^2 3^2$	$2^{10}3^3 5^2$	4	✓
137	t^4+1 $(t+1)^2(t^2+t+1)$	1	1	$2^2 3$	$2^2 3^3 5^2$	3	—
138	$t^4-t^3+t^2-t+1$ $(t+1)^2(t^2+t+1)$	5	5	$2{\cdot}5^2$	$2^4 3^2 5{\cdot}13$	4	—
139	t^4-t^2+1 $(t+1)^2(t^2+t+1)$	2	3	$2^2 3$	$2^3 3^3 5^2$	3	✓
140	$(t-1)^2(t^2+1)$ $t^4+t^3+t^2+t+1$	$2, 5$	$2^3 3{\cdot}5^2$	$2^3 5^2$	$2^8 3^3 5^2 13$	3	—
141	$(t-1)^2(t^2+1)$ $(t+1)^2(t^2-t+1)$	2	3	2^3	$2^9 3^2 5$	3	✓
142	$(t-1)^2(t^2-t+1)$ t^4+2t^2+1	2	$2{\cdot}3$	$2^4 3$	$2^7 3^4 5^2$	3	✓
143	$(t-1)^2(t^2-t+1)$ $(t+1)^2(t^2+1)$	3	3	$2^3 3^2$	$2^{13}3^2 5^2$	5	—
144	$t^4-2t^3+3t^2-2t+1$ $(t+1)^2(t^2+1)$	3	2^2	$2{\cdot}3^2$	$2^7 3^4 5^2$	4	✓
145	$t^4-t^3+t^2-t+1$ $(t+1)^2(t^2+1)$	$2, 5$	$2^3 3{\cdot}5$	$2^2 5$	$2^7 3^3 5{\cdot}13$	3	—
146	$(t-1)^2(t^2-t+1)$ $(t^2+1)(t^2+t+1)$	3	3	$2^2 3^2$	$2^{10}3^3 5^2$	4	✓
147	$t^4-2t^3+3t^2-2t+1$ $(t^2+1)(t^2+t+1)$	2	3^2	2^4	$2^8 3^2 5$	3	✓
148	$(t-1)^2(t^2-t+1)$ $t^4+t^3+t^2+t+1$	5	5	$2{\cdot}5^2$	$2^5 3^2 5{\cdot}13$	4	—
149	$t^4-2t^3+3t^2-2t+1$ $t^4+t^3+t^2+t+1$	1	1	1	1	3	✓
150	$(t-1)^2(t^2-t+1)$ t^4+1	2	2	$2^3 3$	$2^3 3^3 5^2$	3	—
151	$(t-1)^2(t^2-t+1)$ t^4-t^2+1	2	3	$2^3 3$	$2^3 3^3 5^2$	3	✓

Table 1: Degree 4

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
158	$(t^2-t+1)(t^2+1)^2$ $(t^2+t+1)(t^4-t^3+t^2-t+1)$	2	3	2^3	$2^4 3^3 7$	1	✓
162	$(t-1)^2(t+1)^2(t^2+1)$ t^6-t^3+1	3	3	$2{\cdot}3^2$	$2^4 3{\cdot}7^2 13$	1	—
167	$(t-1)^2(t+1)^2(t^2+1)$ $(t^2+t+1)(t^4-t^2+1)$	3	2^2	$2{\cdot}3$	$2^9 3^3 5{\cdot}7^2 13$	1	×
390	$t^6-t^5+t^4-t^3+t^2-t+1$ $(t+1)^2(t^4-t^3+t^2-t+1)$	2, 7	$2^6 3{\cdot}5{\cdot}7^2$	$2^3 7^2$	$2^{14} 3^5 5{\cdot}7^2 19{\cdot}43$	2	×
394	$(t-1)^2(t^2-t+1)(t^2+1)$ $t^6-t^5+t^4-t^3+t^2-t+1$	2	$2^3 3{\cdot}5{\cdot}7$	2^3	$2^8 3^3 5{\cdot}7$	2	✓
437	$t^6+t^5+t^4+t^3+t^2+t+1$ $(t+1)^2(t^2+t+1)^2$	1	1	2	$2^5 3^2$	3	—
468	$t^6-t^5+t^4-t^3+t^2-t+1$ $(t^2+t+1)(t^4+t^3+t^2+t+1)$	1	1	1	1	3	✓
534	$(t^2-t+1)(t^4-t^3+t^2-t+1)$ $t^6+t^5+t^4+t^3+t^2+t+1$	1	1	1	1	3	✓
774	$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$ $t^6-t^5+t^4-t^3+t^2-t+1$	1	1	2	$2^2 3^2$	5	—
819	$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$ t^6-t^3+1	1	1	$2{\cdot}3$	$2^3 3^5 5{\cdot}7^2 13$	6	—
838	$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$ $t^6+t^5+t^4+t^3+t^2+t+1$	1	1	$2{\cdot}7^2$	$2^{15} 3^6 5^2 7^{22} 19{\cdot}43$	7	×

Table 2: Degree 6

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