Mathematisches Forschungsinstitut Oberwolfach

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Lattices, Polytopes and Tilings

27.2. – 4.3.2000

The conference was organized by NIKOLAI P. DOLBILIN (Moscow) and RUDOLF SCHARLAU (Dortmund). There were 23 participants coming from Germany, Russia, USA, Switzerland, Canada and Hungary. The aim was to bring together certain areas of algebra, combinatorics and discrete geometry which could fruitfully interact, but normally are not represented at one meeting. These areas were Geometry of Numbers, more precisely the 'Voronoï-Delone school' of geometry of positive quadratic forms; the theory of tilings; the combinatorial theory of convex polytopes; mathematical crystallography, in particular crystallographic groups; and algorithmic aspects of these fields.

In 19 talks, the participants presented mostly recent results; a few lectures were devoted to surveys.

In an evening problem session, 8 participants proposed research problems of various kinds, which led to lively discussions. Another evening was devoted to software presentations.

After the end of the 'official' lectures on Friday afternoon, there was an informal session on geometry of numbers with contributions by Robert M. Erdahl, Konstantin Rybnikov and Sergey S. Ryshkov.

Many participants met for the first time at this conference and took advantage of the generous hospitality and relaxed atmosphere of the institute which, after the lectures, led to a lot of exchange and discussions and also to informal conversations in small groups.

Author of the report: FRANK VALLENTIN

Abstracts of talks

Similarity submodules and semigroups

MICHAEL BAAKE

Lattices and $\mathbb{Z}$-modules in Euclidean space possess an infinitude of subsets that are images of the original set under similarities. The classification (which is far from being complete) if such self-similar images according to their indices, together with the corresponding semigroups of similarities, has several applications in (quasi-)crystallography, e.g. to the theory of colour symmetries and to inflation symmetries.

Complete results are presented for various lattices and $\mathbb{Z}$-modules of interest in crystal and quasicrystal theory, in particular for those related to root systems in dimensions 2, 3 and 4. Among them are the root systems of the non-crystallographic COXETER groups H_2, H_3 and H_4. The statistics of the similarity submodules (of the corresponding $\mathbb{Z}$-modules generated by the root systems) is encapsulated in terms of DIRICHLET series generating functions, and some of the asymptotic properties are derived.

References for this joint work with R.V. MODDY can be found on the Los Alamos preprint archive[1] (under metric geometry).

Systems of halfspaces and constructions for fundamental domains

LUDWIG BALKE

The well known DIRICHLET-VORONOÏ construction produces a fundamental domain as intersection of halfspaces. Each of these halfspaces has as boundary a bisecting hyperplane between two points. There are natural geometries like $\mathrm{PSL}(2,\mathbb{R})$ in which bisectors exist but the DIRICHLET-VORONOÏ construction can not be applied since there exists no metric producing the bisectors.

In the talk of ANNA PRATOUSSEVITCH, the construction of THOMAS FISCHER for fundamental domains of certain discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ (or a finite cover) was explained. This method yields a fundamental domain as intersection of union of halfspaces whose boundaries are bisectors. I am interested in the question: Why does this construction work and how it is related to the classical DIRICHLET-VORONOÏ procedure?

[1] http://arxiv.org

I suggest the notion of halfspace systems as a framework in which both constructions can be described in a uniform manner. Two observations led to this notion. First, if we describe the DIRICHLET-VORONOÏ cell $D(x)$ as intersection of the halfspaces $H(x, y)$, $x \neq y$, and prove that this is a fundamental domain, we make essential use of the "triangle inequality" $H(x, y) \cap H(y, z) \subset H(x, z)$. The second observation inspired by FISCHER's construction is that in more general situations it is essential to ensure local finiteness, since this is not implied by the discreteness of the point set for which a fundamental domain is produced.

Non-rigidity degree of a lattice and rigid lattices

EVGENII P. BARANOVSKII, VIATCHESLAV P. GRISHUKHIN

VORONOÏ defined a partition of the cone of positive definite quadratic n-ary forms into L-type domains. Each L-type domain is an open polyhedral cone of dimension k, $1 \leq k \leq \frac{n(n+1)}{2}$. We say that a quadratic form f and the corresponding lattice $L(f)$ have *non-rigidity degree* k, if f belongs to an L-type domain of dimension k. A lattice and its forms of minimal non-rigidity degree 1 are called *rigid*.

We prove that the non-rigidity degree of a lattice equals to the corank of a system of equations connecting norms of minimal vectors of cosets of $2L$ in L. Using a list of 84 zone-contracted Voronoi polytopes in $\mathbb{R}^5$ given by P. ENGEL, we find 7 five-dimensional rigid lattices.

0-1 polytopes with many facets

IMRE BARANY

There exist n-dimensional 0-1 polytopes with as many as $(cn/\log n)^{n/4}$ facets. This is our main result. It answers a question of GÜNTER M. ZIEGLER. The construction showing this is random. In particular, we prove that the expected number of facets of a random 0-1 polytope in $\mathbb{R}^n$ with N vertices is at least $(c \log N)^{n/4}$ in the range $\exp(\log^2 n) < N < \exp(n/\log n)$. In the proof extensive use is made of a beautiful result of DYER, FÜREDI and MCDIARMID.

This is a joint work with ATTILA POR.

On a self dual 3-sphere of PETER MCMULLEN

JÜRGEN BOKOWSKI

We study an equifacetted self dual 3-sphere S_{McM} of PETER MCMULLEN, in particular its automorphism group $\mathcal{A}(S_{\mathrm{McM}})$ and its relation to the COXETER group H_4 of the 600-cell. A closely related equifacetted polyhedral 3-sphere (240-cell) with 240 facets and 120 vertices has the same automorphism group. Both these 3-spheres and the polar dual of the last one cannot occur as the boundary complex of a (convex) 4-polytope with $\mathcal{A}(S_{\mathrm{McM}})$ as their full Euclidean symmetry. It is an open problem, whether there exist one of these three 4-polytopes at all. Their combinatorial symmetry would differ from their Euclidean one within their whole realization space, similar to the example given in [2], see also [1]. Tackling these problems with methods from computational synthetic geometry [3] fail because of the large problem size. Therefore, a partial Euclidean symmetry assumption for the questionable polytope is natural. On the other hand, we show that even a certain subgroup of order 5 of the full combinatorial symmetry group $\mathcal{A}(S_{\mathrm{McM}})$ of order 1200 cannot occur as a Euclidean symmetry for MCMULLEN's questionable polytope.

This is a joint work with PHILIPPE CARA and SUSANNE MOCK.

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Tilings as templates for crystal structures

OLAF DELGADO FRIEDRICHS

Chemists and crystallographers often are concerned with what they call the "topology" of a molecule or crystal, where the term "topology" should be understood in its original, informal meaning as introduced by LISTING: "geometry without measurements".

One important question is the classification of all possible crystal networks with, for example, all vertices (representing atoms) incident to exactly 4 edges (representing bonds). Crystal networks, which are infinite periodic graphs, are extremely hard to deal with. In contrast, the combinatorial theory of tilings based on the so-called DELANEY-symbol provides an excellent basis for a systematic and complete treatment of spatial decompositions.

Several crystal structures, most notably certain so-called zeolites, can be readily interpreted as tiling structures. Recent results in tiling theory show that a much larger portion all known crystal networks can be obtained from a relatively simple class of tilings.

Two important open problems in this area are the following:

- How can we construct and characterize a canonical tiling corresponding to a given periodic point set or periodic network?
- What are strong necessary combinatorial conditions for tilings which lead to chemically plausible networks?

Some tiling problems and theorems

NIKOLAI P. DOLBILIN

One of the basic tiling problems is: given a finite set of polyhedra, does it admit at least one tiling. It was proved in 1966 that this problem is computationally undecidable.

We discuss here a solution of a more particular problem: given a convex polyhedron, can it tile space in a translative way? The Extension Theorem provided by the author gives necessary and sufficient conditions under which a well-defined finite polyhedral complex (a "k-corona") assembled with congruent copies of a given polyhedron admits the only isohedral tiling.

The Extension Theorem gives a way to get *all* possible regular tilings with the given polyhedron. The well-known results on fundamental domains in the case of a translation group of a COXETER group generated by mirrors follow from the Extension Theorem, too.

A theory of tilings — An introduction

ANDREAS W.M. DRESS

Based on the fundamental observation that every tiling of a simply connected n-manifold with a symmetry group with compact orbit space can be encoded by a finite set D, an action of $\Sigma_n := \langle \sigma_0, \sigma_1, \ldots, \sigma_n | \sigma_i^2 = 1 \rangle$ on D, and a map of D into the set of $(n+1) \times (n+1)$-COXETER-matrices satisfying certain compatibility conditions, a number of results relating in particular to two-dimensional and three-dimensional tilings and to problems motivated by chemistry and crystallography have been discussed, and a more detailed indication of how the observation above can be used to enumerate for instance all FULLERENE-type molecules with heptagons and pentagons, only, and with (proper) icosahedral symmetry, consisting of exactly 280 C-atoms (minimal size) was presented.

VORONOÏ zonotopes and DELAUNAY dicings

ROBERT M. ERDAHL

We revisit the duality between VORONOÏ polytopes which are zonotopes and DELAUNAY dicings which are dicings. The duality is made transparent by a new construction in which the VORONOÏ polytope is realized as the projection of a parallelopiped in a higher dimensional space, and the DELAUNAY tiling a section of a tiling by the same parallelopiped. This offers a very short, and new, proof of VORONOÏ's conjecture on parallelohedra for the case of zonotopes.

Cocircuit graphs and orientation reconstruction in oriented matroids

KOMEI FUKUDA

We consider the cocircuit graph of an oriented matroid, which is the 1-skeleton of the cell complex formed by the span of the cocircuits. We show that the cell complex is uniquely determined by the cocircuit graph if the oriented matroid is uniform. This answers an open problem posed by CORDOVIL, FUKUDA, and GUEDES DE OLIVEIRA. Furthermore we present a polynomial algorithm for the reconstruction problem, which verifies the correctness of input as well. As a corollary, the face lattice of every cubical zonotope in $\mathbb{R}^d$ is uniquely determined by its dual graph, and can be reconstructed in polynomial time. This partially answers a conjecture of MICHAEL JOSWIG which claims the uniqueness for every cubical polytope. By duality, the face poset of a simple arrangement of $(d-1)$-spheres in S^d is uniquely determined by the 1-skeleton and polynomially computable.

The reconstruction problem for the non-uniform case is also discussed.

This is a joint work with ERIC BABSON and LUKAS FINSCHI.

The fibrifold notation and classification for three dimensional space groups

DANIEL HUSON

We report on joint work whith JOHN H. CONWAY, OLAF DELGADO FRIEDRICHS and BILL THURSTON in which we introduce the concept of "fibrifolds" (fibered orbifolds) and show how it is used to obtain a nice notation and a new simple proof of the classification of space groups in the reducible case.

Torus actions, simple polytopes and coordinate subspace arrangements

TARAS E. PANOV

We show that the cohomology algebra of the complement of a coordinate subspace arrangement in m-dimensional complex space is isomorphic to the bigraded cohomology algebra of STANLEY-REISNER face ring of a certain simplicial complex on m vertices. (The face ring is regarded as a module over the polynomial ring on m generators.) Then we calculate the latter cohomology algebra by means of the standard KOSZUL resolution of polynomial ring. To prove these facts we construct an equivariant with respect to the torus action homotopy equivalence between the complement of a coordinate subspace arrangement and the moment-angle complex defined by the simplicial complex. The moment-angle complex is a certain subset of a unit poly-disk in m-dimensional complex space invariant with respect to the action of an m-dimensional torus. This complex is a smooth manifold provided that the simplicial complex is a simplicial sphere, but otherwise has more complicated structure. Then we investigate the equivariant topology of the moment-angle complex. The very interesting particular case of our constructions is an arrangement defined by the lattice of faces of a simple polytope P^n with m codimension-one faces. The corresponding simplicial complex is the dual to the boundary complex of P^n. In this case the above homotopy equivalence between the complement of the arrangement and the moment-angle complex can be interpreted as the orbit map for a free action of the group $\mathbb{R}^{m-n}$ on the complement of the arrangement. The quotient (i.e. the moment-angle complex) is a smooth manifold $\mathcal{Z}_P$ invested with a canonical action of the compact torus T^m with the orbit space P^n. For each smooth projective *toric variety* M^{2n} defined by a simple polytope P^n with the given lattice of faces there exists a subgroup $T^{m-n} \subset T^m$ acting freely on $\mathcal{Z}_P$ such that $\mathcal{Z}_P/T^{m-n} = M^{2n}$. The cohomology ring of $\mathcal{Z}_P$ is isomorphic to the cohomology ring of the STANLEY-REISNER face ring of P^n regarded as a module over the polynomial ring. In this way the cohomology of $\mathcal{Z}_P$ acquires a *bigraded* algebra structure with bigraded POINCARÉ duality, and the additional grading allows to catch the combinatorial invariants of the polytope.

Crystallographic space groups, classifications in low dimensions

WILHELM PLESKEN

In joint work with T. SCHULZ I have computed the number of affine classes of space groups in dimensions 5 and 6 using the package CARAT. The numbers are 222018 and 28927922 resp. CARAT can count, construct, recognize and interrelate space groups up to degree 6. It contains tables of the conjugacy classes of finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$, $n \leq 6$, and of BRAVAIS groups up to degree 6. It has various algorithms to proceed from there, such as splitting $\mathbb{Q}$-classes in $\mathbb{Z}$-classes, computing normalizers (OPGENORTH) and splitting $\mathbb{Z}$-classes into affine classes (ZASSENHAUS). Its philosophy is to have names for the groups before one has constructed them.

Fundamental domains in finite coverings of $\mathrm{SU}(1,1)$

ANNA PRATOUSSEVITCH

We describe a construction of fundamental domains for the action of a discrete subgroup of a finite covering G of the LIE group $\mathrm{SU}(1,1) = \mathrm{SL}_2(\mathbb{R})$ by left translations, generalizing the construction for the case $G = \mathrm{SU}(1,1)$ suggested by TH. FISCHER 1992. For cocompact fundamental domains with totally geodesic faces with respect to the LORENTZIAN metric on the LIE group G induced by the KILLING form. We give explicit descriptions of the fundamental domains obtained by the construction for some infinite series of discrete subgroups of finite coverings of $\mathrm{SU}(1,1)$ with growing degree of the covering.

The defect of admissible sets in a lattice

ANDREI M. RAIGORODSKII

Let $\Lambda \subset \mathbb{R}^n$ be an n-dimensional lattice in the Euclidean space $\mathbb{R}^n$ containing the origin O. Moreover, we suppose that Λ contains the n-dimensional integer lattice $\mathbb{Z}^n$ as a sublattice. Consider the set of unit coordinate vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \Lambda$. We denote by $\mathcal{E}$ the frame $O\ \mathbf{e}_1, \ldots, \mathbf{e}_n$. Let f be the largest possible number of vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_f}$ in the frame $\mathcal{E}$ that can be complemented to a basis of Λ. By the *defect of the frame* $\mathcal{E}$ *with respect to the lattice* Λ we mean the value $d(\mathcal{E}; \Lambda) = n - f$. Note that $d(\mathcal{E}; \Lambda)$ is bounded below by the smallest number of generating elements in the finite Abelian group $\Lambda/\mathbb{Z}^n$; moreover, there exists a frame $\mathcal{E}' = O\ \mathbf{e}'_1, \ldots, \mathbf{e}'_n$ such that the vectors $\mathbf{e}'_1, \ldots, \mathbf{e}'_n$ make up a basis of the lattice Λ and the quantity $d(\mathcal{E}'; \Lambda)$ is exactly equal to the number of generating elements in the quotient group $\Lambda/\mathbb{Z}^n$. Consider now an arbitrary sequence of such sets $\{\Omega^n_{\mathcal{E}}\}_{n=1}^{\infty}$ that $\pm\mathcal{E} \subset \Omega^n_{\mathcal{E}} \subset [-1,1]^n$ and that, moreover, the section of $\Omega^n_{\mathcal{E}}$ by the coordinate subspace $\mathcal{R}_{i_1,\ldots,i_k}$ of the variables $x_{i_1}, \ldots, x_{i_k}$ coincides with $\Omega^k_{\mathcal{E}}$ $\forall\, 1 \le i_1 < \cdots < i_k \le n$. (Note that the unit octahedron $\mathcal{O}^n_{\mathcal{E}}$, the unit ball $\mathcal{B}^n_{\mathcal{E}}$ or, say, the hyperbolic cross can be considered as examples of the above sets.) A set $\Omega^n_{\mathcal{E}}$ is said to be *admissible with respect to a lattice* Λ if $\Omega^n_{\mathcal{E}} \bigcap \Lambda = \{O, \pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\}$. In the case when the set $\Omega^n_{\mathcal{E}}$ is admissible with respect to the lattice Λ, the quantity $d(\mathcal{E}; \Lambda)$ is called the *defect of the admissible set* $\Omega^n_{\mathcal{E}}$ and is denoted by $d(\Omega^n_{\mathcal{E}}; \Lambda)$. We now define quantities $d_n(\Omega^n_{\mathcal{E}})$ and $d^*_n(\Omega^n_{\mathcal{E}})$, which depend only on the dimension and the set under consideration, as follows: $d_n(\Omega^n_{\mathcal{E}}) = max_{\Lambda}\, d(\Omega^n_{\mathcal{E}}; \Lambda)$, $d^*_n(\Omega^n_{\mathcal{E}}) = max^*_{\Lambda}\, d(\Omega^n_{\mathcal{E}}; \Lambda)$. Here in the case of $d_n(\Omega^n_{\mathcal{E}})$ the maximum is taken over all lattices Λ containing $\mathbb{Z}^n$ as a sublattice such that the set $\Omega^n_{\mathcal{E}}$ is admissible in the lattice Λ; in the case of $d^*_n(\Omega^n_{\mathcal{E}})$ the maximum is taken over all lattices Λ containing $\mathbb{Z}^n$ as a sublattice such that the set $\Omega^n_{\mathcal{E}}$ is admissible in the lattice Λ and the quotient group $\Lambda/\mathbb{Z}^n$ is cyclic, that is, the lattice Λ can be obtained from $\mathbb{Z}^n$ by an addition of a single vector, which means that there exists a vector $\mathbf{a} \in \mathbb{Q}^n$ such that $\Lambda = \langle \mathbb{Z}^n, \mathbf{a} \rangle_{\mathbb{Z}}$.

The present talk is dealt with exhibiting rather good (moreover, sharp in order as $n \to \infty$ for a sufficiently large class of cases) estimates for the values $d_n(\Omega^n_{\mathcal{E}})$ and $d^*_n(\Omega^n_{\mathcal{E}})$ when $\Omega^n_{\mathcal{E}}$ satisfies the conditions mentioned above. Note that the proofs of results involve the use of the combinatorial methods developed for the set covering problem and of an averaging procedure. Note, furthermore, that similar questions in small dimensions have been discussed (however, in a different notation) by T.W. CUSICK, L.J. MORDELL, R. BANTEGNIE, S.S. RYSHKOV.

Triangulations of Polytopes and Polytope Boundaries

JÜRGEN RICHTER-GEBERT

This talk reports on recent reseach work (jointly with A. BELOW and J. DE LOERA) on the algorithmic complexity of finding a minimal triangulation of a polytope or of a polytope boundary. We show that already for 3-polytopes it is $\mathcal{NP}$-hard to find a triangulation with a minimal number of simplices. The same holds for boundaries of 4-polytopes. The result on 3-polytopes is obtained by combining two major ingredients.

- special sub-structures in the face lattice of a 3-polytope may force that in a minimal triangulation certain inner diagonals have to occur.
- these inner structures can "block" each other so that the occurence of one implies the non-occurence of another. This blocking gives the possibility of forming elementary gadgets for logical operations. A certain construction shows that the structure can be modelled in a way that a reduction to the 3-SAT problem is possible.

The result on boundaries of 4-polytopes may be either obtained as a corollary of the previous result. However also a direct approach is possible that shows that finding a minimal triangulation of a 4-polytope boundary is already $\mathcal{NP}$-hard if only cubes, triangular prisms, and pyramids occur as facets.

Tension percolation on a triangular lattice

KONSTANTIN RYBNIKOV

Lattice percolation models play an important role in the study of glasses and ferro-magnetism. For example, the following problem is of considerable interest. Start with an infinite triangular lattice graph in the plane. How many edges must be removed on average so that the resulting graph can be longer support an equilibrium tension? This question is related to the crystalline-glass transition. More formally, if each edge is removed with probability p, what is the critical value of p, so that when exceeded the graph no longer has an infinite component which supports an equilibrium stress positive in each edge? This problem is somewhat similar to the problem of rigidity percolation. For example, every finite subgraph of a triangular (regular or generic) lattice is contained in a finite rigid subgraph; in rigidity percolation one inquires if the graph maintains this property after we remove each edge independently with probability p. Rigidity percolation was intensively studied by physicists and mathematicians over the last 20 years (DUXBURY, JACOBS, THORPE (1983, 1995, 1996, 1999), HOLROYD (1998)). The focus of this problem is the combinatorial rigidity in the plane, whereas the focus of our investigation of tension percolation is the behavior of the planar system in the space, since the property of an infinite graph to be able to support tension implies a sort of strong stability in three-space (CONNELLY, WHITELEY (1996)). We prove that for any value of $p > 0$ there is no equilibrium stress on the altered lattice T_p. Moreover, the complete relaxation of tension occurs in some finite non-random time almost surely. We conjecture that our result holds for a larger class of planar graphs, but does not hold for a spatial lattice based on a regular simplex. (This is a joint work with R. CONNELLY and S. VOLKOV.)

L-simplices of big relative volume and laminar planes in lattices

SERGEY S. RYSHKOV

Let G_2^n be a lattice of dimension $n \geq 3$ with the second perfect quadratic form F_2 as its metric form. Let G_3^8 be the lattice of dimension 8 with the third perfect quadratic form F_3 as its metric form. Let also be $O(F)$ an intersection of small enough neighbourhoods of F with the discriminate surface in the space of coefficients and $O(G)$ the corresponding set of lattices.
In this talk it will be shown that

i) If $n \geq 4$ then an L-subdivision of the lattice G_2^n has no laminar planes; all these lattices have such planes "with zero thickness";

ii) If $n = 4$ then every lattice from $O(G_2^4)$ has at least one laminar plane;

iii) If $n \geq 5$ then there exists a lattice from $O(G_2^n)$ which has at least one laminar plane;

iv) If $n \geq 5$ and n is odd then there exist a lattices from $O(G_2^n)$ which have at least one L-simplex of relative volume $\frac{n-1}{2}$;

v) The lattice G_3^8 and every lattice from $O(G_3^8)$ have no laminar planes.

On the combinatorics of zonotopal lattices

FRANK VALLENTIN

We provide a link between the theory of oriented matroids and the theory of zonotopal lattice tilings and lattice dicings. The main advantage of this approach is the strict separation between combinatorial and metrical data. Firstly, this link was investigated by GERRITZEN and LOESCH ([Ger82], [Loe90]).

Zonotopal lattices are defined as regular sublattices of $\mathbb{Z}^n$ which are embedded in the euclidean $\mathbb{R}^n$. The standard basis of $\mathbb{R}^n$ is not necessarily an orthonormal basis, but it must be an orthogonal basis. The definition of regularity is highly motivated by TUTTE's theory of regular chain groups ([Tut71]).

Let L be a zonotopal lattice. The key observation is the fact that lattice vectors of minimal support with coefficients in $\{-1, 0, +1\}$ are exactly the facet vectors of the lattice's DIRICHLET-VORONOÏ-polytope. Using FARKAS' lemma it follows that the DIRICHLET-VORONOÏ-polytope of L is the orthogonal projection of the cube $[-\frac{1}{2}, \frac{1}{2}]^n$ onto the subspace which is spanned by L.

In this setting two known results get transparent proofs: MCMULLEN's characterization of space tiling zonotopes ([McM75]) and ERDAHL's proof for VORONOÏ's conjecture in the special case of zonotopes ([Erd99]).

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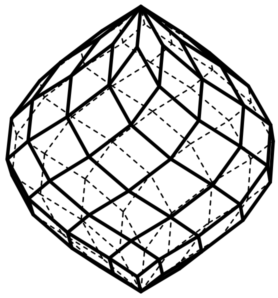
Problem session

Diameter of the dual of a cubical zonotope

KOMEI FUKUDA

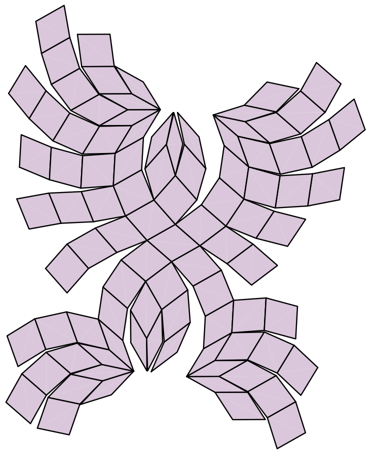


Let Z be a d-zonotope with n zones. We know that the diameter of the 1-skeleton of Z equals n.

Problem. If Z is a cubical zonotope (all facets of Z are $(d-1)$-cubes), then the diameter of its dual Z^* equals $n-d+2$?

This is true for $d \leq 3$. The distance between antipodal vertices in Z^* is exactly $n-d+2$, i.e. $\mathrm{diam}(Z^*) \geq n-d+2$. Is $n-d+2$ only realized by the antipodal vertices?

Equivalently, this problem can be studied in the setting of sphere arrangements or in the setting of oriented matroids. There might be a relationship to the HIRSCH-conjecture.



Hexagonal extensions

ANDREAS W.M. DRESS

Can every fullerene cap (5 pentagons and x hexagons) be extended to a sphere?

Fat DELONE simplices

ROBERT M. ERDAHL, KONSTANTIN RYBNIKOV

DELONE (Uspehi, 1937) asked if DELONE simplices of non-fundamental volume exist. COXETER (CJM, 1951, 1953) noticed that for $n \geq 9$ the existence of such simplices implies that for a given perfect form with fixed homogeneous minimum the lattice from which the minimal vectors are taken need not be unique (see also "Geometry of Numbers" by GRUBER and LEKKERKERKER (1987)). RYSHKOV (Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **33** (1973), 65–71) using the COXETER-BARNES lattices A_n^k, showed that in every dimension $n = 2\tau + 1$ there is a lattice with a DELONE simplex of relative volume τ.

Proposition. There is a metric for which simplex whose vertices are the columns of the following matrix is a DELONE simplex in the lattice $\mathbb{Z}^n$ ($n \geq 4$). Its volume is $n - 3$.

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -(n-3) \end{pmatrix}$$

The proof is based on the theory of dual systems of integer vectors. A good starting point might be the article *Dual systems of integral vectors (general questions and applications to the geometry of positive quadratic forms)* by R.M. ERDAHL and S.S. RYSHKOV (Mat. Sb. 182 (1991) **12**, 1796–1812).

Conjecture. $n - 3$ is the maximal relative volume of a DELONE simplex in dimension n.

Maximal lattice dicings

ROBERT M. ERDAHL, KONSTANTIN RYBNIKOV

A dicing D is an arrangement of hyperplanes in $\mathbb{R}^d$ whose vertices form a lattice $\Lambda(D)$. The arrangement is required to be invariant under all lattice translates. Obviously, the hyperplanes of such an arrangement dice the space into convex lattice polytopes.

A dicing is called maximal if one cannot add a new family of hyperplanes to the dicing without creating new vertices.

Conjecture. Every maximal dicing has a simplicial cell.

This conjecture has been verified for $d \leq 5$ by ERDAHL and RYSHKOV.

This problem is related to the theory of unimodular matrices and to the theory of (regular) oriented matroids. A good starting point might be the articles *On lattice dicing* by R.M. ERDAHL and S.S. RYSHKOV (European J. Combinatorics (1994) **15**, 459–481), and *"Maximal Unimodular Systems of Vectors"* by VLADIMIR DANILOV and VIATCHESLAV GRISHUKIN (*Europ. J. Combinatorics* (1999) **20**, 507–526).

Self-dual polytopes in $\mathbb{R}^4$ with only one facet-type

JÜRGEN BOKOWSKI

Classify all polytopes in $\mathbb{R}^4$ which are self dual and possess only one facet-type. In particular are there others than the simplex and the 24-cell?

Tile-, face-, edge- and vertex-transitive tilings in $\mathbb{E}^3$

OLAF DELGADO FRIEDRICHS

As demonstrated by HEESCH, there are infinitely many tile-transitive tilings of three-dimensional euclidean space. HEESCH's examples, of course, are not strictly convex, but fullfill a necessary local combinatorial condition for strict convexity, namely that each vertex has degree at least 3 relatively to each tile it is contained in. In the following, we shall call such tilings locally convex. It is straightforward to see that local convexity carries over to duals.

DRESS, HUSON and MOLNAR have established in 1993 that there are exactly 7 topological and 88 equivariant types of (2-)face-transitive locally convex tilings in $\mathbb{E}^3$. As in many applications locally non-convex tilings appear naturally, it would be interesting to know whether there are finitely or infinitely many topological types of non-degenerate, but not necessarily locally convex, face-transitive tilings. By non-degenerate, we mean the obvious thing here: each vertex is incident to at least 3 edges in the whole tiling, each edge is incident to at least 3 faces, each faces to at least 3 edges and, finally, each tile to at least 3 faces.

This question seems to be open even in a much stronger version:

> Are there finitely or infinitely many topological types of non-degenerate tilings in $\mathbb{E}^3$ which are tile-, face-, edge- and vertex-transitive at the same time?

There are two such types in the list by DRESS, HUSON and MOLNAR. A third one can be obtained by introducing a curved 2-face into each essential cycle of length 6 in the diamond net. Its tiles are tetrahedra with one additional (i.e. degree 2) vertex on each edge. Faces are paired in such a way that vertices of degree 2 are always matched with vertices of degree 3.

A computer enumeration has shown that there are exactly 9 topological types of such tilings with DELANEY symbols of no more than 20 elements, i.e., with no more than 20 orbits of the symmetry group of the tiling on its flag space.

Torus actions, polytopes, ans subspace arrangements
Some open problems

TARAS E. PANOV

Let K be a simplicial complex on $\{1, \ldots, m\}$, and $\mathbb{K}$ a field. The *face ring* of K is the quotient $\mathbb{K}(K) := \mathbb{K}[v_1, \ldots, v_m]/I$, $I = (v_{i_1}, \ldots, v_{i_k}, i_1 < \ldots < i_k, \{i_1, \ldots, i_k\}$ is not a simplex of $K)$. Define $U(K) = \mathbb{C}^m \setminus \bigcup(z_{i_1} = \ldots = z_{i_k} = 0, \{i_1, \ldots, i_k\}$ is not a simplex of $K)$. (Algebraic) cohomology of $k(K)$ and (topological) cohomology of $U(K)$ are the same:

Theorem. $\mathrm{Tor}_{\mathbb{K}[v_1,\ldots,v_m]}(\mathbb{K}(K), k) = H^*(U(K); \mathbb{K}) = H[\mathbb{K}(K) \otimes \Lambda[u_1, \ldots, u_m], d]; d(u_i) = v_i$, $d(v_i) = 0$.

Problem 1. Does the above identity holds for $\mathbb{Z}$ coefficients?

Suppose now that K is a simplicial sphere of dimension $n+1$. In this case $U(K)$ is homotopically equivalent to a smooth manifold, denoted by $\mathcal{Z}_K$. Let f_i denote the number of i-simplices of K; $(f_0, f_1, \ldots, f_{n-1})$ is called the f*-vector* of K. Define the h*-vector* $(h_0, h_1, \ldots, h_n)$ from

$$h_0 t^n + \cdots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}$$

The POINCARÉ duality for the Betti numbers of $\mathcal{Z}_K$ implies the well-knwon DEHN-SOMMERVILLE equations $h_i = h_{n-i}$, $i = 0, 1, \ldots, n$.

Problem 2. GLB for simplicial spheres: $h_0 \leq h_1 \leq h_2 \leq \ldots \leq h_{[\frac{n}{2}]}$.
In our setting, this is equivalent to certain inequalities for the Betti numbers of $\mathcal{Z}_K$. For instance, $h_1 \leq h_2 \Longleftrightarrow b^3(\mathcal{Z}_K) \leq \binom{f_0 - n}{2}$.

Problem 3. Find some analogues of the DEHN-SOMMERVILLE equations in the case when K is a simplicial manifold (or PL-manifold).

Tetrahedra tiling in 3-space

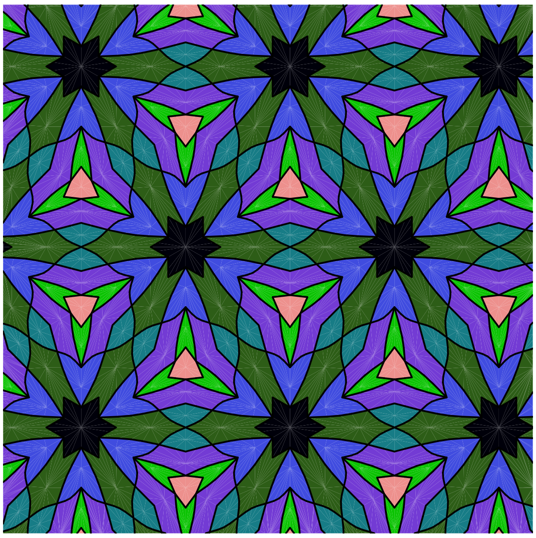
NIKOLAI P. DOLBILIN

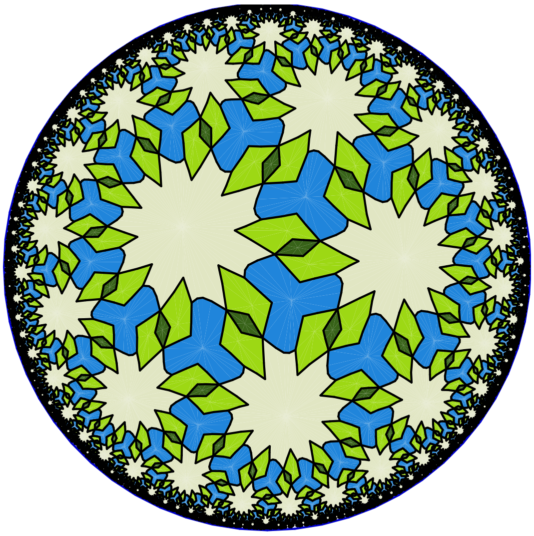
Is it possible to tile 3-space by tetrahedra whose angles are strictly greater than 60 degrees?

Software presentations

Daniel Huson — "2D-Tiler"

The successor of "Reptiles", a program to produce and visualize tilings in euclidean, hyperbolic and spherical 2-space.





Olaf Delgado Friedrichs — "3D-Tiler"

A program to produce and visualize tilings in euclidean 3-space.

Wilhelm Plesken — "CARAT"

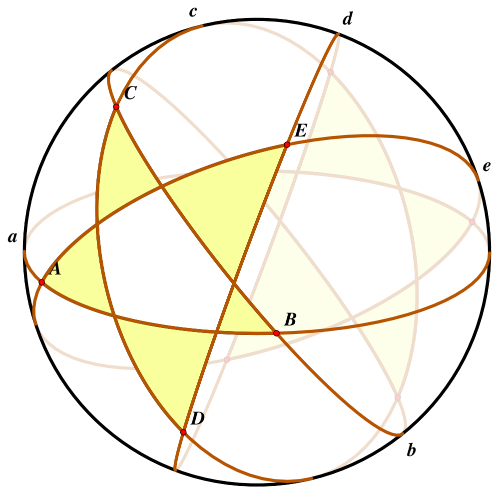
A software package to manage space groups and Bravais groups in low dimensions.

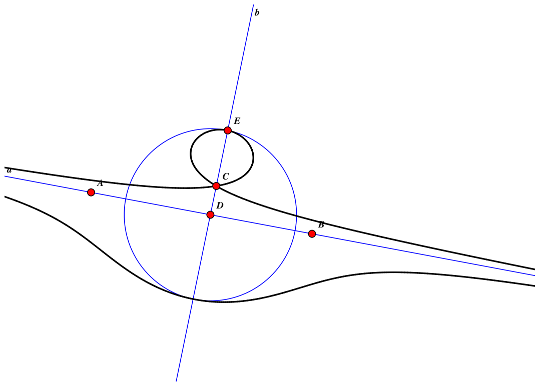
FRANK VALLENTIN — “DVL”

A small program which computes the DIRICHLET-VORONOÏ-polytope of a lattice.

JÜRGEN RICHTER-GEBERT — “Cinderella”

The famous geometry package.





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