Mathematisches Forschungsinstitut Oberwolfach

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Two Hundred Years of Number Theory after Carl-Friedrich Gauß' Disquisitiones Arithmeticæ

June 17th – June 23rd, 2001

The aim of this conference was to bring together mathematicians and historians of mathematics around the specific theme of the development of (algebraic) number theory since 1801. According to the historians' practice, generous slots after all talks were devoted to discussions (in general 25 minutes of discussion for a 50 minutes lecture). These turned out to be very lively and rich in content, but they are unfortunately not reflected in this report, which simply follows the speaker's abstracts.

At the opening of the conference by the organizers (Catherine Goldstein, Norbert Schappacher, Joachim Schwermer), apart from technical remarks, a brief overview of the Disquisitiones Arithmeticæ was given. Then the prehistory of the meeting was recalled:

The conference had been prepared by the RiP workshop „The History of Number Theory (1800–1950)“ held at Oberwolfach in June 1999. The final report of this workshop, along with the first attempt at a bibliography of the subject (a project that has also originated at the workshop) was distributed to all participants at the opening of the conference - along with a small bottle of „Gauß-Geist“ from Göttingen.

Furthermore, in order to capture some major lines of thought from the workshop in a way that could be immediately fruitful for the discussions at the conference, the following short list of sample questions was presented:

1. Which aspects of the Disquisitiones Artihmeticæ were taken up?
   – How? By whom? When? Where?
   For instance:
   - Why were sections I and VII received earlier and more widely?
   - What is the role of numerical examples on late authors?
   - The Disquisitiones Arithmeticæ as a model of rigour?

2. What propels the development of number theory? [and what does not?]
   – Notions? Denkweisen? Problems?
   For instance:
   - Ideals? Forms of degree > 2 ?
   - Geometry (of numbers)?
   - Additive Number Theory: Waring's Problem?

3. Number Theory – a German Topic?
   For instance:
   - Specific philosophy?
   - University reforms?
   - No tradition of analysis?

4. What is not in the Disquisitiones Arithemticæ?
   For instance:
   - Elliptic functions?
   - Function fields?

Abstracts

Section Five of the Disquisitiones Arithmeticæ and the Foundations of Mathematics

Harold M. Edwards

A. Weil called Gauß's exposition of the theory of composition of forms in Section V. of *Disquisitiones Arithmeticæ* a „stumblingblock" until Dirichlet „restored its simplicity", but this view overlooks the fact that Dirichlet, unlike Gauß, did not compose *arbitrary* forms but only „concordant" ones. Therefore, although Dirichlet's method was sufficient to compose proper *equivalence classes* of forms - which is all that is needed for Gauß's applications - it was less general than Gauß's.

Gauß's composition is *not* a binary *operation* but a ternary *relation*. Therefore, to say it is associative is without meaning. Gauß proves, however, that it does imply a binary operation with respect to proper equivalence of forms and that this operation is associative. In modern terms, this is the statement that Gauß's composition of *forms* serves to define a binary operation on *equivalence classes* of forms, but Gauß does not operate with equivalence classes, and I question whether the notion of an equivalence class (an infinite set) as a mathematical entity would have been acceptable to him. I believe that to put his theory in Section V. in a rigorous form Gauß strove to base it on specific calculations and algorithms. This is the conception of mathematical rigor that had prevailed up to Gauß's time and that Kronecker attempted to defend in the generation after Gauß.

The second half of the talk was devoted to an alternative approach to Gauß's composition of forms based on very elementary algorithmic ideas: Let a *hypernumber* for a fixed natural number A, not a square, be an expression $x+y\sqrt{A}$ where x and y are natural numbers (nonnegative integers). Hypernumbers can be added and multiplied in the obvious ways. Given a list $m_1, m_2, \ldots, m_\mu$ of hypernumbers, define $a \equiv b \mod [m_1, m_2, \ldots, m_\mu]$ for hypernumbers a and b to mean $a + \sum c_i m_i = b + \sum c_i' m_i$ for some hypernumbers $c_1, \ldots, c_\mu, c_1', \ldots, c_\mu'$. Define $[m_1, \ldots, m_\mu] = [n_1, \ldots, n_\nu]$ to mean that the two congruence relations are the same. In other words, the *modules* are equal.

There is a natural way to *multiply* modules. Every module is equal to one and only one of the form $\left[ef,\ eg + e\sqrt{A}\right]$ where e, f, g are natural numbers satisfying $ef \neq 0, g < f, g^2 \equiv A \mod f$. Using this canonical form, one can compute with modules - specifically, one can multiply them and determine whether two are equal.

All compositions of two given forms in Gauß's sense can be constructed using this operation of multiplication of modules. Therefore, all of Gauß's computations with forms can be given a clear algorithmic description and his results, such as the law of quadratic reciprocity, can be deduced.

Ein bekanntes unbekanntes Manuskript E. E. Kummers vor seiner Erfindung der idealen Zahlen

Reinhard Bölling

Es wurde ein unbekanntes zahlentheoretisches Manuskript E. E. Kummers vorgestellt, das 1844 als Nachtrag zu seiner Publikation „*De numeris ...*" entstand und damit dem Zeitabschnitt zwischen Kummers Einsicht in den Fehler, den sein noch im April 1844 der

Berliner Akademie eingereichtes Manuskript enthielt, und seiner Erfindung der „idealen complexen Zahlen“ angehört. Kummer beweist darin, dass der λ-te Kreisteilungskörper für $\lambda = 5$ und $\lambda = 7$ normeuklidisch ist. Mit diesem Ergebnis kann Kummer sein ursprüngliches Ziel erreichen, nämlich beweisen, dass alle Primzahlen $p \equiv 1 \pmod{\lambda}$ für $\lambda = 5$ und $\lambda = 7$ im Ring $\mathbb{Z}[\alpha]$ ($\alpha^\lambda = 1$, $\alpha \neq 1$) der ganzen Kreisteilungszahlen in die maximal mögliche Anzahl von $\lambda - 1$ Faktoren zerlegt sind. Im Vortrag ist eine Rekonstruktion von Kummers Vorgehen gegeben worden, die auf den Resultaten in „*De numeris* ...“ basiert (Kummer belässt es in seinem Manuskript bei einem Verweis auf diese Arbeit). Diese $\lambda - 1$ Faktoren sind Primelemente (in Jacobis und Kummers Sprechweise: die „wahren complexen Primzahlen“ von p in $\mathbb{Z}[\alpha]$).

Der Anstoß für die Suche nach solchen Faktoren kam aus der Theorie der Kreisteilung. Gauß, Jacobi und Cauchy hatten unabhängig voneinander gefunden, dass p in $\mathbb{Z}[\alpha]$ auf mehrere Arten als Produkt von zwei Faktoren auftraten, was Jacobi vermuten ließ, dass diese Faktoren in $\mathbb{Z}[\alpha]$ weitere Zerlegung in unzerlegbare Elemente zuließen.

Kummer hatte bei seinen Untersuchungen zur Kreisteilung von Anfang an die Absicht, diese so weit zu entwickeln, um damit die höheren Reziprozitätsgesetze aufzustellen und zu beweisen.

Es wurde auf weitere Arbeiten über euklidische $\mathbb{Z}[\alpha]$ eingegangen. Von den $\mathbb{Z}[\alpha]$ mit der Klassenzahl 1 ist für alle Primzahlen $\lambda \leq 13$ bekannt, dass $\mathbb{Z}[\alpha]$ normeuklidisch ist, während für die verbleibenden Primzahlen $\lambda = 17$ und $\lambda = 19$ die entsprechende Frage noch offen ist.

Eine Publikation des aufgefundenen Manuskriptes ist vorgesehen.

Implizite gruppentheoretische Denkformen in den Disquisitiones Arithmeticæ von C.F. Gauß

HANS WUSSING

Geht man von der methodologischen Grundvorstellung aus, dass sich im Entwicklungsgang der Mathematik in impliziter Form Vorstufen von Denkformen und Begriffen finden, die erst später explizit zu Tage treten, so scheint nach meiner Meinung eine Hauptlinie der Entwicklung der Algebra während des 19. Jahrhunderts darin zu bestehen, dass algebraische Grundstrukturen wie Gruppe, Körper, Ring und hyperkomplexes System in expliziter begrifflicher Form und axiomatischer Fassung herausgearbeitet worden sind.

Die *Disquisitiones Arithmeticæ* bieten in dieser Hinsicht Gelegenheit, implizite gruppentheoretische Denkformen zu untersuchen, insbesondere bei der Behandlung der Theorie der Potenzreste, bei der Behandlung der Kreisteilung und bei der Gaußschen Theorie der Formen. Letztere hat, auf dem Wege über eine Kroneckersche Arbeit von 1870, die sich auf die Komposition der Formen bezieht, zur Herausarbeitung des abstrakten Gruppenbegriffs beigetragen.

Gauß's Influence on Zolotarëv

PAOLA PIAZZA

This talk dealt first with Zolotarëv's theory of ideal numbers for rings of algebraic integers. This work was presented as an alternative to Dedekind's ideal theory, in generalizing Kummer's theory of ideal numbers to every ring of algebraic integers. An explanation was proposed for the poor reception of Zolotarëv's work in Europe, in spite of the fact that this young mathematician was quite well-known and respected during the short period of his mathematical activity (he died still young in a carriage accident). It appears that Dedekind's theory took over quickly.

A few conjectures were made with regard to Gauß's influence on Zolotarëv's work, specifically through Gauß's work on the biquadratic reciprocity and the ring $\mathbb{Z}[i]$. However, it was explained that Zolotarëv developed a local approach to questions of divisibility of algebraic integers, using the norm of algebraic integers. The similarities with Kummer's theory of ideal numbers in the cyclotomic case were stressed.
(Since P. Piazza could not come to Oberwolfach, the manuscript of this talk was read by N. Schappacher)

Gaußian Number Theory and Algebraic Number Theory

Ralf Haubrich

In his *Disquisitiones Arithmeticæ* (1801 - D.A.) Gauß initiated a new mathematical discipline - Gaußian Number Theory - which dominated (many aspects of) nineteenth century number theory up to the end of the century, and which was replaced at about 1900 by algebraic number theory. To substantiate this claim I propose to differentiate six components of a mathematical discipline: subject matter, main concepts and main problems, systematization, statements, language, meta-mathematical views. For instance, the subject matter of Gaußian Number Theory is, as Gauß put it in the preface of the D.A., „about properties special to integers", and the systematization is given by a division into the theory of congruences and the theory of forms. The first appearance of Gaußian Number theory is in the D.A.. However, as early as in the D.A., a conflict between the subject matter of the discipline and its mathematical practice is present, which threatens the very core of Gaußian Number Theory. Gauß proposed several ways how to avoid the conflict. His last and most radical one was published at about 1830 and amounts to an extension of the field of arithmetic, i.e., the subject matter. Several decades later, Gauß's proposal contributed to the development of a new mathematical discipline - algebraic number theory.

Netzwerke des Wissens und Diplomatie des Wohltuns: Alexander von Humboldt und Gauß, Dirichlet, Jacobi, Kummer, Eisenstein

Herbert Pieper

Im Sommer 1801 erschienen in Leipzig die *Disquisitiones Arithmeticæ* (D.A.) von Carl Friedrich Gauß. Als Alexander von Humboldt 1804 von seiner amerikanischen Forschungsreise nach Paris zurückkehrte, fand er den Namen Gauß „wegen der Erscheinung der Zahlentheorie [...] in aller Mund". Ein Jahr später lud der preußische König Friedrich Wilhelm III. Humboldt ein, „in der Berliner Akademie wirksam aufzutreten". Humboldt antwortete dem König, dass sein Erscheinen „sehr unbedeutsam sein" würde, „aber ein Mann könne der Akademie den Glanz wiedergeben", nämlich Gauß. Die Beziehungen zwischen C. F. Gauß und A. v. Humboldt werden an ausgewählten Ereignissen beleuchtet. Es wird gezeigt, wie junge Mathematiker, die auf Grund zahlentheoretischer Arbeiten (die in der Regel an Gauß' Arbeiten, insbesondere die D.A., anknüpften) Gauß oder Humboldt auf sich aufmerksam machten. Mit der Entdeckung der Talente entstand im Umkreis von Gauß und Humboldt ein sich nach und nach vergrößernder Kreis von Zahlentheoretikern, die in Kontakt blieben. Sie bauten ein Kommunikationsnetz auf, das zum einen dem Austausch von Wissen, zum anderen aber auch der Förderung der neu entdeckten Talente und der

Unterstützung der diesem Kreis angehörenden Kollegen „in so manchen Wechselfällen" ihres Lebens und Wirkens diente. Auf der Grundlage sowohl der Korrespondenzen und Gespräche der Mathematiker mit Alexander von Humboldt, als auch ihrer Briefwechsel und Begegnungen untereinander wird ein Einblick in dieses Netzwerk des Wissens und die darin praktizierte „Diplomatie des Wohltuns" gegeben.

Historical Moments: A Look at the Reception of Gauß's Disquisitiones Arithmeticæ in America

Della D. Fenster

When Gauß's *Disquisitiones Arithmeticæ* appeared in 1801, the 25-year-old United States of America consisted of 16 states which geographically spanned the east coast from New Hampshire to Georgia and included Kentucky and Tennessee. Settled primarily by farmers, clergymen, and artisans in the 16th and 17th centuries, the needs for mathematics were very limited. This talk first established the historical context for the American mathematician Leonard Dickson and then examined some of his mathematical contributions to show how Gauß's ideas took shape in this country.

Charles Hermite: A Second Reading of Gauß in France

Catherine Goldstein

Thanks to the early translation into French of the *Disquisitiones Arithmeticæ* (D.A.), Gauß's work was known, discussed and developed by French mathematicians like Legendre, Germain, Lagrange, Poinsot, Cauchy during the first decades of the nineteenth century. But these readings and developments concern mainly the first and seventh sections of the D.A., dealing with congruences and the cyclotomic equations.

In the 40s, however, Charles Hermite (who has studied the D.A. on his own as a youth) began to generalize portions of the fifth section of the D.A., concerning quadratic forms. It is illuminating and characteristic that he described his first move in this direction as inspired by a result of Jacobi concerning analytic functions.

The talk concentrated mainly on the first letter to Jacobi on the theory of numbers (around 1847) and on the „*Introduction des variables continues dans la théorie des nombres*" of 1851 and showed how a deep knowledge of Gauß's D.A. informed these two papers - in particular, Hermite followed closely Gauß's idea of studying the representation of numbers by ternary forms via the representation by binary forms, in order to prove inductively his famous result on the existence of a set of integers $(\alpha, \beta, \ldots, \lambda)$ such that $|f(\alpha, \ldots, \lambda)| < \left(\frac{4}{3}\right)^{\frac{n+1}{2}} \sqrt{n}\, D$ (f quadratic form of discriminant D). He deduced from it results on simultaneous approximations by rational numbers, but also results on the norm of algebraic numbers.

From this point, Hermite developed an alternative program to study algebraic integers, and classify them, through decomposable forms and associated quadratic forms. This program was taken up by several mathematicians including Charve, Jorolan, Poincaré, Humbert in the second half of the nineteenth and in the early twentieth century. It was paradoxically taken by Poincaré in particular as a paradigm for his thesis that continuity and its study (in connection with physics) should be at the forefront of research. The relationship between Gauß's close reading by Hermite and his disinterest for the problems of rigor and foundations (in contrast to what happened for algebraic number theory) was also studied.

The Impact of Gauß's Disquisitiones Arithmeticæ on the Theory of Solvability of Equations by Radicals

Olaf Neumann

The talk was introduced by some remarks on the importance of problems and challenges in mathematics supporting the thesis that mathematics in the long run is developing along problems and the tools aiming at their solution. As an example of a problem intriguing mathematicians over three centuries the question is stressed which equations can be solved by radicals. Gauß's contributions towards a solution are to be found in section VII. of *Disquisitiones Arithmeticæ* (D.A.). It is Gauß's solution of $(X^p - 1)/(X - 1) = 0$ (p a prime) by radicals which inspired Abel's more general theory of equations with the famous commutativity condition. As Gauß himself pointed out his exposition has some gaps. However, those gaps either can be filled in or can be circumvented by his own unpublished calculations of Gaußian sums (Lagrange resolvents for cyclotomic equations) and by the arguments in Abel's paper „*Sur une classe particulière des équations* ...“. Both Abel and Galois quoted Gauß's D.A. as a model for their investigations into equations more general than cyclotomic ones. These facts witness the deep and far-reaching impact of Gauß's D.A. on the theory of equations.

Elliptic Functions and Number Theory

Christian Houzel

Starting from some indications in the *Disquisitiones Arithmeticæ* or in Gauß's *Nachlass* of links between number theory and elliptic functions, I have explained the ulterior developments in the work of Abel, Jacobi, Kronecker and Hermite.

The first topic is the solvability by radicals of the equation of division of periods in the lemniscatic case, related to the presence of complex multiplication (Abel). Abel stated the general problem of complex multiplication of elliptic functions and announced the solvability by radicals of the equation of the singular moduli.

This statement was the point of departure for Kronecker's work on complex multiplication. Kronecker establishes close links between the structure of the equation of singular moduli and the classification of binary quadratic forms.

Another topic is related to the Fourier series of theta-functions and the fundamental identity of Gauß-Jacobi. It was used for applications to arithmetics by Jacobi and Hermite. Kronecker also used such methods.

A Program of Solving the Twelfth Problem of Hilbert

Masahito Takase

There exists the intimate relationship between Gauß's *Disquisitiones Arithmeticæ* (D.A.) and the twelfth problem of Hilbert. There are five threads coming from Gauß's D.A.: Reciprocity Law, Theory of Abelian Equations, Theory of Complex Multiplication, Inverse Problem of Jacobi and Theory of Analytic Functions of Several Complex Variables. They are organically related and they converge on the twelfth problem of Hilbert.

We meet the intriguing phrase „the theory of analytic functions of several complex variables“ in the statement of Hilbert about his twelfth problem.

Gauß's D.A. is the common origin of the theory of analytic functions of several complex variables and the twelfth problem of Hilbert.

The germ of the theory of analytic functions of several complex variables can be found in the inverse problem of Jacobi, which is the theme of the theory of algebraic functions of one complex variable. Owing to Weierstrass and Riemann and some others, we can solve the inverse problem of Jacobi and so we have a general notion of Jacobian functions of arbitrary numbers of variables.

Jacobian functions are transcendent functions, the domains of existence of which are algebraically ramified over Abelian manifolds. We should consider them as true generalizations of elliptic functions. The division theory and the transformation theory should be developed for the Jacobian functions not for the Abelian functions. However, we do not know in detail about them. About fifty years ago the Japanese mathematician Kiyoshi Oka tried to develop the algebraic functions of several complex variables, but he did not succeed in solving it and no one followed up on Oka's idea.

Jacobian functions will play the same role for some specific algebraic number fields as the exponential functions for the rational number field, and as the elliptic modular functions for the imaginary quadratic number field. If we could succeed in finding and discussing such algebraic number fields, the theory of analytic functions of several complex variables would benefit, and we would overcome the difficulties which interrupted Oka about fifty years ago.

Gauß Sums

Samuel James Patterson

Gauß's discovery of the determination of the ‚sign' of the quadratic Gauß sum was made in the middle of May 1801 and included in the *Disquisitiones Arithmeticæ*, Art. 356 even though Gauß had no proof at that time. He only found a proof on 30^{th} August 1805; he described the discovery in a letter to Olbers on the 3^{rd} September 1805. This letter is interesting for a number of reasons. Gauß's proof, which was based on a q-analogue of the identity $\sum(-1)^j\binom{n}{j} = 0$ was presented to the Königliche Akademie in Göttingen in 1808 and published in 1811. The first person to take this then up again was Lejeune Dirichlet in 1835 (publ. 1837) who gave an elegant analytic proof. More or less the major lines of proof were found in 1850 but one method, of significance later, was first announced by E. Hecke in the period 1917-1923. One important consequence of these investigations was the determination of the theta-multiplier (Jacobi, Hermite) which was later to be essential for Siegel's general theory of theta functions and Weil's adelic formulation of it.

Hecke's proof was a consequence of his analytic continuation of ζ- and L-functions, and the product formula of class-field theory. Although Dedekind had introduced his ζ-function of a field as early as 1871 the first results on analytic continuation were found by E. Landau in 1903; these paved the way to Hecke's discoveries and many of the developments of the 20^{th} century.

The Development of the Hauptgeschlechtssatz

Franz Lemmermeyer

Gauß's theory of binary quadratic forms is, in modern textbooks, usually described using the language of quadratic number fields. The principal genus theorem can then be formulated as follows: let $k = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with discriminant d, let $D = d_1 \cdot \ldots \cdot d_t$ be its factorization into prime discriminants, and let $C = Cl^+(k)$ denote the class group in the strict sense. Given an ideal class $c \in C$, write $c = [a]$ for an ideal

a coprime to $2d$, and put $\chi_i(c) = \left(\frac{d_i}{Na}\right)$. These genuscharacters define a homomorphism $\Psi = (\chi_1, \dots, \chi_t) : Cl^+(k) \longrightarrow \mathbb{F}_2^t$, whose kernel is the principal genus. The content of the principal genus theorem is the statement that $ker\Psi = C^2$.

The only pre-Gaußian result in this direction was Euler's conjecture that primes p have the form $p = x^2 - ay^2$ if and only if $p = 4an + r^2$ or $4an + r^2 - a$. Lagrange pointed out that this was false if x, y are to be integers, and more general the counter example $x^2 - 79y^2 \neq 101$, although $101 = 4 \cdot 79 \cdot (-4) + 38^2 - 79$. It has gone unnoticed until now that Euler's conjecture becomes correct if the conditions $x, y \in \mathbb{Z}$ are weakened to $x, y \in \mathbb{Q}$; in fact, the conjecture then becomes equivalent to the principal genus theorem.

The last part of the talk described Kummer's genus theory, Hilbert's reformulation, and finally Noether's Hauptgeschlechtssatz.

The Theory of Algebraic Number Fields in the Period 1930-1952

HELMUT KOCH

The talk begins with some general remarks about the development of theories in five steps: 1. Introductory period, 2. Basic period, 3. Heroic period, 4. Simplifications and complementations, 5. The theory becomes material of other theories. Then we show how these steps give a periodization of the theory of algebraic number fields. It follows the description of the period from 1930 to 1952 as a time of reformulation and simplification of the already established class-field theory. It begins with a series of important papers by Hasse which create local class-field theory and connect it with simple algebras. In 1936 Chevalley introduces ideles (under the name of ideal element) which allows him in 1940 to give a smooth formulation of global class-field theory. In the period 1950-1952 Hochschild and Nakayama come to a foundation of class-field theory by means of the cohomology of groups and this development is complete with Tate's introduction of modified cohomology groups and the formulation of the Artin map as an isomorphism between cohomology groups given by the cupproduct with the canonical class.

Section Eight of the Disquisitiones Arithmeticæ and the Theory on Function Fields

GÜNTHER FREI

Es wurde das achte Kapitel der *Disquisitiones arithmeticae* vorgestellt und dessen Inhalt mit den modernen Begriffsbildungen erklärt. Darin gibt Gauß einen dritten Beweis des Reziprozitätsgesetzes, der vollständig verschieden ist zu den beiden vorhergehenden. Gauß entwickelt darin eine Theorie der Funktionenkörper über einem endlichen Körper $\mathbb{F}_p$, die völlig analog seiner Theorie der Kongruenzen in $\mathbb{Q}$ verläuft. Die analoge Theorie der Kreiskörper führt zum dritten Beweis des Reziprozitätsgesetzes. Dabei spielt der Frobenius-automorphismus eine wichtige Rolle, den Gauß als wichtiges Hilfsmittel einführt und für den er die wichtigsten Eigenschaften herleitet. Daraus ergibt sich u.a. eine vollständige Theorie der endlichen Körper. Um seine Theorie auch auf Funktionenkörper über $\mathbb{F}_{p^q}$ zu erweitern, beweist Gauß das Henselsche Lemma. Seine Fortschritte in der Theorie der Funktionenkörper, die alle 1797 erzielt wurden, lassen sich auch anhand des Tagebuchs sehr schön verfolgen.

Merkwürdige Betrachtungen über Zahlen in der AFAS

AnneMarie Décaillot

Die vorgestellten Arbeiten vor allem in der Association Française pour l'Avancement des Sciences (AFAS) lassen einen breiten Platz für Zahlenproblematik, für „merkwürdige Betrachtungen" über Zahlen, die sowohl Laien als auch Berufsmathematiker anstellten. Die AFAS wurde 1872 von einer Gruppe bekannter Wissenschaftler gegründet (darunter Claude Bernard, Marcelin Berthelot, Louis Pasteur, Jean-Baptiste Dumas, Adolphe Wurtz), die den Wunsch hegten, zu einer moralischen Wiederherstellung des Landes nach dem tragischen Konflikt von 1870 beizutragen. Tatsächlich ist die Zeit nach dem Krieg von 1870 eine für die Verbreitung der Wissenschaften sehr günstige Zeit. „Par la Science, pour la Patrie", so lautet die Devise des Verbandes, der sich während der aufsteigenden Phase der Dritten Republik entwickelt.

Unter den verschiedenen Zahlenproblemen, die in der AFAS vorgestellt wurden, soll hier den Arbeiten des Arithmetikers Edouard Lucas (1842-1891) der Vorrang gegeben werden, und ebenfalls den Arbeiten von Georg Cantor, des ersten deutschen Gelehrten, der zu einem Kongress des Verbandes 1894 nach Caen eingeladen wurde. Diese Einladung stellt schon an sich ein Ereignis dar, denn die AFAS hatte die Idee dieser Einladung aus nationalistischen Gründen bis zu diesem Zeitpunkt verworfen.

On the Composition of Binary Quadratic Forms

Joachim Schwermer
(joint work with Martin Kneser)

The discussion of the composition of binary quadratic forms in Gauß's *Disquisitiones Arithmeticæ* was briefly described in terms of current mathematics. It was pointed out in which generality the results of Gauß are valid, e.g., the restrictions on the discriminants of the quadratic forms involved were stated.

The approach of M. Kneser to the composition of binary modules over an arbitrary integral domain R given in 1982 (J. Number Theory **15**) gives as a special case a very clear structural framework for the considerations (and computations) of Gauß. The constructions and the results of M. Kneser were indicated and the relations with Gauß's work were discussed.

O $\theta\epsilon o\varsigma$ $\alpha\rho\iota\theta\mu\eta\tau\iota\zeta\epsilon\iota$
The Rise of Pure Mathematics as Arithmetic after Gauß

José Ferreirós

Gauß coined the motto „God arithmetizes" on the model of Plato, but showing the changing image of mathematics as based on arithmetic and no longer on the paradigm of geometry. This links to the topic of the rise of pure mathematics in Germany. The talk explored connections between pure mathematics and arithmetic, possible factors influencing this reorientation of German mathematics, and some motives that may have influenced Gauß himself.

After a cursory exposition of the trend of arithmetization after Gauß, some thoughts were offered on the role played by *Disquisitiones Arithmeticæ* as a model. I emphasized a key methodological idea that indicates that Gauß's work may have acted as a model for the „conceptual approach“ to mathematics, within which arithmetization emerged.

Next, connections between Gauß and neohumanism were explored, taking his abovementioned motto as a symbol for them. Gauß's dedication to a pure mathematical topic like number theory fits perfectly the cultural conceptions of neohumanism. At the same time, the promotion of neohumanism in the German reformed universities helps to explain the strikingly strong development of number theory in the next generations.

Finally, on the basis of the 1831 *Selbstanzeige* Gauß's views on the foundations of mathematics were explored. It was shown that they led to an image of „*allgemeine* Arithmetik“ as the complete study of the complex numbers (arithmetical, algebraical, and analytical). This links his work with later authors such as Weierstraß and Dedekind. Possible connections between Gauß's ideas and the philosophy of Kant were also explored.

Über den Einfluss von Gauß' Disquisitiones Arithmeticæ auf Kroneckers Auffassung der Mathematik

JACQUELINE BONIFACE

Die Mathematik im 19. Jahrhundert steht, nach Hilberts Ausdruck, „im Zeichen der Zahl“, und die großen Mathematiker, die ihren Geist geprägt haben, haben zu ihrer „Arithmetisierung“ beigetragen. Dennoch, wenn die Zahl der Mathematik der Zeit ihren besonderen Charakter gibt, ist es eher noch als Gedankenart, denn als Gedankeninhalt. Die Arithmetik dient nämlich als Modell für jede Theorie und die Zahl als Bezugspunkt für jeden mathematischen Gegenstand. Die Arithmetisierung der Analysis erscheint dann nicht nur als eine notwendige Etappe, sondern als das Paradigma der Mathematikentwicklung. Diese Etappe wird immer als eine Erweiterung des Bereiches der mathematischen Gegenstände durch die Vergrößerung des Zahlbegriffs beschrieben. Man weiß aber, dass diese Etappe auf zwei getrennten Wegen vollzogen wurde. Einer dieser Wege ist verkannt geblieben. Dieser wurde von Kronecker aufgenommen. Mit diesem Vortrag möchte ich Kroneckers Zahlauffassung genauer erklären und den Einfluss, den Gauß auf diesen ausgeübt hat, klarstellen.

What is Arithmetization?

NORBERT SCHAPPACHER

An attempt was made to sketch the way in which the term „arithmetic“ (or arithmetization) travelled through time along certain lines of theory-development, between roughly the 1870ies and the 1940ies.

Specifically, were discussed in this order

1. The ways in which „arithmetization“(of analysis) was discussed between 1895 and 1900 in Germany: Klein, Hilbert, Pringsheim; and also Poincaré's reactions.
2. The two actors of arithmetization who were also working arithmeticians (= number theorists): Dedekind and Kronecker.
3. Arithmetizing the notion of real number around 1872: in particular, Dedekind, Weierstrass, Cantor, Méray (and a few others ...).
4. The first spin-off story of 3): Cantor's set theory → paradoxes → Poincaré's talk in Göttingen 1908 → Hermann Weyl during and after WW I, was only hinted at.

5. The intra-arithmetic side of things: 1882 with Kronecker's *Grundzüge* and the article by Dedekind and Weber marks the beginning of another spin-off story in arithmetic theories for algebraic geometry: the digest of Dedekind-Weber by Hensel and Landsberg incorporates elements of Weierstrass's arithmetization. It merges later on into the theory of algebraic function fields of one variable over a finite field (Hasse school) which is strongly marked by a theoretical model in its way of calling theories or proofs „arithmetic".
6. Other lines of developments:
   a) H. Poincaré's research programme on the arithmetic of algebraic curves (1901) aims at an „arithmetization of diophantine analysis",
   b) Zariskis proofs of the resolution of singularities of algebraic surfaces between 1939 and 1941 are called by him „arithmetic", presumably in reference to the Dedekind-Weber tradition on the one hand, and Krull's valuation theory on the other.

„Räumliche Anschauung" in the Work of Minkowski and Others

JOACHIM SCHWERMER

The suggestion made by Gauß in 1840 and pursued by P.G.L. Dirichlet in 1848 for binary forms to interpret quadratic forms as lattices in space was decisive for Minkowski's approach to the theory of quadratic forms. He had seen in the work of Ch. Hermite and others, the fruitful way in which the arithmetic theory of quadratic forms could be used to obtain results in number theory. In particular, the problem of giving a qualitatively good estimate for the ratio of the minimum of a positive definite quadratic form in n variables to the nth root of its determinant by a constant depending only on n played an important role in these applications. Minkowski developed a new field, called „geometry of numbers", by introducing the concept of volume and other geometrical notions as convexitivity into this circle of ideas. The first objective of this talk was to describe how visual thinking has entered (or appears in) Minkowski's early works at a very critical point, and to discuss briefly on which previous contributions (and points of views) of others he relies. The book *Geometry of Numbers* was announced in 1893, and finally published in 1896. In fact, the ideas emerged quite a bit earlier. For (predecessing) sources on Minkowski's development of his ideas toward the „geometry of numbers" I could mostly draw on previously unknown manuscripts. In particular, the manuscript „*Über einige Anwendungen der Arithmetik in der Analysis*" written for his „Probevorlesung" at the occasion of his habilitation 1887 in Bonn is decisive. In the second part the use of the geometry of numbers in Minkowski's various approaches (as shown in unfinished resp. unpublished manuscripts dated 1883, 1885, 1895) to the reduction theory of positive definite quadratic forms was discussed. Even in the final published paper in 1905 the line of argument does not use in a striking way the interpretation in terms of lattices. Later work by Schur and Bieberbach brought this circle of geometric ideas back into play. Finally, a short overview was given of the still lasting influence of the treatment of numbertheoretical questions in a geometric framework as initiated by Minkowski. This forms a basic achievement in the arithmetical theory of algebraic groups, a subject at the frontiers of current research in pure mathematics.

Edited by Martina Schneider

Participants

Dr. Reinhard Bölling
boelling@rz.uni-potsdam.de
Fachbereich Mathematik
Universität Potsdam
Postfach 601553
14415 Potsdam

Prof. Dr. Jacqueline Boniface
boniface@univ-tlse2.fr
52 rue Peyrolieres
31000 Toulouse

Prof. Dr. Anne-Marie Decaillot
deca@math-info.univ-paris5.fr
U.F.R. de Mathematiques,
Statistique et Informatique
Universite Rene Descartes
45, rue des Saints Peres
F-75006 Paris

Prof. Dr. Harold M. Edwards
edwardsh@cims.nyu.edu
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
USA

Prof. Dr. Della D. Fenster
dfenster@richmond.edu
Dept. of Math. and Computer Science
University of Richmond
Richmond, VA 23173
USA

Prof. Dr. Jose Ferreiros
jmferre@cica.es
Dpto. de Filosofia y Logica
Universidad de Sevilla
Camilo Jose Cela, s/n
E-41018 Sevilla

Prof. Dr. Günther Frei
g.frei@active.ch
Lützelstr. 36
CH-8634 Hombrechtikon

Prof. Dr. Catherine Goldstein
catherine.goldstein@math.u-psud.fr
Mathematiques
Universite Paris Sud (Paris XI)
Centre d'Orsay, Batiment 425
F-91405 OrsayCedex

Dr. Ralf Haubrich
rhaubri@gwdg.de
Im Rosengärtchen 44
61440 Oberursel

Prof. Dr. Christian Houzel
houzel@vjf.cnrs.fr
CNRS
Archives de la creation math.
UPS 2065
7, rue Guy Moquet
F-94801 Villejuif

Prof. Dr. Martin Kneser
fullek@uni-math.gwdg.de
Guldenhagen 5
37085 Göttingen

Prof. Dr. Helmut Koch
koch@mathematik.hu-berlin.de
Karl-Egon-Str. 12
10318 Berlin

Dr. Franz Lemmermeyer
franzl@csusm.edu
California State University at
San Marcos
Department of Mathematics
333 South Twin Oaks Valley Rd.
San Marcos, CA 92096-0001
USA

Prof. Dr. Olaf Neumann
neumann@minet.uni-jena.de
Fakultät für Mathematik und
Informatik
Friedrich-Schiller-Universität
Ernst-Abbe-Platz 1-4
07743 Jena

Prof. Dr. Samuel James Patterson
spatter@gwdvms.gwdg.de
sjp@uni-math.gwdg.de
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen

Dr. Herbert Pieper
pieper@bbaw.de
BBAW
Alexander-von-Humboldt-Forschungs-
stelle
Jägerstraße 22/23
10117 Berlin

Prof. Dr. Norbert Schappacher
schappa@math.u-strasbg.fr
U.F.R. de Mathematique et
d'Informatique
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg-Cedex

Martina Schneider
mschnei@wmpi06.math.uni-
wuppertal.de
Seilerstr. 28
42103 Wuppertal

Prof. Dr. Erhard Scholz
Erhard.Scholz@math.uni-wuppertal.de
Fachbereich 7: Mathematik
U-GHS Wuppertal
42097 Wuppertal

Prof. Dr. Joachim Schwermer
joachim.schwermer@univie.ac.at
Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Wien

Prof. Dr. Urs Stammbach
stammb@math.ethz.ch
Mathematik Departement
ETH Zürich
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Masahito Takase
takase@math.kyushu-u.ac.jp
Graduate School of Mathematics
Kyushu University 33
Fukuoka812-8581
JAPAN

Prof. Dr. Hans Wußing
Braunschweiger Str. 39
04157 Leipzig