MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Nonstandard Analysis and Related Methods, and their Applications

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This meeting was organized by Sergio Albeverio (Bonn, Bochum), Leif Arkeryd (Gøteborg), Nigel J. Cutland (Hull), C. Ward Henson (Urbana, Illinois) and Manfred Wolff (Tübingen). The very different research areas of the organizers show partially where Nonstandard Analysis (NSA) is applied in mathematics. The 24 talks of the meeting on Nonstandard Analysis covered mathematical areas such as partial differential equations, additive number theory, Banach space theory, topology, operator theory, measure theory, stochastic calculus and even quantum stochastic calculus. Not all the talks used purely the methods of NSA since there have been some "standard" talks in the spirit of NSA.
The main purpose of NSA is to invent new methods for proving results in an intuitive and easier way. In this spirit continuum problems are reduced formally to discrete finite problems. Another trick is to think in infinitesimal quantities and the infinitesimal behaviour of certain mathematical objects. Such heuristics often used in the physicist's world is made precise in NSA. Thus NSA provides a powerful tool for modern mathematics applied in physics, economics and other sciences.

In PDE's the talks were about several physical important equations (Arkeryd, Brzeźniak, Capinski, Palczewski). C.W. Henson introduced nonstandard hulls and ultraproducts of Banach spaces which were used in further talks (Raynaud, Johnson). In operator theory not only approximation of operators in Hilbert space were discussed (Gordon) but there has been also an informal submeeting of some participants on strongly continuous semigroups and spectral theory. Furthermore another submeeting was about synthetic differential geometry.
The topics of stochastic processes and stochastic calculus were given in some talks (Benoit (formulated in Nelson's IST), Osswald, Russo, von Weizsäcker). The generalization to quantum stochastic calculus was introduced by R.L. Hudson. Most of the material in such areas is based on the Loeb space construction, also discussed and used in measure theory (Ross) and operator-valued measure theory (Ylinen). In topology interesting results were shown using NSA (Loeb, Sari, Zimmer). Modeltheoretic aspects were basic to two talks (Marker, Kanovei). The remaining talks gave applications of NSA to physics (Nakamura, Lumer), probability theory (Sun) and additive number theory (Jin).

In my opinion the most interesting result besides application of NSA to get path space measures in physics (Nakamura) was Jin's usage of NSA in additive number theory. A research area naturally connected to NSA seems to be quantum stochastic calculus (QSC) which was presented by Hudson to show the possibilities where one could construct a nonstandard model of QSC and develope an insightful nonstandard quantum stochastic calculus.

Reporter: Martin Leitz-Martini

Leif Arkeryd

On the stationary Boltzmann equation in a bounded domain in $\mathbb{R}^n$ with given indata on the boundary

Consider the stationary Boltzmann equation in a bounded strictly convex C^1 domain $\Omega \subset \mathbb{R}^n$ with inner normal $n(x)(-n < \beta < 2$ in $|v - v_x|^\beta)$, with Grad's angular cut-off, and with for some $\eta > 0$ a cut-off for velocities smaller than η. The evaporation-condensation boundary problem in this setting is

$$(BE) \begin{cases} v\nabla_x F = Q(F, F), x \in \partial\Omega, v \in \mathbb{R}^n \\ F = kf_b, x \in \partial\Omega, v \cdot n(x) > 0\,. \end{cases}$$

Under some mild restriction on $f_b > 0$ the following holds. There are functions $M_j(m) > 0, j = 1, 2$ (depending on f_b) with $\lim_{m\to 0} M_j(m) = 0, \lim_{m\to\infty} M_2(m) = \infty$, such that for each $m > 0$, the problem (BE) has a solution F with

$$\int_{\Omega\times\mathbb{R}^n} F(x, v)dxdv = M_1(m) \quad \text{and} \quad \int_{\Omega\times\mathbb{R}^n} v^2 F(x, v)dxdv = M_2(m).$$

Nonstandard reasoning was central in the discovery of this result. The proof is based on a general technique introduced by Arkeryd and Nouri, using local aspects of the entropy dissipation integral for an a priori control in nonlinear stationary kinetic problems. In contrast to other problems treated by this technique untill now, the study of (BE) in an essential way uses NSA arguments to established a new separation property fundamental for the proof in the (BE)-case.

Eric Benoit

Measure theory and stochastic processes in the hyperfinite world

Following E. Nelson, we show that it is possible to construct a theory of probability and diffusion processes in the hyperfinite world as rich as the classical one. We use the IST axioms and finite probability theory. We never need classical measure theory on infinite sets. We avoid standardization to stay in the hyperfinite world.

In the first part of the talk we study (hyper)discrete measures on a standard set. We introduce the measure of external subsets. It allows us to define a convenient notion of negligeable sets. We see also that each classical measure could be obtained as a “standardization” of a (hyper)discrete measure.

In a second part we define an equivalence relation on random variables and the macroscopic properties. This notions are adapted to the problem: the macroscopic properties are those we can observe after approximations by infinitesimals and they are compatible with the equivalence relation.

Finally, we apply this ideas to study the brownian motion as the equivalence class of a Wiener walk. We compute also the "density" of a (discrete) diffusion and prove Girsanov's theorem.

Zdzisław Brzeźniak

Stochastic Euler equations

Let $\mathcal{O}$ be a smooth connected open subset of $\mathbb{R}^2$, and let $T \in (0, \infty)$. We are concerned with the existence of a martingale solution to the incompressible stochastic Euler equations

$$\partial_t u + \langle u, \nabla \rangle u + \nabla p = F(t, u) + G(t, u)\dot{W}, \qquad \operatorname{div} u = 0 \qquad (1)$$

with the boundary condition

$$\langle u, \mathbf{n} \rangle = 0 \qquad \text{on} \quad (0, T) \times \partial\mathcal{O} \qquad (2)$$

where $\mathbf{n}$ stands for the unit outward normal to $\partial\mathcal{O}$. We prove existence of a martingale solution to (1)-(2). The constructed solution is a limit as the viscosity converges to zero of a sequence of solutions to modified stochastic Navier-Stokes equations. We study various regularity properties of the solutions, in particular we show the space continuity of the solution to (1)-(2). The methods used in the paper involve a theory of stochastic integration in a certain class of Banach spaces (including L^p–spaces with $p \geq 2$) and adaptation and generalization of some results obtained for classical Euler equations by Kato and Ponce. We study the dependence of the solution $u(t)$ on the initial value u_0. For the stochastic Navier-Stokes equations (SNSE's) we show the continuity of $u_0 \mapsto u(\cdot)$ treated as a mapping from a state space into the space of square integrable random variables. Our results have led us closer to solving the old and important problem of existence of stochastic flow for the SNSE's.

Marek Capinski and Nigel J. Cutland

Existence of global stochastic flow and attractors for Navier-Stokes equations

Suppose that Ω is a probability space with one parameter group $\theta_t : \Omega \to \Omega$ of measure preserving maps, and let $\mathcal{H}$ be a Hilbert space. For a stochastic evolution equation in $\mathcal{H}$ we say that

$$\varphi : [0, \infty) \times \mathcal{H} \times \Omega \to \mathcal{H}$$

is a *flow of solutions* if
(i) for each $\omega \in \Omega$, the function $(t, x) \mapsto \varphi(t, x, \omega)$ is continuous,
(ii) for each $x \in \mathcal{H}$, the process $u(t) = \varphi(t, x, \omega)$ is a solution of the equation in question with the initial value $u(0) = x$ (so $\varphi(0, x, \omega) = x$).
A flow φ is a *crude cocycle* if for all $s \in \mathbb{R}$ there is a full set Ω_s such that for all $\omega \in \Omega_s$

$$\varphi(t + s, x, \omega) = \varphi(t, \varphi(s, x, \omega), \theta_s \omega)$$

hold for each $x \in \mathcal{H}$ and $t \in \mathbb{R}$. A cocycle is *perfect* if Ω_s does not depend on s. Under the condition of invertibility of $\varphi(t, \cdot, \omega)$ in finite dimensional spaces $\mathcal{H}$, Arnold and Scheutzow showed that a crude cocycle can be made perfect, i.e. it may be modified to a stochastically indistinguishable perfect one.
Part of the importance of cocycles stems from the desire to find an appropriate notion of an *attractor* for a stochastically evolving system. The abstract notion formulated by Crauel and Flandoli in the above setting is the following: a *random global attractor* is a random compact subset $A(\omega)$ of $\mathcal{H}$ satisfying

$$\varphi(t, A(\omega), \omega) = A(\theta_t \omega),\ t \geq 0; \quad \lim_{t\to\infty} \text{dist}\,(\varphi(t, x, \theta_{-t}\omega), A(\omega)) = 0\,.$$

Our interest here is in stochastic flows, cocycles and global random attractors for the stochastic Navier-Stokes equations

$$u(t) = u_0 + \int_0^t [-\nu A u(s) - B(u(s)) + f] ds + \int_0^t g(u(s)) d\omega_s\,.$$

In earlier work Brzeźnick, Capinski, Crauel and Flandoli considered special cases of these equations that can be reduced to equations with random coefficients and thus solved pathwise (albeit not simply), and this leads to a pathwise construction of global attractors.

In this work the equations considered are truly stochastic, i.e. irreducible to non-stochastic ones, although it is necessary to impose a certain orthogonality restriction on the noise. This gives some deterministic information about the energy of the solution to the equations which allows the methods of Temam to be used in the stochastic setting. Consequently, we are able to construct a global flow, perfect cocycle and global compact random attractor for the above stochastic Navier-Stokes equations in dimension 2 with periodic boundary conditions.

The paper makes essential use of nonstandard analysis. In particular, the global (standard) stochastic flow and the perfect cocycle are constructed on a Loeb space: whereas it may be possible to obtain a so called crude cocycle on an arbitrary space, it seems that the richness of a Loeb space is needed for a perfect cocycle.

Evgueni I. Gordon

On approximation of operators and groups

A new approach to approximation of operators in the Hilbert spaces of function on locally compact abelian groups is introduced. This approach is based on a sampling procedure applied to the symbols of such operators. To choose the points of sampling we use the approximations of groups by finite groups. In the case of the group $\mathbb{R}^d$ constructed approximations include all finite-differences approximations, which can be obtained by the choice of an appropriate approximation of groups. The convergence of spectra for approximations of Schrödinger type operators with increasing on infinity and periodic potentials is proved.

The problems of approximations of noncommutative topological groups by finite ones are discussed. The approximability of a large class of nilpotent Lie groups and nonapproximability of SO(3) are proved. For the compact groups the equivalence of approximability by finite groups and the approximability of Hopf algebra by finite dimensional Hopf algebras is proved.

C. Ward Henson

Ultraproducts and nonstandard hulls of Banach space structures

This was an expository talk to provide background for the talks of Yves Raynaud and William B. Johnson which followed in the same morning session. The objective was to introduce the two main constructions in model theoretic applications to functional analysis and to show how they are related to each other. The constructions are (1) Banach space ultraproducts (introduced in the 1966 PhD thesis of Krivine) and (2) Nonstandard Hulls (introduced in 1968 by Luxemburg in his influential paper "A general theory of monads"). Consider an index set I and an ultrafilter $\mathcal{U}$ on I. For each set V let ${}^*V$ denote the nonstandard extension of V constructed as the set theoretic ultrapower of V using the ultrafilter $\mathcal{U}$. If $(X(i) : i \in I)$ is a sequence of normed linear spaces indexed over I, then the Banach space ultraproduct of $(X(i) : i \in I)$ is identical to the nonstandard hull of the internal normed space X which corresponds to the set theoretic ultraproduct of $(X(i) : i \in I)$ with respect to $\mathcal{U}$. In other words, the Banach space ultraproduct construction and the nonstandard hull construction are essentially identical.

The speaker gave general background about these constructions for structures based on Banach spaces (lattices, algebras, etc). They apply to any structures in which the basic operations are bounded and uniformly continuous on each bounded set. Emphasis was placed on how classes of the classical Banach spaces and structures based on them are closed under ultraproducts and nonstandard hulls. Emphasis was also placed on the important question for a given class $\mathcal{K}$ of structures, whether or not it is closed under "ultraroots": namely, is it the case that whenever a Banach space ultrapower of X is in $\mathcal{K}$, then X itself must also be in $\mathcal{K}$. Closure under ultraproducts and ultraroots corresponds exactly to the possibility of axiomatizing $\mathcal{K}$ using positive bounded sentences in the speaker's logic for structures from analysis.

Robin L. Hudson

Quantum stochastic calculus for non-standard analysts

Quantum stochastic calculus can be regarded as a noncommutative extension of the Itô calculus of Brownian motion arising from the identification of the L^2–space of Wiener measure with the Fock space over $L^2(\mathbb{R}_+)$.
In this exposition, I emphasise the continuous tensor product structure of the Fock space, which makes the theory of quantum stochastic calculus appear to be a natural candidate

for description by non-standard analysis, generalising the successfull treatment of Itô calculus.
As well as the creation and annihilation processes A^+ and A, linear combinations of which correspond to (mutually noncommuting) Brownian motions, quantum stochastic calculus also uses the preservation process Λ, using which stochastic integrals against the Poisson process are also included in the theory. The theory is developed using two fundamental formulas, which express stochastic integrals and products of stochastic integrals, as Lebesgue integrals, using exponential (factorising) vectors in the Fock space. The fundamental estimate, derived from the second of these formulas, extends integrability to a wide class of operator-valued integrands, permitting definition of iterative integrals and solution of quantum stochastic differential equation by the iterative method.
For multidimensional quantum stochastic calculus, we use the Belavkin notation , which labels the integrator processes by operator valued matrices ofl195 the form
L=

$$\begin{pmatrix} 0 & \bar{u} & \omega \\ 0 & S & v \\ 0 & 0 & 0 \end{pmatrix}$$

$\omega \in \mathbb{C}$ and $S \in B(\mathcal{K})$
where u, v are vectors in a d-dimensional Hilbert space $\mathcal{K}$. By $\mathbb{Z}_2$–grading the space $\mathcal{K}$, we can construct a corresponding $\mathbb{Z}_2$–graded quantum stochastic calculus. In the one-dimensional case $\mathcal{K} = \mathbb{C}$, this construction gives Fermion fields acting in the (Boson) Fock space. We discuss the meaning of the formula

$$dF(M_L) = F(M + dM_L) - F(M_L).$$

Finally we discuss multiple quantum stochastic product integrals of the form

$$\prod_a^b \quad \prod_c^d (1 + \sum c^{jk} dM_{L_j} \otimes dM_{L_k})$$

where $[a, b[$ and $[c, d[$ are disjoint subintervals of $\mathbb{R}_+$.

Renling Jin

Banach density problems in additive number theory

Given a subset A of natural numbers, let

$$\sigma(A) = \inf_{n \geq 1} \frac{|A \cap \{1, 2, \cdots n\}|}{n} \quad \text{and} \quad d^*(A) = \lim_{k \to \infty} \sup_{n-m=k} \quad \frac{|A \cap \{m+1, m+2, \cdots, n\}|}{n-m}$$

$\sigma(A)$ is called the *Shnirel'man density* and $d^*(A)$ is called the *Banach density* of A. Shnirel'man's theorem says that if $\sigma(A) > 0$ and $0 \in A$, then there is a k such that the sum of k–copies of A is the set of all natural numbers. Shnirel'man's theorem about

Banach density says that if $d^*(A) > 0$ and A contains two consecutive numbers, then there ist a k such that the sum of k–copies of A is a thick set. A set A is called *thick* if A contains arbitrarily long (but finite) sequences of consecutive numbers. In the talk, we show three different ways to prove Shnirel'man's theorem about Banach density. All three ways use nonstandard methods and each of those ways reveals a general idea of applying nonstandard methods to additive number theory.

William B. Johnson

Nonlinear Banach space theory: From ultrapowers to $\mathbb{R}^2$

This is a survey of part of the two nonlinear-theories of Banach spaces. In the Lipschitz category the morphisms are Lipschitz mappings between Banach spaces; two spaces are Lipschitz equivalent if there is a bi-Lipschitz homeomorphism between them. In the uniform category the morphisms are uniformly continuous mappings; two Banach spaces are uniformly equivalent if there is a bi-uniformly continuous homeomorphism between them. Differentiation techniques sometimes allows the passage from Lipschitz equivalence to isomorphism while ultraproducts play an important role in relating uniform equivalence to Lipschitz equivalence.
A new development in the nonlinear theories (by S. Bates, J. Lindenstrauss, D. Preiss, G. Schechtman, and the speaker) is the study of Lipschitz quotients and uniform quotients of a given Banach space. Nonlinear quotient mappings have not been previously explored even when the domain is the Banach space $\mathbb{R}^2$; this case, as well as results in the infinite dimensional setting, will be discussed.

Vladimir Kanovei and Michael Reeken

Models of ZFC extendable to models of nonstandard set theories

Necessary and sufficient conditions are presented, for a standard model of ZFC to be extendable to a model of a given nonstandard set theory, for instance, IST, BST, theories of Hrbacek and Kawai.

Peter A. Loeb

The base operator in analysis and topology

The talk is on joint work with Jürgen Bliedtner. We use nonstandard analysis to simplify and further expand the theory of base and antibase operators. This includes the theory of the topologies these operators generate, such as the density topology in measure theory, and even the fine topology in potential theory. A simple example of a base operator is the mapping from sets to their closures in a topological space. Base and antibase operators replace a set with the collection of points where the set is big. With more structure, they are also called upper and lower densities. We give a new, simple description of the topologies these operators generate using a natural extension of the operators from algebras of sets on which they are initially defined to the full power set of the underlying space X. We even extend base operators to bounded real-valued functions on X, capturing in an elementary way topological operations on the functions. Associated with these operators for bounded measurable functions is an optimal differentiation basis. As an application, we easily construct a lifting for the space of bounded measurable functions, assigning to each member f the value c at any point where f is "essential close" to c.

Gunter Lumer

Blow up and residual effects for solutions in parabolic systems with singular boundary interactions

We treat systems governed by parabolic equations, in classical $\Omega \subset \mathbb{R}^N$ and Banach space X, contexts, with boundary operator $B : X \to H \subset X$ and "singular interaction" (si u)(0) $= \sigma$ a distribution or hyperfunction $\in \mathcal{B}(\mathbb{R}, \mathcal{H})$ (– think of $u' = \Delta u + F(t), \quad u(0) = f, \quad u|\partial\Omega = \varphi$ for $t > 0$, written in $X = C(\overline{\Omega})$, $C(\partial\Omega)$ identified via harmonic extensions with $H \subset X, B =$ "$|$" plus harmonic extension, A replacing Δ in the general context $Q(t) =$ "the semigroup generated by A" $= \mathrm{e}^{tA}$, (si u)(0) $= \sigma$ meaning *intuitively* that $\sigma(t)$ is added to $\varphi(t)$ as boundary condition). σ' is always of the form $\sum_{j=1}^{\infty} c_j \delta^{(j)}$ with $c_j \in H$. We use asymptotic methods finding classical C^1 solution $u(t)$ for $t > 0 = \sum_{j=1}^{\infty} Q^{(j)}(t) c_j$ (singular at $t = 0$) when all "nonsingular data" $F = f = \varphi = 0$.
Surprising things happen: (1) $\exists$ nondetectable signal (nds) i.e. $\exists \sigma \neq 0$ for which $u(t) \equiv 0$; (2) such nds however leave a residual effect in the form of a hyperfunction concentrated at 0 (supp = 0), equal to $-Q * \sigma'$ (computed as hyperfunctions convolution in the sense of Komatsu) which can be explicitely computed in terms of the c_j and A (as infinite δ–expansion); we show that if $u(t) \not\equiv 0$ and remains bounded for all $t > 0$ it *must* come from a σ which is a true hyperfunction (i.e. not reducing to a distribution) and such $u(t)$ exists, so you "can see" (film) true hyperfunctions.
Here NSA is used simply to produce one natural way of discovering how to express mathematically the physical content of the physical situation intuitively given and to see why hyperfunctions are indeed needed to capture that full content (by applying to crude nonstandard approximations to the model, the requirement that a physically meaningful description (model) must have (measurable) *standard* data producing in turn *standard* solution).

David Marker

Logarithmic-exponential series

This is joint work with Lou van den Dries and Angus Macintyre. Using fields of generalized power series, we construct $R((t))^{\mathrm{LE}}$ the field of logarithmic-exponential series. This is a quite natural nonstandard model of the theory of the real field with exponentiation which strongly reflects asymptotic properties of definable functions. We use this construction to answer a question of Hardy's by showing that the compositional inverse to $x \mapsto (\log x)(\log\log x)$ is not asymptotic to a composition of exponentials, logarithms and algebraic functions.

The field has a natural derivation satisfy the exponential derivation rule $(e^f)' = f'e^f$ with constant field R. As every element of the field has an anti-derivative, all linear first order differential equations have solutions. The equation $y'' + y = 0$ has no non-trivial solutions. This leads us to conjecture that there is an intimate connection between solving differential equations in $R((t))^{\mathrm{LE}}$ and finding non-oscillating solutions at infinity. We prove a precise version of this conjecture for homogeneous second order equations.

Toru Nakamura

Path space measure for $3+1-\mathrm{D}$ Dirac equation in momentum space

The Green function for the $3+1-\mathrm{D}$ Dirac equation contains not only delta functions as for the case of $1+1-\mathrm{D}$, but also the derivative of delta functions. The fact prevents us away from assigning $*$–measure for each path, consequently from defining $*$–path integral. Therefore, we prefer to work in the momentum space. The electromagnetic potentials cause a momentum of the particle to change from time to time, a stochastic process whose transition probabilities therefore depend on the potentials. These facts are incorporated in the definition of the $*$–measure, to obtain the standard solution as the standard part of the $*$–path sum with respect to the $*$–measure.
It is also shown that a standard measure over a standard path-space can be extracted from the nonstandard one, so that the standard solution is obtained as the standard path integral with respect to the extracted measure. In the construction, nonstandard analysis concretely exposes the structure of the measure, which enables us to derive its fine properties. If the potentials do not depend on time, our result reduces to that of B. Gaveaus in 1984.

Horst Osswald

Extensions of the classical Wiener space and applications

We study shifts on the space $C[0,1]$ of real continuous functions of the form

$$\sigma \circ b(X, \cdot) : [0,1] \ni \tau \mapsto b(X,\tau) + \int_0^\tau \varphi(X,s)d\lambda(s),$$

where φ is a square integrable process on a probability space Ω and b is a Brownian motion on Ω such that $C[0,1]$ coincides with the set of trajectories of b. The function φ is called the kernel of σ.
Using Malliavin calculus combined with NSA we give positive answers for a large class of kernels φ of the following well established question:
Does there exist a measure P absolutely continuous with respect to Wiener measure such that $\sigma \circ b$ follows the law of Brownian motion with respect to P?

Andrzej Palczewski

Convergence of discrete velocity models to kinetic equations

The problem is formulated as follows. Find an approximation to integro- differential equations of kinetic type

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) = \int_{\mathbb{R}^3}\int_{S^2} g(v, v_1, u)\,du\,dv_1$$

by a system of PDE's

$$\frac{\partial f_i}{\partial t} + \quad v_i \cdot \nabla_x f_i \quad = \sum_{j,k,\ell} g(v_i, v_j, u_{ij}^{k\ell})$$

there f is a socalled one-particle distribution function $f = f(x, v, t)$ and g is a nonlinear (bilinear) function of f.

The essential difficulty is the approximation of the fixfold integral by summation over a discrete set of velocities $\{v_i\}$. In solving this problem we introduce in the velocity space $\mathbb{R}^3$ a regular grid $\{v_i : v_i = hn, n \in \mathbb{Z}^3\}$. The problem is then reduced to the analysis of solutions to diophantine equation $x_1^2 + x_2^2 + x_3^2 = m$. To solve the approximation problem we have to know that the member of integer solutions to diophantine equation is tending to infinity for $m \to \infty$, and that these solutions are uniformly distributed on a sphere. Recent results in additive number theory solve both these problems positively (Duke, Iwaniec). Hence we are able to prove that the discrete approximation converges to the integral and estimate the error of approximation (which is of order $h^{2/175}$).

To prove convergence we need the existence and uniqueness of solutions for both our models (continuous and discrete). This can be achieved only for x–independent solutions. Hence from both equations the term $v \cdot \nabla_x f$ is dropped. For such a space-homogeneous equation we can prove the strong convergence of solutions to discrete velocity model to the unique solution of integro-differential equation in the space $L_1(\mathbb{R}^3)$ with the polynomial weight function $(1 + v^2)$. For the space dependent problem we are able to improve the result of S. Mischler showing that with our velocity grid solutions to modified discrete velocity equation converge weakly to the weak solution of DiPerna-Lious of the Boltzmann equation.

Yves Raynaud

Ultrapowers of classical Banach spaces

This talk is intended to give an overview of "concrete" representations of ultrapowers of classical Banach spaces. The heuristics for finding such a representation consists either in considering the given Banach space as a member of a class specified by some additional structure and several axioms which are conserved through the ultrapower process, and using a suitable representation theorem, or in relating the given space to another one, the ultrapowers of which are already known (this is typically done by mean of a locally uniform homeomorphism). Several examples are given to illustrate these two ways, starting from the very classical case of L_p–spaces, examining then Orlicz spaces, vector-valued spaces (in these two cases the class of Banach spaces to deal with has to be strictly enlarged), Calderón interpolation spaces and non-commutative L_p–spaces.

David A. Ross

Some recent applications of Loeb measures

The standard part map is not always as well-behaved as one would like for 'pushing' Loeb measures down to standard spaces. I discuss three recent examples of standard applications where the natural Loeb-measure proofs fail, and need to be supplemented by an unexpected extra argument:

1. In a nonstandard proof of the classical Kolmogorov Existence Theorem (for probability measures on an infinite product $\mathbb{R}^I$), the measure given by the push-down of the natural Loeb measure needs to be constructed on the larger space $\overline{\mathbb{R}}^I$ and then 'traced' onto $\mathbb{R}^I$.
2. To prove the following:
Let $(X_i, \mathcal{B}_i, m_i)(i \in \mathbb{N})$ be a sequence of Borel measure spaces. There is a Borel measure μ on $\Pi_{i\in\mathbb{N}} X_i$ such that if $K_i \subseteq X_i$ is compact for all $i \in \mathbb{N}$ and $\Pi_{i\in\mathbb{N}} m_i(K_i)$ converges then $\mu(\Pi_{i\in\mathbb{N}} K_i) = \Pi_{i\in\mathbb{N}} m_i(K_i)$
the standard part map can only be used to construct local measures, which then need to be pasted together using an internal measure to control the amalgamation.
3. If a metric space X is not locally compact, then the space $\mathcal{F}$ of closed subsets of X (under the usual topology) need not be Hausdorff. Nevertheless, there is a reasonable stand-in x for the standard part map. To prove a Choquet-like result relating precapacities on X to $\mathcal{F}$–valued random variables, a suitable Loeb measure can be constructed on ${}^*\mathcal{F}$ and pushed down via x. However, the argument that x is measurable (and that the resulting image measure on $\mathcal{F}$ has the requisite properties) is difficult, requiring an unusual appeal to the completeness of X.

Francesco Russo

Stochastic calculus with respect to finite quadratic variation processes

We develop a "pathwise stochastic calculus" whose starting point was an article of H. Föllmer (1981). Using discretization techniques we define forward, backward and symmetric integrals and covariation processes. The covariation of a process X with itself is called quadratic variation and it is denoted by $[X, X]$. If $[X, X]$ exists X will be called a finite quadratic variation process.
I provide a large class of examples of finite quadratic variation processes (continuous or with jumps), with some emphasis on Gaussian processes. For such processes X, a calculus is presented with applications to the study of some stochastic differential equations driven by X.

The features of the work are essentially the following.
1) Simplicity. Many rules are directly derived using simple calculus (finite Taylor expansions, uniform continuity ...).
2) The calculus goes beyond semimartingales. Our aim was to understand what could be done when integrators are just Gaussian processes, convolution of martingales as fractional Brownian motion and Dirichlet processes (sum of a local martingale and a zero quadratic variation process).
3) It is a bridge between anticipating and non-anticipating integrals. Our integral helps to relate Skorohod integral with enlargement of filtration techniques.

Tewfik Sari
Semi-continuous set-valued mappings

Set-valued mappings appear naturally in many areas: optimization theory, mathematical economics, dynamical systems .... Their semi-continuity properties are related to the notion of limit of families of subsets of a topological space and also to some topologies on the set of subsets of a topological space. These notions were defined and developed, among others, by Baire, Painlueé, Hausdorff, Vietoris, Kuratowski and Choquet. The aim of the talk is to present the main facts of the theory and to show that the nonstandard approach permits new definitions and new simple and insightful proofs.

Yeneng Sun
Continuum approach to stochastic independence: is it possible?

Continuum methods have provided powerful tools in mathematics and its applications. However, there was no progress on the study of independence via continuum methods. A reason behind this phenomenon is that independence and joint measurability are never compatible with each other in the continuum setting except for some trivial cases.

Here we use a larger measure-theoretic framework based on Loeb product spaces to bypass this measurability difficulty. Distinct new phenoma arise naturally. In particular we characterize the satisfiability of various versions of the law of large numbers via almost independence. This provides a rigorous foundation for the exact cancellation of idiosyncratic risks underlying many economic models.

Completely new connections between various basic probabilistic concepts are also discovered. It is common sense that pairwise independence and mutual independence are distinct in the finite setting. It is also well known that some weaker properties involving the multiplication of generating functions, characteristic functions and distribution functions are different. However, it is found that all these concepts are almost equivalent in the ideal setting. In addition, the duality between independence and another basic probabilistic concept - exchangeability is established.

Heinrich von Weizsäcker

Asymptotic separation of measures in Markov-chains

For each $N \in \mathbb{N}$, let $P_0^{(N)}, P_1^{(N)}$ be two probability measures over a common space Ω_N. We are interested in the rate of the exponential decay of the *overlap*

$$\|P_0^{(N)} \wedge P_1^{(N)}\| = \frac{1}{2}(2 - \|P_0^{(N)} - P_1^{(N)}\|),$$

(minimal Bayes risk in statistical language), i.e. we want to evaluate the limit

$$r = \lim_{N\to\infty} \frac{1}{N} \log \|P_0^{(N)} \wedge P_1^{(N)}\|$$

Let for $i = 0, 1$ the measure $P_i^{(N)}$ be the law of an irreducible Markov chain over a finite set, governed by the transition matrix π_i. Then

$$r = \inf_{0<t<1} \log \rho(\pi_t)$$

where the (substochastic) matrix π_t is given by $\pi_t(x,y) = (\pi_0(x,y))^{1-t}(\pi_1(x,y))^t$ and ρ denotes spectral radius (Scheffel 97).

This has been extended in (Scheffel-Weizsäcker, Math. Meth. of Stat. 1997/98). The proof uses large deviations of the empirical pair measure (and an extension thereof), the variational formula for the spectral radius of nonnegative matrices and a minimax argument. The talk also mentioned recent related results about optimal information gain in the continuous parameter problem and some guess for the similar rate in continuous space time models.

Kari Ylinen

Aspects of operator measures and bimeasures susceptible to a nonstandard treatment

Complex bimeasures are separately σ–additive functions defined on the Cartesian product of two σ–algebras. In case they take only nonnegative real values, they can under general regularity conditions be extended to measures on the product σ–algebra, but general $\mathbb{C}$–valued bimeasures do not have this property, and their integration theory meets with serious difficulties. The author has proved a representation theorem (Stud. Math. 104 (1993), p. 272) for complex bimeasures in terms of spectral measures. It is therefore conceivable that a nonstandard treatment of spectral measures and more generally positive operator measures could shed light on the bimeasure integration problem. More motivation for such a treatment comes from the recent development by the author with some collaborators (P. Lahti, M. Maczynski, J.-P. Pellonpää) of integration theory with respect to positive operator measures and its use in quantum mechanics. As a first step towards a nonstandard study of positive operator measures, the construction of the corresponding Loeb space was announced. If one begins with a standard positive operator measure, the Loeb operator measure will have its values acting on the original Hilbert space. It is also possible to start with an internal finitely additive operator measure relative to an internal inner product space H, and then the values of the Loeb operator measure will act in the nonstandard hull $\widehat{H}$. As an application, the semispectral measure corresponding to a semispectral function on the real line was constructed.

G. Beate Zimmer

Small norm isomorphisms of $C(K)$–spaces

In a joint paper with Nigel Cutland (Bull. Austr. Math. Soc., Vol. 57 (1998), p.55-58), we used nonstandard peak functions, that is, internal functions of norm one that are supported in one monad to obtain a new proof of the Banach-Stone theorem. Using nonstandard peak functions, we were able to extend Banach's elegant original proof, which was for the special case of compact metric spaces to the general setting of compact Hausdorff spaces. We characterized peak functions in a way that is invariant under isometries. A linear isometry then takes a peak function to a peak function. This induces a bijection between the underlying compact Hausdorff spaces.

Recently I have been looking into generalizations of this result. One possible generalization would be to look at complex $C(K)$–spaces. Our results for the isometric case did use the order structure of the reals heavily. I did prove some results on images of peak functions in the complex case.

It is well-known that a more general version of the Banach-Stone Theorem holds. Proofs for this generalization were first given by Amir and Cambern in 1965. If X and Y are compact Hausdorff spaces and if there is a linear bijection $T : C(X) \to C(Y)$ such that $\|T\| \cdot \|T^{-1}\| = \alpha < 2$, then X and Y are homeomorphic. The nonstandard extension of the operator T (after suitable normalization) then maps a peak function f_x to a function Tf_x, whose values are greater or equal to $\frac{\alpha}{2}$ in exactly one monad. Showing that this induces a homeomorphism between X and Y is a little more work than in the isometric case. I have a complete proof for this situation, but still hope to make it look a bit less technical.

Finally I am beginning to look into near-isometries of subspaces of $C(K)$–spaces. As every Banach space can be represented as a subspace of a $C(K)$–space, this is naturally a very interesting topic to anybody in functional analysis or operator theory.

I would very much like to get a chance to work some more with Nigel Cutland on this and am sure that this would be produce very interesting results.

Sergio Albeverio	sergio.albeverio@rz.ruhr-uni-bochum.de, albeverio@uni-bonn.de
Robert Anderson	anderson@econ.berkeley.edu
Leif Arkeryd	arkeryd@math.chalmers.se
Eric Benoit	ebenoit@math.univ-lr.fr
Zdzisław Brzeźniak	z.brzezniak@maths.hull.ac.uk
Marek Capinski	capinski@im.uj.edu.pl
Nigel. J. Cutland	n.j.cutland@maths.hull.ac.uk
Thomas Elsken	thomas.elsken@mathematik.uni-rostock.de
Paolo Giordano	giordano@mat.unimi.it
Evgueni I. Gordon	gordon@irina.nnov.ru
C. Ward Henson	henson@math.uiuc.edu
Robin L. Hudson	rlh@maths.ntu.ac.uk
Renling Jin	jinr@cofc.edu
William B. Johnson	johnson@math.tamu.edu
Vladimir G. Kanovei	kanovei@math.uni-wuppertal.de, kanovei@mech.math.msu.su
Martin Leitz-Martini	martin.leitz@uni-tuebingen.de
Tom Lindstrøm	lindstro@math.uio.no
Peter A. Loeb	loeb@math.uiuc.edu
Gunter Lumer	dufour@umh.ac.be
David Marker	marker@math.uic.edu
Toru Nakamura	tnakamur@tsuda.ac.jp
Juha Oikkonen	oikkonen@cc.helsinki.fi
Horst Osswald	horst.osswald@mathematik.uni-muenchen.d400.de
Andrzej Palczewski	apalczew@mimuw.edu.pl
Hans Ploss	ploss@radon.mat.univie.ac.at
Yves Raynaud	yr@moka.ccr.jussieu.fr
Michael Reeken	reeken@wmra1.math.uni-wuppertal.de
Hermann Render	render@math.uni-duisburg.de
David A. Ross	ross@math.hawaii.edu
Francesco Russo	russo@math.univ-paris13.fr
Sergei N. Samborski	samborsk@matin.math.unicaen.fr
Tewfik Sari	T.sari@univ-mulhouse.fr
Yeneng Sun	matsuny@math.nus.edu.sg
Heinrich von Weizsäcker	weizsaecker@mathematik.uni-kl.de
Manfred Wolff	manfred.wolff@uni-tuebingen.de
Jiang-Lun Wu	jiang-lun.wu@rz.ruhr-uni-bochum.de
Kari Ylinen	ylinen@utu.fi
Beate Zimmer	bzimmer@sunmuw1.muw.edu

Tagungsteilnehmer

Prof.Dr. Sergio Albeverio
Mathematisches Institut
Universität Bonn
Wegelerstr. 10

53115 Bonn

Prof.Dr. Robert Anderson
Department of Economics
University of California
549 Evans Hall #3880

Berkeley , CA 94720-3880
USA

Prof.Dr. Leif Arkeryd
Department of Mathematics
Chalmers University of Technology
at Göteborg

S-412 96 Göteborg

Prof.Dr. Eric Benoit
Department de Mathematiques
Universite de la Rochelle
Avenue Marillac

F-17042 La Rochelle Cedex

Prof.Dr. Zdzislaw Brzezniak
Dept. of Pure Mathematics
University of Hull
Cottingham Road

GB-Hull , HU6 7RX

Prof.Dr. Marek Capinski
Graduate Business
School - NLV
Zielona 27

33300 Nowy Sacz
POLAND

Prof. Nigel J. Cutland
Dept. of Pure Mathematics
University of Hull
Cottingham Road

GB-Hull , HU6 7RX

Thomas Elsken
Fachbereich Mathematik
Universität Rostock
Universitätsplatz 1

18055 Rostock

Prof.Dr. Paolo Giordano
via S. Caboto 23

I-20094 Corsico (MI)

Prof.Dr. Evgueni Israel Gordon
Department of Mechanics and
Mathematics
Nizhni Novgorod State University
Gagarin pr. 23

603600 Nizhni Novgorod
RUSSIA

Prof.Dr. C. Ward Henson
Department of Mathematics
University of Illinois
273 Altgeld Hall MC-382
1409, West Green Street

Urbana , IL 61801-2975
USA

Prof.Dr. Robin L. Hudson
Department of Mathematics
Nottingham Polytechnic
Burton Street

GB-Nottingham NG1 4BU

Prof.Dr. Renling Jin
Department of Mathematics
College of Charleston

Charleston , SC 29424
USA

Prof.Dr. William B. Johnson
Department of Mathematics
Texas A & M University

College Station , TX 77843-3368
USA

Prof.Dr. Vladimir G. Kanovei
Department of Mathematics
Caltech

Pasadena , CA 91125
USA

Prof.Dr. Martin Leitz-Martini
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10

72076 Tübingen

Prof.Dr. Tom Lindstrom
Institute of Mathematics
University of Oslo
P.O. Box 1053

N-Oslo Blindern

Prof.Dr. Peter A. Loeb
Department of Mathematics
University of Illinois
273 Altgeld Hall MC-382
1409, West Green Street

Urbana , IL 61801-2975
USA

Prof.Dr. Gunter Lumer
Dept. de Mathematiques
Faculte des Sciences
Universite de l'Etat a Mons
15, avenue Maistriau

B-7000 Mons

Prof.Dr. David Marker
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street

Chicago , IL 60607-7045
USA

Prof.Dr. Toru Nakamura
Sundai Preparatory School
2-5-17
Kanda-Surugadai
Chiyoda-ku

Tokyo 101-8313
JAPAN

Prof.Dr. Juha Oikkonen
Dept. of Mathematics
University of Helsinki
P.O. Box 4

SF-00014 Helsinki

Prof.Dr. Horst Osswald
Mathematisches Institut
Universität München
Theresienstr. 39

80333 München

Prof.Dr. Andrzej Palczewski
Institute of Mathematics
University of Warsaw
Banacha 2

02-097 Warszawa
POLAND

Dr. Hans Ploss
Institut für Mathematik
Universität Wien
Strudlhofgasse 4

A-1090 Wien

Prof.Dr. Yves Raynaud
Equipe d'Analyse, T. 46, 4e etage
Universite Pierre et Marie Curie
(Universite Paris VI)
4, Place Jussieu

F-75252 Paris Cedex 05

Prof.Dr. Michael Reeken
Fachbereich 7: Mathematik
U-GHS Wuppertal

42097 Wuppertal

Dr. Hermann Render
Fachbereich Mathematik
Universität-GH Duisburg

47048 Duisburg

Prof.Dr. David A. Ross
Dept. of Mathematics
University of Hawaii
2565 The Mall

Honolulu , HI 96822
USA

Dr. Francesco Russo
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
Av. J.-B. Clement

F-93430 Villetaneuse

Prof.Dr. Sergei N. Samborski
Dept. de Mathematiques
Universite de Caen

F-14032 Caen Cedex

Prof.Dr. Tewfik Sari
Laboratoire de Mathematique
Universite de Haute Alsace
4, rue des Freres Lumiere

F-68093 Mulhouse Cedex

Prof.Dr. Yeneng Sun
Department of Mathematics
National University of Singapore
10 Kent Ridge Cresent

Singapore 119260
SINGAPORE

Prof.Dr. Heinrich von Weizsäcker
Fachbereich Mathematik
Universität Kaiserslautern

67653 Kaiserslautern

Prof.Dr. Manfred Wolff
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10

72076 Tübingen

Prof.Dr. Jiang-Lun Wu
c/o Prof. Dr. S. Albeverio
Ruhr Universität Bochum
NA 3/5

44780 Bochum

Prof.Dr. Kari Ylinen
Institute of Mathematical Sciences
University of Turku

SF-20014 Turku

Beate Zimmer
Mississippi University for Women
Box W-100

Columbus , MS 39701
USA