Mathematisches Forschungsinstitut Oberwolfach

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Schrödinger Operators

May 13th – May 19th, 2001

The conference was organized by Volker Enß (Aachen) and Christian Gérard (Palaiseau), the 44 participants came from twelve countries.

Both random and deterministic Schrödinger and Dirac operators were studied including interactions with electric and magnetic fields and with quantized fields. Special topics of the talks were the integrated density of states for random Schrödinger operators, the photoelectric effect, Born-Oppenheimer approximation, quantum field theory and semi-classical analysis, high energy asymptotics and long range scattering theory and resolvent estimates.

Abstracts

Ohmic behaviour in a Hamiltonian model

Stephan De Bièvе

(joint work with L. Bruneau)

Many simple microscopic or macroscopic systems obey a phenomenological equation of the type

$$\ddot{q}(t) + \nabla V(q(t)) + \gamma\dot{q}(t) = 0, \quad (\gamma > 0).$$

Examples include the motion of electrons in a metal, or of a small particle in a viscous medium. The energy loss (at a rate $-\gamma\dot{q}^2$) implied by the above equation leads to several well-known phenomena: for potentials bounded below, the particle will come to a stop at one of the critical points of the potential. If the critical point is a minimum, it will do so exponentially fast. If, on the other hand, $\nabla V(q) = -F$, the particle will reach a limiting speed $v(F)$, proportional to the applied field. This is at the origin of Ohmic law.

The phenomenological friction term $-\gamma\dot{q}$ summarizes the reaction of the environment of the particle to the motion of the latter. The energy lost by the particle is transferred to the medium in which it moves and one expects a Hamiltonian treatment of the combined system to be possible.

We have presented such a model, which is of the Pauli-Fierz type and vaguely related to the Caldeira-Leggett model as well. Its Hamiltonian is

$$H(q,p;\Phi,\pi) = \frac{p^2}{2} + V(q) + \iint \mathrm{d}x\,\mathrm{d}y\big[c^2(\nabla_y\Phi)^2(x,y) + \pi(x,y)^2\big] + \int \mathrm{d}x\,\mathrm{d}y\rho(x-q,y)\Phi(x,y).$$

Here ρ is a "form factor" ($\rho \in C_0^\infty, \rho \geq 0$) describing the extension of the particle. The corresponding equation of motion are

$$(\partial_t^2 - c^2\Delta_y)\Phi(x,y) = -\rho(x-q(t),y)$$
$$\ddot{q} + \nabla V(q) = -\int_{\mathbb{R}^3}\mathrm{d}x\int_{\mathbb{R}^3}\mathrm{d}y\,\rho(x-q,y)(\nabla_x\Phi)(x,y).$$

Staring at there for a while, one realises that one can see the wave field $\Phi(x,y)$ as representing for each x an "obstacle" in the form of a three-dimensional oscillating medium. When the particle passes at x, it pumps energy into the field $\Phi(x,\cdot)$.

For this model we have proven that, if c is large enough and if the initial condition does not have to much energy, then, for all $\nabla V = -F$ not too large

$$|q(t) - (q_\infty + v(F)t)| \to 0$$

where $v(F)$ is an explicitly computable function with $v'(0) \neq 0$. We have also proven the expected results for confining potentials.

Singular Lagrangian manifolds and semi-classical analysis

Yves Colin de Verdière

We want to describe microlocal solutions of a family of Schrödinger equations $\hat{H}_{E,t}u = O(h^\infty)$, where

$$\hat{H}_{E,t} = -h^2\partial_x^2 + V_t(x) - E$$

and t is a parameter in $\mathbb{R}^d$, $x \in \mathbb{R}$. The potential V_t is smooth w.r. to (t,x), *uniformly* w.r. to (t,x).

We look at *generic* (non removable) singularities of the family of curves $\xi^2+V_t(x)-E=0$. We study the problem on the classical level describing normal forms and universal unfoldings. Typical simple examples are the Morse case (see YCdV and B. Parisse, CMP 205, 459–500) and the cusp. We describe the microlocal normal forms using Fourier Integral operators. From this normal form, we get a microlocal scattering matrix near each singular point. We can then study the global problem leading to the "singular Bohr-Sommerfeld rules" for the eigenvalue problem. There is a preprint available on `http://www-fourier.ujf-grenoble.fr/~ycolver`. The higher dimensional case (integrable case) is studied in the PhD thesis of San Vũ Ngọc (CPAM 53, 143–217) and in the preprint YCdV and SVN (prépublications Institut Fourier no. 508).

Spectral methods in the study of the return to equilibrium

Jan Derezinski

The problem of the return to equilibrium for W*-dynamical systems can be reduced to the study of the spectrum of certain self-adjoint operators, called Liouvilleans. I describe some methods due to V. Jakši ć, C. A. Pillet and myself, with can be used to study Pauli-Fierz Liouvilleans and to prove the return to equilibrium for Pauli-Fierz systems at various temperatures.

Pauli operator and Aharonov Casher theorem for measure valued magnetic fields

László Erdős

(joint work with Vitali Vougalter)

We define the two-dimensional Pauli operator and identify its core for magnetic fields that are regular Borel measures. The magnetic field is generated by a scalar potential hence we bypass the usual $A \in L^2_{\rm loc}$ condition on the vector potential which does not allow to consider such singular fields. We extend the Aharonov-Casher theorem for magnetic fields that are measures with finite total variation and we present a counterexample in case of infinite total variation. One of the key technical tools is a weighted L^2 estimate on a singular integral operator.

On the spectral theory of some quantum field hamiltonians

Vladimir Georgescu

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{C}$ a unital C*-algebra on $\mathcal{H}$ without non-zero finite rank projections. If $N<\infty$, let $\Gamma_N(\mathcal{H})$ be the truncated symmetric Fock space associated to $\mathcal{H}$, and $a_N(u)$ the truncated annihilation operators. Denote by $\mathcal{C}_N$ the C*-algebra on $\Gamma_N(\mathcal{H})$ generated by the operators $a_N(u)$ with $u \in \mathcal{H}$ and $\Gamma_N(s)$ with $s \in \mathcal{C}$.

The main result presented in this talk says that there is a unique morphism $\mathcal{C}_N \to \mathcal{C} \otimes \mathcal{C}_{N-1}$ such that the image of $a_N(u)$ is $1 \otimes a_{N-1}(u)$ and that of $\Gamma_N(s)$ is $s \otimes \Gamma_{N-1}(s)$. Moreover, the kernel of this morphism is the algebra of compact operators on $\Gamma_N(\mathcal{H})$.

This fact allows one to compute the essential spectrum and to prove the Mourre estimate for boson field hamiltonians with a particle number cut-off and one-boson kinetic energy affiliated to $\mathcal{C}$.

Rigorous Results on Molecular Propagation: Past, Present, and Hopes for the Future

George Hagedorn

The time-dependent Born-Oppenheimer Approximation is the main source of information about molecular propagation. It leads to an asymptotic expansion in powers of ϵ for solutions to the molecular Schrödinger equation. Here ϵ^4 is the electron mass divided by the mean nuclear mass. By applying an optimal truncation technique to this expansion, we obtain an approximation whose errors are bounded by $C_1 \exp(-C_2/\epsilon^2)$ with $C_2 > 0$.

A basic assumption of Born-Oppenheimer Approximations is that the electron energy level of interest stays well separated from the rest of the electronic spectrum. The simplest violations of this assumption occur at crossings and avoided crossings. We describe the effects of these phenomena on the propagation of molecular wave packets.

To do better than the exponential estimates described above, one must inject new physics into the approximations. We describe conjectures about how one might describe molecular propagation when non-adiabatic behavior of electrons is involved.

Wegner estimate and integrated density of states for random operators with nonsign definite potentials

Peter David Hislop

We study the IDS of random Schrödinger and wave operators with Anderson type potentials. The new result is that we can treat single-site potentials with no sign conditions. We prove a Wegner estimate at energies below $\inf \sigma(H_0)$ and, provided the disorder is small, at internal band edges. These results apply to a family of Schrödinger operators with random magnetic fields. The proof is based on the vector field method of Klopp and the L^p estimate on the spectral shift function of Combes, Hislop, Nakamura.

QFT for scalar particles in external forces on Riemannian manifolds

Hiroshi Isozaki

We introduce a class of noncompact Riemannian manifolds on which we can discuss QFT in external forces. This class contains physically important examples such as Euclidean space, hyperbolic space and, by passing to conformal change, the Schwarzschild metric. The S-matrix for massive Klein-Gordon equation is unitarily implemented on the Fock space.

We can also do the same thing for the massless case provided the space is asymptotically flat or it is hyperbolic.

Sharp spectral asymptotics for operators with irregular coefficients. Pushing the limits.

Victor Ivrii

For operators with first derivatives of coefficients continuous with continuity modulus

$$O((\log|x-y|)^{-1}) \quad \text{resp.} \quad o((\log|x-y|)^{-1})$$

we prove spectral asymptotics with the remainder estimate $O(h^{1-d})$ (resp. $o(h^{1-d})$ under billiard condition). The advance is based on the logarithmic uncertainty principle. One can use microlocal analysis as soon as

$$\rho^{\gamma} \geq C_{\rho} h|\log h|$$

where ρ, γ are scales with respect to ξ, x respectively.

Semiclassical resolvent estimates for Schrödinger matrix operators

Thierry Jecko

For the semiclassical Schrödinger operator $-h^2\Delta_x I_2 + M(x)$ in $L^2(\mathbb{R}^n, \mathbb{C}^2)$ with smooth long range potential, we investigate the semiclassical Mourre method to get the resolvent estimates $R(\lambda \pm \mathrm{i}0) = O(h^{-1})$ as bounded operator from $L^2_s(\mathbb{R}^n, \mathbb{C}^2)$ in $L^2_{-s}(\mathbb{R}^n, \mathbb{C}^2)$ for $s > \frac{1}{2}$. If the eigenvalues of M do not cross, it suffices to require a non-trapping condition on the eigenvalues of the symbol at energy λ, but if the eigenvalues of M cross in a codimension 2 submanifold, then an obstruction can occur at the crossing (that might generates resonances close to the real axis). However, if this obstruction do not occur and a non-trapping condition holds, one can perform the semiclassical Mourre method under a further, quite restrictive assumption on the Hamiltonian flows of the eigenvalues of the symbol.

The High-Energy Asymptotics of One-Dimensional Dirac Scattering

Wolf Jung

Denote the transmission amplitude in one-dimensional quantum scattering by $\tau = \mathrm{e}^{-\mathrm{i}\delta}$. Now δ has an asymptotic expansion in inverse powers of the momentum q or of the energy E. Standard techniques for obtaining these expansions are related to Krein's spectral shift function $\xi(E)$ or to the WKB method. I proposed a different technique, which is simpler and has a direct physical interpretation:

1. construct a convergent series (Born or Jost)
2. the terms consist of oscillatory integrals, which yield an asymptotic expansion by partial integration
3. knowing that an asymptotic expansion exists, obtain the coefficients from an ansatz and recursion relations.

The method included transforming both the Schrödinger- and the Dirac equation to a massless Dirac-type equation with energy-dependent potential matrix. I have learned from the audience that the results and the techniques of the above are well-known in the theory of solitons, at least for energy-independent potentials. New aspects are the parallel treatment of Schrödinger- and Dirac equations, a modified Born series taking into account the largest term in the energy-dependent potential, the physical interpretation in terms of superposition and interference of Feynman amplitudes, and a conjecture about interchanging the limits $q \to \infty$ and $\hbar \to 0$. A preprint on these topics will be available from mp_arc in autumn.

The photoelectric effect for an atom coupled to a second quantized electromagnetic field

Fréderic Klopp

(joint work with V. Bach, H. Zenk)

We consider an atom with a single bound state (that is the ground state) coupled to a quantized electromagnetic field. We consider the ionization of the atom by an incident photon cloud consisting of N involved photons. We prove that the total ionized charged is additive in the N involved photons. Furthermore, Einstein's prediction for the photoelectric effect is quantitatively and qualitatively correct to leading order in the coupling parameter; that is ionization only happen if the single photons have momentum large enough to cross the energy gap, and the kinetic energy of the ejected electron is then given by the difference of the photon energy of each single photon (in the cloud) and the ionization energy.

Regularity of the surface density of states for random Schrödinger operators

Vadim Kostykin

(joint work with Robert Schrader)

We consider random Schrödinger operators with the interaction localized at a hypersurface $\mathbb{R}^{\nu_1}$ in $\mathbb{R}^{\nu}$, $\nu_1 \leq \nu$. An important quantity related to such a kind of operators is the surface density of states, which measures a density of states per unit surface. So far it was known that this quantity is a distribution of order at most 3. Using the concept of the spectral shift density and applying the theory of the spectral shift function we prove that the surface density of states belongs to L^p_{loc} for any $1 \leq p < \infty$. In the case of Jacobi matrices (discrete Schrödinger operators) the result is sharper: the surface density of states is uniformly bounded by one.

Spectral shift function and semi-classical asymptotics for trapping perturbations

Vesselin Petkov

(joint work with V. Bruneau)

We examine the representation of the derivative $\xi'(\lambda, h)$ of the spectral shift function related to two self-adjoint operators $L_j(h)$, $j = 1, 2$ for $\lambda \in [a, b]$, $0 < a < b$, $0 < h \leq h_0$. The operators

$$L_j(h) = \sum_{|\nu| \leq 2} a_{j,\nu}(x, h)(hD)^{\nu}$$

are long-range perturbations of $-h^2\Delta$ and the distribution $\xi(\lambda; h) \in \mathcal{D}'(\mathbb{R})$ is defined by the trace

$$C_0^{\infty}(\mathbb{R}) \ni \varphi \mapsto \operatorname{tr}(\varphi(L_1(h)) - \varphi(L_2(h)) = \langle \xi'(\lambda, h), \varphi(\lambda) \rangle_{\mathcal{D}', \mathcal{D}}.$$

Our main result says that $\xi'(\lambda,h)$ admits a representation

$$(1)\qquad \xi'(\lambda,h)=\Big[\sum_{w\in\mathrm{Res}L_j\cap\Omega,\mathrm{Im}w\neq 0}-\frac{\mathrm{Im}}{\pi|\lambda-w|^2}+\sum_{w\in\mathrm{Res}L_j\cap J}\delta(\lambda-w)\Big]_{j=1}^2+\frac{1}{\pi}r(x,h),$$

where $\Omega\subset\subset\mathbb{C}$ is a compact domain, $J=\Omega\cap\mathbb{R}^+$, $\mathrm{ResL_j}$ denotes the set of resonances of L_j in $\{\mathrm{Im}w\leq 0\}$ and $|r(\lambda,h)|\leq C(w)h^{-n}$ for $W\subset\subset\Omega$, $\lambda\in I=W\cap\mathbb{R}^+$, while $[a_j]_{j=1}^2=a_2-a_1$. Applying the representation (1), we obtain a local trace formula and a Breit-Wigner approximation of $\mathcal{E}'(\lambda,h)$.

Finite gap potentials and WKB asymptotics for 1D Schrödinger operators

Christian Remling

(joint work with Thomas Kriecherbauer)

Consider a one-dimensional Schrödinger operator $H=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)$ with power decaying potential $V(x)=O(x^{-\alpha})$. We construct examples which show that a previously obtained dimensional bound on exceptional sets is optimal in its whole range of validity. This construction uses finite gap potentials and relies on pointwise bounds on these potentials. The main part of the argument consists of an analysis of the so-called Jacobi inversion problem.

Relativistic Hamiltonian for Electron in Heavy Atoms

Heinz Siedentop

(joint work with Raymond Brummelhuis and Edgardo Stockmeyer)

Jansen and Heß — correcting an earlier paper of Douglas and Kroll — have derived a (pseudo-)relativistic energy expression successfully describing heavy atoms. It is an approximate no-pair Hamiltonian in the Furry picture. We present a recent result on the boundedness of the energy jointly obtained with Brummelhuis and Stockmeyer: the corresponding (one-particle) quadratic form is bounded from below if and only if $0\leq\alpha Z\leq 1.006$. This allows to define a distinguished self-adjoint operator for the same range of coupling constants.

Long-range three-body scattering

Erik Skibsted

We consider the problem of asymptotic completeness (AC) for a system of three quantum mechanical particles with pair interactions

$$\partial^\beta V^\alpha(x^\alpha)=O(|x^\alpha|^{-\mu-\|\beta\|}).$$

With the assumption of spherical symmetry and a negative upper bound at infinity, AC holds in the regime $\mu\in(\frac{1}{2},\sqrt{3}-1]$. In one dimension and under further concavity conditions, AC holds in $(0,\frac{1}{2}]$ as well.

Another new result is AC for a class of potentials with a certain positive lower bound at infinity, again in the regime $\mu\in(0,\frac{1}{2}]$.

Adiabatic decoupling and the time-dependent Born-Oppenheimer theory

STEFAN TEUFEL

(joint work with Herbert Spohn)

We consider a molecular Hamiltonian of the form

$$H = \frac{\hbar^2}{2M}\Big(- i\nabla_x + A_{\text{ext}}(x)\Big)^2 + \frac{\hbar^2}{2m}\Big(- i\nabla_y - A_{\text{ext}}(y)\Big)^2 + V(x,y) \tag{2}$$

on $\mathcal{H} = L^2(\mathbb{R}^{3n}_x)\otimes L^2(\mathbb{R}^{3e}_y)$. Here n is the number of nuclei, e is the number of electrons, M is the mass of the nuclei and m the mass of the electrons. Since the ratio $\varepsilon^2 := \frac{m}{M} \approx 10^{-4}$ is typically small, it can be used as an expansion parameter. Going to atomic units $\hbar = m = 1$ and rescaling the potentials turns (2) into

$$H^\varepsilon = \frac{\varepsilon^2}{2}\Big(- i\nabla_x + A_{\text{ext}}(x)\Big)^2 + H_{\text{e}}(x) \tag{3}$$

with

$$H_{\text{e}}(x) := \frac{1}{2}\Big(- i\nabla_y - A_{\text{ext}}(y)\Big)^2 + V(x,y)\,.$$

We show that on subspaces belonging to isolated energy bands $E(x)$, i.e. $H_{\text{e}}(x)\psi_E(x,y) = E(x)\psi_E(x,y)$ and $E(x)$ separated from the rest of $\operatorname{spec}(H_{\text{e}}(x))$ by a gap uniformly for $x\in\mathbb{R}^{3n}$, the full time evolution is well approximated by an effective one generated by

$$H^\varepsilon_{\text{BO}} = \frac{\varepsilon^2}{2}\Big(- i\nabla_x + A_{\text{ext}}(x) + A_{\text{geo}}(x)\Big)^2 + E(x)$$

for the nuclei only. $A_{\text{geo}}(x) := -i\langle\psi_E(x), \nabla_x\psi_E(x)\rangle$ is known as the Berry connection and the replacement $H^\varepsilon \to H^\varepsilon_{\text{BO}}$ as Peierl's substitution.

For the precise statement let $\mathcal{U}:\mathcal{H}\to L^2(\mathbb{R}^{3n})$, $\psi\mapsto\langle\psi_E(x),\psi(x)\rangle_{L^2(\mathbb{R}^{3e})}$. We show that there is a constant $c<\infty$ such that

$$\Big\| \Big(e^{-iH^\varepsilon \frac{t}{\varepsilon}} - \mathcal{U}^*\, e^{-iH^\varepsilon_{\text{BO}}\frac{t}{\varepsilon}}\,\mathcal{U}\Big)\; P_* \Big\|_{\mathcal{B}(\mathcal{H})} \le c\,\varepsilon\,(1+|t|)$$

with $P_* := \int^\oplus dx\,\langle\psi_E(x),\cdot\rangle\psi_E(x)$. We also prove a version of the above statement which holds "locally" in the configuration space of the nuclei.

The Wegner estimate and the common density of the Anderson coupling constants

IVAN VESELIĆ

Wegner's estimate (Z.Phys B44, 1981) plays a crucial role in the analysis of random Schödinger operators from solid state physics. On the one hand, it is part of the existence proof of pure point spectrum in certain energy regions. On the other, it supplies information about the regularity properties of the integrated density of states.

The talk presents a new proof of Wegner's estimate, which is valid also for certain indefinite Anderson models. A new aspect is the use of the common density of the Anderson coupling constants instead of the often used conditional density.

Resolvent estimates on infinite volume Riemannian manifolds with cusps

GEORGI VODEV

The limiting absorption principle is proved for the Laplace-Beltrami operator on infinite volume Riemannian manifolds which may have cusps. Moreover, an exponential bound (with respect to the spectral parameter) of the norm of the limiting operator (on the real axis) is proved. This extends a previous result by Burq to more general manifolds. As a consequence, a free of resonances region of the form

$$|\mathrm{Im}\lambda| \leq \mathrm{e}^{-C_1|\lambda|}, \quad |\lambda| \geq C_2, \quad C_1, C_2 > 0,$$

is obtained for two dimensional Riemann surfaces of the form

$$M = Z \cup X_1 \cup \cdots \cup X_I \cup Y_1 \cup \cdots \cup Y_J, \quad I \geq 0, J \geq 1,$$

where Z is a compact Riemannian manifold and

$$X_i = [a_i, \infty)_r \times (\mathbb{R} \setminus h_i\mathbb{Z})_t, \quad a_i, h_i > 0, \quad \text{with metric } \mathrm{d}r^2 + \mathrm{e}^{-2r}\mathrm{d}t^2$$
$$Y_i = [b_i, \infty)_r \times (\mathbb{R} \setminus \ell_i\mathbb{Z})_t, \quad b_i, \ell_i > 0, \quad \text{with metric } \mathrm{d}r^2 + \cosh^2 r\mathrm{d}t^2,$$

and the resonances are defined as being the poles of the meromorphic continuation of the resolvent

$$R(\lambda) := \Big(\Delta_M - \lambda^2 - \frac{1}{4}\Big)^{-1} : L^2_{\mathrm{comp}}(M, \mathrm{dvol}_g) \longrightarrow L^2_{\mathrm{loc}}(M, \mathrm{dvol}_g)$$

from $\mathrm{Im}\lambda < 0$ to the whole complex plane $\mathbb{C}$.

Lifshits Tails in Magnetic Fields

SIMONE WARZEL

(joint work with Thomas Hupfer and Hajo Leschke)

We investigate the leading low-energy fall-off of the integrated density of states of a charged quantum particle subject to a constant magnetic field and repulsive impurities randomly located according to Poisson's distribution.

This so-called magnetic Lifshits tail is determined for the case of two space dimensions with a perpendicular magnetic field and for all single-impurity potentials with either super-Gaussian, Gaussian or regular sub-Gaussian decay at infinity. While the result for regular sub-Gaussian decay coincides with the corresponding classical one, the Lifshits tail caused by super-Gaussian decay exhibits a universal quantum behaviour. As a consequence, Gaussian decay is proven to discriminate between quantum and classical tailing.

In the case of three space dimensions, the magnetic Lifshits tail is investigated for all impurity potentials with super-Gaussian or Gaussian decay. Its precise form is determined for all impurity potentials with stretched (sub-) Gaussian decay. In this case it turns out that the tail is independent of the magnetic field and coincides, up to a logarithmic acceleration, with that for one dimension and not too slowly decaying impurity potentials.

High energy asymptotics of the scattering amplitude

DIMITRIJ YAFAEV

We find an explicit expression for the kernel of the scattering matrix containing at high energies all terms of power order. It turns out that the same expression gives a complete description at the diagonal singularities of the kernel in the angular variables. Both short- and long-range electric as well as magnetic potentials are considered.

Behavior at infinity of fundamental solution of time dependent Schrödinger equations

Kenji Yajima

Let $E(t,s,x,y)$ be the distribution kernel of the propagator for the time dependent Schrödinger equation in $\mathbb{R}^n$,

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\triangle u + V(t,x)u, \qquad u(s,x) = \phi(x). \tag{4}$$

We prove that the asymptotic behavior at infinity of E is stable under the subquadratic perturbations. $\partial_x^2 V(t,x)$ is the Hessian of V wrt. x and

$$\mathcal{I}_{\varepsilon,T} = \{(t,s) : 0 < |t-s| < T,\ |t-s-m\pi| > \varepsilon,\ \forall m \in \mathbb{Z}\setminus\{0\}\}$$

Theorem 1. (a) *Let V satisfy the condition*

$$\text{(SQ)} \qquad \lim_{|x|\to\infty} \sup_{t\in\mathbb{R}^1} |\partial_x^2 V(t,x)| = 0, \quad |\partial_x^\alpha V(t,x)| \le C_\alpha, \quad |\alpha| \ge 3$$

Then, for any $0 < \pm(t-s) < T$, E is C^∞ wrt. (x,y) and may be written in the form

$$E(t,s,x,y) = \frac{e^{\mp in\pi/4}}{(2\pi|t-s|)^{n/2}} e^{iS(t,s,x,y)} a(t,s,x,y),$$

where S is real smooth and, as $x^2+y^2\to\infty$, uniformly wrt. $0<|t-s|<T$,

$$\partial_x^\alpha \partial_y^\beta \left(S(t,s,x,y) - \frac{(x-y)^2}{2(t-s)} \right) \to 0, \quad |\alpha+\beta| \ge 2, \tag{5}$$

$$\partial_x^\alpha \partial_y^\beta (a(t,s,x,y) - 1) \to 0, \quad |\alpha+\beta| \ge 0. \tag{6}$$

(b) *Let $V(t,x) = \frac{1}{2}x^2 + W(t,x)$ and W satisfies (SQ). Then, for any $(t,s)\in\mathcal{I}_{\varepsilon,T}$, E is C^∞ wrt. (x,y) and may be written in the form, for $0 < t-s-m\pi < \pi$, $m\in\mathbb{Z}$,*

$$E(t,s,x,y) = \frac{i^{-m} e^{-in\pi/4}}{(2\pi|\sin(t-s)|)^{n/2}} e^{iS(t,s,x,y)} a(t,s,x,y),$$

where S is real smooth and, as $x^2+y^2\to\infty$, uniformly wrt. $(t,s)\in\mathcal{I}_{\varepsilon,T}$

$$\partial_x^\alpha \partial_y^\beta \left(S(t,s,x,y) - \frac{(x^2+y^2)\cos(t-s) - 2xy}{2\sin(t-s)} \right) \to 0, \quad |\alpha+\beta| \ge 2, \tag{7}$$

$$\partial_x^\alpha \partial_y^\beta (a(t,s,x,y) - 1) \to 0, \quad |\alpha+\beta| \ge 0. \tag{8}$$

Spectral Asymptotics of Weyl type for Schrödinger operators with discrete spectrum

Lech Zielinski

Let A be a self-adjoint operator in $L^2(\mathbb{R}^d)$ formally given by

$$\sum_{1\le j,k\le d} D_j(a_{jk}(x)D_k) + v(x)$$

where $D_j = -\mathrm{i}\frac{\partial}{\partial x_j}$, $a_{jk} = \overline{a}_{kj} \in L^\infty(\mathbb{R}^d)$ such that

$$a_0(x,\xi) = \sum_{1\le j,k\le d} a_{jk}(x)\xi_j\xi_k \ge c|\xi|^2, \quad (c>0)$$

$$(1+|x|)^c \le v(x) \le C(1+|x|)^c. \quad (C,c>0)$$

Let $\rho, r \in]0,1]$ be fixed and assume that

$$\begin{aligned}
|x-y| \le cv(x)^3 \quad &\Rightarrow \quad c \le \frac{v(x)}{v(y)} \le C \\
|x-y| \le 1 \quad &\Rightarrow \quad |v(x)-v(y)| \le Cv(x)^{1-\rho}|x-y|r \\
|x-y| \le 1 \quad &\Rightarrow \quad |a_{jk}(x)-a_{jk}(y)| \le Cv(x)^{-\rho}|x-y|r
\end{aligned}$$

for some $C, c > 0$. Let $\rho' > 0$, $r' > 0$ be arbitrary fixed numbers satisfying

$$\rho' < \rho \quad \text{and} \quad r' < r$$

and $h(x,\xi) = v(x)^{-\rho'}(1+|x|)^{-r'}$. Then there exists $\overline{C} > 0$ large enough to ensure the estimate (for $\lambda \ge \overline{C}$)

$$\big|\mathcal{N}(A,\lambda) - \int_{a(x,\xi)<\lambda} \mathrm{d}x\, \frac{\mathrm{d}\xi}{(2\pi)^d}\big| \le \overline{C} \int_{a(1-\overline{C}h)<\lambda<a(1+\overline{C}h)} \mathrm{d}x\,\mathrm{d}\xi$$

where $\mathcal{N}(A,\lambda)$ is the counting function of A and $a(x,\xi) = a_0(x,\xi) + v(x)$. In the case $(1+|x|)^m \le v(x) \le C(1+|x|)^m$ the above result gives

$$\mathcal{N}(A,\lambda) = \int_{a(x,\xi)<\lambda} \mathrm{d}x\, \frac{\mathrm{d}\xi}{(2\pi)^d}\big(1 + O(\lambda^{\rho'-r'/2})\big), \quad (\rho'<\rho, r'<r).$$

Edited by Olaf Post

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