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IGNACIO NAHUEL ZURRIÁN

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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THE ALGEBRA OF DIFFERENTIAL OPERATORS FOR A GEGENBAUER WEIGHT MATRIX.

IGNACIO NAHUEL ZURRIÁN

Abstract. In this work we study in detail the algebra of differential operators $\mathcal{D}(W)$ associated with a Gegenbauer matrix weight. We prove that two second order operators generate the algebra, indeed $\mathcal{D}(W)$ is isomorphic to the free algebra generated by two elements subject to certain relations. Also, the center is isomorphic to the affine algebra of a singular rational curve. The algebra $\mathcal{D}(W)$ is a finitely-generated torsion-free module over its center, but it is not flat and therefore neither projective. After [Tir11], this is the second detailed study of an algebra $\mathcal{D}(W)$ and the first one coming from spherical functions and group representation theory.

1. Introduction

For a given matrix weight $W(x)$ of size N on the real line one can consider the associated algebra of differential operators

$$\mathcal{D}(W) = \{D : Q_w D = \Lambda_w(D) Q_w,\ \Lambda_w(D) \in \mathbb{C}^{N\times N} \text{ for all } w \geq 0\},$$

where $\{Q_w\}_{w\geq 0}$ is the sequence of monic orthogonal polynomials with respect to W, see [GT07] for basic definitions and main results. Starting with [GPT02, GPT03, Grü03, DG04] one has a growing collection of weight matrices $W(x)$ for which the algebra $\mathcal{D}(W)$ is not trivial, i.e. does not only consist of scalar multiples of the identity operator.

In [Tir11] we can find the only deep study of an algebra $\mathcal{D}(W)$. In that case the author considered the simplest possible example, which was one of the five examples included earlier in [CG06].

In this work we will consider the first example coming from group representation theory. More precisely, our weight W and its orthogonal polynomials were studied in [PZ15] based on [TZ14]. Namely,

$$W(x) = W_{p,n} = (1-x^2)^{\frac{n}{2}-1} \begin{pmatrix} p\,x^2 + n - p & -nx \\ -nx & (n-p)x^2 + p \end{pmatrix}, \quad x \in [-1,1],$$

for real parameters $0 < p < n/2$ and the monic orthogonal polynomials $\{Q_w\}_{w\geq 0}$ are explicitly given by

$$Q_w = \frac{w!}{2^w \left(\frac{n+1}{2}\right)_w} \begin{pmatrix} C_w^{\frac{n+1}{2}}(x) + \frac{n+1}{p+w}\, C_{w-2}^{\frac{n+3}{2}}(x) & \frac{n+1}{p+w}\, C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{n+1}{n-p+w}\, C_{w-1}^{\frac{n+3}{2}}(x) & C_w^{\frac{n+1}{2}}(x) + \frac{n+1}{n-p+w}\, C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix},$$

where $(a)_w = a(a+1)\dots(a+w-1)$ denotes the Pochhammer's symbol and $C_w^{\lambda}(x)$ denotes the w-th Gegenbauer polynomial.

This algebra is of interest by itself, in spite of the fact that our algebra is not commutative and our family of eigenfunctions is discrete, there are strong motivations coming from the classical commuting operators

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theory. Among all the equations that are considered in the framework of the inverse scattering method we can mention the Korteweg-de Vries equation, its two dimensional generalization Kadomtsev-Petviashvili equation, the non-linear Schrödinger equation, the sine-Gordon equation and many other fundamental equations of the modern mathematical physics. All these equations can be represented as compatibility conditions for an overdetermined system of auxiliary linear problems. For example, for the KdV equation, this system has the form

$$D\psi = 0, \qquad \psi_t = E\psi,$$

with

$$D = -\tfrac{\partial^2}{\partial x^2} + u(x,t), \qquad E = \tfrac{\partial^3}{\partial x^3} - \tfrac{3}{2}\, u\, \tfrac{\partial}{\partial x} - \tfrac{3}{4}\, u_x.$$

The compatibility of this system implies

$$\left[\tfrac{\partial}{\partial t} - E, D\right] = 0 \qquad \Longleftrightarrow \qquad D_t = [E, D]\,.$$

This operator equation is called "Lax equation"; each Lax equation is an infinite-dimensional analogue of the completely integrable systems. For the KdV equation, Novikov considered the restriction to the space of stationary solutions, which is essentially equivalent to consider a particular case of the more general problem of the classification of commuting ordinary differential operators with scalar coefficients. As a pure algebraic problem it was considered by Burchnall and Chaundy in [BC23, BC28]; given two commuting operators D and E of order s and ℓ, respectively, there exists a polynomial $R(\lambda,\mu)$ in two variables such that

$$R(D,E) = 0.$$

If s and ℓ are coprime then for each point $q = (\lambda,\mu)$ of the curve Γ, that is defined in $\mathbb{C}^2$ by the equation $R(\lambda,\mu) = 0$, there corresponds a unique (up to constant factor) common eigenfunction $\psi(x,q)$ of D and E:

$$D\psi(x,q) = \lambda\psi(x,q) \qquad E\psi(x,q) = \mu\psi(x,q).$$

These commuting pairs are classified by a set of algebro-geometric data, see [Kri81, Mum77, Tak05].

Of course, working in a scalar context is crucial, as one would expect; if we are dealing with a noncommutative ring of coefficients, such as the ring of matrices, things are more complicated. By considering only commuting differential operators of order two one has a very illuminating contrast: in the scalar case the coefficients of the highest derivatives of the commuting operators have to be the same up to constant factor, then one is essentially dealing with an operator of order two and other of order one. In the matrix case this is completely different.

Anyhow, if one wants to study matrix commuting differential operators with common eigenfunctions, a very good natural source is the algebra $\mathcal{D}(W)$, paying special attention to its center or even to the abelian maximal subalgebras.

Now, we proceed to describe the content of each section. In Section 2 we give a brief recall of some general notions and properties related to the algebras $\mathcal{D}(W)$.

In Section 3 we introduce our Gegenbauer example. We prove that our weights $W_{p,n}$, $0 < p < n/2$, are non-similar each other. This case is very different from the case considered in [Tir11], therefore any attempt for reproducing exactly the same computations applied there is useless. Anyway, we take advantage of some of the tools already used by Tirao in order to simplify some conditions for the operators in $\mathcal{D}(W)$ and at the begining we follow the line of argument ideated in [Tir11].

In Section 4 we prove that for any operator

$$D = \sum_{\ell=0}^{s} \partial^\ell F_\ell(x) \in \mathcal{D}(W),$$

if $F_s \neq \mathbf{0}$ then $\deg(F_s) = s$ and s is even. Even more, $\{D_1, D_2, D_3, D_4\}$ generates the algebra $\mathcal{D}(W)$ with

$$\begin{aligned}
D_1 &= \partial^2 \left(\begin{smallmatrix} x^2 & x \\ -x & -1 \end{smallmatrix}\right) + \partial \left(\begin{smallmatrix} (n+2)x & n-p+2 \\ -p & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} p\,(n-p+1) & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
D_2 &= \partial^2 \left(\begin{smallmatrix} -1 & -x \\ x & x^2 \end{smallmatrix}\right) + \partial \left(\begin{smallmatrix} 0 & p-n \\ p+2 & (n+2)x \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & (p+1)(n-p) \end{smallmatrix}\right),\\
D_3 &= \partial^2 \left(\begin{smallmatrix} -x & -1 \\ x^2 & x \end{smallmatrix}\right) + \partial \left(\begin{smallmatrix} -p & 0 \\ 2(p+1)x & p+2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ p(p+1) & 0 \end{smallmatrix}\right),\\
D_4 &= \partial^2 \left(\begin{smallmatrix} x & x^2 \\ -1 & -x \end{smallmatrix}\right) + \partial \left(\begin{smallmatrix} n-p+2 & 2(n-p+1)x \\ 0 & p-n \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & (n-p)(n-p+1) \\ 0 & 0 \end{smallmatrix}\right),
\end{aligned}$$

and the corresponding eigenvalues are

$$\Lambda_w(D_1) = \left(\begin{smallmatrix} (w+p)(w+n-p+1) & 0 \\ 0 & 0 \end{smallmatrix}\right), \qquad \Lambda_w(D_2) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & (w+p+1)(w+n-p) \end{smallmatrix}\right),$$
$$\Lambda_w(D_3) = \left(\begin{smallmatrix} 0 & 0 \\ (w+p)(w+p+1) & 0 \end{smallmatrix}\right), \qquad \Lambda_w(D_4) = \left(\begin{smallmatrix} 0 & (w+n-p)(w+n-p+1) \\ 0 & 0 \end{smallmatrix}\right).$$

In Section 5 we prove that $\mathcal{D}(W)$ is isomorphic to the free algebra generated by two elements, A and B, with to the following relations

$$B^2A - AB^2 = 0, \qquad BA^2 + A^2B - 2\,ABA - B = 0,$$
$$BAB + A^3 - 2\,AB^2 - A = 0, \qquad B^3 - 2A^2B + ABA = 0.$$

In this section we also prove that all these non-similar weights $W_{p,n}$ have isomorphic algebras $\mathcal{D}(W_{p,n})$.

In Section 6 we focus our attention on the center $\mathcal{Z}(W)$ of the algebra $\mathcal{D}(W)$, proving that it is generated by two operators C_1 and C_2 of order four and six, respectively. Even more, the center of the algebra $\mathcal{D}(W_{p,n})$ is isomorphic to the affine algebra of the following singular rational curve:

$$x^3 - y^2 = (n - 2p)\,x\,y.$$

Finally, we prove that the algebra $\mathcal{D}(W)$ is a finitely-generated torsion-free module over the ring $\mathcal{Z}(W)$. But it is not flat and therefore neither projective.

On the one hand, we are convinced that the study of these algebras may be useful for a better understanding of the theory of matrix orthogonal polynomials that satisfy differential equations. In that direction there is a forthcoming work [Zur15], which includes a very simple and direct criterion for the reducibility of a matrix weight W based on the algebra $\mathcal{D}(W)$; it is worth to mention that recently Tirao developed a completely different criterion, with no relation to the operator algebra, in [Tir15]. On the other hand, we approach the natural next step which is to start a classification of matrix commuting differential operators in algebro-geometric terms.

2. Preliminaries

Let $W = W(x)$ be a weight matrix of size N on the real line. By this we mean a complex $N \times N$-matrix valued integrable function on the interval (a, b) such that $W(x)$ is positive definitive almost everywhere and with finite moments of all orders. From now on we shall denote by M_N the algebra of all $N \times N$ matrices over $\mathbb{C}$ and $M_N[x]$ will denote the algebra over $\mathbb{C}$ of all polynomials in the undetermined x with coefficients in M_N. With the symbol $\mathbf{I}$ we will denote the identity of M_N and T^* will denote the conjugate transpose of the matrix T . We introduce as in [Kre49] and [Kre71] the following matrix valued Hermitian sesquilinear form in the linear space $M_N[x]$:

$$(P, Q) = \int_a^b P(x)W(x)Q(x)^* dx.$$

Then it follows that $M_N[x]$ is a left inner product M_N-module and that there exists a unique sequence $\{Q_w\}_{w\geq 0}$ of monic orthogonal polynomials. More generally by definition a sequence $\{P_w\}_{w\geq 0}$ of matrix orthogonal polynomials is a sequence of elements $P_w \in M_N[x]$ such that P_w is of degree w, its leading coefficient is a non singular matrix and $(P_w', P_w) = 0$ for all $w' \neq w$. Then any sequence $\{P_w\}_{w\geq 0}$ of matrix orthogonal polynomials is of the form $P_w = A_w Q_w$ where $A_w \in GL_N(\mathbb{C})$ is arbitrary for each $w \geq 0$. We come now to the notion of a differential operator with matrix coefficients acting on matrix valued polynomials, i.e. elements of $M_N[x]$. These operators could be made to act on our functions either on the left or on the right. One finds a discussion of these two actions in [Dur97]. The conclusion there is that if one wants to have matrix weights W that are not direct sums of scalar one and that have matrix polynomials as their eigenfunctions with a matrix eigenvalue acting on the left, one should settle for right-hand-side differential operators. We agree now to say that D given by

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx},$$

acts on $P(x)$ by means of

$$PD(x) = \sum_{i=0}^{s} \partial^i (P)(x) F_i(x), \quad \partial = \frac{d}{dx}.$$

One could make D act on P on the right as defined above, and still write down the symbol DP for the result. The advantage of using the notation PD is that it respects associativity: if D_1 and D_2 are two differential operators we have $P(D_1 D_2) = (PD_1)D_2$. We have a right module. The following three propositions are taken from [GT07].

Proposition 2.1 ([GT07], Proposition 2.6)**.** *Let $W = W(x)$ be a weight matrix of size N and let $\{Q_w\}_{w\geq 0}$ be the sequence of monic orthogonal polynomials in $M_N[x]$. If*

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx},$$

is a linear right-hand side ordinary differential operator of order s such that

$$Q_w D = \Lambda_w Q_w, \qquad \text{for all} \quad w \geq 0, \tag{1}$$

with $\Lambda_w \in M_N$, then $F_i = F_i(x) \in M_N[x]$ and $\deg F_i \leq i$. Moreover D is determined by the sequence $\{\Lambda_w\}_{w\geq 0}$.

We could have written the eigenvalue matrix Λ_w to the right of the matrix valued polynomials Q_w above. However, as shown in [Dur97], this only leads to uninteresting cases where the weight matrix is diagonal. To ease the notation if $\nu \in \mathbb{C}$ let

$$[\nu]_i = \nu(\nu-1)\cdots(\nu-i+1), \qquad [\nu]_0 = 1.$$

Proposition 2.2 ([GT07], Proposition 2.7)**.** *Let $D = \sum_{i=0}^{s} \partial^i F_i(x)$ satisfy* (1)*, with*

$$F_i(x) = \sum_{j=0}^{i} F_j^i(D).$$

Then

$$\Lambda_w = \sum_{j=0}^{s} [w]_i F_i^i(D) \qquad \text{for all} \quad w \geq 0. \tag{2}$$

Hence, $w \to \Lambda_w$ is a matrix-valued polynomial function of degree less or equal to the order of D.

Given a sequence of matrix orthogonal polynomials $\{Q_w\}_{w\geq 0}$ we are interested in the algebra $\mathcal{D}(W)$ of all right-hand side differential operators with matrix-valued coefficients that have the polynomials Q_w as eigenfunctions. Notice that if $Q_w D = \Lambda_w Q_w$ for some eigenvalue matrix $\Lambda_w \in M_N$, then Λ_w is uniquely determined by D. In such a case we write $\Lambda_w(D) = \Lambda_w$. Thus

$$\mathcal{D}(W) = \{D : Q_w D = \Lambda_w(D) Q_w,\ \Lambda_w(D) \in M_N \text{ for all } w \geq 0\}. \tag{3}$$

First of all we observe that the definition of $\mathcal{D}(W)$ depends only on the weight matrix $W = W(x)$ and not on the sequence $\{Q_w\}_{w\geq 0}$.

Proposition 2.3 ([GT07], Proposition 2.8). *Given a sequence $\{Q_w\}_{w\geq 0}$ of orthogonal polynomials let us consider the algebra $\mathcal{D}(W)$ defined in* (3)*. Then $D \to \Lambda_w(D)$ is a representation of $\mathcal{D}(W)$ into M_N, for each $w \geq 0$. Moreover the sequence of representations $\{\Lambda_w\}_{w\geq 0}$ separates the elements of $\mathcal{D}(W)$.*

In particular, if $\{Q_w\}_{w\geq 0}$ is the sequence of monic orthogonal polynomials we have a homomorphism

$$\Delta = \prod \Delta_w : \mathcal{D}(W) \to {M_N}^{\mathbb{N}_0}$$

of $\mathcal{D}(W)$ into the direct product of $\mathbb{N}_0$ copies of M_N. Moreover Δ is injective.

In Section 3 of [GT07] the ad-conditions coming from the bispectral pairs (L, D), where L is the difference operator associated to the three term recursion relation satisfied by the sequence of monic orthogonal polynomials and $D \in \mathcal{D}(W)$, are used to described the image $\Delta(W)$ of $\mathcal{D}(W)$ by the eigenvalue isomorphism Δ. This gives a completely different presentation of $\mathcal{D}(W)$ and will be used frequently to simplify several computations. Let us mention that the ad-conditions were first introduced in [DG86].

3. The 2×2 Gegenbauer Example

From now on we will consider the matrix valued orthogonal polynomials going along with the 2×2 weight matrix given by

$$W(x) = W_{p,n} = (1-x^2)^{\frac{n}{2}-1} \begin{pmatrix} p\,x^2 + n - p & -nx \\ -nx & (n-p)x^2 + p \end{pmatrix}, \quad x \in [-1,1], \tag{4}$$

for real parameters $0 < p < n/2$.

The orthogonal polynomials arising here are discussed in full in [PZ15]. As we said in the introduction the purpose of this paper is to compute in this case the algebra $\mathcal{D}(W)$ and to study its structure. The monic orthogonal polynomials $\{Q_w\}_{w\geq 0}$ with respect to the weight matrix $W(x)$ are explicitly given by

$$Q_w = \frac{w!}{2^w \left(\frac{n+1}{2}\right)_w} \begin{pmatrix} C_w^{\frac{n+1}{2}}(x) + \frac{n+1}{p+w}\, C_{w-2}^{\frac{n+3}{2}}(x) & \frac{n+1}{p+w}\, C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{n+1}{n-p+w}\, C_{w-1}^{\frac{n+3}{2}}(x) & C_w^{\frac{n+1}{2}}(x) + \frac{n+1}{n-p+w}\, C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix}, \tag{5}$$

where $(a)_w = a(a+1)\dots(a+w-1)$ denotes the Pochhammer's symbol and $C_n^{\lambda}(x)$ denotes the n-th Gegenbauer polynomial

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n,\ n+2\lambda \\ \lambda + 1/2 \end{matrix}; \frac{1-x}{2}\right).$$

As usual, we assume $C_n^{\lambda}(x) = 0$ if $n < 0$; recall that C_n^{λ} is a polynomial of degree n, with leading coefficient $\frac{2^n(\lambda)_n}{n!}$. Recall that (see [AS65, p 561])

$$C_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha) k! (n-2k)!} (2z)^{n-2k}. \tag{6}$$

Let us recall the concept of similarity for matrix weights. Two weights W and $\widetilde{W}$ are said to be *similar* if there exists a nonsingular matrix M, which does not depend on x, such that

$$\widetilde{W}(x) = MW(x)M^*, \qquad \text{for all } x \in (a,b).$$

Lemma 3.1. *If $(p,n) \neq (\tilde{p}, \tilde{n})$ then $W_{p,n}$ is not similar to $W_{\tilde{p},\tilde{n}}$.*

Proof. If $W_{p,n}$ is similar to $W_{\tilde{p},\tilde{n}}$ then there exists a nonsingular matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $\widetilde{W} = MWM^*$, therefore it is clear that $n = \tilde{n}$. Hence we have that the matrix $\left(\begin{smallmatrix} \tilde{p}\,x^2+n-\tilde{p} & -nx \\ -nx & (n-\tilde{p})x^2+\tilde{p} \end{smallmatrix}\right)$ is of the form

$$\left(\begin{smallmatrix} \left(|a|^2p+|b|^2n-|b|^2p\right)x^2+\left(-\overline{a}bn-\overline{b}an\right)x+|a|^2n-|a|^2p+|b|^2p & \left(\overline{c}ap+(bn-bp)\overline{d}\right)x^2+\left(-bn\overline{c}-an\overline{d}\right)x+(an-ap)\overline{c}+\overline{d}bp \\ \left(\overline{a}cp+(dn-dp)\overline{b}\right)x^2+\left(-dn\overline{a}-cn\overline{b}\right)x+(cn-cp)\overline{a}+\overline{b}dp & \left(|c|^2p+|d|^2n-|d|^2p\right)x^2+\left(-\overline{c}dn-\overline{d}cn\right)x+|c|^2n-|c|^2p+|d|^2p \end{smallmatrix}\right).$$

Then, looking at the entry $(1,2)$ in the matrix above, we have

$$\left(\overline{c}ap + (bn-bp)\,\overline{d}\right) = 0 \qquad\qquad \text{and} \qquad\qquad (an-ap)\,\overline{c} + \overline{d}bp = 0,$$

but since M is inversible and $0 < p < n/2$ we have that M has to be the identity matrix. Hence $(p,n) = (\tilde{p},\tilde{n})$. $\square$

Let us denote

$$Q_w = \sum_{i=0}^{w} x^i B_i^w.$$

From (5) and (6) we obtain

$$B_{w-2k}^w = \frac{w!(-1)^k 2^{-2k}}{(\frac{n+1}{2}+w-k)_k k!(w-2k)!} \begin{pmatrix} \frac{p+w-2k}{p+w} & 0 \\ 0 & \frac{n-p+w-2k}{n-p+w} \end{pmatrix}, \tag{7}$$

$$B_{w-2k-1}^w = \frac{w!(-1)^k 2^{-2k}}{(\frac{n+1}{2}+w-k)_k k!(w-2k-1)!} \begin{pmatrix} 0 & \frac{1}{p+w} \\ \frac{1}{n-p+w} & 0 \end{pmatrix}. \tag{8}$$

We will use some tools already used by Tirao in order to simplify some conditions for the operators in $\mathcal{D}(W)$. We quote a result given in Tirao's paper, changing some parameter names (see Proposition 3.4 and equation (25) in [Tir11]).

Proposition 3.2. *Let $D = \sum_{j=0}^{s} \partial F_j(x)$, with $F_j = \sum_{i=0}^{j} x^i F_i^j$. Then $D \in \mathcal{D}(W)$ if and only if*

$$\sum_{i=0}^{s} B_{w-m+i}^w \left(\sum_{\ell=0}^{s} [w-m+i]_\ell F_{\ell-i}^\ell\right) - \left(\sum_{\ell=0}^{s} [w]_\ell F_\ell^\ell\right) B_{w-m}^w = 0$$

for all $0 \le m \le w$, $0 \le w$.

The following lemma states a nice property for our weight W that can be easily proved.

Lemma 3.3. *Given*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have that

$$W(-x) = TW(x)T.$$

Definition 3.4. Given $D = \sum_{0 \le i \le s} \partial^i F_i \in \mathcal{D}$ let

$$\widetilde{D} = \sum_{0 \le i \le s} \partial^i (-1)^i T \widetilde{F}_i T \in \mathcal{D},$$

where $\widetilde{F}_i(x) = F_i(-x)$.

Proposition 3.5 ([Tir11], Proposition 3.9). *If $D \in \mathcal{D}(W)$ then $\widetilde{D} \in \mathcal{D}(W)$. Moreover, if $\{Q_w\}_{w\geq 0}$ is the sequence of monic orthogonal polynomials with respect to W and $\{\Lambda_w(D)\}_{w\geq 0}$ is the corresponding eigenvalue sequence, then*

$$\Lambda_w\left(\widetilde{D}\right) = T\Lambda_w(D)T.$$

Theorem 3.6 ([Tir11], Theorem 3.10). *The map $D \to \widetilde{D}$ defines an involutive automorphism of the algebra $\mathcal{D}(W)$.*

Now we give a lemma with some results belonging to a previous work with Pacharoni.

Lemma 3.7 ([PZ15], Corollary 5.3). *The set of differential operators of order at most two in $\mathcal{D}(W)$ is a five-dimensional vector space generated by the identity* $\mathbf{I}$ *and*

$$D_1 = \partial^2 \begin{pmatrix} x^2 & x \\ -x & -1 \end{pmatrix} + \partial \begin{pmatrix} (n+2)x & n-p+2 \\ -p & 0 \end{pmatrix} + \begin{pmatrix} p\,(n-p+1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \partial^2 \begin{pmatrix} -1 & -x \\ x & x^2 \end{pmatrix} + \partial \begin{pmatrix} 0 & p-n \\ p+2 & (n+2)x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (p+1)(n-p) \end{pmatrix},$$

$$D_3 = \partial^2 \begin{pmatrix} -x & -1 \\ x^2 & x \end{pmatrix} + \partial \begin{pmatrix} -p & 0 \\ 2(p+1)x & p+2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ p(p+1) & 0 \end{pmatrix},$$

$$D_4 = \partial^2 \begin{pmatrix} x & x^2 \\ -1 & -x \end{pmatrix} + \partial \begin{pmatrix} n-p+2 & 2(n-p+1)x \\ 0 & p-n \end{pmatrix} + \begin{pmatrix} 0 & (n-p)(n-p+1) \\ 0 & 0 \end{pmatrix}.$$

The corresponding eigenvalues are given by

$$\begin{aligned} &\Lambda_w(D_1) = \begin{pmatrix} (w+p)(w+n-p+1) & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_w(D_2) = \begin{pmatrix} 0 & 0 \\ 0 & (w+p+1)(w+n-p) \end{pmatrix}, \\ &\Lambda_w(D_3) = \begin{pmatrix} 0 & 0 \\ (w+p)(w+p+1) & 0 \end{pmatrix}, \quad \Lambda_w(D_4) = \begin{pmatrix} 0 & (w+n-p)(w+n-p+1) \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{9}$$

It can be easily checked that

$$\widetilde{D}_1 = D_1, \qquad \widetilde{D}_2 = D_2, \qquad \widetilde{D}_3 = -D_3 \qquad \widetilde{D}_4 = -D_4,$$

It is worth to pay special attention to the explicit expression of the matrix eigenvalues in Lemma 3.7, we will use them in some proofs in this paper. Also, they give us the following remark.

Remark 3.8. We have the following relations among the differential operators D_1, D_2, D_3, D_4.

$$\begin{aligned} &D_1D_2 = 0, && D_1D_3 = 0, && D_1D_4 = D_4D_2 - (n-2p)D_4, \\ &D_2D_1 = 0, && D_2D_3 = D_3D_1 + (n-2p)D_3, && D_2D_4 = 0, \\ &D_3D_2 = 0, && D_3D_3 = 0, && D_3D_4 = D_2^2 - (n-2p)D_2, \\ &D_4D_1 = 0, && D_4D_3 = D_1^2 + (n-2p)D_1, && D_4D_4 = 0. \end{aligned}$$

4. The Order of the Operators in $\mathcal{D}(W)$

In this section we prove that any operator

$$D = \sum_{\ell=0}^{s} \partial^\ell F_\ell(x) \in \mathcal{D}(W)$$

is of even order and that if $F_s \neq \mathbf{0}$ then $\deg(F_s) = s$. Furthermore, $\{D_1, D_2, D_3, D_4\}$ generates the algebra $\mathcal{D}(W)$. We start by considering operators D such that $\widetilde{D} = D$, later we prove the same results for all the operators.

From (7) and (8) we have

$$\left(B^w_{w-2(k-i)}\right)_{11} = \frac{p+w-2k+2i}{p+w-2k} \frac{(-1)^i 2^{2i} \left(\frac{n+1}{2}+w-k\right)_i [k]_i}{[w-2k+2i]_{2i}} \left(B^w_{w-2k}\right)_{11}, \tag{10}$$

$$\left(B^w_{w-2k-1}\right)_{12} = \frac{w-2k}{p+w-2k} \left(B^w_{w-2k}\right)_{11}. \tag{11}$$

Let us assume that $\widetilde{D} = D$ and that $s = 2r+1$, $r \in \mathbb{N}$. If, for $0 \leq \ell \leq s$, we denote $F_\ell = \sum_{i=0}^{\ell} x^i F_i^\ell$ then from our hypothesis we have

$$F^\ell_{\ell-2i} = \begin{pmatrix} u^\ell_{\ell-2i} & 0 \\ 0 & v^\ell_{\ell-2i} \end{pmatrix}, \qquad F^\ell_{\ell-2i+1} = \begin{pmatrix} 0 & y^\ell_{\ell-2i+1} \\ z^\ell_{\ell-2i+1} & 0 \end{pmatrix}.$$

If we put $m = 2k$, Proposition 3.2 gives us that

$$\sum_{i=0}^{r} B^w_{w-2k+2i} \left(\sum_{\ell=0}^{s} [w-2k+2i]_\ell F^\ell_{\ell-2i} \right) + \sum_{i=1}^{r+1} B^w_{w-2k+2i-1} \left(\sum_{\ell=0}^{s} [w-2k+2i-1]_\ell F^\ell_{\ell-2i-1} \right) - \left(\sum_{\ell=0}^{s} [w]_\ell F^\ell_\ell \right) B^w_{w-2k} = 0$$

for $0 \leq 2k \leq w$.

Then, looking at the entry $(1,1)$ and using (10) and (11) we have

$$\begin{aligned} &\sum_{i=0}^{r} \frac{p+w-2k+2i}{p+w-2k} \frac{(-1)^i 2^{2i} \left(\frac{n+1}{2}+w-k\right)_i [k]_i}{[w-2k+2i]_{2i}} \left(B^w_{w-2k}\right)_{11} \left(\sum_{\ell=2i}^{s} [w-2k+2i]_\ell \, u^\ell_{\ell-2i} \right) \\ &+ \sum_{i=1}^{r+1} \frac{w-2k+2i}{p+w-2k} \frac{(-1)^i 2^{2i} \left(\frac{n+1}{2}+w-k\right)_i [k]_i}{[w-2k+2i]_{2i}} \left(B^w_{w-2k}\right)_{11} \left(\sum_{\ell=2i-1}^{s} [w-2k+2i-1]_\ell \, z^\ell_{\ell-2i+1} \right) \\ &- \sum_{\ell=0}^{s} [w]_\ell \, u^\ell_\ell \left(B^w_{w-2k}\right)_{11} = 0. \end{aligned}$$

for $0 \leq 2k \leq w$. Hence

$$\begin{aligned} &\sum_{i=0}^{r} (p+w-2k+2i)(-1)^i 2^{2i} \left(\tfrac{n+1}{2}+w-k\right)_i [k]_i \left(\sum_{\ell=2i}^{s} [w-2k]_{\ell-2i} \, u^\ell_{\ell-2i} \right) \\ &+ \sum_{i=1}^{r+1} (-1)^i 2^{2i} \left(\tfrac{n+1}{2}+w-k\right)_i [k]_i \left(\sum_{\ell=2i-1}^{s} [w-2k]_{\ell-2i+1} \, z^\ell_{\ell-2i+1} \right) - \sum_{\ell=0}^{s} [w]_\ell \, u^\ell_\ell (p+w-2k) = 0. \end{aligned}$$

for $0 \leq 2k \leq w$.

Let us consider $w = \widetilde{m} + 2k$, therefore for any $\widetilde{m}$ and k in $\mathbb{N}_0$ we have

$$\begin{aligned} &\sum_{i=0}^{r} (-1)^i 2^{2i} \left(\tfrac{n+1}{2}+\widetilde{m}+k\right)_i [k]_i (p+\widetilde{m}+2i) \sum_{\ell=2i}^{s} [\widetilde{m}]_{\ell-2i} \, u^\ell_{\ell-2i} \\ &+ \sum_{i=1}^{r+1} (-1)^i 2^{2i} \left(\tfrac{n+1}{2}+\widetilde{m}+k\right)_i [k]_i \sum_{\ell=2i-1}^{s} [\widetilde{m}]_{\ell-2i+1} \, z^\ell_{\ell-2i+1} - (p+\widetilde{m}) \sum_{\ell=0}^{s} [\widetilde{m}+2k]_\ell \, u^\ell_\ell = 0. \end{aligned} \tag{12}$$

If we consider (12) as a polynomial in k then we obtain

$$z_0^s = 0 = u_s^s, \tag{13}$$

by looking at the terms of degree $s+1$ and s.

We observe that (12) is a polynomial equation in two variables $\widetilde{m}$ and k, its term of highest total degree gives us that the following equation holds.

$$\sum_{i=1}^{r}(-1)^i 2^{2i}(\widetilde{m}+k)^i k^i \widetilde{m}^{s-2i+1}\ \left(u_{s-2i}^s + z_{s-2i+1}^s\right) = 0, \tag{14}$$

$$\sum_{i=1}^{r}\sum_{j=0}^{i}\binom{i}{j}(-1)^i 2^{2i}(\widetilde{m})^{i-j} k^j k^i \widetilde{m}^{s-2i+1}\ \left(u_{s-2i}^s + z_{s-2i+1}^s\right) = 0,$$

$$\sum_{i=1}^{r}\sum_{j=0}^{i}\binom{i}{j}(-1)^i 2^{2i} k^{j+i} \widetilde{m}^{s-i-j+1}\ \left(u_{s-2i}^s + z_{s-2i+1}^s\right) = 0.$$

Then, for any $1 \le q \le r$ we observe that the term of degree q in the variable k is

$$\sum_{\frac{q}{2}\le i\le q}\binom{i}{q-i}(-1)^i 2^{2i} k^q \widetilde{m}^{s-q+1}\ \left(u_{s-2i}^s + z_{s-2i+1}^s\right) = 0.$$

Hence, for any $1 \le q \le r$ we have

$$\sum_{\frac{q}{2}\le i\le q}\binom{i}{q-i}(-1)^i 2^{2i}\left(u_{s-2i}^s + z_{s-2i+1}^s\right) = 0. \tag{15}$$

From considering $q=1$ in (15) we have $u_{s-2}^s + z_{s-1}^s = 0$, then from considering $q=2$ also in (15) we have $u_{s-4}^s + z_{s-3}^s = 0$. Inductively we have

$$u_{s-2i}^s = -z_{s-2i+1}^s, \ \text{for } 1 \le i \le r. \tag{16}$$

In a completely analogous way, one find that

$$y_0^s = 0 = v_s^s \tag{17}$$

and that

$$v_{s-2i}^s = -y_{s-2i+1}^s, \ \text{for } 1 \le i \le r. \tag{18}$$

On the other hand, form Proposition 3.5, we know that $\Lambda_w(D)$ is a diagonal matrix for any w. Then, D commutes with D_1. By looking at the highest order derivative in the equation $DD_1 = D_1D$ we obtain

$$\begin{pmatrix} x^2 & x \\ -x & -1\end{pmatrix} F_s = F_s \begin{pmatrix} x^2 & x \\ -x & -1\end{pmatrix},$$

i.e.

$$\begin{pmatrix} x^2 & x \\ -x & -1\end{pmatrix} \sum_{k=0}^{s} x^k F_k^s = \sum_{k=0}^{s} x^k F_k^s \begin{pmatrix} x^2 & x \\ -x & -1\end{pmatrix},$$

then, for any $0 \le k \le s$, we have

$$\left(\begin{smallmatrix} 0 & 0 \\ 0 & -1\end{smallmatrix}\right) F_k^s + \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right) F_{k-1}^s + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right) F_{k-2}^s = F_k^s \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1\end{smallmatrix}\right) + F_{k-1}^s \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right) + F_{k-2}^s \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right). \tag{19}$$

If k and s are of the same parity, say $k = s-2i+2$, then (19) becomes

$$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right) F_{s-2i+1}^s = F_{s-2i+1}^s \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$$

whence, looking at the entry $(1,1)$, we obtain

$$z^s_{s-2i+1} = -y^s_{s-2i+1}, \text{ for } 1 \le i \le r+1. \tag{20}$$

If k and s are of different parity, say $k = s-2i+1$, then the entry $(1,2)$ of the matrix equation (19) gives us

$$-z_{s-2i+1} + u_{s-2i} = -v_{s-2i} + z_{s-2i-1},$$

which combined with (16), (18) and (20) becomes

$$u^s_{s-2i} = -u^s_{s-2i-2}. \tag{21}$$

Furthermore, using (16), (18) and (20) it is clear that the values $u^s_1, u^s_3, \dots, u^s_s$ determine F_s. Using (21) it follows straight forward that the value of u^s_s determines all the other entries of F_s and that, since $u^s_s = 0$, we have $F_s = \mathbf{0}$. Therefore, there is no operator D of odd order in $\mathcal{D}(W)$ such that $\widetilde{D} = D$.

Even more, since F_s is zero (12) turns out to be

$$\sum_{i=0}^{r} (-1)^i 2^{2i} \left(\tfrac{n+1}{2} + \widetilde{m} + k\right)_i [k]_i (p + \widetilde{m} + 2i) \sum_{\ell=2i}^{s-1} [\widetilde{m}]_{\ell-2i}\, u^\ell_{\ell-2i}$$
$$+ \sum_{i=1}^{r} (-1)^i 2^{2i} \left(\tfrac{n+1}{2} + \widetilde{m} + k\right)_i [k]_i \sum_{\ell=2i-1}^{s-1} [\widetilde{m}]_{\ell-2i+1}\, z^\ell_{\ell-2i+1} - (p+\widetilde{m}) \sum_{\ell=0}^{s-1} [\widetilde{m}+2k]_\ell\, u^\ell_\ell = 0.$$

If we denote $s' = s-1$ and observe the term of highest total degree in the variables $\widetilde{m}$ and k, we obtain the following equation,

$$\sum_{i=1}^{r} (-1)^i 2^{2i} (\widetilde{m}+k)^i k^i \widetilde{m}^{s'-2i+1} \left(u^{s'}_{s'-2i} + z^{s'}_{s'-2i+1}\right) - \widetilde{m}(\widetilde{m}+2k)^{s'} u^\ell_\ell = 0.$$

If $u^{s'}_{s'} = 0$ then this equation is exactly the same equation that (14) with s' instead of s, therefore with the same procedure as before we obtain an equation similar to (16) but with s' instead of s; if $v^{s'}_{s'} = 0$, then we have also an equation similar to (18). The equation (19) and its consequent equations (20) and (21) are obtained independently of the parity of s, therefore they remain valid for s'. Having then that if $u^{s'}_{s'} = v^{s'}_{s'} = 0$ then $F_{s'} = \mathbf{0}$.

Finally, for a given operator $D = \sum_{\ell=0}^{s} \partial^\ell F_\ell(x) \in \mathcal{D}(W)$, with $\widetilde{D} = D$, of even order $s = 2r$ we can consider the operator

$$D - u^s_s D^r_1 - v^s_s D^r_s,$$

which has to be of order $s-2$. Inductively it can be easily seen that the set of operators

$$\{D_1, D_2\}$$

generates the subalgebra of all the operators in $\mathcal{D}(W)$ such that $\widetilde{D} = D$.

The following theorem summarizes all the results obtained so far in this section.

Theorem 4.1. *Any operator $D = \sum_{\ell=0}^{s} \partial^\ell F_\ell \in \mathcal{D}(W)$ of order s, such that $\widetilde{D} = D$, is of even order and the polynomial F_s is of degree s.*

Furthermore, the operators D_1 and D_2 generate the subalgebra of $\mathcal{D}(W)$

$$\left\{ D \in \mathcal{D}(W) : \widetilde{D} = D \right\},$$

where

$$D_1 = \partial^2 \begin{pmatrix} x^2 & x \\ -x & -1 \end{pmatrix} + \partial \begin{pmatrix} (n+2)x & n-p+2 \\ -p & 0 \end{pmatrix} + \begin{pmatrix} p\,(n-p+1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \partial^2 \begin{pmatrix} -1 & -x \\ x & x^2 \end{pmatrix} + \partial \begin{pmatrix} 0 & p-n \\ p+2 & (n+2)x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (p+1)(n-p) \end{pmatrix}.$$

Corollary 4.2. *Any operator $D = \sum_{\ell=0}^{s} \partial^\ell F_\ell \in \mathcal{D}(W)$ of order s, such that $\widetilde{D} = -D$, is of even order and the polynomial F_s is of degree s.*

Proof. Given an operator $D = \sum_{\ell=0}^{s} \partial^\ell F_\ell \in \mathcal{D}(W)$ of order s such that $\widetilde{D} = -D$ let us consider the operator

$$D_0 = D(D_3 + D_4).$$

Since $\widetilde{D}_3 = -D_3$ and $\widetilde{D}_4 = -D_4$ we have that $\widetilde{D}_0 = D_0$. Also, the highest order term of $D_3 + D_4$ is $\partial^2 (x^2 - 1) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, therefore the order of D_0 is $s+2$, which implies, by Theorem 4.1, that $s+2$ is even and that if $D_0 = \sum_{\ell=0}^{s+2} \partial^\ell G_\ell$ then $G_{s+2} = (1 - x^2) F_s$ is a polynomial of degree $s+2$, hence F_s is a polynomial of order s. ☐

Theorem 4.3. *The algebra $\mathcal{D}(W)$ is generated by $\{D_1, D_2, D_3, D_4\}$. If $D = \sum_{\ell=0}^{s} \partial^\ell F_\ell \in \mathcal{D}(W)$ with $F_s \neq \mathbf{0}$ then s is even and F_s is of degree s.*

Proof. For any $D \in \mathcal{D}(W)$ we have that $D = E_1 + E_2$, with $\widetilde{E}_1 = E_1$ and $\widetilde{E}_2 = -E_2$. Then, it only remains to prove that if $D = \sum_{\ell=0}^{s} \partial^\ell F_\ell \in \mathcal{D}(W)$,with $\widetilde{D} = -D$, then it is generated by $\{D_1, D_2, D_3, D_4\}$. Let $F_s = \sum_{j=0}^{s} F_j^s$. From Corollary 4.2 we know that s is even and from $\widetilde{D} = -D$ it follows that $(F_s^s)_{11} = (F_s^s)_{22} = 0$. Is $s \leq 2$ it is already proved by Lemma 3.7, if $s \geq 4$ we consider the operator

$$D_0 = D - (D_1 + D_2)^{\frac{s}{2}-1} \left((F_s^s)_{21} D_3 + (F_s^s)_{12} D_4 \right).$$

It can be easily checked that $\widetilde{D}_0 = -D_0$ and that $D_0 = \sum_{\ell=0}^{s} \partial^\ell G_\ell$ with G_s is of degree lesser than s and therefore, by Corollary 4.2, $G_s = 0$, then D_0 is an operator of order at most $s-2$; inductively we obtain the statement of the theorem. ☐

With a very similar proof we obtain the following corollary.

Corollary 4.4. *The vector space $\mathcal{D}(W)$ is spanned by*

$$\left\{ (D_1 + D_2)^i D_1, (D_1 + D_2)^i D_2, (D_1 + D_2)^i D_3, (D_1 + D_2)^i D_4, \mathbf{I} : i \in \mathbb{N}_0 \right\}.$$

Proof. Let $D \in \mathcal{D}(W)$, then it is of the form

$$D = \sum_{\ell=0}^{2r} \partial^\ell F_\ell,$$

for some $r \in \mathbb{N}_0$, with $F_s = \sum_{j=0}^{s} F_j^s$. Let us consider the operator

$$D_0 = D - (D_1 + D_2)^{k-1} \left((F_s^s)_{11} D_1 + (F_s^s)_{22} D_2 + (F_s^s)_{21} D_3 + (F_s^s)_{12} D_4 \right).$$

It can be easily checked that that if $D_0 = \sum_{\ell=0}^{s} \partial^\ell G_\ell$ then G_s is of degree lesser than s, and therefore by Theorem 4.3 D_0 is an operator of order at most $s-2$; inductively we obtain the statement of the theorem. ☐

Corollary 4.5. *$\mathcal{D}(W)$ is a free left-module over the subalgebra generated by $D_1 + D_2$. The following set is a basis*

$$\{\mathbf{I}, D_2, D_3, D_4\}.$$

As a direct consequence of combining Proposition 2.2 and Theorem 4.3 we also have.

Corollary 4.6. *For any operator $D \in \mathcal{D}(W)$, the order of D is equal to the degree of $\Lambda_w(D)$ as a polynomial on w.*

5. Structure of $\mathcal{D}(W)$

In this section we go deeper into the structure of the algebra $\mathcal{D}(W)$. Also, we prove that the non-similar weights $W_{p,n}$ have isomorphic algebras $\mathcal{D}(W_{p,n})$.

Theorem 5.1. *The algebra $\mathcal{D}(W)$ is generated by*

$$\{D_1 + D_2, D_3 + D_4\}.$$

Proof. From Remark 3.8 we have that

$$(D_3 + D_4)^2 = (D_1 + D_2)^2 + (n - 2p)(D_1 - D_2),$$

therefore $\{D_1 + D_2, D_3 + D_4\}$ generates D_1 and D_2.

From

$$(D_3 + D_4)D_1 = D_3D_1 = D_2D_3 - (n - 2p)D_3 = D_2(D_3 + D_4) - (n - 2p)D_3$$

we have that $\{D_1 + D_2, D_3 + D_4\}$ generates also D_3 and D_4. The theorem follows form Theorem 4.3. □

Let us define

$$A = (D_1 + D_2)/(2n - p), \qquad B = (D_3 + D_4)/(2n - p).$$

Then we have these expressions for our operators D_1, D_2, D_3, D_4 (see Remark 3.8)

$$\begin{aligned} D_1 &=(B^2 - A^2 + A)(n - 2p)/2, & D_2 &=(A^2 - B^2 + A)(n - 2p)/2,\\ D_3 &=(-BA + AB + B)(n - 2p)/2, & D_4 &=(BA - AB + B)(n - 2p)/2. \end{aligned} \tag{22}$$

Let us call $\mathcal{A}$ to the subalgebra generated by A.

Lemma 5.2. *The set $S = \{\mathbf{I}, B^2, B, BA\}$ also is a basis of $\mathcal{D}(W)$ as left-module over $\mathcal{A}$.*

Proof. It is clear from (22) and Corollary 4.5 that S generates $\mathcal{D}(W)$. Let assume that

$$m_1 I + m_2 B^2 + m_3 B + m_4 BA = 0$$

for some $m_1, m_2, m_3, m_4 \in \mathcal{A}$. From (22) we can observe that

$$B^2 = A^2 + A - 2(n - 2p)^{-1} D_2, \qquad BA = (D_4 - D_3)(n - 2p)^{-1} + A(D_3 + D_4),$$

then we have

$$(m_1 + m_2 A^2 I + A)\mathbf{I} + m_2(-2(n - 2p)^{-1})D_2 + \big(m_3 + m_4(-(n - 2p)^{-1}\mathbf{I} + A)\big) D_3 + \big(m_3 + m_4((n - 2p)^{-1}\mathbf{I} + A)\big) D_4 = 0,$$

whence follows straightforward that $m_1 = m_2 = m_3 = m_4 = 0$. □

It is worth to observe that the following relations hold:

$$\begin{aligned} B^2A - AB^2 &= 0,\\ BA^2 + A^2B - 2\,ABA - B &= 0,\\ BAB + A^3 - 2\,AB^2 - A &= 0,\\ B^3 - 2A^2B + ABA &= 0. \end{aligned} \tag{23}$$

We now shall prove that the algebra $\mathcal{D}(W)$ can be described as the complex algebra generated by A and B subject to the relations given in (23). Let V be a two dimensional complex vector space and let $T(V)$ be the corresponding tensor algebra. The identity of $T(V)$ will be also denoted by I. We choose a basis $\{\alpha, \beta\}$

of V and define $I(V)$ to be the two sided ideal of $T(V)$ generated by the elements on the left-hand side of the following four equations, obtained by replacing A by α and B by β in (23):

$$\begin{aligned} \beta^2\alpha - \alpha\beta^2 &= 0, \\ \beta\alpha^2 + \alpha^2\beta - 2\,\alpha\beta\alpha - \beta &= 0, \\ \beta\alpha\beta + \alpha^3 - 2\,\alpha\beta^2 - \alpha &= 0, \\ \beta^3 - 2\alpha^2\beta + \alpha\beta\alpha &= 0. \end{aligned} \tag{24}$$

Theorem 5.3. *The algebra $\mathcal{D}(W)$ is isomorphic to the quotient algebra $T(V)/I(V)$.*

Proof. For simplicity we will denote with the same letter an element $T(V)$ and its canonical projection on the quotient algebra $T(V)/I(V)$. The homomorphism ξ from the quotient algebra $T(V)/I(V)$ into $\mathcal{D}(W)$ defined by $\xi(\alpha) = A$ and $\xi(\beta) = B$ is surjective. Let us denote by $\mathcal{A}'$ the subalgebra of $T(V)/I(P)$ generated by α. It is clear that $\mathcal{A}'$ is isomorphic to $\mathcal{A}$. The point now is to prove that ξ is injective. Let us call S to the left-module over $\mathcal{A}$ generated by

$$\{\mathbf{I}, \beta^2, \beta, \beta\alpha\}.$$

Since ξ transform S into the basis $\{\mathbf{I}, B^2, B, BA\}$ it is enough to prove that $S = T(V)/I(V)$. First notice that since α and β satisfy (24) we have that S is invariant under right multiplication by α and left multiplication by β. If we prove that S is also left invariant under multiplication by β we reach the statement of theorem.

We will prove by induction on k that $\beta\,\alpha^k\,(\alpha\beta)$ belongs to S for any $k \in \mathbb{N}_0$. For $k = 0$ we have $\beta(\beta\alpha) = \beta^2\alpha = \alpha\beta^2$ (first equation in (24)). For $k = 1$ we have $\beta\alpha(\beta\alpha)$, but we can write $\beta\alpha\beta$ as a linear combination of β^2 and $\mathbf{I}$ (third equation in (24)) which are in the center of the algebra, and then we can write $\beta\alpha\beta\alpha$ also as a linear combination of β^2 and $\mathbf{I}$. Assume that $\beta\alpha^k(\beta\alpha)$ can be written as a linear combination of $\mathbf{I}$ and β^2 for $k \geq 1$ and let us consider the case $k+1$; since $\beta\alpha\alpha$ can be written as $p(\alpha)\beta + q(\alpha)\beta\alpha$ with p, q polynomials (last equation in (24)), we have

$$\beta\alpha^{k+1}\beta\alpha = (p(\alpha)\beta + q(\alpha)\beta\alpha)\alpha^{k-1}\beta\alpha = p(\alpha)\beta\alpha^{k-1}\beta\alpha + q(\alpha)\beta\alpha^k\beta\alpha,$$

which, due to the inductive hypothesis, is a linear combination of $\mathbf{I}$ and β^2. Then $\beta\,\alpha^k\,(\alpha\beta)$ belongs to S for any $k \in \mathbb{N}_0$.

In a very similar way it can be proved that $\beta\,\alpha^k\,(\beta), \beta\,\alpha^k\,(\beta^2)$ and $\beta\,\alpha^k\,(\mathbf{I})$ belongs to S for any $k \in \mathbb{N}_0$. Therefore, S is stable under left multiplication by β and the theorem is proved. □

In spite of that Lemma 3.1 states that for $(p, n) \neq (p', n')$ the weights $W_{p,n}$ and $W_{p',n'}$ are non-similar, the relations (24) do not depend on the parameters n and p. We have then the following result.

Theorem 5.4. *Given two matrix weights $W = W_{p,n}$ and $W' = W_{p',n'}$ we have that the algebras $\mathcal{D}(W)$ and $\mathcal{D}(W')$ are isomorphic.*

Notice that if one has two similar weights W and $\widetilde{W}$, then the corresponding differential operators algebras are isomorphic: Let M an inversible matrix such that $\widetilde{W} = MWM^*$, it can be easily checked that $D \to MDM^{-1}$ is an isomorphism from $\mathcal{D}(W)$ onto $\mathcal{D}(\widetilde{W})$.

6. The center of $\mathcal{D}(W)$

In this section we will study the center of our algebra and we will analyze the structure of $\mathcal{D}(W)$ as a module over its center.

Let us define

$$C_1 = (D_3 + D_4)^2, \qquad C_2 = D_3 D_1 D_4 + D_4\,(D_2 - (n-2p)\mathbf{I})\,D_3.$$

From Lemma 3.7 it follows that

$$(25)\qquad \begin{aligned}\Lambda_w(C_1) =& p_1(w)\mathbf{I} = (w+p)(w+p+1)(w+n-p+1)(w+n-p)\mathbf{I},\\ \Lambda_w(C_2) =& p_2(w)\mathbf{I} = (w+p)\,(w+p+1)^2\,(w+n-p+1)\,(w+n-p)^2\,\mathbf{I},\end{aligned}$$

whence it is clear that C_1 and C_2 are in the center of $\mathcal{D}(W)$. Also, we observe that they are not algebraically independent since the following relations holds:

$$C_1^3 - C_2^2 = (n-2p)\,C_1\,C_2.$$

Theorem 6.1. *The center $\mathcal{Z}(W)$ of the algebra $\mathcal{D}(W)$ is generated by C_1 and C_2 and it is isomorphic to the affine algebra of the following singular algebraic curve:*

$$x^3 - y^2 = (n-2p)\,x\,y.$$

Proof. Let assume that X is an element in $\mathcal{Z}(W) \subset \mathcal{D}(W)$, let s be the order of X. Since $\Lambda_w(X)\Lambda_w(D_1) = \Lambda_w(D_1)\Lambda_w(X)$ we have that $\Lambda_w(X)$ is a diagonal matrix for every w (see Theorem 3.7 for the expression of $\Lambda_w(D_1)$). From $\Lambda_w(X)\Lambda_w(D_3) = \Lambda_w(D_3)\Lambda_w(X)$ we obtain that $\Lambda_w(X)$ is a scalar matrix, i.e.

$$\Lambda_w(X) = p_X(w)\mathbf{I}$$

with $p_X(w)$ a polynomial on w. From Corollary 4.6 we have that the degree of p_X is equal to the order of X, s, which has to be even. If $s=0$ we have nothing to prove; it is clear that $s \neq 2$ since all the operators of order 2 in $\mathcal{D}(W)$ are a linear combination of $\{D_1, D_2, D_3, D_4, \mathbf{I}\}$ (see Theorem 3.7). We will conclude the proof by induction on the even number s: assume that $s \geq 4$, since the polynomials p_1 and p_2 are of degree 4 and 6 respectively we can find integers $s_1 \in \mathbb{N}_0$ and $s_2 \in \{0,1\}$ such that the polynomial $p_{\widetilde{X}} = p_X - p_1^{s_1}p_2^{s_2}$ is of degree less to s. Therefore, by inductive hypothesis, the operator $\widetilde{X} = X - {C_1}^{s_1}{C_2}^{s_2}$ is generated by C_1 and C_2, and then also the operator X. ☐

Corollary 6.2. *For every operator $D \in \mathcal{Z}(W)$ there exist a unique pair of polynomials p and q such that*

$$D = p(C_1) + q(C_1)C_2.$$

Theorem 6.3. *The algebra $\mathcal{D}(W)$ is a finitely-generated torsion-free module over the ring $\mathcal{Z}(W)$, but it is not flat and therefore neither projective.*

Proof. Torsion free is clear since for every operator D in $\mathcal{Z}(W)$ the eigenvalue $\Lambda_w(D)$ is a scalar matrix. From Corollary 4.4, since $C_1 = (D_3+D_4)^2 = (D_1+D_2)^2 + (n-2p)(D_1-D_2)$ (see Remark 3.8), it is clear that $\mathcal{D}(W)$ is finitely generated over the subalgebra generated by C_1 and therefore by $\mathcal{Z}(W)$. Given that the center $\mathcal{Z}(W)$ is a Noetherian ring, $\mathcal{D}(W)$ is projective if and only if it is locally free if and only if it is flat.

We will prove that $\mathcal{D}(W)$ is not locally free. Let $\mathfrak{m}$ the maximal ideal generated by C_1 and C_2, then the complement S are the operators of the form $p(C_1) + q(C_1)C_2$ with p and q polynomials such that the constant term of p is nonzero; i.e. S is the set of operators D in $\mathcal{Z}(W)$ such that the polynomial $\Lambda_w(C_1)$ does not divides $\Lambda_w(D)$ (recall from (25) that $\Lambda_w(C_1)$ divides $\Lambda_w(C_2)$).

Let us assume that $\{A_i\}_{j\in J}$ is a basis of the localization of $\mathcal{D}(W)$ with respect to $\mathfrak{m}$, called $\mathcal{D}(W)_{\mathfrak{m}}$, over the ring $\mathcal{Z}(W)_{\mathfrak{m}}$, the localization of $\mathcal{Z}(W)$ by $\mathfrak{m}$. Therefore,

$$(26)\qquad D_3 = \sum_{j\in J} C_{1,j}A_j\,, \qquad D_2D_3 = \sum_{j\in J} C_{2,j}A_j,$$

with $C_{i,j}$ of the form

$$C_{i,j} = \frac{p_{i,j}(C_1) + q_{i,j}(C_1)C_2}{p'_{i,j}(C_1) + q'_{i,j}(C_1)C_2},$$

with $p_{i,j}, q_{i,j}, p'_{i,j}, q'_{i,j}$ polynomials for all $i = 1,2$ and $j \in J$, such that the constant coefficient of $p'_{i,j}$ is nonzero.

Notice that there exists a j_0 such that p_{2,j_0} has a nonzero constant term, in other words p_{j_0} is in S: from considering the eigenvalues in (26) we have

$$\Lambda_w(D_2)\Lambda_w(D_3)\prod_{j\in J}(p'_{2,j}(\Lambda_w(C_1))+q'_{2,j}(\Lambda_w(C_1))\Lambda_w(C_2))$$
$$=\sum_{j\in J}\left(\prod_{k\neq j}(p'_{2,k}(\Lambda_w(C_1))+q'_{2,k}(\Lambda_w(C_1))\Lambda_w(C_2))\right)\Big(p_{2,j}(\Lambda_w(C_1))+q_{2,j}(\Lambda_w(C_1))\Lambda_w(C_2))\Big)\Lambda_w(A_j),$$

if $p_{2,j}$ has a zero constant term for every j then we would have that the polynomial $w+n-p+1$ divides the right-hand side of the equation above because it divides $\Lambda_w(C_1)$ and $\Lambda_w(C_2)$, hence we would have that it also divides the left-hand side which is clearly false (see (9).

Given that

$$C_1D_2D_3-C_2D_3=0,$$

we have

$$C_1\sum_{j\in J}C_{2,j}A_j-C_2\sum_{j\in J}C_{1,j}A_j=0,$$

and then, denoting $p_{i,j},q_{i,j},p'_{i,j},q'_{i,j}$ the respective polynomials $p_{i,j}(C_1),q_{i,j}(C_1),p'_{i,j}(C_1),q'_{i,j}(C_1)$ (in order to simplify the notation in this part of the proof) and using that $C_2^2=C_1^3+(2n-p)C_1C_2$, we have

$$\begin{aligned}
C_1C_{2,j_0}-C_2C_{1,j_0} &=0,\\
C_1(p_{2,j_0}+q_{2,j_0}C_2)/(p'_{2,j_0}+q'_{2,j_0}C_2) &=C_2(p_{1,j_0}+q_{1,j_0}C_2)/(p'_{1,j_0}+q'_{1,j_0}C_2),\\
C_1(p_{2,j_0}+q_{2,j_0}C_2)(p'_{1,j_0}+q'_{1,j_0}C_2) &=(p_{1,j_0}C_2+q_{1,j_0}C_2^2)(p'_{2,j_0}+q'_{2,j_0}C_2),\\
C_1(p_{2,j_0}+q_{2,j_0}C_2)(p'_{1,j_0}+q'_{1,j_0}C_2) &=(p_{1,j_0}C_2+q_{1,j_0}(C_1^3-(n-2p)C_1C_2))(p'_{2,j_0}+q'_{2,j_0}C_2),\\
C_1(p_{2,j_0}+q_{2,j_0}C_2)(p'_{1,j_0}+q'_{1,j_0}C_2) &=(p_{1,j_0}C_1^3+(q_{1,j_0}-p_{1,j_0}(n-2p)C_1)C_2)(p'_{2,j_0}+q'_{2,j_0}C_2),
\end{aligned}$$
$$p'_{1,j_0}C_1p_{2,j_0}+q'_{1,j_0}C_1q_{2,j_0}C_1^3+C_1(p'_{1,j_0}q_{2,j_0}+q'_{1,j_0}p_{2,j_0}+(n-2p)q'_{1,j_0}q_{2,j_0})C_2=$$
$$p_{1,j_0}C_1^3p'_{2,j_0}+(q_{1,j_0}-p_{1,j_0}(n-2p)C_1)q'_{2,j_0}C_1^3+$$
$$(p_{1,j_0}C_1^3q'_{2,j_0}+(q_{1,j_0}-p_{1,j_0}(n-2p)C_1)p'_{2,j_0}+(q_{1,j_0}-p_{1,j_0}(n-2p)C_1)q'_{2,j_0}(n-2p)C_1)C_2,$$

but this contradicts Corollary 4.6: let us focus on the term without C_2, on the left-hand side of the last equation we have a polynomial on (C_1) with nonzero term of degree one but on the right-hand side we have a polynomial with null term of order one.

This contradiction is a consequence of assuming that there exists a basis. Therefore, $\mathcal{D}(W)_{\mathfrak{m}}$ is not free. Then $\mathcal{D}(W)$ is not locally free over $\mathcal{Z}(W)$. $\square$

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CIEM-FaMAF, Universidad Nacional de Córdoba, Córdoba 5000, Argentina

E-mail address: zurrian@famaf.unc.edu.ar