Mathematisches Forschungsinstitut Oberwolfach

Tagunsbericht 16/2000

Diophantische Approximation

9.4.-15.4.2000

Die Tagung fand unter der Leitung von

Hugh L. Montgomery (Ann Arbor),
Yuri V. Nesterenko (Moskau),
Hans Peter Schlickewei (Marburg),
Robert Tijdeman (Leiden)

statt.
Die Schwerpunkte der Vorträge lagen in folgenden Gebieten der Diophantischen Approximation:

> Rationale Diophantische Approximation, Diophantische Gleichungen, Maßtheoretische Diophantische Approximation, Linearformen in Logarithmen Algebraischer Zahlen, Transzendenztheorie, Analytische Methoden in der Theorie der Transzendenten Zahlen.

Neben dem Schwerpunkt der Tagung "Diophantische Approximation" beschäftigte sich eine kleinere Gruppe von Teilnehmern mit einem Problemkreis aus dem Gebiet der Analytischen Zahlentheorie, der Selberg Klasse.
In sämtlichen Vorträgen wurden neue Ergebnisse zu den oben genannten Themen präsentiert. Insbesondere wurden gänzlich neue Resultate bezüglich quantitativen Fragestellungen bewiesen.
Neben dem ausgezeichneten Vortragsprogramm fand ein reger Austausch von Ideen der Teilnehmer statt, der sich bestimmt in der nachfolgenden Zeit in weiteren Fortschritten niederschlagen wird.
Die besonders angenehme und anregende Athmospäre am Institut wurde von allen Tagungsteilnehmern hervorgehoben. Die Tagungsleiter und die Teilnehmer danken hierfür dem Land Baden-Württemberg, Herrn Prof. Kreck, dem Direktor des Instituts, und dem gesamten Institutspersonal.

Vortragsauszüge

A relative Dobrowolski's lower bound over abelien extensions

Francesco Amoroso, Caen

(joint work with Umberto Zannier)
Let K be any number field and let L be any abelien extension of K. We prove the following theorem, which generalizes Dobrowolski's theorem:
Let $\alpha \in \overline{\mathbb{Q}}^* \setminus \overline{\mathbb{Q}}^*_{\mathrm{tor}}$. Then

$$\mathrm{h}(\alpha) \geq \frac{C(K)}{D} \left(\frac{\log(2D)}{\log\log(5D)} \right)^{-13},$$

where $D = [L(\alpha) : L]$ and $C(K) > 0$.

Sums and Products of Numbers with Restricted Partial Quotients

Steve Astels, Athens

For any set B of positive integers we let

$$F(B) = \{x \in \mathbb{R}; x = [a_0, a_1, \dots] \text{ with } a_0 \in \mathbb{Z} \text{ and } a_i \in B \text{ for } i \geq 1\}$$

We will discuss sums, differences, products and quotients of different $F(B)$'s.

On some exponential equations of S. S. Pillai

Michael A. Bennett, Urbana

If a, b and c are positive integers with $a, b \geq 2$, the diophantine equation

$$a^x - b^y = c$$

has at most two solutions in positive integers (x, y) and, if c is either "suitable small" or "suitable large" relative to a and b, at most one such solution. These results sharpen and generalize work of Pillai, Le, LeVeque and Terai. The principal technique is a routine application of linear form machinery, with a little twist.

A generalisation of the Khintchine-Groshev theorem for non-degenerate manifolds

Victor Beresnevich, Minsk

Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be monotonic, $||x||$ denote the distance from x to $\mathbb{Z}$. The Khintchine-Groshev theorem states that if the sum

$$\sum_{q=1}^{\infty} q^{n-1} \psi(q) \tag{1}$$

converges then for almost all $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ the inequality

$$||a_n y_n + \cdots + a_1 y_1|| < \psi(|\boldsymbol{a}|) \tag{2}$$

has only finitely many solutions $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, on the other hand if (1) diverges then for almost all $\boldsymbol{y} \in \mathbb{R}^n$ inequality (2) has infinitely many solutions $\boldsymbol{a} \in \mathbb{Z}^n$. We will deal with generalisations of this for the case of $y_1, \dots, y_n$ being functions of $d < n$ variables. ($\boldsymbol{y}$ lying on a submanifold of $\mathbb{R}^n$). Such generalisations are known for some classes of manifolds.
We prove an analoque of the convergence part of the Khintchine-Groshev theorem for any non-degenerate manifold.

Lower bounds for the number of rational points close to a smooth manifold

Vasilii Bernik, Minsk

M. Huxley found a few estimates for the upper bounds for the upper number of rational points close to a curve. Let $f(x)$ be a real function twice continuously differentiable for $0 \leq x \leq 1$ with

$$c^{-1} \leq f(x) \leq c, \quad c > 0$$

and let $R(Q)$ be the number of points $(\frac{a}{q}, \frac{b}{q})$ with a, b, q integers, $Q \leq q \leq 2Q$ with

$$|\frac{b}{q} - f(\frac{a}{q})| \leq Q^{-\mu-1}, \quad \mu > 0.$$

Then as Q tends to infinity, we have bounds

$$R(Q) \ll_{\varepsilon} \min(Q^{2-\frac{\mu}{2}}, Q^{2-\mu} + Q^{\frac{11}{6}}, Q^{2-\mu+\varepsilon} + Q)$$

We found the lower bounds for $R(Q)$

$$R(Q) \gg \begin{cases} Q^{2-\mu} & \text{if} \quad 0 < \mu < \frac{1}{3} \\ Q^{\frac{3}{2}} & \text{if} \quad \frac{1}{3} \leq \mu < \frac{1}{2} \\ Q^{3-3\mu} & \text{if} \quad \frac{1}{2} \leq \mu < 1 \end{cases}$$

There were use the ideas of metric diophantine approximation on manifolds and Khinchine transference principle.

Siegel-Shidlovsky theorem, after Yves André, and exponents of differential operators.

Daniel Bertrand, Paris

Most of the arithmetics in Yves André's new proof (Annals of Maths, to appear) of the theorem of Siegel and Shidlovsky on E-functions is contained in the following consequence of Chudnovsky's theorem on G-functions.

Theorem (André): *let F be an E-function lying in $\mathbf{Q}[[z]]$, and let $D_F \in \mathbf{C}(z)[d/dz]$ be the differential operator of minimal order annihilating F. Then D_F has at worst apparent singularities outside* $\{0, \infty\}$.

The proof is then based on the construction of an auxiliary function, using merely linear algebra, and the conclusion follows from Fuchs' formula on exponents, as generalized to the case of irregular singularities in joint works of F. Beukers, G. Laumon and myself.

Yves André's method also applies to the study of Chirsky's Euler-type functions, but not to its effective aspects. The formula on exponents, on the other hand, provides an effective version of Shidlovsky's lemma, as needed in the classical proofs of independence for E, G or 3-functions.

Effective Siegel's theorem for modular curves

Yuri F. Bilu, Basel

Let C be a projective curve defined over $\bar{\mathbb{Q}}$ and $x \in \bar{\mathbb{Q}}(C)$ a non-constant rational function. Assume that the pair (C, x) satisfies the assumption of Siegel's theorem (that is, either $\mathbf{g}(C) \geq 1$ or x has at least three poles). We say that *Siegel's theorem is effective for the pair* (C, x) if for any number field K such that (C, x) is definable over K, for any K-model of (C, x), and for any finite set S of places of K, the set

$$C(\mathcal{O}_S, x) := \{P \in C(K) : \ x(P) \in \mathcal{O}_S\}$$

of S-integral points is effectively bounded. It is known that Siegel's theorem is effective when $\mathbf{g}(C) \leq 1$ [Baker and Coates (1970)], and in some other cases.
Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, X_Γ the corresponding modular curve, and j the rational function on X_Γ defined by the modular invariant. It is well-known that the pair (X_Γ, j) is definable over a number field, and one may wonder whether Siegel's theorem is effective for it.

Theorem *Let Γ be a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *such that the assumption of Siegel's theorem holds for the pair (X_Γ, j). Then Siegel's theorem is effective for this pair.*

Particular cases were done by Kubert and Lang (1981) and myself (1995).
The proof involves Baker's method and elementary group theory. If the time permits, I will outline the proof for the most important particular case $\Gamma = \Gamma_0(p)$.
Notice that, up to conjugation, there are only 17 congruence subgroups Γ (containing the minus identity matrix) such that the assumption of Siegel's theorem does not hold for the pair (X_Γ, j). They include $\Gamma_0(N)$ with
$N \in \{1, 2, 3, 5, 7, 13\}$ and 11 more groups without standard names.

Linear independence of Gamma Values with Various Base Fields

Dale W. Brownawell, University Park

D.S. Thakur defined a function field Gamma function

$$\Gamma_q(z) = \frac{1}{z} \prod_{n \in \mathbb{F}_q[t], n \text{ monic}} (1 + \frac{z}{n})^{-1}$$

in analogy with the classical Euler Gamma function. Matt Papanikolas and I recently have shown that all $\overline{\mathbb{F}_q(t)}$-linear relations on $\Gamma_q(s)$, $s \in \mathbb{F}_q(t)$, follow from the known ones. Moreover there are no $\overline{\mathbb{F}_q(t)}$-linear relations between such $\Gamma_q(s)$ and $\Gamma_{q'}(s')$ when $q \neq q'$.

1. A new method for binary additive problems
2. Almost periodic functions

Jörg Brüdern, Stuttgart

A sequence $\mathcal{S}$ of natural numbers is *distributed* if for any $a \in \mathbb{Z}$, $q \in \mathbb{N}$ there is an asymptotic formula

$$\#\{s \in \mathcal{S} \cap [1, x] : s \equiv a \bmod q\} \sim \rho g(q, a) x$$

as $x \to \infty$. We normalize by $g(1, 0) = 1$ whence ρ is the density of $\mathcal{S}$. The series

$$\mathfrak{s} = \sum_{q=1}^{\infty} \sum_{a \bmod q} \Big| \sum_{b \bmod q} g(q, b) e(ab/q) \Big|^2$$

converges whenever $\rho > 0$. If $\rho\mathfrak{s} = 1$ we call $\mathcal{S}$ extremal. There are many examples of extremal sequences: arithmetic progressions, k-free numbers, more generally multiplicative sequences with positive density. They also have nice additive properties, depending on the following characterisation of extremal sequences.
Theorem 1. Let $Q(x) \to \infty$ as $x \to \infty$ with $1 \le Q \le \sqrt{x}$. Let $\mathfrak{m} = \{\alpha \in [0, 1] : \|q\alpha\| \le Q/x \Rightarrow q > Q\}$. Then

$$\int_{\mathfrak{m}} \Big| \sum_{s \in \mathcal{S} \cap [1, x]} e(\alpha s) \Big|^2 d\alpha = o(x).$$

From this result one readily derives an asymptotiv formula for the number of representations of an integer n as the sum $n = u + v$ with $u \in \mathcal{U}$, $v \in \mathcal{V}$ where $\mathcal{U}$ is distributed and $\mathcal{V}$ is extremal. One also finds asymptotic formulae for the number of k-tuples in an extremal sequence, and for the number of arithmetic progressions of length $r \ge 3$ in such a sequence. In the light of this, it is of interest to give a characterization of extremal sequences independent from the framework of the circle method. J.C. Puchta has recently found the following.

Theorem 2. A distributed sequence $\mathcal{S}$ is extremal if and only if $\rho > 0$ and its characteristic function is almost periodic in the space of arithmetic functions equipped with the semi-norm

$$||f||^2 = \limsup x^{-1} \sum_{n \le x} |f(n)|^2.$$

This implies, for example, that the intersection of two extremal sequences either has density 0 or is extremal. This can be used to construct further families of extremal sequences such as r-tuples of k-free numbers.

On integral-valued entire functions

Peter Bundschuh, Köln

We will explain fairly more than the following
Theorem (M. Welter). Let $q \in \mathbf{Z}\backslash\{0, \pm 1\}$ and suppose that the entire transcendental function f satisfies $f^{(\sigma)}(q^\nu) \in \mathbf{Z}$ for any $(\nu, \sigma) \in \mathbf{N}_0^2$. Then

$$(*) \qquad \tilde{\rho}(f) \ := \ \limsup_{r \to \infty} \frac{\log\log |f|_r}{\log^2 r} \ \geq \ \frac{1}{4 \log |q|}\,.$$

This is a special case of a result of the following type: Let K denote an algebraic number field with $d := [K : \mathbf{Q}]$, and suppose $(u_n)_{n=0,1,\ldots}$ to be an infinite sequence of distinct elements of K with controlled houses and denominators. Let f be an entire function satisfying $f^{(\sigma)}(u_\nu) \in K$ for any $(\nu, \sigma) \in \mathbf{N}_0^2$, again with controlled houses and denominators, and with $\tilde{\rho}(f) < (4\, d \log q)^{-1}$, then f is a polynomial. Here the real number $q > 1$ is a parameter having to do with the above-mentioned quantitative controls. - The proof uses Gel'fond's transcendence method.
It should be pointed out that $(*)$ can be shown to be best possible under the remaining hypotheses of the Theorem. For this, our method with Shiokawa [Arch.Math. 65, 32-35 (1995)] is used on which we reported here in 1993.

Approximation by algebraic numbers

Yann Bugeaud, Strasbourg

Let $n \geq 2$ be an integer. Forty years ago, Wirsing proved that every real number ξ not algebraic of degree $\leq n$ can be approximated at order $n/2$ (he obtained in fact a slightly better exponent) by infinitely many algebraic number numbers α of degree $\leq n$. We examinate wether one can impose various conditions on the heights and of the degrees of the approximants α. For instance, we show that the above result is still true when the α's are required to be of degree exactly n.

The Subspace Theorem and Transcendence

Pietro Corvaja, Udine

(After a joint work with Umberto Zannier)
The starting point of this work is a new proof of a theorem of Mahler concerning the transcendency of

$$\sum_{n=0}^{\infty} \alpha^{2^n}$$

where α is algebraic with $0 < |\alpha| < 1$.

This new proof makes use of the Subspace Theorem; roughly speaking, we exploit the fact that a power sum cannot be to close to an integer, unless it is itself an integer.
As a second instance we can prove that given a Laurent series $f(z) = \sum_{i=-k}^{\infty} a_i z^i$ with algebraic coefficients $(a_i)_i$, and an algebraic number q with $0 < |q| < 1$ the set of positive integers n such that

$$f(q^n) \in \mathbb{Z}$$

is finite, unless f is Laurent polynomial.
We obatained the following Transcendence Criterion:
Theorem: Let K be a number field; let $\alpha \in K^*$ and $c_1, c_2, \dots$ be a sequence in K^*. Let $m_1 < m_2 < \cdots$ be an increasing sequence of positive integers. Assume that

$$\mathrm{h}(c_i) = o(m_i)$$
$$\sup_N \limsup_p \frac{m_{p+N}}{m_p} = \infty$$

there exist $h > 0$ and $\varepsilon > 0$ s.t.

$$m_{i+h} > (1+\varepsilon)m_i$$

Consider the number $\gamma = \sum c_i \alpha^{m_i}$ and the function $f(X) = \sum_{i=0}^{\infty} c_i X^{m_i}$. If γ is algebraic then one may split $f(X)$ as a sum of polynomials

$$Q_l(x) = \sum_{i=j_l}^{j_{l+1}-1} c_i X^{m_i}$$

where $0 < j_0 < j_1 < \cdots <$ is a sequence of integers s.t. $j_{l+1} < j_l + h$ and

$$Q_l(\alpha) = 0 \text{ for } l > 0.$$

Moments of L-functions and symmetry

Brian Conrey, Stillwater

(with D. Farmer)
In these talks we describe conjectures for the general structure of moments of L-functions. Classically we know that

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^2 \mathrm{dt} \sim \log T \quad \text{(Hardy, Littlewood)}$$

and

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^4 \mathrm{dt} \sim \frac{1}{2\pi^2}\log^4 T \quad \text{(Ingham)}$$

More recently, Conrey and Ghosh conjectured that

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^6 \mathrm{dt} \sim \frac{42}{91}\prod_p (1-\frac{1}{p})^4(1+\frac{4}{p}+\frac{1}{p^2})\log^9 T$$

and Conrey and Gonek conjectured that

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^8 \mathrm{dt} \sim \frac{24024}{16!}\prod_p (1-\frac{1}{p})^9(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3})\log^{16} T$$

Then Keating and Smith applied methods of random matrix theory to obtain a full conjecture:

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^{2k}\mathrm{dt} \sim g_k a_k \frac{(\log T)^{k^2}}{\Gamma(1+k^2)}$$

where

$$a_k = \prod_p (1-\frac{1}{p})^{k^2} \sum_{j=0}^{\infty} \frac{d_k(p^j)^2}{p^j}$$

and g_k can be determined by computing moments of characteristic polynomials of unitary matrices:

$$\int_{U(N)} |\det(A - e^{i\theta}I)|^{2k}\mathrm{dH} \sim \frac{g_k N^{k^2}}{\Gamma(1+k^2)}$$

with $g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$ for integral k. Here $U(N)$ is the compact group of $N \times N$ unitary matrices and dH is the Haar measure.

We conjecture that for any primitive F of the Selberg Class that a similar formula will hold:

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^{2k}\mathrm{dt} \sim g_k a_k(F) \frac{(\log T)^{k^2}}{\Gamma(1+k^2)}$$

where g_k is as above and where $a_k(F)$ is an Euler product. We say that any L-function in t-aspect shows a “unitary” symmetry.

If one considers quadratic Dirichlet L-functions, then we conjecture that

$$\frac{1}{D^*}\sum_{|d|<D} L(\frac{1}{2},\chi_d)^k \sim g_k(s) a_k(\quad) \frac{(\log D)^{k(k+1)/2}}{\Gamma(1+k(k+1)/2}$$

where $D^* = \sum_{|d|<D}$ and $g_k(s) = (\frac{k(k+1)}{2})! \prod_{l=1}^{k-1} \frac{l!}{(2l)!}$ and

$$a_k = \prod \frac{(1-\frac{1}{p})^{k(k+1)/2}}{(1+\frac{1}{p})} \left(\frac{(1-\frac{1}{\Gamma p})^{-k} + (1+\frac{1}{\Gamma p})^{-k}}{2} + \frac{1}{p} \right)$$

The $g_k(S)$ is computing by integrating characteristic polynomials of Symplectes matrices with respect to the Haar measure.

A third example was presented for a family of L-functions which exhibit an orthogonal symmetry.

This idea of families was introduced by Katy and Sarnak', Conrey and Farmer realized that the symmetry type could be determined from the structure of moments.

Convex theorems and the Selberg Class

Brian Conrey, Stillwater

(with D. Farmer)

The elements of the Selberg Class have degree d associated to them. It is known that if $d > 0$ then $d \geq 1$. All of the elements of the Selberg Class of degree 1 have been classified. In these talk we present a first attempt to classify degree 2 elements.

Suppose that $F \in \mathcal{S}^*(2)$ with a functional equation

$$(*) \qquad \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)F(s) = \Phi(s) = \pm\Phi(R-S).$$

The object is to try to show that

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz) \in \mathcal{S}_k(\Gamma_0(N))$$

where $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. We are forced to assume an Euler product condition stronger what is normally considered in $\mathcal{S}$. Namely, we let $F(s) = \prod_p F_p(s)$ where $F_p(s) = (1 - \frac{a_p}{ps} + p^{k-1-2s})^{-1}$ for $p \not| N$, $F_p(s) = (1 - p^{k-1-2s})^{-1}$ if $p||N$ and $F_p(s) = 1$ if $p^2|N$. We show, under these assumptions, that $f(z) \in \mathcal{S}_k(\Gamma_0(N))$ for $5 \leq N \leq 17$, $(N \neq 13)$ and $N = 23$.
(Note: Hecke's convex theorem gives this result for $1 \leq N \leq 4$. Weil had considered and given a convex theorem where an assumption of $(*)$ plus a functional equation for $F_\chi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\chi(n)} n^s$ with primitive mod q where $(q, N) = 1$. We replace twists by Eulerproducts.)

Small solutions of quadratic diophantine equations

Rainer Dietmann, Basel

We consider quadratic diophantine equations, which take the shape

$$(*) \qquad Q(x_1, \dots, x_s) = 0$$

for $Q(X_1, \dots, X_s) \in \mathbb{Z}[X_1, \dots, X_s]$ of degree at most 2. Let H be an upper bound for the absolute values of coefficients of Q, and assume that the homogeneous quadratic part of Q is non-singular. We prove for all $s \geq 3$ the existence of polynomial bounds $\Lambda_s(H)$ with the following property: If $(*)$ has a solution $\boldsymbol{x} \in \mathbb{Z}^s$ at all, then it has one with

$$|x_i| \leq \Lambda_s(H) \quad (1 \leq i \leq s).$$

For $s = 3$ and $s = 4$ no polynomial bounds $\Lambda_s(H)$ has been known so far, and for $s \geq 5$ we could quite significantly improve the existing bounds.

Diophantine approximation on curves and Hausdorff dimension

Maurice M. Dodson, Heslington, Yorkshire

Only estimates are known for the Hausdorff dimension of sets of simultaneously very well approximable points on smooth curves. The dimension of 'very closely' approximable points on the unit circle has been determined recently (using Phytagorean triples and ubiquity) and progress has been made for other curves. The results depend on the nature of the curve and constrast with the dual form of approximation in which the Hausdorff dimension of the very well approximable points depends solely on the degree of the approximation.

The Remak height: conjugate algebraic numbers on two circles

Arturas Dubickas, Vilnius

In a joint work (with Chris Smyth) we investigate certain height of a polynomial, which was first appeared in a paper of Robert Remak (1952). This quantity is bounded above by the Mahler measure and below (essentially) by its square root. It turns out that the upper bound is an equality when the algebraic number (polynomial) is either cyclotomic or ± a Salem number. We also prove an "if and only if" type result for the lower bound to become an equality. This involves certain Pisot numbers, and also algebraic numbers closely related to Salem numbers.

Points on subvarieties on tori

Jan-Hendrik Evertse, Leiden

This is joint work with Hans Peter Schlickewei, Marburg.

Let X be a subvariety of the N-dimensional torus $\mathbb{G}_m^N$ defined over an algebraically closed field K of characteristic 0. Let Γ be finitely generated subgroup of $\mathbb{G}_m^N(K) = (K^*)^N$. Let $\overline{\Gamma} = \{\boldsymbol{x} \in \mathbb{G}_m^N(K) : \exists\, m \in \mathbb{N} \text{ with } \boldsymbol{x}^m \in \Gamma\}$ be the group of division points of Γ. We prove the following quantitative version of a theorem of Laurent (which was in turn a special case of one of Lang's conjectures).

Theorem 1: $X \cap \overline{\Gamma}$ is conatained in some finite union $\mu_1 M_1 \cup \cdots \cup \mu_t M_t$, where each $\mu_i M_i$ is a translate of an irreducible algebraic subgroup of $\mathbb{G}_m^N$ which is contained in X, and where

$$t \le c(n,d)^{r+1} \quad \text{with } n = \dim X,\ d = \deg\, X,\ r = \operatorname{rank} \Gamma,\ c(r,d) = 2^{2^{6d\binom{n+d}{d}}}.$$

(Here we embed $\mathbb{G}_m^N$ into $\mathbb{P}^N$ by means of $i : (x_1, \ldots, x_N) \longmapsto (1 : x_1 : \cdots : x_N)$ and define $\deg X := \deg \overline{i(X)}^{\mathrm{Zar}}$).

Let $V(n,d)$ be the collection of subvarieties of $\mathbb{G}_m^N$ defined over K of dimension n and degree d. Two varieties $X_1, X_2 \in V(n,d)$ are called $\overline{\Gamma}$-equivalent if we have $X_2 = \boldsymbol{\mu} * X_1 = \{\boldsymbol{\mu} * \boldsymbol{x}_1 : \boldsymbol{x}_1 \in X_1\}$ for some $\boldsymbol{\mu} \in X_1$. Then we have

Theorem 2: For every finitely generated subgroup Γ of $\mathbb{G}_m^N(K)$, there are finitely many $\overline{\Gamma}$-equivalence classes of varieties in $V(r,d)$ such that the following holds:

if $X \in V(r,d)$ lies outside these equivalence classes than there are subvarieties $Y_1, .., Y_t$ of X with $\dim Y_i = n-1$, $\deg Y_i \le d^2$, $t \le d^2 2^{d^2\binom{n+d}{d}^2}$, such that

$$X \cap \overline{\Gamma} \subseteq Y_1 \cup \cdots \cup Y_t.$$

Arithmetic of toric deformations

Roberto Ferretti, Zürich

The approach of Faltings-Wüstholz for diophantine approximations of hyperplane cuts on non-linear projective varieties, leads naturally to the study of the "degree of contact. This is a sort of multiplicity, first introduced by Mumford in the context of geometric invariant theory. We show how to compute this quantity using the combinatorial techniques of toric deformations of Kaponov-Sturmfels-Zelevinsky. Further, we apply them in order to find systems of diophantine approximations that are out of reach of the Schmidt's Subspace Theorem. Moreover, we show how these computations may even give new results in Arakelov Geometry.

On an exponential sum related to cryptography

John Friedlander, Toronto

We discuss joint work with R. Canetti, S. Konyagin, P. Liemann and I. Shparlinksi on the sum

$$S = \sum_{x=1}^{t} |\sum_{y=1}^{t} e_p(a\theta^y + b\theta^{xy})|^4$$

where p is a prime, θ is an element of $\mathbb{F}_p^*$ of order t, a and b are in $\mathbb{F}_p$ and both not zero. We obtain the bound $S \ll pt^{11/3}$ which improves the trivial bound provided $t > p^{3/4+\varepsilon}$.

We apply the result to study the uniform distribution $\mod p$ of the triples $(\theta^x, \theta^y, \theta)$ where x and y run through various interesting subsets of the period and also to study the similar question for the RSA sequences of pseudorandom numbers.

On index form equations

Kalman Győry, Debrecen

Index form equations (IFE's) play an important role in algebraic number theory. In 1976 we developed a method for solving effectively such equations. We first reduced the IFE to an appropriate system of unit equations (UE's) and then we applied Baker's method to derive bounds for the solutions of these UE's. This general approach was later successfully combined with other results on UE's to give upper bounds for the number of solutions, and to solve completely certain concrete IFE's. Recently we gave a significant refinement of the method by reducing IFE's to certain special UE's which have much fewer unknown exponents. This made it possible to improve considerably the previous bounds on the sizes and the number of solutions of IFE's. Further, this enabled us to solve concrete IFE's in any number field of degree ≤ 5. In my talk the refined version of the method and some of its applications will be presented.

Lower bounds for the difference of almost perfect powers

Lajos Hajdu

The results presented in the talk are joint with Yann Bugeaud.
Let a, b, x, y, n and m be non-zero integers such that

$$n \geq 2, m \geq 2, |y| \geq 2 \text{ and } ax^n \neq by^m. \tag{3}$$

The first lower bound independent of x and y for $|ax^n - by^m|$ was provided by Shorey (1980). However, his bound was not completely explicit. The first explicit lower bound in this direction is due to Turk (1986), but only in the special case $(a, b) = (1, 1)$, i.e. concerning $|x^n - y^m|$.
Using a result of Brindza, Evertse and Győry (1991) for bounding the solutions of some diophantine equations in terms of the discriminants, Bugeaud (1996) proved that if x, y, n and m are as in (3) then

$$|x^n - y^m| \geq m^{2/5n} n^{-5} 2^{-6-42/n}.$$

In our talk a similar result in the general case (i.e. for arbitrary a and b) is presented. We note that our estimate is sharper than the one of Bugeaud even for $(a, b) = (1, 1)$. Moreover, we provide an explicit lower bound for $|ax^n - by^m|$ which is independent of y and m. The latter estimate improves and extends the similar results of Turk (1986) and Bugeaud (1996).
As a natural application, we also mention some straightforward consequences of our results concerning Pillai's equation.

Diophantine Approximation and the Calculation of some volumes

Stefan Kühnlein, Karlsruhe

Let $C \subseteq \mathbb{R}^2$ be the cone $\{(x,y) | \, x \geq 0, \, |y| \leq x\}$ and $F : C \to \mathbb{R}_{\geq 0}$ the homogenous function $F(x, y) = |yx^2|^{2/3}$. For $a \in \mathbb{R}$ let Λ be the lattice generated by $(0, y)$ and $(1, a)$. Then, if a is irrational algebraic, we have

$$(*) \qquad \mathrm{vol}(\{(x,y) \in C \mid F(x,y) \leq 1\}) = \lim_{r \to \infty} N(r)/r^2,$$

where $N(r) = \sharp\{(x,y) \in \Lambda \cap C \mid F(x,y) \leq r^2\}$. The proof uses the Thue-Siegel-Roth theorem. The theorem is a version of Dirichlet's expanding domains principle for an unbounded region. For non-algebraic a the right hand side of $(*)$ can be any number $\geq LHS$, including (of course) infinity.

Approximation by algebraic numbers with large degree

Michel Laurent, Marseille

We extend a well known theorem of Wirsing about the approximation of a complex number by algebraic numbers with bounded degree. In some sense we allow variable degrees in the statement. The depency upon the degree is fairly precise and reduces exactly to Wirsing for constant degrees.

Bounds for the number of simultaneous rational approximations

Helmut Locher, Marburg

Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ be algebaic numbers such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. and let $\delta > 0$. In 1970 W.M. Schmidt proved that

$$|y\alpha_i - x_i| < y^{-(1/n)-\delta} \quad y, x_1, \ldots, x_n \in \mathbb{Z}, y > 0, \tag{1}$$

has only finitely many solutions. So far, the problem of proving upper and lower bounds for the number of solutions of (1) is completely open. In 1991 W.M. Schmidt was able to prove upper and lower bounds for the number of subspaces containing the solutions of (1).
We consider the to (1) dual problem

$$|\alpha_1 y_1 + \cdots \alpha_n y_n - x| < y^{-n-\delta},\ x, y_1, \ldots, y_n \in \mathbb{Z},\ y = \max\{|y_1|, \ldots, |y_n|\} > 0 \tag{2}$$

We will present analogous results about (2) which show that even (1) and (2) are dual in a qualitative sense (Khintchine's transference principle) they are no longer completely dual in a quantitative sense.

The size of the coefficients of cyclotomic polynomials

Helmut Maier, ULm

Let $\Phi_n(z) = \sum_{m=0}^{\varphi(n)} a(m,n) z^m$ be the n-th cyclotomic polynomial,

$$\begin{aligned} A(n) &= \max |a(m,n)| \\ a(m) &= \max_n |a(m,n)| \end{aligned}$$

We represent results of Montgomery-Vaughan and Bachman on $a(m)$ and own results on $A(n)$. The results of Montgomery-Vaughan and Bachman give asymptotics for $\log a(m)$.
Our results imply that $n^{\varepsilon(n)} \le A(n) \le n^{\psi(n)}$ for almost all n, where $\varepsilon(n) \to 0$ and $\psi(n) \to \infty$ for $n \to \infty$.
A more recent result states that for almost all integers with
$\omega(n) \ge C \log\log n$ we have

$$A(n) \ge \exp((\log n)^{c/2-\varepsilon}).$$

Counting points with multiplicatively dependent coordinates on a curve

David Masser, Basel

This is joint work with Bombieri and Zannier.
Let C be a curve in affine n-space defined over a field K of characteristic zero. Under reasonable assumptions we prove that there at most finitely many points on C where coordinates are non-zero and generate a (non-fixed) multiplicative group of rank at most $n-2$. This extends earlier work of the author for number fields K. We also discuss what happens with the rank $n-1$.

On the equation $\frac{x^n-1}{x-1} = y^q$.

Maurice Mignotte, Strasbourg

This equation was first studied by Nagell and Ljunggren who solved completely the cases $q = 2$ and $3|n$, $4|n$. Then Shorey and Tijdeman, using Baker's theory, obtained finiteness results under various hypotheses.

In this talk, we shall present complete solutions of different cases. This work was obtained in collaberation with Y. Bugeaud, G. Hamot, Y. Roy and T. Shorey. The most striking, obtained with Y. Bugeaud, is the following: among the numbers with in decimal notation 11, 111,..., 11...11,... none is a prime power; this solves a conjecture 50 years old.

Beyond Pair Correlation

Hugh L. Montgomery, Michigan

Holdston and Montgomery showed that if the Riemann Hypothesis is true then the Pair Correlation Conjecture ($F(\alpha,T) \sim 1$ as $T \to \infty$ uniformly for $1 \le \alpha \le A$) is equivalent to the assertion that

$$\int_1^x (\psi(\kappa+h) - \psi(\kappa) - h)^2 \mathrm{d}\kappa \sim hx \log x/h$$

for $x^\varepsilon \le h \le x^{1-\varepsilon}$. In joint work with K. Sundararajan is shown that a sharp quantitative form of the prime k-tuple hypothesis suggests that

$$\int_1^x (\psi(\kappa+h) - \psi(\kappa) - h)^k \mathrm{d}\kappa \sim (m_k + o(1))X(h \log X/h)^{k/2}$$

for any fixed positive integer k. Where $m_k = \frac{k!}{(k/2)!2^{k/2}}$ if k is even, and $m_k = 0$ if k is odd. Since there are the moments of the normalized normal variable, the consequent conjecture is that $\psi(\kappa+h) - \psi(\kappa)$ is approximately normally distributed with mean $\sim h$ and variance $\sim h \log x/h$.

Greedy Sums of Distinct Squares

Hugh L. Montgomery, Michigan

This is a report on joint work with Ulrike Vorhauer (Universität Ulm). Let s_1^2 be the largest square not exceeding n, let s_2^2 be the largest square not exceeding $n - s_1^2$, and so on, so that

$$n = s_1^2 + s_2^2 + \cdots + s_r^2.$$

The summands are strictly decreasing for the remainders ≥ 9, but

$$\begin{aligned}
8 &= 4+4\\
7 &= 4+1+1+1\\
6 &= 4+1+1\\
5 &= 4+1\\
4 &= 4\\
3 &= 1+1+1\\
2 &= 1+1\\
1 &= 1\\
0 &= 0
\end{aligned}$$

thus the summands are distinct in 4 of these 9 cases. Following Mike Shepard, who first considered these numbers, we say that n is a "greedy sum of distinct squares" if the summands are distinct. One might expect that this set has density 4/9, but the numerical evidence of Shepard suggests an asymptotic density of 1/2. We show that the asymptotic density does not exist. Instead, there is a non-constant continuous function of period 1 such that

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} \frac{A(4\exp(2^{k+\kappa}))}{4\exp(2^{k+\kappa})} = f(\kappa)$$

for any fixed κ, where $A(u)$ denotes the number of the greedy sums of distinct squares not excedding u. The proof depends on an analysis of a solution of a non-linear difference-differential equation.

Mahler's Method (after Drew Vandeth)

Alf van der Poorten, Sydney

A theorem of Paul-Georg Becker had generalised Nishioka's work on Mahler's method in transcendence theory by dealing with transformations T, algebraic over $\mathbb{Q}(z)$, and holomorphic in a nbhd of the origin and $\mathrm{ord}_{z=0}T \geq 2$, generalising $z \to z^t$. The theorem provides a condition whereby $\mathrm{ord}_{z=0}T$ must exceed some function of the degrees of the defining polynomial of T and of the functional equation $P(z, g(z), g(Tz))$ satisfied by transcendental function g in order that one may conclude that $g(\alpha)$ is transcendental for appropriate algebraic α near the origin.
In his PhD thesis (Macquorie University, Sydney 2000) Drew Vandeth points out that Becker's result deals with the case $\tau > 1$: here $g(z) = \sum a_n z^n$ and $h(a_n) < n^z$. He refines underlying results of Gramain, Mignotte and Waldschmidt (Acta Arith., 1986), in order to obtain the other half of Becker's theorem - to iost that using $0 < \tau < 1$. In that case $g(\alpha)$ is transcendental without further condition on $\mathrm{ord}_{z=0}T$, implying the proof of the Mahler-Manin conjecture à la that of equipe Stéphanoise.

Normalized heights on semi-abelian varieties

Patrice Philippon, Paris

(Joint work with Simon David)
We define normalized heights on semi-abelian varieties (extensions of abelian varieties by torii) and establish a Bogomolov type property for this height, determining the torsion subvarieties.
We also prove a geometric lower bound for the heights of subvarieties.

Counting translates in Lang's conjecture

Gaël Rémond, Paris

I deal with the following setting: let A be an abelian variety over $\bar{\mathbb{Q}}$ and Γ a finite rank subgroup of $A(\bar{\mathbb{Q}})$. The objects of study are intersections $X(\bar{\mathbb{Q}}) \cap \Gamma$ where X is a subvariety of A.
For example, the simplest non-trivial case is that of a curve $X = C$ defined over a number field K with $A = \mathbb{R}Jac(C)$ and $\Gamma = A(K)$: here $X(\bar{\mathbb{Q}}) \cap \Gamma = C(K)$ and Faltings' result (Mordell's conjecture) is that $C(K)$ is finite. I focus on quantitative versions of finiteness statements (*e. g.* upper bound for $\#C(K)$).
In the general case, the finiteness result (as conjectured by Lang and proven by Faltings) asserts that $X(\bar{\mathbb{Q}}) \cap \Gamma$ is contained in a finite union of sets $(x + B)(\bar{\mathbb{Q}})$ where $x \in X(\bar{\mathbb{Q}}) \cap \Gamma$, B an abelian subvariety of X and $x + B \subset X$.
My main result gives an upper bound for the number of such translates of abelian subvarieties. Features of this bound are first that it is independent on the height of X; the group Γ occurs only through its rank; moreover the bound is monomial in the degree of X with an explicit exponent and it should be

possible to compute explicitly the multiplicative constant (it involves a work in progress by David and Philippon).
It might be of interest to note that, even in the case of curves mentionned above, the method gives a more explicit bound for $\#C(K)$ than was previously known (as in Bombieri's paper on Mordell's conjecture). Proofs use euclidean geometry in $\Gamma \otimes \mathbb{R}$, with Néron-Tate metrics, an idea that can be traced back to Mumford's paper (1965) and which Vojta used to reprove Mordell's conjecture. Results of both authors (for curves) are generalised to higher dimensional varieties, partly by making Faltings' arguments for these explicit.

Interpolation fomulas and auxiliary functions

Damien Roy, Ottawa

The purpose of an interpolation formula is to control the growth of an analytic function in terms of the values of the function and some of its derivatives on a given set of points. This is usefull in transcendental number theory in connection with the so-called auxiliary functions. This talk is centered on a new interpolation formula for "semi-cartesian products". We discuss its relevance to the Schwarz lemma and conjectures about the more general situation. We also present applications of this interpolation formula, notably on a construction of auxiliary function related to the four exponentials conjecture.

On Siegel's lemma

Andrzej Schinzel, Warszawa

Let for a vector $\boldsymbol{n} = [n_1, \dots, n_k] \in \mathbb{Z}^k :\ \mathrm{h}(n) = \max |n_i|$ and let

$$c(k) = \sup_{\boldsymbol{n} \in \mathbb{Z}^{k+1} - \{0\}} \inf_{\boldsymbol{x} \in \mathbb{Z}^{k+1} - \{0\}} \frac{\mathrm{h}(\boldsymbol{n})}{\mathrm{h}(\boldsymbol{n})^{1/k}}$$

The following results will be proved.
Theorem:

$$c(k)^{-k} = \inf \Delta(O_{\alpha_1, \dots, \alpha_{k-2}}),$$

where $O_{\alpha_1, \dots, \alpha_{k-2}}$ is a generalized octahedron given by

$$|x_i| \le 1 \quad (1 \le i \le k), \qquad |\sum_{i=1}^{k-2} \alpha_i x_i + x_{k-1} + x_k| \le 1$$

and $(\alpha_1, \dots, \alpha_{k-2}$ runs through all $(k-2)$-tuples of rational numbers from the interval $[0, 1]$.
Corollary 1 (Chatadus-Schinzel) $c(2) = \sqrt{\frac{4}{3}}$
Corollary 2 (Aliev) $c(3) = \sqrt[3]{\frac{27}{19}}$

Mahler, Masser, S-units and Mixing

Klaus Schmidt, Wien

Let $T : \gamma \to T_\gamma$ be a measure-preserving action of a countable group Γ on a probability space (X, S, μ). The action T is k-mixing if, for all $B_1, \dots, B_k \in S$, the sets $T_{\gamma_1}B_1, T_{\gamma_2}B_2, \dots, T_{\gamma_k}B_k$ become independent as $\gamma_i, \gamma_i^{-1}\gamma_i \to \infty$ forall i, j.

$$(i.e. \lim_{\substack{\gamma_i \to \infty \\ \gamma_i^{-1}\gamma_j \to \infty}} \mu(\cup_{i=1}^k T_{\gamma_i}^{-1} B_i) = \prod_{i=1}^k \mu(B_i))$$

For $\Gamma = \mathbb{Z}$ it is not known if 2-mixing implies 3-mixing.

In 1978 Ledrappier gave a simple example of a $\mathbb{Z}^2$-action by automorphisms of a compact abelien group X which is 2-mixing, but not 3-mixing.

Further analysis of the mixing behaviour of such actions revealed an intimate connection with a result of Mahler (1935), an unpublished theorem by Masser (1985), the recent S-unit theorem by Evertse, Schlickewei and Schmidt, and a conjecture about linear relations in fields of characteristic $p > 0$.

Coding Theory and Uniform Distribution

Maxim Skriganov, Kiel

We show that every uniformly distributed point sets have a rich combinatorial structure. Namely, we show that such sets can be completely characterized as maximum distance separable codes with respect to a non-Hamming metric in the vector space over finite field.

Developing such an approach William Chen and the author could recently give explicit constructions of point distributions in all dimensions which have minimal order (Roths bound) of the L_2-discrepancy. Untill now, explicit constructions of such distributions were known only in dimensions $n = 2$.

Transcendence of a certain power series

Taka-aki Tanaka, Yokohama

In what follows, let α be an algebraic number with $0 < |\alpha| < 1$ and d an integer greater than 1.

Theorem. *Let $\{r_k\}_{k \geq 0}$ be a sequence of positive integers such that $\lim_{k \to \infty} r_{k+1}/r_k = d$. Suppose that there exists a positive number M such that $r_{k+1} \geq dr_k - M$ for all $k \geq 0$. Let $f(z) = \sum_{k=0}^{\infty} z^{r_k}$. Then the number $f(\alpha)$ is transcendental.*

Example 1. The number

$$\sum_{k=0}^{\infty} \alpha^{kd^k}$$

is transcendental.

Example 2. The numbers

$$\sum_{k=0}^{\infty} \alpha^{2kd^k + (-d)^k}, \quad \sum_{k=0}^{\infty} \alpha^{[\omega d^k + \eta]}, \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha^{k \cdot \binom{2k}{k}}$$

are transcendental, where $\omega > 0$ is irrational, $\eta \geq 0$, $[x]$ denotes the largest integer not exceeding a real number x, and $\binom{m}{n}$ is the binomial coefficient.

Norm Form Inequalities

Jeff L. Thunder, DeKalb

Let $F(\boldsymbol{X}) \in \mathbb{Z}[\boldsymbol{X}]$ be a decomposable form in n variables of degree d, and consider the inequality

$$(*) \qquad |F(\boldsymbol{X})| \leq m$$

for m some positive parameter. We will discuss how well the number of integral solutions to $(*)$ is approximated by the total volume of all real solutions, with special attention paid to the case when $d = n + 1$.

Approximation measures for logarithms of algebraic numbers

Carlo Viola, Pisa

We present new applications of a method introduced by Rhin and Viola in 1996, based on the study of suitable transformations, and of related permutation groups, acting on integrals of Beukers' type. These applications are:

i) An improvement on the best irrationality measure of $\zeta(3)$. We obtain $\mu(\zeta(3)) < 5.513891$ (joint work with G. Rhin).

ii) New results on $\mathbb{Q}(\alpha)$-irrationality measure of $\log \alpha$, for several algebraic numbers $\alpha \in \mathbb{C}$ (joint work with F. Amoroso).

Rigidity in the Selberg Class. I. Continuous families

Ulrike Vorhauer, Ulm

This is joint work with Eduard Wirsing, Ulm.
Many deep and interesting analytic number theory are connected with a class of Dirichlet series $\mathcal{S}$ that Atle Selberg defined in 1989 essentially by a certain type of functional equation similar to the Riemann Zeta-function and the property of having a Euler product.
A primitive function in this class is a non-constant element of $\mathcal{S}$ that cannot be factored into a product of non-constant functions in $\mathcal{S}$. For entire $F(s) \in \mathcal{S}$, $F(s+i\theta)$ $(\theta \in \mathbb{R})$ is called a shifted function. It is also a member of $\mathcal{S}$. Functions $F(s,\xi) \in \mathcal{S}$ depending on a real parameter ξ are called continuous (holomorphic) family if $F(s,\xi)$ is continuous (holomorphic) as a function of ξ. We say a family is primitive, if $F(s,\xi)$ is primitive for each ξ. There are many conjectures connected with the Selberg class. Here we are concerned with the following three:
Selberg's Orthonormality Conjecture (SOC). For primitive functions P, P' we have

$$\sum_{p \leq x} \frac{a(p)\overline{a'(p)}}{p} = \delta(P,P') \log\log x + O(1) \quad (x \to \infty)$$

where $\delta(P,P') = 1$ if $P = P'$ and $\delta(P,P') = 0$ otherwise.
For our proofs in fact $o(\log\log x)$ instead of $O(1)$ is sufficient. We will refer to this as "weak SOC".
Countability Conjecture (CC). There are only countably many shift classes of primitive functions in $\mathcal{S}$.
Sarnak's Rigidity Conjecture. (i) Any continuous primitive family on an interval $J \subseteq \mathbb{R}$ is obtained by continuously shifting some primitive element of $\mathcal{S}$, that is to say, it is of the form $P(s,\xi) = P(s + ih(\xi))$ where P is primitive and $h : J \to \mathbb{R}$ is continuous. (ii): Any continuous family can be factored into primitive continuous families.
For continuous families we showed the following result: Assume weak Selberg's Orthonormality Conjecture and the Countability Conjecture, then Sarnak's Rigity Conjecture is true.

Since locally factorization is unique by SOC, it is easy to see that the continuous factorization given by the theorem is unique on an interval as long as all factors are distinct there. Furthermore it is obvious that at any point of factors coincidence we might switch factors between the coinciding factors without violating continuity. Hence factorization of continuous families is not unique in general.
For the proof of our theorem one important tool is the following topological lemma due to Sierpinski: Let J be an interval in $\mathbb{R}^1$ and suppose that in J a system of at most countably many disjoint relativeely closed sets $F_n \neq \emptyset$ is given, such that $\cup F_n = J$. Then the system consists of the one Element J.

Multiple Zeta Values and Euler-Zagier Numbers

Michel Waldschmidt, Paris

For positive integers $s_1, \dots, s_k$ with $s_1 \geq 2$, the series

$$\sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k}$$

converges and its sum is denoted by $\zeta(s_1, \dots, s_k)$. In case $k = 1$ this is nothing else than the value of the Riemann zeta function at the point s_1. What are the algebraic relations between these numbers? The product of two multiple zeta values is again a multiple zeta value: this is easily checked by multiplying the two series, and using a "shuffle" product. An example is

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5),$$

There are other quadratic relations between these numbers, which arise from another expression of $\zeta(\underline{s}) = \zeta(s_1, \dots, s_k)$ as a (Chen) iterated integral:

$$\zeta(\underline{s}) = \int_{\Delta_p} \omega_{\epsilon_1}(t_1) \cdots \omega_{\epsilon_p}(t_p),$$

with the following notation: $p = s_1 + \cdots + s_k$, the sequence $(\epsilon_1, \dots, \epsilon_p)$ of elements in $\{0,1\}$ is defined by writing the word

$$x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 = x_{\epsilon_1} \cdots x_{\epsilon_p}$$

on the alphabet $X = \{x_0, x_1\}$, the differential forms ω_ϵ are defined by

$$\omega_0(t) = \frac{dt}{t} \quad \text{et} \quad \omega_1(t) = \frac{dt}{1-t}$$

and Δ_p is the following simplex in $\mathbb{R}^p$:

$$\Delta_p = \{\underline{t} \in \mathbb{R}^p \ ; \ 1 > t_1 > \cdots > t_p > 0\}.$$

Again this yields an expression of the product of two multiple zeta values as a linear combination of multiple zeta values, a simple example being

$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1).$$

The main conjecture is that these two types of relations are sufficient to describe all algebraic relations between these numbers.
This subject has deep connections with many other mathematical topics: combinatoric (the theory of quasisymmetric functions, Radford's Theorem and Lyndon words), Lie and Hopf algebras, Écalle's theory of resurgent series, Goncharov's work on mixed Tate motives on Spec$\mathbb{Z}$, polylogarithms, monodromy of differential equations, the fundamental group of the projective line minus three points and Belyi's Theorem, the absolute Galois group of $\mathbb{Q}$, the group of Grothendieck-Teichmüller, knots theory and Vassiliev invariants, K-theory, Feynman diagrams and quantum field theory, quasi-triangular quasi-Hopf algebras and Drinfeld's associator Φ_{KZ} (related to the connexion of Knizhnik-Zamolodchikov).

Rigidity in the Selberg Class. II. Holomorphic families

Eduard Wirsing, Ulm

This is joint work with Ulrike Vorhauer, Ulm.
If a family $F(s,\xi)$ in the Selberg Class defined for ξ in an interfal I depends holomorphically on the parameter ξ then, assuming SOC and CC, the factorization $F(s,\xi) = \prod_{\kappa=1}^{k} P_\kappa(s + ih_{\kappa(\xi)})$ that is guaranteed by the above theorem on the continuous case can be chosen such that the $h_\kappa(\xi)$ are holomorphic on I.
After fixing values $s = s_1, \dots, s_k$ one has to solve the k ensuing equations for the variables $\eta_\kappa = h_\kappa(\xi)$. In general, however, the solutions invariant under certain permutations, so the implicit function theorem fails to work directly. The difficulty is overcome by factoring out symmetry i.e. by solving for the elementary symmetric functions $\sigma_1, \dots, \sigma_k$ of the η_κ in a first step.
Using the fact that for sets of primitive elements of the Selberg Class the logarithms and their derivatives are linearly independent we can choose the $s_1, \dots, s_k$ so as to make the relevant Jacobian nonzero.
It remains to show that with a monic polynomial whose coefficients are holomorphic functions on an interval and whose zeros can be represented as holomorphic functions of the parameter on that interval.

Theta constants and differential equations

W. Zudilin, Moscow

For any genus g consider the *theta constants* (*thetanulls*, or *Thetanullwerte*)

$$\vartheta_{\mathbf{a}}(\tau) = \vartheta_{(\mathbf{a}',\mathbf{a}'')}(\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\bigl(\pi i\,{}^t\!(\mathbf{n}+\tfrac{1}{2}\mathbf{a}')\tau(\mathbf{n}+\tfrac{1}{2}\mathbf{a}') + \pi i\,{}^t\!(\mathbf{n}+\tfrac{1}{2}\mathbf{a}')\mathbf{a}''\bigr)$$

attached to the even 2-characteristics $\mathbf{a} = (\mathbf{a}',\mathbf{a}'')$ from $\mathfrak{K}_+ \subset (\mathbb{Z}/2\mathbb{Z})^{2g} = \{\mathbf{a} : {}^t\mathbf{a}'\mathbf{a}'' \equiv 0 \bmod 2\}$, where $\tau = (\tau_{jl})_{j,l=1,\dots,g}$ is a symmetric $g \times g$-matrix from Siegel half space $\mathfrak{H}_g = \{\tau : \operatorname{Im}\tau > 0\}$ of degree g. We generalize well-known results of M. Halphen, R. A. Rankin, K. Mahler for theta constants of genus 1 and their derivatives to the case of arbitrary genus.
Consider partial derivations δ,

$$\delta_{jj} = \frac{1}{\pi i}\frac{\partial}{\partial\tau_{jj}}, \quad j = 1,\dots,g, \qquad \delta_{jl} = \frac{1}{2\pi i}\frac{\partial}{\partial\tau_{jl}} = \delta_{lj}, \quad j,l = 1,\dots,g,\ j \neq l,$$

and logarithmic derivatives of the theta constants

$$\psi_{\mathbf{a},jl} = \psi_{\mathbf{a},jl}(\tau) := \frac{\delta_{jl}\vartheta_{\mathbf{a}}}{\vartheta_{\mathbf{a}}} = \psi_{\mathbf{a},lj}, \qquad \mathbf{a} \in \mathfrak{K}_+, \quad j,l = 1,\dots,g.$$

To the ψ-functions, we assign the quadratic forms

$$\psi_{\mathbf{a}} = \psi_{\mathbf{a}}[\mathbf{z}] := \sum_{j,l=1}^{g} \psi_{\mathbf{a},jl} \cdot z_j z_l;$$

then we define quartic forms

$$\delta\psi_{\mathbf{a}} = \delta\psi_{\mathbf{a}}[\mathbf{z}] := \sum_{j,l,m,p=1}^{g} \delta_{jl}\psi_{\mathbf{a},mp} \cdot z_j z_l z_m z_p.$$

Theorem 1. *The logarithmic derivatives of theta constants satisfy the system of partial differential equations*

$$\vartheta_{\mathbf{a}}^4 \cdot \delta\psi_{\mathbf{a}} = \frac{1}{2^{g-2}} \sum_{\mathbf{b}\in\mathfrak{K}_+} (-1)^{<\mathbf{a},\mathbf{b}>} \vartheta_{\mathbf{b}}^4 \cdot \psi_{\mathbf{b}}^2 - 2\vartheta_{\mathbf{a}}^4 \cdot \psi_{\mathbf{a}}^2, \qquad \mathbf{a} \in \mathfrak{K}_+,$$

where $<\mathbf{a},\mathbf{b}> = {}^t\mathbf{a}'\mathbf{b}'' - {}^t\mathbf{b}'\mathbf{a}'' \bmod 2$.
Consider the rings

$$Q_g = \mathbb{Q}[\vartheta_{\mathbf{a}}, \psi_{\mathbf{a},jl}]_{\mathbf{a}\in\mathfrak{K}_+;j,l=1,\dots,g} \quad \text{and} \quad Q'_g = \mathbb{Q}[\psi_{\mathbf{a},jl}]_{\mathbf{a}\in\mathfrak{K}_+;j,l=1,\dots,g}.$$

Theorem 1 then implies that the *fraction field* of Q_g is stable under δ_{jl}-derivations.
Theorem 2. *The ring Q_g has transcendence degree $2g^2 + g$ over $\mathbb{Q}$.*
For small genus $g = 1, 2, 3$ the results above can be sharpened as follows:

i) theta constants in genus ≤ 3 are algebraic over the fraction field of Q'_g;

ii) the rings Q'_1 and Q'_2 (and the fraction field of Q'_3) are δ-stable.

In particular, the rings Q'_1, Q'_2, and Q'_3 have transcendance degree 3, 10, and 21 over $\mathbb{Q}$ respectively. The proof of Theorem 1 lies on Riemann's relations for abelian theta functions and the heat equation, while the proof of Theorem 2 combines results from differential Galois theory with a modular interpretation of the period matrix of abelian varieties. The constant $2g^2 + g$ is $\dim Sp_{2g}(\mathbb{C})$, and the sympectic group $Sp_{2g}(\mathbb{C})$ occurs as a differential Galois group of a Picard–Vessiot extension for a system of *linear* partial differential equations.

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