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SUPERTROPICAL QUADRATIC FORMS I

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. We initiate the theory of a quadratic form q over a semiring R. As customary, one can write

$$q(x+y) = q(x) + q(y) + b(x,y),$$

where b is a **companion bilinear form**. But in contrast to the ring-theoretic case, the companion bilinear form need not be uniquely defined. Nevertheless, q can always be written as a sum of quadratic forms $q = \kappa + \rho$, where κ is **quasilinear** in the sense that $\kappa(x+y) = \kappa(x) + \kappa(y)$, and ρ is **rigid** in the sense that it has a unique companion. In case that R is a supersemifield (cf. Definition 4.1 below) and q is defined on a free R-module, we obtain an explicit classification of these decompositions $q = \kappa + \rho$ and of all companions b of q.

As an application to tropical geometry, given a quadratic form $q : V \to R$ on a free module V over a commutative ring R and a supervaluation $\varphi : R \to U$ with values in a supertropical semiring [5], we define – after choosing a base $\mathcal{L} = (v_i \mid i \in I)$ of V – a quadratic form $q^{\varphi} : U^{(I)} \to U$ on the free module $U^{(I)}$ over the semiring U. The analysis of quadratic forms over a supertropical semiring enables one to measure the "position" of q with respect to $\mathcal{L}$ via φ.

Contents

Introduction and basic notions

Our objective in this paper is to lay a general foundation of the theory of quadratic forms over semirings, in particular, semifields; in other words, the scalars are not required to have negation. Examples of interest include the positive rational numbers, more generally

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the set of sums of squares in a field, and rational functions with positive coefficients. But the main motivation for us is tropical (and supertropical) mathematics, which is done over the max-plus algebra. Thus, we are concerned especially with quadratic forms on modules over idempotent semirings. In this paper all semirings have a unit element and are tacitly assumed to be commutative. Thus here a **semiring** R is a set R equipped with addition and multiplication, such that both $(R,+)$ and $(R,\cdot)$ are abelian monoids[1] with elements $0 = 0_R$ and $1 = 1_R$ respectively, and multiplication distributes over addition in the usual way. We call a semiring R a **semifield**, if every nonzero element of R is invertible, hence $R \setminus \{0\}$ is an abelian group.

A **module** V over R is an abelian monoid $(V,+)$ equipped with a scalar multiplication $R \times V \to V$, $(a, v) \mapsto av$, such that exactly the same rules hold as customary for modules if R is a ring: $a(bv) = (ab)v$, $a(v+w) = av+aw$, $(a+b)v = av+bv$, $1_R \cdot v = v$, $0_R \cdot v = 0_V = a \cdot 0_V$ for all $a, b \in R$, $v, w \in V$. Most often we write 0 for both 0_V and 0_R and 1 for 1_R.

We call an R-module V **free**, if there exists a family $(\varepsilon_i \mid i \in I)$ in V such that every $x \in V$ has a unique presentation $x = \sum\limits_{i \in I} x_i \varepsilon_i$ with scalars $x_i \in R$ and only finitely many x_i nonzero, and we then call $(\varepsilon_i \mid i \in I)$ a (classical) **base** of the R-module V. In particular, the R-module $R^{(I)}$ consisting of all tuples $(x_i \mid i \in I)$ with $x_i \in R$, almost all $x_i = 0$, is free. It has the **standard base** consisting of the tuples with precisely one coordinate 1 and the other coordinates 0. In contrast to the field case, many modules over a semifield are not free.

We carry over the customary definition of a quadratic form: If V is a module over a semiring R, then a **quadratic form on** V is a function $q : V \to R$ with $q(ax) = a^2 q(x)$ for $a \in R$, $x \in V$, such that there exists a symmetric bilinear form $b : V \times V \to R$ with

$$q(x + y) = q(x) + q(y) + b(x, y)$$

for $x, y \in V$. Every such bilinear form b will be called a **companion** of q, and the pair (q, b) will be called a **quadratic pair** on V. If R happens to be a ring, then, of course, q has just one companion, namely, $b(x, y) := q(x+y)-q(x)-q(y)$, but if R is a semiring which cannot be embedded into a ring, this most often is wrong, and much of this paper is concerned with investigating these companions, especially in the first three chapters.

Quadratic form theory over a semiring in general is an arid area. But from Chapter 4 on we focus on a rather special class of semirings, the so-called "*supertropical semirings*", which have been designed to enrich the algebraic toolbox for working in tropical mathematics, in particular, tropical geometry, as described e.g. in [3], [18].

These semirings are closely related to (totally) ordered monoids. Any ordered monoid gives rise to the familiar **max-plus algebra**, whose multiplication is the original monoid operation, and whose addition is the maximum in the given ordering. The ensuing algebraic structure is that of a (commutative) **bipotent semiring**. Conversely, in the notation customarily used in tropical geometry, bipotent semirings appear as ordered additive monoids with absorbing element $-\infty$. Thus the primordial object here is the bipotent semifield $T(\mathbb{R}) = \mathbb{R} \cup \{-\infty\}$, cf. e.g. [3, §1.5].

In [4] the first author had introduced a cover of $T(\mathbb{R})$, graded by the multiplicative monoid $(\mathbb{Z}_2, \cdot)$, which was dubbed the *extended tropical arithmetic*. Then, in [12] and [13], this structure was amplified to the notion of a **supertropical semiring**. A supertropical semiring R is equipped with a "ghost map" $\nu = \nu_U : R \to R$, which respects addition and multiplication and is idempotent, i.e., $\nu \circ \nu = \nu$. Moreover, in this semiring $a + a = \nu(a)$ for

[1] A monoid means a semigroup which has a neutral element.

every $a \in R$ (cf. [5, §3]). This replaces the rule $a+a=a$ taking place in the usual max-plus (or min-plus) arithmetic. We call $\nu(a)$ the "**ghost**" of a, and we call the non-ghost elements of R "**tangible**". (The element 0 is regarded both as tangible and ghost.) R then carries a multiplicative idempotent $e = e^2$ such that $\nu(a) = ea$ for every $a \in R$. The image eR of the ghost map, called the **ghost ideal** of R, is itself a bipotent semiring.

Supertropical semirings allow a refinement of valuation theory to a theory of "*supervaluations*", the basics of which can be found in [5]–[7]. Supervaluations may provide an enriched version of tropical geometry, cf. [5, §9,§11] and [12], as well as of tropical matrix theory and the associated linear tropical algebra, as introduced in [13]–[15] and [9]. We recall the initial definitions.

An **m-valuation** (= monoid valuation) on a semiring R is a multiplicative map $v : R \to M$ to a bipotent semiring M with $v(0) = 0$, $v(1) = 1$, and

$$v(x+y) \leq v(x) + v(y) \quad [= \max(v(x), v(y)], \tag{0.1}$$

cf. [5, §2]. We call v a **valuation** if in addition the semiring M is **cancellative**, by which we mean that $M \setminus \{0\}$ is closed under multiplication and is a cancellative monoid in the usual sense. If R happens to be a (commutative) ring, these valuations coincide with the valuations of rings defined by Bourbaki [1] (except that we switched from additive notation there to multiplicative notation here), and if R is a field, we are back to Krull valuations.

Given an m-valuation $v : R \to M$ there exists multiplicative mappings $\varphi : R \to U$ into various supertropical semirings U with $\varphi(0) = 0$, $\varphi(1) = 1$, such that M is the ghost ideal of U and $\nu_U \circ \varphi = v$. These are the **supervaluations** covering v, cf. [5, §4]. We then write $v = e\varphi$.

Assume that R is a ring and $q : V \to R$ is a quadratic form on a free R-module V. Assume further that $\varphi : R \to U$ is a supervaluation. We will describe below (§7) a process of "supertropicalization" which assigns to q *after choice of a base* $\mathcal{L} = (v_i \mid i \in I)$ of V a quadratic form $q^\varphi : U^{(I)} \to U$ over U, and further assigns to the companion b of q a companion b^φ of q^φ.

What can be the merits of such a supertropicalization (q^φ, b^φ) of the datum $(q, \mathcal{L})$? An important point seems to be the fact, proved in §6 (cf. Theorem 6.6), that the free U-module $U^{(I)}$ essentially has only one base, its standard base $(\varepsilon_i \mid i \in I)$; namely, we obtain all other bases of $U^{(I)}$ by permuting the ε_i and multiplying them by units of U. Also b^φ can be read off from q^φ in a simple way (cf. §7). Thus we can concentrate on q^φ alone. But changing the base $\mathcal{L}$ of V may alter q^φ considerably (cf. [8]).

It seems to be appropriate to regard the isomorphism class of a given quadratic form $q^\varphi : U^{(I)} \to U$ as a sort of invariant of the pair $(q, \mathcal{L})$ measuring the "position" of q with respect to $\mathcal{L}$ by means of the supervaluation φ.

Concerning explicit computation, we emphasize that every base $\mathcal{L}$ of V is admitted here, while in usual quadratic form theory, even over a field R, often first a base of V has to be established fitting the form q and the problem (e.g., an orthogonal base when char $R \neq 2$).

Before we say more about the contents of the papers, we give a precise definition of bipotent and supertropical semirings completing the more intuitive description of these semirings above.

Definition 0.1 (cf. [5, §1]). *A semiring R is **bipotent**, if for any $x, y \in R$*

$$x + y \in \{x, y\}. \tag{0.2}$$

If R is bipotent then a total ordering $\leq$ on R is given by the rule

$$x \leq y \quad \Leftrightarrow \quad x+y=y. \tag{0.3}$$

Clearly, this ordering $\leq$ is compatible with addition and multiplication, and $0 \leq x$ for every $x \in R$.

The notion of bipotence is foreign to rings, in which $a+b=b$ implies $a=0$, so one would expect a different flavor from the classical theory of quadratic forms.

Definition 0.2 (cf. [5, §3]). *A semiring R is **supertropical**, if $e := 1+1$ is an idempotent element (i.e., $1+1 = 1+1+1+1$) and the following axioms hold for all $x, y \in R$.*

$$\text{If } ex \neq ey, \quad \text{then} \quad x+y \in \{x,y\}. \tag{0.4}$$

$$\text{If } ex = ey, \quad \text{then} \quad x+y = ey. \tag{0.5}$$

If R is supertropical, then $x+y \in \{x,y\}$ for any $x, y \in eR$, and thus the ideal eR is a bipotent semiring with unit element e. Moreover, (0.4) sharpens to the following rule $(x, y \in R)$:

$$\text{If } ex < ey, \quad \text{then} \quad x+y=y \tag{0.6}$$

(cf. [5, §3]). Rules (0.5) and (0.6) show that the addition on R is determined by the ordering of the bipotent semiring eR, the idempotent e, and the map $\nu_R : R \to eR$, $x \mapsto ex$. ν_R is the ghost map, and eR is the ghost ideal of R mentioned above.

Notice also that $ex = 0$ implies $x = 0$, as follows from (0.5), applied to the elements x and 0.

After providing basic notation and definitions about quadratic forms in §1, we study in §2–§4 the set of companions of a given quadratic form $q : V \to R$ over a supertropical semiring R, arriving in §4 at explicit and complete results in the case that V is free and R is a **tangible supersemifield**, which means that every nonzero tangible $a \in R$ is invertible in R and moreover $eR = (e\mathcal{T}) \cup \{0\}$, whence every nonzero $b \in eR$ is invertible in eR.[2]

We call a quadratic form q **quasilinear** if the zero bilinear form $b = 0$ is a companion of q, i.e., $q(x+y) = q(x)+q(y)$ for all $x, y \in V$, and we call q **rigid** if q has only one companion.

It turns out (Theorem 2.14 below), that for V a free module with base $(\varepsilon_i \mid i \in I)$ a quadratic form q on V is rigid iff $q(\varepsilon_i) = 0$ for all $i \in I$, a fact of central importance for the whole paper.

In §5 we study presentations of a quadratic form $q : V \to R$ as a sum $q = \kappa + \rho$ (i.e., $q(x) = \kappa(x) + \rho(x)$ for every $x \in V$) with κ quasilinear and ρ rigid. If R is supertropical and the R-module V is free, it turns out that such a presentation is always possible, and κ is uniquely determined by q. We call κ the **quasilinear part** of q, and write $\kappa = q_{\mathrm{QL}}$, and we call any rigid ρ with $q = q_{\mathrm{QL}} + \rho$ a **rigid complement** of q_{QL} in q. In the special case that R is a nontrivial tangible supersemifield, the results in §4 allow a precise description of all rigid complements for a given q. The interplay between quasilinear and rigid forms will also be a major theme in [8].

Quasilinear parts and rigid complements comprise a subject completely alien to quadratic forms over rings. Nothing similar to §5 seems to be possible in classical quadratic form theory.

If R is a supertropical semiring, then every R-module V carries a natural partial ordering, called the **minimal ordering of** V, which is defined simply as follows. If $x, y \in V$, then

$$x \leq y \quad \Leftrightarrow \quad \exists z \in V : x+z=y, \tag{0.7}$$

[2]Discarding an uninteresting case, we also assume in §4 that R is **nontrivial**, i.e. $eR \neq \{0, e\}$.

as will be explained in §6. (For $V = R$ this was already observed in [5, §11].) This ordering is compatible with addition and scalar multiplication in an obvious sense.

The minimal ordering seems to be instrumental for understanding various somewhat combinatorial facets of supertropical quadratic form theory, which have no counterparts in the quadratic form theory over rings. In particular, the minimal ordering on a free module V over a supertropical semiring R may be regarded as responsible for the uniqueness result about bases of V mentioned above.

Finally, in §7, we tie the theory to the classical theory of quadratic forms. We spell out the concept of supertropicalization of a quadratic form on a free module over a ring by a supervaluation. But exploiting this concept in depth requires more theory of quadratic forms and pairs over a supertropical semiring, some of which will be presented in [8].

We advocate that in a full fledged supertropical quadratic form theory it is mandatory to admit also certain semirings, which are related to supertropical semirings, but themselves are not supertropical, e.g., polynomial function semirings in any number of variables over a supertropical semiring, cf. [12, §4].

A semiring R is called **upper bound** (abbreviated u.b.) if the relation (0.7) for $V = R$ is a partial ordering on R, called again the *minimal ordering on* R, cf. [5, Definition 11.5]. Polynomial function semirings are an example.

A modest theory of quadratic forms and quadratic pairs over an u.b. semiring seems to be a reasonable general frame in which to place supertropical quadratic form theory. Several u.b.-results along these lines can be found in §3 and §5.

Notations 0.3. *Let* $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *If R is a semiring, then R^* denotes the group of units of R. If R is a supertropical semiring, then*

- $\mathcal{T}(R) := R \setminus eR =$ *set of tangible elements* $\neq 0$,
- $\mathcal{G}(R) := eR \setminus \{0\} =$ *set of ghost elements* $\neq 0$,
- ν_R *denotes the ghost map* $R \to eR$, $a \mapsto ea$.

When there can be no confusion, we write $\mathcal{T}$, $\mathcal{G}$, ν instead of $\mathcal{T}(R)$, $\mathcal{G}(R)$, ν_R.
For $a \in R$ we also write $ea = \nu a = \nu(a) = a^\nu$.

1. Quadratic forms over a semiring

To begin with, we assume that R is a semiring (always commutative) and V is an R-module.

Definition 1.1. *A **quadratic form** q on V is a function $q : V \to R$ such that*

$$q(ax) = a^2 q(x) \tag{1.1}$$

for any $a \in R$, $x \in V$, and there exists a symmetric bilinear form $b : V \times V \to R$ (not necessarily uniquely determined by q) such that for any $x, y \in V$

$$q(x + y) = q(x) + q(y) + b(x, y). \tag{1.2}$$

*We then also say that (q, b) is a **quadratic pair** on V.*

Given a quadratic pair (q, b) on V, it follows from (1.1) and (1.2) with $x = y$ that

$$4q(x) = 2q(x) + b(x, x). \tag{1.3}$$

Definition 1.2. *We call a quadratic pair (q, b) on V **balanced** if for any $x \in V$*

$$b(x, x) = 2q(x). \tag{1.4}$$

Remark 1.3. *If R is a ring, then a quadratic pair (q,b) is determined by the quadratic form q alone, namely*

$$b(x,y) = q(x+y) - q(x) - q(y), \tag{1.5}$$

and, moreover, (q,b) is balanced. If, in addition, 2 is a unit in R, then we have a bijection between quadratic forms q and symmetric bilinear forms b on V via (1.5) *and*

$$q(x) = \frac{1}{2} b(x,x), \tag{1.6}$$

as is very well known.

But later our main interest will be in the case that R is a supertropical semiring (or even a "supersemifield", cf. Definition 4.1 below). Then quadratic forms and symmetric bilinear forms will be only loosely related, and bilinear forms which are not symmetric will play a major role. Also we will meet many quadratic pairs which are not balanced.

Example 1.4. *Any bilinear form $B : V \times V \to R$ on V (not necessarily symmetric) gives us a balanced quadratic pair (q,b) on V as follows $(x,y \in V)$:*

$$q(x) := B(x,x), \tag{1.7}$$

$$b(x,y) := B(x,y) + B(y,x). \tag{1.8}$$

We exhibit a special class of quadratic forms which are much easier to handle than quadratic forms in general.

Definition 1.5. *A quadratic form $q : V \to R$ is called* ***quasilinear***[3] *if q, together with the null form $b : V \times V \to R$, $b(x,y) = 0$ for all $x,y \in V$, is a quadratic pair. This means that for any $x,y \in V$, $a \in R$,*

$$q(ax) = a^2 q(x), \qquad q(x+y) = q(x) + q(y). \tag{1.9}$$

Example 1.6. *If $B : V \times V \to R$ is a symmetric bilinear form, then the quadratic form $q(x) := B(x,x)$ together with the bilinear form $b(x,y) = 2B(x,y)$ is a balanced quadratic pair (cf. Example 1.4). If in addition $2 = 0$ in R, then the form q is quasilinear.*

Quasilinear forms over a supertropical semifield have been considered in [10, §5]. We spell out what the equations (1.3) and (1.4) mean if the semiring R is bipotent or supertropical.

Remark 1.7. *Let (q,b) be a quadratic pair on V.*

(i) *Suppose that R is bipotent. Then $2 \cdot 1_R = 1_R$, and thus* (1.3) *reads*

$$q(x) = q(x) + b(x,x),$$

which means that[4]

$$b(x,x) \le q(x). \tag{1.10}$$

The pair (q,b) is balanced iff for all $x \in V$

$$b(x,x) = q(x). \tag{1.11}$$

[3] In the case that R is a valuation domain, cf. [16, I,§6]. In [2] these forms are called "totally singular".

[4] Recall that a bipotent semiring has a natural total ordering: $x \le y$ iff $x + y = y$.

(ii) *Assume that R is supertropical. Now $2 \cdot 1_R = e$. Thus (1.3) reads*

$$eq(x) = eq(x) + b(x, x)$$

which means that

$$eb(x, x) \leq eq(x). \tag{1.12}$$

The pair (q, b) is balanced iff

$$b(x, x) = eq(x) \tag{1.13}$$

for all $x \in V$.

We return to an arbitrary semiring R and now focus on the case that V is a free R-module with (classical) base $\varepsilon_1, \dots, \varepsilon_n$ and describe a quadratic pair (q, b) in explicit terms. We write elements x, y of V as

$$x = \sum_{i=1}^{n} x_i \varepsilon_i, \qquad y = \sum_{i=1}^{n} y_i \varepsilon_i,$$

with $x_i, y_i \in \mathbb{R}$. We use the abbreviations

$$\alpha_i := q(\varepsilon_i), \qquad \beta_{i,j} := b(\varepsilon_i, \varepsilon_j). \tag{1.14}$$

Applying the rules (1.1) and (1.2) and iterating, we obtain first

$$q(x) = q\left(\sum_{i=1}^{n-1} x_i \varepsilon_i \right) + \alpha_n x_n^2 + \sum_{i=1}^{n-1} \beta_{i,n} x_i x_n,$$

and finally

$$q(x) = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j} \beta_{i,j} x_i x_j. \tag{1.15}$$

We further have

$$b(x, y) = \sum_{i,j=1}^{n} \beta_{i,j} x_i y_j, \tag{1.16}$$

with $\beta_{i,j} = \beta_{j,i}$, and, in consequence of (1.3),

$$4\alpha_i = 2\alpha_i + \beta_{i,i}. \tag{1.17}$$

If q is quasilinear, then

$$q(x) = \sum_{i=1}^{n} \alpha_i x_i^2. \tag{1.18}$$

Definition 1.8. *We call the α_i and the $\beta_{i,j}$ the* ***coefficients*** *of the quadratic pair (q, b) with respect to the base $\varepsilon_1, \dots, \varepsilon_n$.*

Proposition 1.9. *The quadratic pair (q, b) is balanced iff $\beta_{i,i} = 2\alpha_i$ $(1 \leq i \leq n)$.*

Proof. The equations $\beta_{i,i} = 2\alpha_i$ are special cases of the balancing rule (1.4). Assume that these equations hold. Then indeed, for any $x = \sum_i x_i \varepsilon_i$,

$$\begin{aligned} b(x,x) &= \sum_{i,j} \beta_{i,j} x_i x_j \\ &= \sum_i \beta_{i,i} x_i^2 + 2 \sum_{i<j} \beta_{i,j} x_i x_j \\ &= 2\Big[\sum_i \alpha_i x_i^2 + \sum_{i<j} \beta_{i,j} x_i x_j \Big] = 2q(x). \end{aligned}$$

□

If R is not a ring but only a semiring, then often different polynomials in $x_1, \dots, x_n$ give the same function on V. In order not to complicate our setting too much at present, we now redefine (in the case of a free module V) a quadratic form as a polynomial on V, a notion which with sufficient care can be regarded as independent of the choice of the base $\varepsilon_1, \dots, \varepsilon_n$. Accordingly, we also view a bilinear form on V as a polynomial in variables $x_1, \dots, x_n$, $y_1, \dots, y_n$.

Whenever necessary, we specify the quadratic forms and pairs in this polynomial sense **formal quadratic forms** and pairs, while for the forms and pairs in Definition 1.1 we use the term "**functional**". The dependence of a formal pair (q, b) on a given base of the free R-module is not a conceptual problem, since it is obvious how to rewrite q and b with respect to another base.

We now work with formal quadratic forms and pairs, keeping fixed the base $\varepsilon_1, \dots, \varepsilon_n$ of V. In the following we often denote a bilinear form b on V by its "Gram-matrix"

$$b = \begin{pmatrix} \beta_{1,1} & \dots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \dots & \beta_{n,n} \end{pmatrix}, \tag{1.19}$$

$\beta_{i,j} := b(\varepsilon_i, \varepsilon_j)$, and we denote by ${}^{\mathrm{t}}b$ the bilinear form given by the transpose of the matrix b. On the functional level ${}^{\mathrm{t}}b(x, y) = b(y, x)$.

Definition 1.10. *We call a bilinear form b on V **triangular**, if all coefficients $\beta_{i,j}$ with $i > j$ are zero; hence*

$$b = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,n} \\ & \beta_{2,2} & & \vdots \\ & & \ddots & \beta_{n-1,n} \\ 0 & & & \beta_{n,n} \end{pmatrix}. \tag{1.20}$$

This property strongly depends on the choice of the base $\varepsilon_1, \dots, \varepsilon_n$ of V.

Definition 1.11. *Given a quadratic form q and a bilinear form B on V, we say that q **admits** B, or that B **expands** q, if $B(x,x) = q(x)$ (cf. Example 1.4). In explicit terms this means that, if $B = (\beta_{i,j})$, then*

$$q(x) = \sum_{i,j=1}^{n} \beta_{i,j} x_i x_j. \tag{1.21}$$

The following is now obvious.

Proposition 1.12. *A given quadratic form q on V with coefficients α_i $(1 \le i \le n)$, and $\beta_{i,j}$ $(1 \le i,j \le n)$, (cf. Definition 1.8) expands to a unique triangular bilinear form, denoted by ∇q, namely,*

$$\nabla q = \begin{pmatrix} \alpha_1 & \beta_{1,2} & \dots & \beta_{1,n} \\ & \alpha_2 & & \vdots \\ & & \ddots & \beta_{n-1,n} \\ 0 & & & \alpha_n \end{pmatrix}. \tag{1.22}$$

The symmetric bilinear form

$$b_q = \begin{pmatrix} 2\alpha_1 & \beta_{1,2} & \dots & \beta_{1,n} \\ \beta_{1,2} & 2\alpha_2 & & \vdots \\ \vdots & & \ddots & \beta_{n-1,n} \\ \beta_{1,n} & \dots & \beta_{n,n-1} & 2\alpha_n \end{pmatrix} \tag{1.23}$$

is the unique one which completes the formal quadratic form q to a balanced quadratic pair (q, b_q).

Notations 1.13.

(i) *We denote the quadratic form q with coefficients α_i $(1 \le i \le n)$ and $\beta_{i,j}$ $(1 \le i,j \le n)$ by the triangular scheme*[5]

$$q = \begin{bmatrix} \alpha_1 & \beta_{1,2} & \dots & \beta_{1,n} \\ & \alpha_2 & \ddots & \vdots \\ & & \ddots & \beta_{n-1,n} \\ & & & \alpha_n \end{bmatrix}, \tag{1.24}$$

showing all coefficients of the homogeneous polynomial $q(x)$. It is the matrix of ∇q in square brackets. We also say that (1.25) *(or* (1.15)*) is a presentation of the functional quadratic form q.*

(ii) *The quasilinear quadratic form* (1.18) *now reads*

$$q = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ & \alpha_2 & \ddots & \vdots \\ & & \ddots & 0 \\ & & & \alpha_n \end{bmatrix}.$$

For this form, we usually switch to simpler notation with one row:

$$q = [\alpha_1, \alpha_2, \dots, \alpha_n]. \tag{1.25}$$

The notation of (1.24) describes a (formal) quadratic form q on V by parameters α_i, $\beta_{i,j}$, which are uniquely determined by q and the given base of V.

We return to functional quadratic forms and pairs.

Definition 1.14. *Let q be a functional quadratic form on the R-module V. We call a symmetric bilinear form b on V a* ***companion*** *of q (or say that b* ***accompanies*** *q), if b completes q to a functional quadratic pair (q,b). When the pair (q,b) is balanced, we call b a* ***balanced companion*** *of q.*

[5]Such triangular schemes have already been used in the literature in the case that R is a ring, cf., e.g., [17, I,§2].

As a consequence of Proposition 1.12 we state

Corollary 1.15. *Every functional quadratic from q on a free R-module has a balanced companion b (perhaps not unique, cf. §2).*

In the entire paper our main interest will be in functional quadratic forms and pairs, while formal forms and pairs will usually serve to denote functional forms and pairs in a precise and efficient way (in the case that the R-module V is free). Most often we will not specify whether the forms and the pairs at hand are formal or functional. This should always be clear from the context.

Occasionally we need to extend the notation of (1.13) to the case that V is a free R-module with an infinite base $(\varepsilon_i \mid i \in I)$. Then we choose a total ordering of I and define a formal quadratic form q on V as a triangular table

$$q = \big[\alpha_{i,j} \mid i \leq j\big] \tag{1.26}$$

with entries $\alpha_{i,j} \in R$. The associated functional quadratic form $\bar{q} : V \to R$ is

$$\bar{q}\bigg(\sum_{i\in I} x_i\varepsilon_i\bigg) = \sum_{i\leq j} \alpha_{i,j}x_ix_j, \tag{1.27}$$

with $x_i \in R$, and $x_i \neq 0$ for only finitely many i. A formal bilinear form B on V is an $I \times I$ matrix

$$B = \big(\beta_{i,j} \mid (i,j) \in I \times I\big) \tag{1.28}$$

with $\beta_{i,j} \in R$. The associated functional bilinear form $\bar{B} : V \times V \to R$ reads

$$\bar{B}\bigg(\sum_{i\in I} x_i\varepsilon_i, \sum_{i\in I} y_i\varepsilon_i\bigg) = \sum_{(i,j)\in I\times I} \beta_{i,j}x_ix_j. \tag{1.29}$$

Notice that $\bar{B}$ determines the matrix B uniquely, since $\beta_{i,j} = \bar{B}(\varepsilon_i, \varepsilon_j)$. Thus in contrast to the case of quadratic forms, *formal bilinear forms are the same objects as functional ones* (after fixing an ordered base of V). Consequently, we write again B instead of $\bar{B}$. We often also write q instead of $\bar{q}$, being aware that this imprecise.

We have the following complete analogy of Proposition 1.12 for a free vector space V with an infinite ordered base $(\varepsilon_i \mid i \in I)$.

Proposition 1.16. *Let $q = [\alpha_{i,j} \mid i \leq j]$ be a formal quadratic form on V. The triangular bilinear form*

$$\nabla q = \big(\gamma_{i,j} \mid (i,j) \in I \times I\big) \tag{1.30}$$

with $\gamma_{i,j} = \alpha_{i,j}$ for $i \leq j$, $\gamma_{i,j} = 0$ for $i > j$ expands $\bar{q} : V \to R$. The symmetric bilinear from

$$b_q = \big(\beta_{i,j} \mid (i,j) \in I \times I\big) \tag{1.31}$$

with $\beta_{i,i} = 2\alpha_{i,i}$, $\beta_{i,j} = \beta_{j,i} = \alpha_{i,j}$ for $i < j$, is a balanced companion of $\bar{q}$. On the formal level, ∇q is the unique triangular bilinear form expanding q, and b_q is the unique balanced companion of q.

2. Generalities on partial companions and rigidity

As before, R denotes an arbitrary semiring. Let q be a functional quadratic form on an R-module V. Recall the definition of "companion" (Definition 1.14). We fix a companion $b_0 : V \times V \to R$ of q (which exists by Definition 1.1), and now search for other companions of q in a systematic way. Here the notion of a "*partial companion*" of q will turn out to be helpful.

Definition 2.1. *Let $b : V \times V \to R$ be a symmetric bilinear form. We say that b is a **companion of** q **at** a point $(x, y) \in V \times V$ (or, that b **accompanies** q **at** (x, y)), if*

$$q(x + y) = q(x) + q(y) + b(x, y), \tag{2.1}$$

which can be rephrased as

$$q(x) + q(y) + b(x, y) = q(x) + q(y) + b_0(x, y). \tag{2.2}$$

*If this happens to be true for every point (x, y) of a set $T \subset V \times V$, we call b a **companion of** q **on** T; and in the subcase $T = S \times S$, where $S \subset V$, we say more briefly that b is a **companion of** q **on** S. We then also say that b **accompanies** q on T, resp. on S.*

Remark 2.2. *The bilinear form b accompanies q on an R-submodule W of V iff $b|W \times W$ is a companion of $q|W$.*

Companionship of the null bilinear form deserves special attention. This is reflected in the following terminology.

Definition 2.3. *We say that q is **quasilinear on** a set $T \subset V \times V$, if $b = 0$ accompanies q on T, i.e., for any $(x, y) \in T$*

$$q(x + y) = q(x) + q(y);$$

and we say that q is quasilinear on $S \subset V$ if this happens on $T = S \times S$.

This terminology refines Definition 1.5. The term "quasilinear" there means "quasilinear on V".

Example 2.4. *If $4 \cdot 1_R = 2 \cdot 1_R$, for example, if R is supertropical, then every quadratic form on V is quasilinear on the diagonal of $V \times V$.*

Example 2.5. *Assume that $(\lambda + \mu)^2 = \lambda^2 + \mu^2$ for all $\lambda, \mu \in R$ (e.g., R is supertropical). Let $x_0 \in V$. A symmetric bilinear form b on V accompanies $q : V \to R$ on Rx_0 iff for all $\lambda, \mu \in R \setminus \{0\}$:*

$$(\lambda^2 + \mu^2)q(x_0) = (\lambda^2 + \mu^2)q(x_0) + \lambda\mu b(x_0, x_0). \tag{2.3}$$

Example 2.6. *Let $x_0, y_0 \in V$. A symmetric bilinear form b accompanies $q : V \to R$ on $Rx_0 \times Ry_0$ iff for all $\lambda, \mu \in R \setminus \{0\}$:*

$$\lambda^2 q(x_0) + \mu^2 q(y_0) + \lambda\mu b_0(x_0, y_0) = \lambda^2 q(x_0) + \mu^2 q(y_0) + \lambda\mu b(x_0, y_0). \tag{2.4}$$

(Here R may be any semiring.)

In the following, q always denotes a quadratic form on an R-module V and b a symmetric bilinear form on V. Given a set $T \subset V \times V$ on which b accompanies q, we look for ways to obtain a bigger set $T' \supset T$ on which b accompanies q. As above b_0 denotes a fixed companion of q.

Lemma 2.7. *If b accompanies q at the points $(x, y), (x', y)$, and (x, x'), then b accompanies q at the point $(x + x', y)$.*

Proof. We will use the equations

$$q(x)+q(y)+b(x,y)=q(x)+q(y)+b_0(x,y), \qquad (\alpha)$$

$$q(x')+q(y)+b(x',y)=q(x')+q(y)+b_0(x',y), \qquad (\beta)$$

$$q(x)+q(x')+b(x,x')=q(x)+q(x')+b_0(x,x'), \qquad (\gamma)$$

indicated by (2.2), and furthermore

$$q(x+x')=q(x)+q(x')+b(x,x'), \qquad (\delta)$$

$$q(x+x')=q(x)+q(x')+b_0(x,x'). \qquad (\varepsilon)$$

Applying (δ) we obtain

$$q(x+x')+q(y)+b(x+x',y)=q(x)+q(x')+q(y)+b(x,x')+b(x,y)+b(x',y).$$

Applying (α), (β), and (γ) to the right hand side, we replace b by b_0 throughout. Finally, applying (ε) we obtain

$$q(x+x')+q(y)+b(x+x',y)=q(x+x')+q(y)+b_0(x+x',y),$$

as desired. □

Lemma 2.8. *Assume that b accompanies q on a set $S\subset V$. Then b accompanies q on the submonoid $\langle S\rangle$ of $(V,+)$ generated by S.*

Proof. We have

$$\langle S\rangle=\{x_1+\dots+x_r \mid r\in\mathbb{N},\ x_i\in S\}\cup\{0\}.$$

It is trivial that b accompanies q on $V\times\{0\}$. We prove by induction on r that b accompanies q at every point $(x_1+\dots+x_r,y)$ with $x_i,y\in S$. This is true by assumption for $r=1$. Assuming the claim for sums $x_1+\dots+x_s$ with $s<r$, we see that b accompanies q at $(x_1+\dots+x_{r-1},x_r)$. We conclude by Lemma 2.7 that b accompanies q at $(x_1+\dots+x_r,y)$. Thus b accompanies q on $\langle S\rangle\times S$. By an analogous induction we conclude that b accompanies q on $\langle S\rangle\times\langle S\rangle$. □

Proposition 2.9. *Assume that $(\varepsilon_i \mid i\in I)$ is a subset of V which generates V as an R-module, and assume that b accompanies q on the set $S=\bigcup_{i\in I}R\varepsilon_i$. Then b is a companion of q.*

Proof. Now $\langle S\rangle=V$. Lemma 2.8 applies. □

We also address the question of uniqueness of a partial companion of q.

Definition 2.10. *We say that q is **rigid at a point** (x,y) of $V\times V$, if $b_1(x,y)=b_2(x,y)$ for any two companions b_1,b_2 of q at (x,y). We rephrase this as*

$$b(x,y)=b_0(x,y) \qquad (2.5)$$

*for any companion b of q at (x,y), where, as before, b_0 denotes a given companion of q (on V). If q is rigid at every point (x,y) of a set $T\subset V\times V$, we say that q is **rigid on** T. Finally, if q is rigid on $S\times S$ for some $S\subset V$, we more briefly say that q is **rigid on** S, and in the subcase $S=V$ we say that q is a **rigid quadratic form**.*

Later we will make use of the following easy fact.

Proposition 2.11. *Assume that S_1,S_2 are subsets of V, and suppose that q is rigid on $S_1\times S_2$. Let W_i denote the R-submodule of V generated by S_i $(i=1,2)$. Then q is rigid on $W_1\times W_2$.*

Proof. Let b be a companion of q on $S_1 \times S_2$. Then $b|S_1 \times S_2 = b_0|S_1 \times S_2$. From this one concludes easily by bilinearity that $b|W_1 \times W_2 = b_0|W_1 \times W_2$. $\square$

Corollary 2.12. *Assume that $(\varepsilon_i \mid i \in I)$ generates the R-module V, and suppose that q is rigid at $(\varepsilon_i, \varepsilon_j)$ for any $i, j \in I$. Then q is a rigid quadratic form.*

Remarkably, the following rigidity result holds over any semiring R.

Proposition 2.13. *A quadratic form q on an R-module V is rigid on the submodule W of V generated by the set $\{x \in V \mid q(x) = 0\}$.*

Proof. Let $S := \{x \in V \mid q(x) = 0\}$. Given two companions b_1 and b_2 of q on S, we have

$$b_1(x, y) = q(x + y) = b_2(x, y)$$

for every $(x, y) \in S \times S$. Thus q is rigid on S. By Proposition 2.11 it follows, that q is rigid on W. $\square$

Under a mild condition on the semiring R (from the viewpoint of supertropical algebra) we obtain an explicit description of all rigid quadratic forms on a free R-module V.

Theorem 2.14. *Assume that V is free with base $(\varepsilon_i \mid i \in I)$, and that R satisfies the following rules for all $a, b \in R$:*

(i) $2a = 0 \Rightarrow a = 0$,

(ii) $(a + b)^2 = a^2 + b^2$.

Then a quadratic form q on V is rigid iff $q(\varepsilon_i) = 0$ for every $i \in I$.

Proof. If all $q(\varepsilon_i) = 0$, we conclude by Proposition 2.13 that q is rigid. Now assume that q is rigid, but that there exists some $k \in I$ with $q(\varepsilon_k) \neq 0$. Then the balanced companion b_q described in Proposition 1.16 (after choosing a formal representative of q, again denoted by q) has value $b_q(\varepsilon_k, \varepsilon_k) = 2q(\varepsilon_k) \neq 0$. On the other hand, the null form $b = 0$ accompanies q at $(\varepsilon_k, \varepsilon_k)$, as follows from property (ii) above (cf. Example 2.5). Since q is rigid at $(\varepsilon_k, \varepsilon_k)$, and both b_q and $b = 0$ accompany q at $(\varepsilon_k, \varepsilon_k)$, this is a contradiction. $\square$

An important example, where the rules (i) and (ii) above hold, is the semiring R of polynomial functions on K^n for K a supertropical semiring and any $n \in \mathbb{N}$ (cf. [12]), and of course the semiring K itself.

3. Companions on a free module

We allow R to be any semiring, but now always assume that V is a *free* R-module with a fixed base $(\varepsilon_i \mid i \in I)$, and that q is a functional quadratic form on V.

Definition 3.1. *For any $(i, j) \in I \times I$, let $C_{i,j}(q)$ denote the subset of R consisting of the values $b(\varepsilon_i, \varepsilon_j)$ for all companions b of q on $R\varepsilon_i \times R\varepsilon_j$, cf. Definition 2.1. We call the family of subsets of R*

$$C(q) := (C_{i,j}(q) \mid (i, j) \in I \times I) \tag{3.1}$$

*the **companion table** of q. Notice that $C_{i,j}(q) = C_{j,i}(q)$. In the case $I = \{1, 2, \dots, n\}$, we usually write $C(q)$ as a symmetric $n \times n$-matrix,*

$$C(q) := \begin{pmatrix} C_{1,1}(q) & \dots & C_{1,n}(q) \\ \vdots & \ddots & \vdots \\ C_{n,1}(q) & \dots & C_{n,n}(q) \end{pmatrix}, \tag{3.2}$$

and also speak of the ***companion matrix*** *of q. (Of course, if I is infinite, we may again view $C(q)$ as an $I \times I$-matrix after choosing a total ordering on I.)*

As in §2, we choose a fixed companion b_0 of q. We put

$$\beta^0_{i,j} := b_0(\varepsilon_i, \varepsilon_j). \tag{3.3}$$

Looking at Example 2.6, we obtain the following description of $C_{i,j}(q)$.

Proposition 3.2. *$C_{i,j}(q)$ is the set of all $\beta \in R$ with*

$$\lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\beta = \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\beta^0_{i,j}, \tag{3.4}$$

for all $\lambda, \mu \in R \setminus \{0\}$.

Theorem 3.3. *If $(\beta_{i,j} \mid (i,j) \in I \times I)$ is a family in R with $\beta_{i,j} \in C_{i,j}(q)$ and $\beta_{i,j} = \beta_{j,i}$, then the bilinear form b with $b(\varepsilon_i, \varepsilon_j) = \beta_{i,j}$ is a companion of q. This establishes a bijection of the set of these families $(\beta_{i,j})$ with the set of companions of q.*

Proof. Given such a family $(\beta_{i,j})$, let b denote the (unique) symmetric bilinear form with $b(\varepsilon_i, \varepsilon_j) = \beta_{i,j}$. Then b is a companion of q on the set $\bigcup_{i,j} R\varepsilon_i \times R\varepsilon_j$ by definition of the sets $C_{i,j}(q)$. We conclude by Proposition 2.9 that b is a companion of q (on V). The last assertion of the theorem is now obvious. □

Corollary 3.4. *$C_{i,j}(q)$ is the set of values $b(\varepsilon_i, \varepsilon_j)$ with b running through all companions of q.*

Proof. We choose a total ordering on I and fix a pair $(i,j) \in I \times I$ with $i \leq j$. Given an element $\beta_{i,j}$ of $C_{i,j}(q)$, we have to find a companion b of q with $b(\varepsilon_i, \varepsilon_j) = \beta_{i,j}$. By Theorem 3.3 this is easy: We choose for every $(k,\ell) \in I \times I$ with $k \leq \ell$ and (k,ℓ) different from (i,j) an element $\beta_{k,\ell}$ of $C_{k,\ell}(q)$. By the theorem, there exists a (unique) companion b of q such that $b(\varepsilon_k, \varepsilon_\ell) = \beta_{k,\ell}$ for all $(k,\ell) \in I \times I$ with $k \leq \ell$. In particular, $b(\varepsilon_i, \varepsilon_j) = \beta_{i,j}$. □

As a consequence of this corollary, we state

Proposition 3.5. *q is rigid iff every set $C_{i,j}(q)$ consists of only one element.*

Remark 3.6. *It follows from Proposition 3.2 that for any (i,j) in $I \times I$*

$$C_{i,j}(q) = C_{i,j}(q \mid R\varepsilon_i + R\varepsilon_j). \tag{3.5}$$

More generally, $C_{i,j}(q) = C_{i,j}(q|W)$ for any free submodule W of V which contain $\varepsilon_i, \varepsilon_j$ as part of a base.

In the case $I = \{1, \dots, n\}$, Proposition 3.2 and Theorem 3.3 read as follows:

Scholium 3.7. *Assume that $(\varepsilon_i \mid 1 \leq i \leq n)$ is a base of the free module V. Assume further that in this base q has the triangular scheme*

$$q = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ & \ddots & \vdots \\ 0 & & \alpha_{n,n} \end{bmatrix}.$$

We can choose for b_0 the balanced companion of q with diagonal coefficients $2\alpha_{i,i}$ and upper diagonal coefficients $\alpha_{i,j}$, cf. (1.23). Thus the companions of q are the bilinear forms

$$b = \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{pmatrix}$$

with coefficients $\beta_{i,j} = \beta_{j,i}$ *satisfying*

$$(\lambda + \mu)^2 \alpha_{i,i} = (\lambda^2 + \mu^2)\alpha_{i,i} + \lambda\mu\beta_{i,i}, \tag{3.6}$$

for $1 \leq i \leq n$, *and*

$$\lambda^2 \alpha_{i,i} + \mu^2 \alpha_{j,j} + \lambda\mu\alpha_{i,j} = \lambda^2 \alpha_{i,i} + \mu^2 \alpha_{j,j} + \lambda\mu\beta_{i,j}, \tag{3.7}$$

for $1 \leq i < j \leq n$, *with both* λ, μ *running through* $R \setminus \{0\}$.

Our main focus will be a precise description of the sets $C_{i,j}(q)$ in the case that R is a "supersemifield" (cf. §4). In preparation for this, we now exhibit facts which hold over more general semirings of interest in supertropical algebra.

Definition 3.8 (cf. [5, §11])**.** *A semiring R is called **upper bound** (abbreviated u.b.) iff the binary relation* $(x, y \in R)$

$$x \leq y \quad \Leftrightarrow \quad \exists z \in R : x + z = y, \tag{3.8}$$

*on the set R is a partial ordering. This ordering is then called the **minimal ordering** on R.*

This terminology alludes to the fact that R is u.b. iff R carries some partial ordering $\leq'$ compatible with addition, such that $0 \leq' x$ for every $x \in R$, and any two elements x, y of R have an upper bound with respect to $\leq'$ (which then can be chosen as $x + y$). Any such partial ordering $\leq'$ will be a refinement of the minimal ordering.

Clearly the minimal ordering itself respects addition and multiplication:

$$x \leq y \quad \Rightarrow \quad x + z \leq y + z, \ xz \leq yz, \text{ for any } z \in R. \tag{3.9}$$

For example, the semiring of polynomials over a supertropical semiring R in any set of variables is u.b., cf. [5, Proposition 11.8], and of course R itself.

Proposition 3.9. *If R is u.b., then every set $C_{i,j}(q)$ is convex in R. In other words, if $\beta_1 \leq \gamma \leq \beta_2$ and $\beta_1, \beta_2 \in C_{i,j}(q)$, then $\gamma \in C_{i,j}(q)$.*

Proof. We invoke Proposition 3.2. For any $\lambda, \mu \in R \setminus \{0\}$

$$\begin{aligned} \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\beta_1 &\leq \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\gamma \\ &\leq \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\beta_2. \end{aligned}$$

Now the first sum equals the third sum, and hence equals also the second sum. □

In the same vein we obtain

Proposition 3.10. *Assume that R is a semiring and $q : V \to R$ is quasilinear on a set $T \subset V \times V$ (cf. Definition 2.3). Let b_1 and b_2 be symmetric bilinear forms on V.*

a) *If b_1 and b_2 are companions of q on T, then $b_1 + b_2$ is a companion of q on T.*

b) *Assume that R is u.b. If $b_1 + b_2$ is a companion of q on T, then both b_1 and b_2 are companions of q on T.*

Proof. Let $(x, y) \in T$. We have $q(x + y) = q(x) + q(y)$.

a): From $q(x) + q(y) = q(x) + q(y) + b_i(x, y)$ $(i = 1, 2)$, we obtain

$$q(x) + q(y) = q(x) + q(y) + b_2(x, y) = q(x) + q(y) + b_1(x, y) + b_2(x, y).$$

b): We have for $i = 1, 2$

$$q(x) + q(y) \leq q(x) + q(y) + b_i(x, y) \leq q(x) + q(y) + b_1(x, y) + b_2(x, y).$$

The first sum equals the third sum, hence also equals the second sum. □

Let R_0 denote the prime supertropical semiring, $R_0 = \{0, 1, e\}$.

Corollary 3.11. *Assume that R is u.b. and contains R_0 as a subsemiring. Let V be any R-module. Then a symmetric bilinear form b on V is a companion of a quasilinear form $q : V \to R$ iff eb is a companion of q.*

Proposition 3.12. *Assume that f is an element of R with $f + f = f$ (e.g., R contains $R_0 = \{0, 1, e\}$ and $f = e$). Assume furthermore that $C_{i,j}(q) \cap (Rf) \neq \emptyset$. Then $C_{i,j}(q)$ is closed under addition.*

Proof. We again invoke Proposition 3.2. Let $\beta_0 \in C_{i,j}(q) \cap Rf$. Pick $\beta_1, \beta_2 \in C_{i,j}(q)$. For any $\lambda, \mu \in R$ the three sums $\lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\beta_k$ $(k = 0, 1, 2)$ are equal. From this we obtain that

$$\begin{aligned} \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu(\beta_1 + \beta_2) &= \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu(\beta_0 + \beta_0) \\ &= \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu\beta_0, \end{aligned}$$

since $\beta_0 + \beta_0 = \beta_0$. We conclude that $\beta_1 + \beta_2 \in C_{i,j}(q)$. □

Proposition 3.13. *Assume that R is u.b., and that $(a + b)^2 = a^2 + b^2$ for any $a, b \in R$. In particular $(a = b = 1)$, $2 \cdot 1_R = 4 \cdot 1_R$. (For example, R could be u.b., containing the prime supertropical semiring R_0). Then, with $e := 2 \cdot 1_R$,*

$$C_{i,i}(q) = \{\beta \in R \mid 0 \leq \beta \leq eq(\varepsilon_i)\} \tag{3.10}$$

for every $i \in I$.

Proof. Due to Remark 3.6, we may assume that $I = \{1\}$, $V = R\varepsilon_1$, and then may simplify to $V = R$, $\varepsilon_1 = 1_R$. Let $\alpha := q(\varepsilon_1)$. This means that $q(x) = \alpha x^2$ $(x \in R)$. We have

$$q(x + y) = \alpha x^2 + \alpha y^2 = \alpha x^2 + \alpha y^2 + e\alpha xy.$$

Thus both linear forms $b_0(x, y) := 0$, $b_1(x, y) := e\alpha xy$ are companions of q, i.e., $0 \in C_{1,1}(q)$ and $e\alpha \in C_{1,1}(q)$. We know by Proposition 3.9 that $C_{1,1}(q)$ is convex; hence $C_{1,1}(q)$ contains the interval

$$[0, e\alpha] := \{\beta \in R \mid 0 \leq \beta \leq e\alpha\}.$$

On the other hand, if b is any companion of q, then

$$eq(\varepsilon_1) = q(e\varepsilon_1) = q(\varepsilon_1 + \varepsilon_1) = eq(\varepsilon_1) + b(\varepsilon_1, \varepsilon_1);$$

hence $b(\varepsilon_1, \varepsilon_1) \leq eq(\varepsilon_1)$. Trivially, $0 \leq b(\varepsilon_1, \varepsilon_1)$. Thus $b(\varepsilon_1, \varepsilon_1)$ is contained in $[0, e\alpha]$. □

Theorem 3.14. *Assume that R is a supertropical semiring with $e\mathcal{T} = \mathcal{G}$. Assume further that $b : V \times V \to R$ is a symmetric bilinear form that accompanies q on the set $\bigcup_{i \in I} \mathcal{T}\varepsilon_i$. Then b is a companion of q.*

Proof. By Proposition 2.9 it suffices to verify that b accompanies q on $R\varepsilon_i \times R\varepsilon_j$ for any two indices $i, j \in I$. This means that we have to verify (cf. (2.4))

$$\lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu b(\varepsilon_i, \varepsilon_j) = \lambda^2 q(\varepsilon_i) + \mu^2 q(\varepsilon_j) + \lambda\mu b_0(\varepsilon_i, \varepsilon_j), \tag{3.11}$$

for any $\lambda, \mu \in R \setminus \{0\}$, knowing that (3.11) holds for $\lambda, \mu \in \mathcal{T}$. We start with (3.11) for fixed $\lambda, \mu \in \mathcal{T}$. Adding $\lambda^2 q(\varepsilon_i) + \lambda\mu b(\varepsilon_i, \varepsilon_j)$ on both sides, we obtain (3.11) with λ replaced by $e\lambda$. In an analogous way we obtain (3.11) with μ replaced by $e\mu$. Multiplying (3.11) by the scalar e, we obtain (3.11) with λ, μ replaced by $e\lambda, e\mu$. The claim is proved. □

Remark 3.15. *As the proof reveals, the theorem holds more generally if V is any R-module and $(\varepsilon_i \mid i \in I)$ generates V.*

4. The companions of a quadratic form over a tangible supersemifield

In this section, we start a study of quadratic forms and their companions over a "*tangible supersemifield*", as defined below.

Recall that a semiring R is called a *semifield*, if all non-zero elements of R are invertible; hence $R \setminus \{0\}$ is the group R^* of units of R (here always assumed to be abelian).

Definition 4.1. *A supertropical semiring R is a **supersemifield**, if all tangible elements of R are invertible in R and all ghost elements $\neq 0$ of R are invertible in the bipotent semiring eR. Then eR is a semifield, and $\mathcal{T}(R)$ is the group of units of R, except in the degenerate case $\mathcal{T}(R) = \emptyset$, whereby $R = eR$.*

Definition 4.2. *We say that a supertropical semiring R is **tangible**, if R is generated by $\mathcal{T}(R)$ as a semiring. Clearly, this happens iff $e\mathcal{T}(R) = \mathcal{G}(R)$.* [6]

Remarks 4.3.

a) *If R is a supertropical semiring and $\mathcal{T}(R) \neq \emptyset$, then the set*
$$R' := \mathcal{T}(R) \cup e\mathcal{T}(R) \cup \{0\}$$
is the biggest subsemiring of R which is tangible supertropical.

b) *A tangible supertropical semiring R is a supersemifield iff every element of $\mathcal{T}(R)$ is invertible in R. Indeed, if $x \in \mathcal{T}(R)$ is invertible in R, then ex is invertible in eR.*

In many arguments we can retreat from a supersemifield R to the tangible supersemifield R' by omitting the "superfluous" ghost elements of R. But for categorical reasons we do not always exclude non-tangible supersemifields from our study.

As in the classical quadratic form theory of fields the "*square classes*" of a semiring R will be important for understanding quadratic forms over R.

Definition 4.4. *We call two elements x, y of a semiring R **square equivalent**, and write $x \sim_{\mathrm{sq}} y$, iff there exists a unit z of R* [7] *with $xz^2 = y$. The set $x(R^*)^2$ consisting of all $y \in R$ with $x \sim_{\mathrm{sq}} y$ will be called the **square class** of x (in R).*

Remarks 4.5. *Assume that R is a supertropical semiring.*

a) *Every square class different from $\{0\}$ consists either of elements of $\mathcal{T}(R)$ or of elements of $\mathcal{G}(R)$.*

b) *If two elements of eR are square equivalent in R, then they are square-equivalent in eR.*

c) *If R is a tangible supersemifield, then the square class of any $x \in R$ is the set $x(\mathcal{T}(R))^2$. Moreover two elements of eR, which are square equivalent in eR, are square equivalent in R.*

Definition 4.6. *A **multiquadratic extension** of a supersemifield R is a supersemifield R_1 containing R as a subsemiring such that $x^2 \in R$ for every $x \in R_1$.*

Given a supersemifield R, it is always possible to construct a multiquadratic extension R_1 of R – not necessarily unique – such that every $x \in R$ is a square in R_1. This (easy) fact will

[6] In [5], the tangible supersemifields have been called "supertropical semifields" (loc. cit. §3). We avoid this term here, since these semirings are not semifields in the technical sense.

[7] Then certainly $z \in \mathcal{T}(R)$.

be explained in a sequel of this paper. If the supersemifield R is tangible, then clearly R_1 is again tangible.

Until the end of this section, we fix a tangible supersemifield R. We discard the "trivial" supersemifields where $\mathcal{G}(R) = \{e\}$.

Notations 4.7. *We write $\mathcal{T} := \mathcal{T}(R)$, $\mathcal{G} := \mathcal{G}(R)$. We choose a multiquadratic extension of R, denoted by $R^{1/2}$, such that every $x \in R$ is a square in $R^{1/2}$ and write $\mathcal{T}^{1/2} := \mathcal{T}(R^{1/2})$, $\mathcal{G}^{1/2} := \mathcal{G}(R^{1/2})$. Notice that $\mathcal{G}^{1/2}$ is an ordered abelian group containing $\mathcal{G}$ as a subgroup, and that for every $x \in \mathcal{G}$ there exists a unique $z \in \mathcal{G}^{1/2}$ with $z^2 = x$. Indeed, if z_1, z_2 are elements of $\mathcal{G}^{1/2}$ with $z_1 < z_2$, then $z_1^2 < z_1 z_2 < z_2^2$. We denote this unique square root z of $x \in \mathcal{G}$ by $\sqrt{x}$. (Elements of $\mathcal{T}$ may have different square roots in $\mathcal{T}^{1/2}$.)*

At present the extension $R^{1/2}$ of R will only serve as a tool to ease notation. Later this will change; then we will need a precise theory of multiquadratic extensions, to be given in a sequel of this paper.

Terminology 4.8.

a) *Concerning the totally ordered group $\mathcal{G}$, we have a dichotomy:*

*Either $\mathcal{G}$ is **densely ordered**, i.e., for any two elements $x_1 < x_2$ of $\mathcal{G}$ there exists some $y \in \mathcal{G}$ with $x_1 < y < x_2$. Then also $eR = \mathcal{G} \cup \{0\}$ is densely ordered, since $\mathcal{G}$ does not have a smallest element.*

*Or $\mathcal{G}$ is **discrete**, i.e., for every $x \in \mathcal{G}$ there exists a biggest $x' \in \mathcal{G}$ with $x' < x$. For short, we say that R **is dense** and that R **is discrete**, respectively.*

b) *If R is discrete, we fix some $\pi \in \mathcal{T}$ such that $e\pi$ is the biggest element z of eR with $z < e$, and call π a **prime element** of R.*

c) *Notice that if R is dense, then $R^{1/2}$ also is dense; while if R is discrete, then $R^{1/2}$ is discrete. In the discrete case, $\sqrt{e\pi}$ is the biggest element of $\mathcal{G}^{1/2}$ smaller than e. Then we choose a prime element z of $R^{1/2}$ with $z^2 = \pi$, and denote this element z by $\sqrt{\pi}$.*

If V is a free module over the tangible supersemifield R with base $(\varepsilon_i \mid i \in I)$, and $q : V \to R$ is a quadratic form on V, we want to determine the sets $C_{i,j}(q)$ defined in §3. Recall that $C_{i,j}(q)$ coincides with $C_{i,j}(q \mid R\varepsilon_i + R\varepsilon_i)$, as observed in §3 (Remark 3.6); hence it suffices to study the case $I = \{1, 2\}$.

Thus we now focus on the following case: V is free with base $\varepsilon_1, \varepsilon_2$, and

$$q = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix} \tag{4.1}$$

for given $\alpha_1, \alpha_2, \alpha \in R$. (More precisely, q is the functional quadratic form on V represented by this triangular scheme.)

We know already (Proposition 3.13) that

$$C_{i,i}(q) = [0, e\alpha_i] \qquad (i = 1, 2). \tag{4.2}$$

Here, as in §3, we use interval notation with respect to the minimal ordering of R; namely for $c \in R$

$$[0, c] := \{x \in R \mid 0 \le x \le c\}.$$

Our problem is to determine $C_{1,2}(q)$. Certainly $\alpha \in C_{1,2}(q)$, cf. Proposition 1.12 or Proposition 1.16.

Lemma 4.9. *$C_{1,2}(q)$ is the set of all $\beta \in R$ with*

$$\lambda\alpha_1 + \lambda^{-1}\alpha_2 + \alpha = \lambda\alpha_1 + \lambda^{-1}\alpha_2 + \beta \tag{4.3}$$

for every $\lambda \in \mathcal{T}$.

Proof. By Scholium 3.7 and Theorem 3.14, we know that $C_{1,2}(q)$ is the set of all $\beta \in R$ with

$$\lambda^2\alpha_1 + \mu^2\alpha_2 + \lambda\mu\alpha = \lambda^2\alpha_1 + \mu^2\alpha_2 + \lambda\mu\beta$$

for all $\lambda, \mu \in \mathcal{T}$. Dividing by $\lambda\mu$, we obtain condition (4.3), there with λ/μ instead of λ; hence the claim. □

Proposition 4.10. *Assume that $\alpha_1 = 0$ or $\alpha_2 = 0$. Then $C_{1,2}(q) = \{\alpha\}$.*

Proof. We may assume that $\alpha_2 = 0$. Now condition (4.3) reads

$$\lambda\alpha_1 + \alpha = \lambda\alpha_1 + \beta.$$

If this holds for every $\lambda \in \mathcal{T}$, then $\alpha = \beta$,[8] and we conclude by Lemma 4.9 that $C_{1,2}(q) = \{\alpha\}$. □

In the following, we use the "**ν-notation**". For $a, b \in R$, we say that a is **ν-equivalent** to b, and write $a \cong_\nu b$ if $ea = eb$. We say that a is **ν-dominated** by b (resp. **strictly ν-dominated**) by b, and write $a \leq_\nu b$ (resp. $a <_\nu b$), if $ea \leq eb$ (resp. $ea < eb$). We say that $a \in R$ is a **ν-square** (in R), iff $a \cong_\nu b^2$ for some $b \in R$.

We now assume that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.

Convention 4.11.

a) *If $\alpha_1\alpha_2$ is a ν-square, we choose some $\xi \in \mathcal{T}$ with $\alpha_1\xi^2 \cong_\nu \alpha_2$. Otherwise we choose $\xi \in \mathcal{T}^{1/2}$ with $\alpha_1\xi^2 \cong_\nu \alpha_2$. Thus $\alpha_1\xi \cong_\nu \xi^{-1}\alpha_2$ (in $R^{1/2}$) in both cases. If $\alpha_1, \alpha_2 \in \mathcal{T}$, we may think of $\xi\alpha_1$ as a sort of "tangible geometric mean" of α_1, α_2, since $e\xi\alpha_1 = \sqrt{e\alpha_1 \cdot e\alpha_2}$.*

b) *We distinguish the following three cases.*
   Case I: *$\xi \in \mathcal{T}$, i.e., $\alpha_1\alpha_2$ is a ν-square.*

   Case II: *R is dense, $\xi \notin \mathcal{T}$.*

   Case III: *R is discrete, $\xi \notin \mathcal{T}$.*

c) *In Case III, we choose $\sigma, \tau \in \mathcal{T}$ with $e\tau < e\xi < e\sigma$ and with no element of $\mathcal{G}$ between $e\tau$ and $e\sigma$. In other terms, employing the prime element π of R,*

$$\tau \cong_\nu \pi\sigma, \qquad \xi \cong_\nu \sqrt{\pi}\sigma.$$

   Replacing σ by $\xi^2\tau^{-1}$, we may assume in addition (for simplicity) that $\sigma\tau = \xi^2$.

Theorem 4.12. *Assume that $\alpha^2 >_\nu \alpha_1\alpha_2$. Then $C_{1,2}(q) = \{\alpha\}$, except in the following case: R is discrete, $\alpha_1\alpha_2$ is not a ν-square, and $\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2$. Then, if $\alpha_1 \in \mathcal{G}$ and $\alpha_2 \in \mathcal{G}$,*

$$C_{1,2}(q) = [0, e\alpha], \tag{4.4}$$

while, if $\alpha_1 \in \mathcal{T}$ or $\alpha_2 \in \mathcal{T}$,

$$C_{1,2}(q) = \{\beta \in R \mid \beta \cong_\nu \alpha\}. \tag{4.5}$$

[8]Here we need the nontriviality assumption that $\mathcal{G} \neq \{e\}$.

Proof. The case $\alpha_1\alpha_2 = 0$ was covered by Proposition 4.10. Assume now that $\alpha_1\alpha_2 \neq 0$. We again rely on Lemma 4.9. If we are in Case I, we insert $\lambda = \xi$ in condition (4.3). Since $\xi\alpha_1 \cong_\nu \xi^{-1}\alpha_2$ and $\alpha >_\nu \xi\alpha_1$, we obtain

$$\alpha = e\xi\alpha_1 + \alpha \overset{!}{=} e\xi\alpha_1 + \beta.$$

This forces $\beta = \alpha$; hence $C_{1,2}(q) = \{\alpha\}$. If we are in Case II or III, then no $\lambda \in \mathcal{T}$ is ν-equivalent to ξ. If $\lambda >_\nu \xi$ then $\lambda\alpha_1 >_\nu \lambda^{-1}\alpha_2$, and condition (4.3) reads

$$\lambda\alpha_1 + \alpha = \lambda\alpha_1 + \beta. \tag{4.6}$$

If $\lambda <_\nu \xi$ then $\lambda\alpha_1 <_\nu \lambda^{-1}\alpha_2$, and (4.3) reads

$$\lambda^{-1}\alpha_2 + \alpha = \lambda^{-1}\alpha_2 + \beta. \tag{4.7}$$

Assume now that we are in Case II. Since R is dense, we can choose $\lambda \in \mathcal{T}$ with

$$\xi\alpha_1 <_\nu \lambda\alpha_1 <_\nu \alpha.$$

For such λ condition (4.6) reads $\alpha = \lambda\alpha_1 + \beta$. This forces $\beta = \alpha$, and we conclude again that $C_{1,2}(q) = \{\alpha\}$.

We are left with Case III. Now $\alpha^2 \geq_\nu \pi^{-1}\alpha_1\alpha_2$, since $e\alpha_1\alpha_2$ is not a square. We have $\alpha_1\alpha_2 \cong_\nu \xi^2\alpha_1 \cong_\nu \pi\sigma^2\alpha_1^2$; hence $\alpha^2 \geq_\nu \sigma^2\alpha_1$, i.e. $\alpha \geq_\nu \sigma\alpha_1$. If $\alpha >_\nu \sigma\alpha_1$, i.e., $\alpha^2 >_\nu \pi^{-1}\alpha_1\alpha_2$, we insert $\lambda = \sigma$ into condition (4.6) and obtain

$$\alpha = \sigma\alpha_1 + \alpha \overset{!}{=} \sigma\alpha_1 + \beta.$$

This forces $\beta = \alpha$; hence $C_{1,2}(q) = \{\alpha\}$.

Assume finally that $\alpha \cong_\nu \sigma\alpha_1$, i.e., $\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2$. Inserting $\lambda = \sigma$ into (4.6), we obtain

$$e\sigma\alpha_1 = \sigma\alpha_1 + \alpha \overset{!}{=} \sigma\alpha_1 + \beta.$$

If $\alpha_1 \in \mathcal{T}$, this forces $\beta \cong_\nu \sigma\alpha_1 \cong_\nu \alpha$; while if $\alpha_1 \in \mathcal{G}$, this only forces $\beta \leq_\nu \sigma\alpha_1 \cong_\nu \alpha$.

Thus we have found the constraints for $\beta \in C_{1,2}(q)$ that $\beta \cong_\nu \alpha$ if $\alpha_1 \in \mathcal{T}$, resp. $\beta \leq_\nu \alpha$ if $\alpha_1 \in \mathcal{G}$. Of course, these constraints are also valid if $\alpha_2 \in \mathcal{T}$, resp. $\alpha_2 \in \mathcal{G}$, since we may interchange the base vectors ε_1 and ε_2 of V.

If $\beta \cong_\nu \alpha$, it is easily checked that (4.6) holds for all $\lambda \in \mathcal{T}$ with $\lambda \geq \sigma$, and (4.7) holds for all $\lambda \in \mathcal{T}$ with $\lambda \leq \tau$. Thus

$$C_{1,2}(q) = \{\beta \in R \mid \beta \cong_\nu \alpha\}$$

if $\alpha_1 \in \mathcal{T}$ or $\alpha_2 \in \mathcal{T}$. If both α_1 and α_2 are ghost and $\beta \leq_\nu \alpha$, again an easy check reveals that (4.6) holds for all $\lambda \in \mathcal{T}$ with $\lambda \geq \sigma$, and (4.7) for all $\lambda \in \mathcal{T}$ with $\lambda \leq \tau$. We conclude that now $C_{1,2}(q) = [0, e\alpha]$. $\square$

Theorem 4.13. *Assume that $\alpha^2 \leq_\nu \alpha_1\alpha_2$.*

a) *Then*

$$C_{1,2}(q) = \{\beta \in R \mid \beta^2 \leq_\nu \alpha_1\alpha_2\} \tag{4.8}$$

except in the case that R is discrete, $\alpha_1\alpha_2$ is not a ν-square and both α_1 and α_2 are ghost. Then

$$C_{1,2}(q) = \{\beta \in R \mid \beta^2 \leq_\nu \pi^{-1}\alpha_1\alpha_2\}. \tag{4.9}$$

b) *Using Convention 4.11, this means more explicitly:*

In Case I: $C_{1,2}(q) = [0, e\xi\alpha_1] = [0, e\xi^{-1}\alpha_2]$.

In Case II: $C_{1,2}(q) = [0, e\xi\alpha_1[:= \{\beta \in R \mid 0 \leq_\nu \beta <_\nu e\xi\alpha_1\}$.

In Case III: $C_{1,2}(q) = [0, e\tau\alpha_1] = [0, e\sigma^{-1}\alpha_2]$ *if* $\alpha_1 \in \mathcal{T}$ *or* $\alpha_2 \in \mathcal{T}$, *whereas* $C_{1,2}(q) = [0, e\sigma\alpha_1] = [0, e\tau^{-1}\alpha_2]$ *if* $\alpha_1 \in \mathcal{G}$ *and* $\alpha_2 \in \mathcal{G}$.

Proof. Now $\alpha \leq_\nu \xi\alpha_1 \cong_\nu \xi^{-1}\alpha_2$. We again exploit Lemma 4.9. If $\lambda \in \mathcal{T}$ and $\lambda >_\nu \xi$, then condition (4.3) there reads

$$\lambda\alpha_1 = \lambda\alpha_1 + \beta, \tag{4.6$'$}$$

and if $\lambda \in T$, $\lambda <_\nu \xi$, then (4.3) reads

$$\lambda^{-1}\alpha_2 = \lambda^{-1}\alpha_2 + \beta. \tag{4.7$'$}$$

Assume first that we are in Case I. Inserting $\lambda \in T$ with $\lambda \cong_\nu \xi$ into (4.3), we obtain for $\beta \in C_{1,2}(q)$ the necessary condition

$$e\xi\alpha_1 = e\xi\alpha_2 + \beta,$$

which means that $\beta \leq_\nu \xi\alpha_1$. This constraint for β implies that for $\lambda >_\nu \xi$ we have $\lambda\alpha_1 >_\nu \beta$; hence (4.6$'$) holds, while for $\lambda <_\nu \xi$ we have $\lambda^{-1}\alpha_2 >_\nu \beta$, and hence (4.7$'$) holds. Thus $\beta \leq_\nu \xi\alpha_1$ implies that $\beta \in C_{1,2}(q)$. We conclude that in Case I

$$C_{1,2}(q) = [0, e\xi\alpha_1] = [0, e\xi^{-1}\alpha_2].$$

In Case II (R dense), condition (4.6$'$) for all $\lambda \in \mathcal{T}$ with $\lambda >_\nu \xi$ forces $\beta \leq_\nu \xi\alpha_1$, and hence $\beta <_\nu \xi\alpha_1$, while in Case III we obtain the constraint $\beta \leq_\nu \sigma\alpha_1$.

Assume that $\beta <_\nu \xi\alpha_1$. Then (4.6$'$) holds for all $\lambda >_\nu \xi$. If $\lambda <_\nu \xi$ and $\lambda \in \mathcal{T}$ we have $\lambda^{-1}\alpha_2 >_\nu \xi^{-1}\alpha_2 = \xi\alpha_1 >_\nu \beta$; hence (4.7$'$) is valid. Thus $\beta \in C_{1,2}(q)$. We conclude that in Case II

$$C_{1,2}(q) = [0, \xi\alpha_1[.$$

We turn to Case III and assume the constraint $\beta \leq_\nu \sigma\alpha_1$. For $\lambda >_\nu \sigma$ condition (4.6$'$) is evident. For $\lambda \cong_\nu \sigma$ condition (4.6$'$) gives the constraint

$$\sigma\alpha_1 \overset{!}{=} \sigma\alpha_1 + \beta.$$

If $\alpha_1 \in \mathcal{G}$, this holds. If $\alpha_1 \in \mathcal{T}$, we obtain the stronger constraint

$$\beta <_\nu \sigma\alpha_1, \quad \text{i.e.,} \quad \beta \leq_\nu \tau\alpha_1.$$

Now assume that $\lambda \in \mathcal{T}$ and $\lambda <_\nu \xi$. We have $\lambda \leq_\nu \tau$, hence

$$\lambda^{-1}\alpha_2 \geq_\nu \tau^{-1}\alpha_2 \cong_\nu \sigma\alpha_1 \geq_\nu \beta.$$

If $\alpha_2 \in \mathcal{G}$, this implies (4.7$'$); while if $\alpha_2 \in \mathcal{T}$, we obtain the constraint $\beta <_\nu \tau^{-1}\alpha_2 \cong_\nu \sigma\alpha_1$, i.e., again $\beta \leq_\nu \tau\alpha_1$. We conclude that if $\alpha_1 \in \mathcal{G}$ and $\alpha_2 \in \mathcal{G}$, then

$$C_{1,2}(q) = [0, \sigma\alpha_1] = [0, \tau^{-1}\alpha_2];$$

while if $\alpha_1 \in \mathcal{T}$ or $\alpha_2 \in \mathcal{T}$, then

$$C_{1,2}(q) = [0, e\tau\alpha_1] = [0, e\sigma^{-1}\alpha_2].$$

We have proved part b) of the theorem. It is immediate that this is equivalent to part a). □

We state three consequences of Proposition 4.10 and Theorems 4.12 and 4.13.

Corollary 4.14. *Let $\alpha_1, \alpha_2,\ \alpha \in R$. The quadratic form*

$$q = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix}$$

is quasilinear iff either $\alpha^2 \leq_\nu \alpha_1\alpha_2$, or R is discrete where $\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2$ and both α_1, α_2 are ghost.

Proof. q is quasilinear iff q is quasilinear at $(\varepsilon_1, \varepsilon_2)$, iff $0 \in C_{1,2}(q)$. The assertion can be read off from Proposition 4.10 and Theorems 4.12 and 4.13. (Observe that if R is discrete and $\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2$, then $\alpha_1\alpha_2$ is not a ν-square, and hence we are in Case III.) $\square$

Corollary 4.15. *Assume that $\alpha_1, \alpha_2,\ \alpha \in R$, where $\alpha^2 \leq_\nu \alpha_1\alpha_2$ if R is dense, and $\alpha^2 \leq_\nu \pi^{-1}\alpha_1\alpha_2$ if R is discrete. Then the triangular schemes*

$$\begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix}, \qquad \begin{bmatrix} \alpha_1 & e\alpha \\ & \alpha_2 \end{bmatrix}$$

represent the same quadratic form on $V = R\varepsilon_1 + R\varepsilon_2$.

Proof. Let q be the functional form represented by $\left[\begin{smallmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{smallmatrix}\right]$. The triangular scheme $\left[\begin{smallmatrix} \alpha_1 & e\alpha \\ & \alpha_2 \end{smallmatrix}\right]$ represents q iff $e\alpha \in C_{1,2}(q)$. The claim can again be read off from Theorems 4.12 and 4.13. (Observe that if R is discrete, then either $\alpha^2 \leq_\nu \alpha_1\alpha_2$, or we are in Case III.) $\square$

N.B. A big part of Corollary 4.15 is obvious from Corollaries 4.14 and 3.11.

Corollary 4.16. *Let $\alpha_1, \alpha_2,\ \alpha \in R$ and $q = \left[\begin{smallmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{smallmatrix}\right]$.*

a) *When R is dense, then q is rigid at $(\varepsilon_1, \varepsilon_2)$ iff $\alpha_1\alpha_2 <_\nu \alpha^2$.*

b) *When R is discrete, then q is rigid at $(\varepsilon_1, \varepsilon_2)$ iff $\alpha_1\alpha_2 <_\nu \pi\alpha^2$.*

Proof. Browsing through Proposition 4.10 and Theorems 4.12 and 4.13, we see that $C_{1,2}(q) = \{\alpha\}$ precisely in these cases. $\square$

5. The quasilinear part of a form and rigid complements

We assume in the whole section that V *is a free module* over a semiring R satisfying the following rules for all $a, b \in R$:

$$2a = 0 \Rightarrow a = 0, \tag{5.1}$$

$$(a+b)^2 = a^2 + b^2. \tag{5.2}$$

In the following $q : V \to R$ is always a (functional) quadratic form on V. Our results on companions in §2 and §3 give some insight into the set of presentations of q as a sum $q = \kappa + \rho$ with κ a quasilinear and ρ a rigid form. Here the sum is meant in the sense of functions on V, i.e., $q(x) = \kappa(x) + \rho(x)$ for every $x \in V$.

First recall from §2 that q itself is rigid iff $q(\varepsilon_i) = 0$ for every $i \in I$ (Theorem 2.14).

Proposition 5.1. *Let b be a companion of $q : V \to R$. We define quadratic forms κ and ρ on V by the formulas ($x = \sum_{i \in I} x_i\varepsilon_i \in V$)*

$$\kappa(x) = \sum_{i \in I} q(\varepsilon_i)x_i^2 \tag{5.3}$$

$$\rho(x) = \sum_{i<j} b(\varepsilon_i, \varepsilon_j)x_i x_j, \tag{5.4}$$

where we have chosen a total ordering on the set of indices I. Then κ is quasilinear, ρ is rigid, and $q = \kappa + \rho$.

Proof. It is obvious that $q = \kappa + \rho$ (cf. (1.15)). Clearly $\rho(\varepsilon_i) = 0$ for all $i \in I$, whence ρ is rigid. As a consequence of rule (5.2), we have

$$\kappa(x + y) = \kappa(x) + \kappa(y) \quad \text{for all} \quad x, y \in V,$$

whence κ is quasilinear. □

Proposition 5.2. *If $q = \kappa + \rho$ with κ quasilinear and ρ rigid, then κ satisfies the formula* (5.3) *for all $x \in V$, and hence κ is uniquely determined by q. In particular, κ does not depend on the choice of the base (ε_i).*

Proof. For every $i \in I$ we have $\rho(\varepsilon_i) = 0$, and hence $q(\varepsilon_i) = \kappa(\varepsilon_i)$. Since κ is quasilinear, this implies (5.3). □

Definition 5.3. *We call κ the* ***quasilinear part*** *of q, and write $\kappa = q_{\mathrm{QL}}$. Further we call any rigid form ρ on V with $q_{\mathrm{QL}} + \rho = q$ a* ***rigid complement of*** q_{QL} ***in*** q*, or more briefly, a* ***rigid complement in*** q*, justified by the fact that q_{QL} is uniquely determined by q. We denote the set of all rigid complements in q by* $\mathrm{Rig}(q)$.

Rigid complements in q are closely related to certain companions of q.

Definition 5.4. *We call a companion b of q* ***off-diagonal*** *if $b(\varepsilon_i, \varepsilon_i) = 0$ for every $i \in I$.*

Remark 5.5. *We have $0 \in C_{ii}(q)$ for every $i \in I$, since q is quasilinear on $R\varepsilon_i$ due to* (5.2)*. Thus every form q on V possesses off-diagonal companions.*

Proposition 5.6. *The rigid complements ρ in q correspond bijectively with the off-diagonal companions b of q by the formulas* (5.4) *and*

$$b(\varepsilon_i, \varepsilon_j) = \rho(\varepsilon_i + \varepsilon_j) \tag{5.5}$$

for $i, j \in I$. The bilinear form b is the (unique) companion of ρ.

Proof. Given a companion b of q we obtain a rigid complement ρ in q by formula (5.4), as observed in Proposition 5.1, and (5.4) implies (5.5). If moreover b is off-diagonal, then b is uniquely determined by (5.5).

Conversely, if $q = \kappa + \rho$ with κ quasilinear and ρ rigid, then κ has the companion $b_0 = 0$, while ρ has a (unique) companion b. Thus $b_0 + b = b$ is a companion of $\kappa + \rho = q$. We have

$$4\rho(\varepsilon_i) = \rho(\varepsilon_i + \varepsilon_i) = \rho(\varepsilon_i) + \rho(\varepsilon_i) + b(\varepsilon_i, \varepsilon_i),$$

and conclude that $b(\varepsilon_i, \varepsilon_i) = 0$, since $\rho(\varepsilon_i) = 0$. Thus b is off-diagonal. □

For later use we record the following facts, which are now obvious.

Scholium 5.7.

a) *A form κ on V is quasilinear iff*

$$\kappa\Big(\sum_i x_i \varepsilon_i\Big) = \sum_i \kappa(\varepsilon_i) x_i^2. \tag{5.6}$$

b) *A form ρ on V is rigid iff*

$$\rho\Big(\sum_i x_i \varepsilon_i\Big) = \sum_{i<j} \rho(\varepsilon_i + \varepsilon_j) x_i x_j. \tag{5.7}$$

Notation 5.8. *We denote the set of all quadratic forms on V by $\mathrm{Quad}(V)$, and view this set as an R-module of R-valued functions in the obvious way:*

$$(q_1 + q_2)(x) := \quad q_1(x) + q_2(x),$$
$$(\lambda q)(x) := \quad \lambda \cdot q(x).$$

for every $x \in V$ ($q_1, q_2, q \in \mathrm{Quad}(V)$, $\lambda \in R$). We further denote the set of quasilinear forms on V by $\mathrm{QL}(V)$ and the set of rigid forms on V by $\mathrm{Rig}(V)$.

Remarks 5.9.

a) *It is evident that both $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ are submodules of the R-module $\mathrm{Quad}(V)$.*

b) *As a consequence of Propositions 5.1, 5.2, we have*

$$\mathrm{QL}(V) + \mathrm{Rig}(V) = \mathrm{Quad}(V), \tag{5.8}$$
$$\mathrm{QL}(V) \cap \mathrm{Rig}(V) = \{0\}. \tag{5.9}$$

c) *For any $q_1, q_2 \in \mathrm{Quad}(V)$*

$$(q_1 + q_2)_{\mathrm{QL}} = (q_1)_{\mathrm{QL}} + (q_2)_{\mathrm{QL}}, \tag{5.10}$$

and

$$\mathrm{Rig}(q_1) + \mathrm{Rig}(q_2) \subset \mathrm{Rig}(q_1 + q_2). \tag{5.11}$$

d) *For any $q \in \mathrm{Quad}(V)$ and $\lambda \in R$*

$$(\lambda q)_{\mathrm{QL}} = \lambda \cdot q_{\mathrm{QL}}, \tag{5.12}$$

and

$$\lambda \cdot \mathrm{Rig}(q) \subset \mathrm{Rig}(\lambda q). \tag{5.13}$$

Here the assertions about rigid complements in c), d) are obvious from Definition 5.3, while the assertions about quasilinear parts follow from the formula (5.3) *describing $\kappa = q_{\mathrm{QL}}$.*

From now on, *we assume that the semiring R is u.b.* (cf. Definition 3.8) and obeys rule (5.2).[9] The minimal ordering $\leq$ on R gives us a natural partial ordering on the module of R-valued functions $\mathrm{Quad}(V)$ as follows: If $q, q' \in \mathrm{Quad}(V)$, then

$$q \leq q' \quad \Leftrightarrow \quad \forall x \in V : q(x) \leq q'(x). \tag{5.14}$$

This ordering is compatible with addition and scalar multiplication.

The ordering (5.14) leads to a new characterization of quasilinear parts.

Lemma 5.10. *If q, q' are quasilinear forms on V with $q \leq q'$, then $(q)_{\mathrm{QL}} \leq (q')_{\mathrm{QL}}$.*

Proof. For $x = \sum_i x_i \varepsilon_i \in V$ we have

$$q_{\mathrm{QL}}(x) = \sum_i x_i^2 q(\varepsilon_i) \quad \leq \quad \sum_i x_i^2 q'(\varepsilon_i) = q'_{\mathrm{QL}}(x),$$

due to formula (5.3) describing $\kappa = q_{\mathrm{QL}}$. □

Theorem 5.11. *If q is any quadratic form on V, then q_{QL} is the unique maximal quasilinear form κ on V with $\kappa \leq q$; in more explicit terms, $q_{\mathrm{QL}} \leq q$ and $\chi \leq q_{\mathrm{QL}}$ for every quasilinear form $\chi \leq q$.*

[9]The rule (5.1) is a consequence of upper boundedness.

Proof. If χ is quasilinear and $\chi \leq q$, then we conclude from Lemma 5.10 that $\chi = \chi_{\mathrm{QL}} \leq q_{\mathrm{QL}}$. $\square$

We now look at rigid complements in terms of the ordering (5.14).

Theorem 5.12. *Assume that R is u.b. and obeys* (5.2).

a) *The R-module* $\mathrm{Rig}(V)$ *is a lower set in* $\mathrm{Quad}(V)$, *i.e., if χ and ρ are quadratic forms on V with ρ rigid and $\chi \leq \rho$, then χ is rigid.*
b) *For any quadratic form q on V, the set* $\mathrm{Rig}(q)$ *of rigid complements b in q is convex in* $\mathrm{Quad}(V)$.
c) *If q is quasilinear, then* $\mathrm{Rig}(q)$ *is a lower set in* $\mathrm{Quad}(V)$ *closed under addition.*

Proof.

a): This is obvious from the fact that a form ρ on V is rigid iff $\rho(\varepsilon_i) = 0$ for every $i \in I$.
b): This follows from Proposition 5.6 and the fact that in the companion matrix of q the off-diagonal entries $C_{ij}, i \neq j$, are convex in R (Proposition 3.10).
c): $\mathrm{Rig}(q)$ is a lower set, since $\mathrm{Rig}(q)$ is convex and $0 \in \mathrm{Rig}(q)$. It follows from Propositions 5.6 and 3.10, that $\mathrm{Rig}(q)$ is closed under addition.

$\square$

We turn to the seemingly more subtle question of whether the convex set $\mathrm{Rig}(q)$ has maximal or minimal elements, and how many. In the case that R is a nontrivial tangible supersemifield we can resort to the explicit determination of the companion matrix in §4 to get an answer.

Theorem 5.13. *Assume that R is a nontrivial tangible supersemifield, and q is a quadratic form on a free R-module V.*

i) *For any $\rho \in \mathrm{Rig}(q)$, there exists a minimal element ρ' in $\mathrm{Rig}(q)$ with $\rho' \leq \rho$.*
ii) *If R is dense, then* $\mathrm{Rig}(q)$ *has a unique minimal element.*
iii) *If R is discrete, and ρ_1, ρ_2 are minimal elements of* $\mathrm{Rig}(q)$, *then $\rho_1(x) \cong_\nu \rho_2(x)$ for every $x \in V$.*
iv) *If R is discrete then* $\mathrm{Rig}(q)$ *has a unique maximal element, while if R is dense it can happen that* $\mathrm{Rig}(q)$ *has no maximal element.*

Proof. All assertions follow from Proposition 5.6 by a sharp look at the description of the off-diagonal entries $C_{ij}(q)$, $(i \neq j)$, of the companion matrix of q via Proposition 4.10 and Theorems 4.12 and 4.13. Note that in the delicate case that formula (4.5) in Theorem 4.12 comes into play, the set $C_{12}(q) = \{\beta \in R \mid \beta \cong_\nu \alpha\}$ has the unique maximal element $e\alpha$ and all other elements of $C_{12}(q)$ are minimal, while in Theorem 4.13.b, Case II, where R is dense, $C_{12}(q)$ has no maximal element, but has the minimal element zero. $\square$

6. The minimal ordering on a free R-module

In this section R is a supertropical semiring. If V is any module over R, we define on V a binary relation $\leq_V$ as follows:
For any $x, y \in U$,

$$x \leq_V y \quad \rightleftharpoons \quad \exists z \in V : x + z = y. \tag{6.1}$$

This relation is clearly reflexive ($x \le x$) and transitive ($x \le y, y \le z \Rightarrow x \le z$). It is also antisymmetric, *hence is a partial ordering on the set* V. Indeed, assume that $x + z = y$ and $y + w = x$. This implies $x + z + w = x$, $y + z + w = y$, and then

$$x + e(z + w) = x, \quad y + e(z + w) = y.$$

Adding z at both sides of the first equation, and using that $z + ez = ez$, we obtain

$$y = x + e(z + w) = x,$$

as desired.

Clearly, our partial ordering $\le_V$ obeys the rules ($x, y, z \in V$)

$$0 \le z, \tag{6.2}$$

$$x \le y \quad \Rightarrow \quad x + z \le y + z. \tag{6.3}$$

(Thus, $x \le y$, $x' \le y' \Rightarrow x + x' \le y + y'$.) It is now obvious that any partial ordering $\le'$ on V with the properties (6.2), (6.3), is a refinement of $\le_V$: If $x \le_V y$, then $x \le' y$.

Definition 6.1. *We call $\le_V$ the* ***minimal ordering*** *on the R-module V.*

Notation 6.2. *As long as no other orderings of V come into play, we usually write $x \le y$ instead of $x \le_V y$. But notice that if W is a submodule of V, it may happen for $x, y \in W$ that $x \le_V y$ but not $x \le_W y$.*

As usual, $x < y$ means that $x \le y$ and $x \ne y$.

In particular, R itself carries the minimal ordering $\le_R$. It already showed up in §3, cf. Definition 3.8. Again, we usually write $\lambda \le \mu$ instead of $\lambda \le_R \mu$.

Scalar multiplication is compatible with these orderings on R and V :

$$\lambda \le \mu, \ x \le y \quad \Rightarrow \quad \lambda x \le \mu y \tag{6.4}$$

for all $\lambda, \mu \in \mathbb{R}$, $x, y \in V$.

Before moving on to details about minimal orderings, we hasten to point out that these orderings are relevant for the geometry in a supertropical quadratic space. This is apparent already from the definition of quadratic forms (Definition 1.1).

Remark 6.3. *As before, let V be a module over a supertropical semiring R. If (q, b) is a quadratic pair on V, then for all $x, y, z, w \in V$ the following holds:*

$$x \le_V z \quad \Rightarrow \quad q(x) \le_R q(z), \tag{6.5}$$

$$x \le_V z, \ y \le_V w \quad \Rightarrow \quad b(x, y) \le_R b(z, w), \tag{6.6}$$

$$b(x, y) \le_R q(x + y). \tag{6.7}$$

The minimal ordering of R has the following detailed description in terms of the ν-dominance relation and the sets eR and $\mathcal{T} = R \setminus (eR)$.

Proposition 6.4.

a) *Assume that $x \in eR$. Then x is comparable (in the minimal ordering) to every $y \in R$. More precisely, using the ν-notation (§4)*

$$x < y \quad \Leftrightarrow \quad x <_\nu y, \tag{6.8}$$

$$y < x \quad \Leftrightarrow \quad \text{either } y <_\nu x \text{ or } y \in \mathcal{T} \text{ and } y \cong_\nu x. \tag{6.9}$$

b) *Assume that $x \in \mathcal{T}$, $y \in R$. Then*

$$x < y \quad \Leftrightarrow \quad \text{either } x <_\nu y \text{ or } x \cong_\nu y \text{ and } y \in eR, \tag{6.10}$$

$$y < x \quad \Leftrightarrow \quad y <_\nu x. \tag{6.11}$$

Thus x and y are incomparable iff $y \in \mathcal{T}$ and $x \neq y$, but $x \cong_\nu y$.

Proof. All this can be read off from the description of the sum $x + y$ of $x, y \in R$ in terms of the ν-dominance relation, recalled from [12, §2].[10] □

Assume now that V is a free R-module with base $(\varepsilon_i \mid i \in I)$. If x, y are vectors in V with coordinates $(x_i \mid i \in I)$, $(y_i \mid i \in I)$, i.e.,

$$x = \sum_{i \in I} x_i \varepsilon_i \quad y = \sum_{i \in I} y_i \varepsilon_i,$$

where $x_i \neq 0$ or $y_i \neq 0$ only for finitely many $i \in I$, then clearly

$$x \leq_V y \quad \Leftrightarrow \quad \forall i \in I \quad x_i \leq_R y_i. \tag{6.12}$$

We ask for the relation of a second base $(\eta_k \mid k \in K)$ of V to the given base $(\varepsilon_i \mid i \in I)$.

Lemma 6.5. *Let $\mathfrak{B}$ denote the set of all $x \in V$ such that $y \in Rx$ for every $y \in R$ with $y \leq x$. Then*

$$\bigcup_{i \in I} R^* \epsilon_i \subset \mathfrak{B} \subset \bigcup_{i \in I} R \epsilon_i. \tag{6.13}$$

Proof. If $x = \lambda \varepsilon_i$ with $\lambda \in R^*$, $i \in I$ and $y \leq x$, it is clear from (6.12) that $y = \mu \varepsilon_i$ with $\mu \in R$ (and $\mu \leq \lambda$). Thus $y = \mu \lambda^{-1} x$. This proves that $x \in \mathfrak{B}$.

Assume now that $x \in V$ but $x \notin \bigcup_{i \in I} R\varepsilon_i$. We can pick two scalars $\lambda, \mu \in R$ with $0 < e\lambda \leq e\mu$ and two indices $i \neq j$, such that

$$x = \lambda \varepsilon_i + \mu \varepsilon_j + z$$

with $z \in \sum_{k \neq i,j} R\varepsilon_k$. Then $\lambda \varepsilon_i < x$. Suppose there exists some $\alpha \in R$ with $\lambda \varepsilon_i = \alpha x$. Then

$$\lambda \varepsilon_i = \alpha \lambda \varepsilon_i + \alpha \mu \varepsilon_j + \alpha z.$$

We read off that $\lambda = \alpha \lambda$, $0 = \alpha \mu$. From $0 < e\lambda \leq e\mu$ we conclude that $0 \leq e\alpha\lambda \leq e\alpha\mu = 0$, hence $e\alpha\lambda = 0$, hence $\lambda = \alpha\lambda = 0$, a contradiction. Thus $x \notin \mathfrak{B}$. □

Theorem 6.6. *Let $(\varepsilon_i \mid i \in I)$ and $(\eta_k \mid k \in K)$ be bases of V. Then there exists a bijection $\phi : I \to K$ and a family $(u_i \mid i \in I)$ of units of R such that for every $i \in I$*

$$\eta_{\phi(i)} = \mu_i \varepsilon_i.$$

In short, all bases of V arise from $(\varepsilon_i \mid i \in I)$ by relabeling the ε_i and multiplying them with units of R.

Proof. Given $i \in I$, we conclude by Lemma 6.5 that there exists a unique $k \in K$ and $t \in R$ with $\varepsilon_i = t\eta_k$, and further a unique $j \in I$ and $u \in R$ with $\eta_k = \mu\varepsilon_j$. It follows that $\varepsilon_i = t\mu\varepsilon_j$, hence $j = i$ and $t\mu = 1$. Thus we have an injection $\phi : I \to K$ and a family of units $(\mu_i \mid i \in I)$ of R such that $\eta_{\phi(i)} = \mu_i \varepsilon_i$ for every $i \in I$. Consequently, the vectors $\eta_{\phi(i)}$ generate the R-module V. Thus ϕ is also surjective. □

[10]The general assumption in [12], that the monoid $(eR, \cdot\,)$ is cancellative, is not needed here. It is only relevant if products xy are involved.

For R a tangible supersemifield and I finite, this result already appears in [13] with a very different proof [loc cit., essentially Proposition 3.9].

7. The supertropicalizations of a quadratic form

Let R be a ring, V a free R-module, and $q : V \to R$ a quadratic form. Assume further that U is a supertropical semiring with ghost ideal $M := eU$, and $\varphi : R \to U$ is a supervaluation (cf. Introduction and [5]). (The case of primary interest that we have in mind is that R is a field and U is a supertropical semifield, and hence $e\varphi : R \to M$ is a Krull valuation, in multiplicative notation.) We describe a procedure to associate to q a quadratic form over U in various ways.

First we choose a base $(v_i \mid i \in I)$ of the free R-module V.

Let $U^{(I)}$ denote the free U-module consisting of the tuples $x = (x_i \mid i \in I)$ with $x_i \in U$, almost all $x_i = 0$. It has the standard base $(\varepsilon_i \mid i \in I)$ consisting of the tuples with one coordinate 1, all other coordinates 0. Thus

$$x = \sum_{i \in I} x_i \varepsilon_i. \tag{7.1}$$

We "extend" $\varphi : R \to U$ to a map from V to $U^{(I)}$, denoted by the same letter φ, by the formula ($a_i \in R$)

$$\varphi\left(\sum_{i \in I} a_i v_i\right) := \sum_{i \in I} \varphi(a_i)\varepsilon_i. \tag{7.2}$$

Notice, that for $a \in R$, $v \in V$

$$\varphi(av) = \varphi(a)\varphi(v). \tag{7.3}$$

Definition 7.1. *If $B : V \times V \to R$ is a bilinear form on the free R-module V, we define a bilinear form*

$$B^\varphi : U^{(I)} \times U^{(I)} \to U$$

on the U-module $U^{(I)}$ by stating

$$B^\varphi(\varepsilon_i, \varepsilon_j) = \varphi(B(v_i, v_j)) \tag{7.4}$$

for any two indices $i, j \in I$. We call B^φ the supertropicalization (or "stropicalization" for short) of B under φ with respect to the base $(v_i \mid i \in I)$ of V.

Thus, if $I = \{1, \dots, n\}$ and, in matrix notation,

$$B = \begin{pmatrix} \beta_{11} & \dots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \dots & \beta_{n,n} \end{pmatrix}$$

with $\beta_{i,j} := B(v_i, v_j)$, then

$$B^\varphi = \begin{pmatrix} \varphi(\beta_{1,1}) & \dots & \varphi(\beta_{1,n}) \\ \vdots & \ddots & \vdots \\ \varphi(\beta_{n,1}) & \dots & \varphi(\beta_{n,n}) \end{pmatrix}. \tag{7.5}$$

Remark 7.2. *It follows from (7.4) that for any two vectors $a, b \in \bigcup_{i \in I} Rv_i$*

$$B^\varphi(\varphi(a), \varphi(b)) = \varphi(B(a, b)). \tag{7.6}$$

For other vectors a, b in V this often fails.

We are now ready to define the supertropicalizations of a given quadratic form $q : V \to R$. We choose a total ordering of the index set I, and then have the unique triangular bilinear form

$$B := \nabla q : V \times V \to R$$

at our disposal, which expands q (cf. (1.30)). It gives us a triangular form B^φ on $U^{(I)}$.

Definition 7.3. *We define the* ***stropicalization*** *(=**supertropicalization**)* $q^\varphi : U^{(I)} \to U$ ***of*** q ***under*** φ ***with respect to the ordered base*** $\mathcal{L} := (v_i \mid i \in I)$ *of* V *by the formula*

$$q^\varphi(x) := B^\varphi(x, x) \tag{7.7}$$

for $x \in U^{(I)}$*. By this definition*

$$\nabla(q^\varphi) = (\nabla q)^\varphi. \tag{7.8}$$

If $I = \{1, 2, \dots, n\}$ *and*

$$q = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \vdots \\ & & a_{n,n} \end{bmatrix}, \tag{7.9}$$

then

$$q^\varphi = \begin{bmatrix} \varphi(a_{1,1}) & \dots & \varphi(a_{1,n}) \\ & \ddots & \vdots \\ & & \varphi(a_{n,n}) \end{bmatrix}. \tag{7.10}$$

In other terms, if we write q as a polynomial with variables $\lambda_1, \dots, \lambda_n$,

$$q = \sum_{i \le j} a_{i,j} \lambda_i \lambda_j \in R[\lambda_1, \dots, \lambda_n],$$

then

$$q^\varphi = \sum_{i \le j} \varphi(a_{i,j}) \lambda_i \lambda_j \in U[\lambda_1, \dots, \lambda_n].$$

This means that q^φ is the supertropicalization of the polynomial q, as defined in [12] and [5, §9]. More precisely, q^φ is the functional quadratic form on $U^{(n)}$ represented by this polynomial.

As a consequence of Definition 7.3 and Remark 7.2, we can state

Remark 7.4. *For any vector* a *in* $\bigcup_{i \in I} Rv_i$*, we have*

$$q^\varphi(\varphi(a)) = \varphi(q(a)),$$

while for other vectors in V *this may fail.*

The stropicalization q^φ comes with the balanced companion

$$\beta := B^\varphi + ({}^{\mathrm{t}}B)^\varphi, \tag{7.11}$$

where $B := \nabla q$ (cf. Example 1.4). For $i \neq j$, we have

$$\beta(\varepsilon_i, \varepsilon_j) = \varphi(a_{i,j}), \tag{7.12}$$

while

$$\beta(\varepsilon_i, \varepsilon_i) = \varphi(a_{i,i}) + \varphi(a_{i,i}) = e\varphi(a_{i,i}). \tag{7.13}$$

Remark 7.5. *This balanced companion β can be different from the stropicalization b^{φ} of the companion b of q. Indeed,*

$$b^{\varphi}(\varepsilon_i, \varepsilon_j) = \beta(\varepsilon_i, \varepsilon_j) = \varphi(a_{i,j}) \tag{7.14}$$

for $i \neq j$, but

$$b^{\varphi}(\varepsilon_i, \varepsilon_i) = \varphi(2a_{i,i}), \tag{7.15}$$

which may very well be tangible, even if 2 is not a unit in the valuation ring of $e\varphi$.

Very roughly, the isomorphism class of the free supertropical quadratic module $(U^{(I)}, q^{\varphi})$ may be viewed as an invariant of the pair $(q, \mathcal{L})$ measuring the "position" of the quadratic form q relative to the ordered base $\mathcal{L}$ of the free R-module V by the supervaluation $\varphi : R \to U$. This suggests itself, in view of the fact that the base $(\varepsilon_i \mid i \in I)$ of $U^{(I)}$ can be changed only in a very minor way: We only may multiply the ε_i by units of U, cf. Theorem 6.6. In imaginative terms, the base $\mathcal{L}$ becomes "frozen" in a free quadratic U-module obtained from (V, q) by a kind of degenerate scalar extension $\varphi : R \to U$. $\{\varphi$ is multiplicative, but respects addition only in a very weak way.$\}$

On the other hand, while q has a unique companion b, q^{φ} may have companions different from b^{φ}, which we are free to use.

Which advantages should we expect by passing from (V, q) to a supertropicalization $(U^{(I)}, q^{\varphi})$? Something can be located already at the present early stage of a theory of supertropical quadratic forms.

Let us write more briefly $(\widetilde{V}, \tilde{q})$ instead of $(U^{(I)}, q^{\varphi})$. The quadratic module $(\widetilde{V}, \tilde{q})$ allows arguments of a combinatorial flavour, due to the fact that $\widetilde{V}$ comes with its minimal ordering (which is of a strikingly simple nature: $x \leq y$ iff y is obtained from x by adding some vector z, cf. §6). For example, we can search for $\tilde{q}$-minimal vectors in $\widetilde{V}$. These are vectors $x \in \widetilde{V}$ such that $q(x) < q(y)$ for all $y \in \widetilde{V}$ with $y < x$, (cf. [8]). We also can pass from $\tilde{q}$ to its quasilinear part $[\tilde{q}]_{\mathrm{QL}}$ (cf. §5). Nothing like this is possible in (V, q) itself.

By use of the minimal ordering it is also possible to exhibit various natural U-submodules of $\widetilde{V}$ of interest. To give just one example: After choosing a companion $\tilde{b}$ of $\tilde{q}$, it turns out[11], that for any $c \in U \setminus \{0\}$ and $x \in \widetilde{V}$ the set

$$B_c(x) := \{y \in \widetilde{V} \mid \tilde{b}(x, y)^2 \leq c\tilde{q}(x)\tilde{q}(y)\},$$

is a U-submodule of $\widetilde{V}$.

The isomorphism class of $(\widetilde{V}, \tilde{q})$ itself is an invariant of $(q, \mathcal{L})$, which perhaps is too clumsy for practical purposes if $\dim(V)$ is big. But we can look for quotients of submodules of $\widetilde{V}$ by equivalence relations, which are compatible with $\tilde{q}$ in a suitable sense, and thus gain supertropical quadratic modules to be used as invariants of $(q, \mathcal{L})$, which can be handled more easily. The family $(B_c(x) \mid c \in R \setminus \{0\})$, (with fixed $x \in \widetilde{V}$) is a kind of filtration of the quadratic module $(\widetilde{V}, \tilde{q})$, which well deserves to be studied under this perspective.

Note that submodules of $\widetilde{V}$, and, all the more, quotients of these, most often are not free. Already this indicates the need for a supertropical quadratic form theory admitting U-modules of a rather general nature, not just free ones.

[11] To be proved in a sequel to this paper.

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