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Positivität von Polynomen

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The meeting was organized by Eberhard Becker (Dortmund), Christian Berg (København) and Alexander Prestel (Konstanz). The abstracts of the talks are listed below. The list begins with Becker's introductory talk and continues in alphabetical order according to the speakers' last names.

The concept of the meeting

Eberhard Becker

The meeting brings together researchers from various areas in mathematics to discuss recent results and future directions in the study of positive polynomials. More precisely, the theory of moments in functional analysis, real algebraic geometry, optimization theory, applications in engineering as well as symbolic algorithms and complexity issues in computational real algebraic geometry formed the topics of this meeting. K. Schmüdgen's solution of the moment problem for compact semialgebraic sets in 1990 and the subsequent refinements by Putinar et al., the algebraic approaches by Wörmann, Prestel/Jacobi were the topic of many talks. On the other hand, modern optimization and its application to minimizing polynomials on semialgebraic sets, where the former methods are applied, took a great part in the meeting. All was supplemented by applications to control theory and algorithmic issues.

Abstracts

Representation of positive functions by analytic or smooth functions

Francesca Acquistapace

(joint work with Carlos Andradas and Fabrizio Broglia)

For a not bounded basic closed set $X = \{t_1 \geq 0, \ldots, t_r \geq 0\}$ and a polynomial p verifying $p > 0$ on X we find a representation

$$p = \sigma_0 + \sigma_1 t_1 + \cdots + \sigma_r t_r$$

where $\sigma_0, \sigma_1, \ldots, \sigma_r$ are analytic functions strictly positive on $\mathbb{R}^n$, hence squares.

This is a consequence of a strict positivstellensatz for the ring $\mathcal{O}(\mathbb{R}^n)$ of global analytic functions. A similar result can be proved also for the ring $\mathcal{C}^k(\mathbb{R}^n)$, $0 \leq k \leq \infty$ and for the ring $\mathcal{D}^k(\mathbb{R}^n)$, $0 \leq k < \infty$ of definable functions on a o(rder)–minimal structure expanding $(\mathbb{R}, \exp)$.

Barrier functions for positive matrices and polynomials

Andreas Bernig

(joint work with Eberhard Becker and Antonio Diaz Cano)

After the definition of barrier functions, I study a simple example (the cone of positive definite symmetric matrices) from the differential–geometric viewpoint. This yields to a very well–known symmetric Riemannian manifold of rank n. Concerning the cone of positive polynomials, I propose a good candidate for a barrier function, satisfying at least $3\frac{1}{2}$ of 4 required properties.

Algebraic Varieties Arising in Truncated Complex Moment Problems

Raúl E. Curto

Given complex numbers

$$\gamma \equiv \gamma^{(4)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40},$$

with $\gamma_{ij} = \overline{\gamma_{ji}}$, the quartic complex moment problem for γ entails finding conditions for the existence of a positive Borel measure μ, supported in $\mathbb{C}$, such that

$$\gamma_{ij} = \int \overline{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 4).$$

In joint work with Lawrence A. Fialkow we have recently obtained a complete solution to the quartic moment problem in the case when the associated moment matrix $M(2)(\gamma)$ is singular. Each representing measure satisfies card supp $\mu \geq$ rank $M(2)$, and we have developed concrete necessary and sufficient conditions for the existence and uniqueness of representing measures, particularly minimal ones.

We show that rank $M(2)$–atomic minimal representing measures exist in the case the moment problem is subordinated to an ellipse, parabola, a non–degenerate hyperbola. If the quartic moment problem is subordinated to a pair of intersecting lines, minimal representing measures sometimes require more than rank $M(2)$ atoms, and those problems subordinated to a general intersection of two conics may not have any representing measure at all. As an application, we describe in detail the minimal quadrature rules of degree 4 for arclength on a parabolic arc.

We then extend our results to solve the so–called parabolic MP, that is, one in which the columns of the associated real moment matrix $M_{\mathbb{R}}(n)$ (for arbitrary $n \geq 1$) satisfy $Y = X^2$. We do this by appealing to a crucial estimate linking the rank of $M_{\mathbb{R}}(n)$ and the cardinality of the associated algebraic variety. Many of the results extend to other general quartic MP, associated to the prototypical column relations $YX = 1$ and $YX = 0$.

Truncated Multivariable Moment Problems

Lawrence Fialkow

For complex numbers $\gamma \equiv \gamma^{(2n)} = \{\gamma_{ij}\}_{0 \leq i+j \leq 2n}$ and $K \subset \mathbb{C}$ (closed), the moment problem entails finding a positive Borel measure μ, supp $\mu \subset K$ so that $\gamma_{ij} = \int \overline{z}^i z^j \, d\mu$ $(0 \leq i+j \leq 2n)$. In collaboration with R. E. Curto, we study conditions for the existence of (finitely atomic) representing measures in terms of positivity and extension properties of the moment matrix $M(n)(\gamma)$ associated to γ. Necessary conditions are that $M(n)$ is positive, recursively generated, and that card $V(\gamma) \geq$ rank $M(n)$, where $V(\gamma)$ is the variety associated to γ. We study polynomials $p(z, \overline{z})$, $\deg p \leq n$, such that $\gamma^{(2n)}$ has a measure whenever the above conditions are satisfied and there is a dependence relation $p(z, \overline{z}) = 0$ in the column space of $M(n)$. Exactly which polynomials have this property is an open question; examples include: any analytic polynomial $p(z)$; $y = x^2$; $\overline{z}z = a + bz + c\overline{z} + dz^2 + e\overline{z}z$.

Hyperbolic Polynomials: theory and applications

Osman Güler

These polynomials originated in partial differential equations. Such a polynomial $p(x)$ has a convex cone associated with it, called the hyperbolicity cone. We show that $-\log p(x)$ is a self–concordant barrier, with striking properties which are useful for designing long–step interior point methods. Many practical problem classes in convex programming can be looked at from this point of view, such as linear programming, semi–definite programming, etc. There are also potentially useful problem classes that need future development such as programming over some symmetric functions. Also, we discuss the roots of such polynomials: they satisfy many inequalities similar to the ones satisfied by the eigenvalues of symmetric matrices. We end the talk with a speculation that something like the Horn conjecture (recently solved) might be true for the roots of hyperbolic polynomials.

Barrier Functions for Symmetric Cones

Raphael Hauser

Self–scaled barrier functions are fundamental objects in the theory of interior–point methods for linear optimization over symmetric cones, a special class of cones of positive polynomials.

Symmetric cones can be classified in terms of a decomposition into irreducible components. We show that self–scaled barriers allow a similar classification: Any self–scaled barrier on a symmetric cone K can be decomposed into irreducible components that are affine transformations of the universal barrier on the irreducible components of K.

Solving some global optimization problems via positive polynomials

Jean B. Lassere

We consider the global minimization problem $\mathbb{P}$ of minimizing a polynomial f over the set $K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$ where the g_i's are all real–valued polynomials. We define a sequence $\{\mathbb{Q}_i\}$ of positive semi–definite relaxations of $\mathbb{P}$. Then under the condition that the Jacobi–Prestel–Putinar "linear" representation holds for polynomials f strictly positive on K, we prove that $\inf \mathbb{Q}_i \uparrow \inf \mathbb{P}$ as $i \to \infty$. In many cases, the global optimal value is obtained at a particular relaxation (when the representation holds for $f - \inf \mathbb{P}$). Several other issues are discussed.

Optimization of polynomials using partial moment sequences

Murray Marshall

Let $\mathbb{R}[\underline{x}]$ denote the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$. Fix a finite set $S = \{g_1, \ldots, g_s\}$ in $\mathbb{R}[\underline{x}]$, let $K_S = \{p \in \mathbb{R}^n \mid g_i(p) \geq 0, i = 1, \ldots, s\}$ and let M_S denote the quadratic module in $\mathbb{R}[\underline{x}]$ generated by S. Fix $f = \sum_\gamma f(\gamma) x^\gamma \in \mathbb{R}[\underline{x}]$ and assume f attains a finite minimum value $f^\star$ on K_S. Let $\Lambda(d)$ denote the set of n–tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i integers, $\sum \alpha_i \leq d$ and let $\mathcal{M}_{2d} = \{y \in \mathbb{R}^{\Lambda(2d)} \mid y_0 = 1\}$. Define f_d to be the minimum value of $\lambda = \sum_\gamma f(\gamma) y_\gamma$, y running through $\mathcal{M}_{2d}$ (with $2d \geq \deg(f)$) subject to the constraints:

(1) For each $i = 0, \ldots, s$ the symmetric matrix $((g_i * y)_{\alpha+\beta})$, $\alpha, \beta \in \Lambda(d - \frac{v_i}{2})$ is PSD where $v_i = \deg(g_i)$ and $g_0 := 1$.

Then $\{f_d\} \nearrow$ and $f^{\text{sos}} \leq \lim_{d\to\infty} f_d \leq f^\vee \leq f^*$ where $f^{\text{sos}} = \sup\{\lambda \mid f - \lambda \in \mathcal{M}_S\}$, $f^\vee = \sup\{\lambda \mid f - \lambda \in \overline{\mathcal{M}_S}\}$, where $\mathcal{M}_S$ is the closure of $\mathcal{M}_S$. The exact relationship between f^{sos}, $\lim_{d\to\infty} f_d$, and $f^\vee$ is not well–understood. In the special case where K_S is compact and $\exists r \in \mathbb{R}$ such that $r - \|\underline{x}\|^2 \in \mathcal{M}_S$, Lasserre used a result of Jacobi–Putinar to prove that $f^{\text{sos}} = \lim_{d\to\infty} f_d = f^\vee = f^*$. If the moment problem fails for $\mathcal{M}_S$ then $f^\vee < f^*$ in general.

Now define $\bar{f}_d$ to be the minimum value of λ such that $\exists y \in \mathcal{M}_{2d}$ satisfying:

(2) For each $e \in \{0,1\}^s$, the symmetric matrix $((g^e * y)_{\alpha+\beta})$, $\alpha, \beta \in \Lambda(d - \frac{v_e}{2})$ is PSD and

(3) For each $e \in \{0,1\}^s$, the symmetric matrix $((g^e(\lambda - f) * y)_{\alpha+\beta})$, $\alpha, \beta \in \Lambda(d - \frac{v_e}{2} - \frac{\deg f}{2})$ is PSD.

Here, $g^e := g_1^{e_1} \cdots g_s^{e_s}$, $v_e := \deg(g^e)$. The sequence $\{\bar{f}_d\}$ is increasing. Using the Positivstellensatz, $\bar{f}_d = f^*$ holds for any d sufficiently large (but depending only on the degrees of f and the g_i).

Positive Polynomials and Optimization

Yurii Nesterov

We consider some questions related to convex representation of positive polynomials of one and two variables. We show that in one dimension the condition number of Hankel matrix grows exponentially with dimension. For polynomials of two variables we show that some simple techniques (passing to polar coordinates, fixing the signs of variables) strictly increase the set of positive polynomials representable as a sum of squares.

The Positivstellensatz and Semidefinite Programming
Pablo A. Parrilo

We discuss the application of semidefinite programming techniques to problems in semialgebraic geometry. In particular, we presented a methodology for finding a priori bounded certificates to the Positivstellensatz equation to prove emptiness of semialgebraic sets. A partial comparison with alternative representations of non-negativity is made. Additionally, a simple constructive solution to the problem of finding linear representations of nonnegative polynomials over finite varieties was presented.

Representations of Real Rings and Positive Polynomials
Alexander Prestel

Let A be a commutative ring with 1. A subset P of A is called a *preordering* of A if $P+P \subset P$, $P \cdot P \subset P$, $PA^2 \subset P$, $-1 \notin P$. P is called *archimedean* if to every $a \in A$ there exists $n \in \mathbb{N}$ s.t. $n - a \in P$. If P is a maximal preordering, it also satisfies $P \cup -P = A$ and $P \cap -P$ is a prime ideal of A. Denote by $X_T^{\max}$ the set of maximal preorderings of A containing a given archimedean preordering T of A. For $P \in X_T^{\max}$, the homomorphism $\varphi_P : A \to \overline{A} := A/(P \cap -P)$ maps into $\mathbb{R}$ with $\varphi_P(P) \subset \mathbb{R}^+$. φ_P is continuous in the canonical topology of $X_T^{\max}$.

Real Representation Theorem: The map defined by $\Phi_T(a) = \hat{a}$ with $\hat{a}(P) = \varphi_P(a)$ is a homomorphism $\Phi_T : A \to \mathcal{C}(X_T^{\max}, \mathbb{R})$ such that $\Phi_T(A)$ is dense in $\mathcal{C}(X_T^{\max}, \mathbb{R})$ and

$$\hat{a} \geq 0 \text{ on } X_T^{\max} \iff na + 1 \in T \text{ for all } n \in \mathbb{N}.$$

We explained the history of the theorem, its proof, and applications to the representation of positive polynomials, strictly positive on a compact semi–algebraic subset of $\mathbb{R}^n$ (Schmüdgen's Theorem).

Quadrature domains and some of their applications
Mihai Putinar

The L–problem of moments studied by A. A. Markov leads, when extended to several variables, to extremal solutions of the form

$$d\mu = \chi_{\{p<0\}} dx,$$

where p is a polynomial and dx is Lebesgue measure. It was shown by M. Krein that these solutions, i.e. semi–algebraic sets, are characterized by finitely many moments.

It remains an open question to understand the algebraic/differential mechanism which explains this finite determination.

In the case of 2 real variables an exponential transform of the generating function of moments "linearizes" and explains via some positive definite kernel, this finite determination phenomenon. The resulting planar domains are the quadrature domains introduced by D. Aharonov and H. S. Shapiro in 1971 in connection with some conformal mapping problems.

These domains naturally appear in fluid mechanics, potential theory and operator theory.

A View of Interior–Point Methods for Convex Optimization

James Renegar

The principal mathematical ideas underlying interior–point methods for general convex optimization problems are presented. The ideas are developed from the perspective of Riemannian geometry, the local inner product being induced by the Hessians of a barrier functional whose domain is the feasible region of the optimization problem to be solved.

Cones of positive semidefinite and sums of squares of forms and duality

Bruce Reznick

Let $P_{n,m}$ and $\Sigma_{n,m}$ denote the cones of forms of degree m in n variables which are positive semidefinite and a sum of squares respectively. A great unsolved mystery is why $P_{n,m} \supsetneq \Sigma_{n,m}$ for sufficiently large (n, m), while a psd form is a sum of squares of rational functions. To understand the differences between these cones, we use the venerable inner product familiar from 19th c. apolarity and 20th c. harmonic analysis. Under this inner product, $P^*_{n,m}$ is the cone of sums of mth powers of linear forms and $\Sigma^*_{n,m}$ is the cone of forms whose associated generalized Hankel matrix is psd. The inner product has many algebraic properties, and these should be exploited too.

Computational problems related to positive polynomials

Fabrice Rouillier

Deciding if a semi-algebraic set is empty or not is critical for the study of problems related to positive polynomials. Only few implemented algorithms exist for this purpose : the Cylindrical Algebraic Decomposition (CAD) is the main one. Unfortunately, only small problems (with few variables and low degrees) can be solved using such methods.

On the other hand, many algorithms with a good asymptotic complexity are proposed in the literature. Most of them are based on the so called Critical Points Method, for computing at least one point on each semi-algebraically connected component of a real algebraic set, used as a black box for deciding if a semi-algebraic set is empty or not. Unfortunately, due to the use of various tricks for keeping a good theoretical complexity (sum of squares, infinitesimal deformations, etc.), straightforward implementations of these algorithms are inefficient.

We propose a new version of the Critical Points Method using the distance function to one (well chosen) point. Given any algebraic set V, we define an algebraic set $\mathcal{C}(V, A)$ that contains these critical points and a sub-algebraic variety of V. Our main result consists in proving that a good point A may be chosen so that $\mathcal{C}(V, A)$ is the disjoint union of a finite set of points and a sub-algebraic variety W of V with smaller dimension than V, without any restriction neither on the variety (does not need to be smooth or compact) nor on the set of polynomials used in the computations for the definition of V (for example, the generated ideal does not need to be prime).

We are thus led to compute the isolated points of $\mathcal{C}(V, A)$ and to study, in the same way, the sub-variety W. We therefore obtain an algorithm without any infinitesimal deformation whose proof is simply based on the fact that the dimension of the studied varieties strictly decreases at each step.

The limitations of such an algorithm are pointed out and solved (number of determinants) : we show how to use the theory of polynomial triangular sets to optimize the computations. We finally present some practical experiments which illustrate the practical behavior of our algorithm. It shows the interest of our approach and justifies our choices.

Degree bounds for Positivstellensatz

Marie-Françoise Roy

Stengle's positivstellensatz (1976) is the following statement:

Let F, G, H be 3 families of polynomials. Let $\mathcal{M}(F)$ be the monoid generated by F, $\mathcal{C}(F \cup G)$ the cone generated by $F \cup G$, $\mathcal{I}(H)$ the ideal generated by H. Then

$$\{x \in \mathbb{R}^n \mid \forall f \in F\ f(x) > 0,\ \forall g \in G\ g(x) \geq 0,\ \forall h \in H\ h(x) = 0\} = \emptyset$$
$$\iff \qquad \exists m \in \mathcal{M}\ \exists c \in \mathcal{C}\ \exists i \in \mathcal{I} \quad m + c + i = 0$$

It can be seen as a way of providing algebraic certificates for emptiness.

The first proof is based on Zorn's lemma.

Explicit bounds were given by H. Lombardi in 1993, they are not elementary recursive.

Elementary recursive bounds for the degree (a tower of 3 exponents) can be obtained by a method for constructing identities through case by case reasoning using

- algebraic identities for Hankel matrices (1 level of exponents)
- cylindrical decomposition method (2 levels of exponents).

Stable preorders and the non–compact moment problem

Claus Scheiderer

A preorder $P \subset \mathbb{R}[x_1, \ldots, x_n]$, generated by $g_1, \ldots, g_r$, is said to be stable if for every $d \in \mathbb{N}$ there is $N = N(d) \in \mathbb{N}$ such that every $f \in P$ with $\deg(f) \leq d$ has a representation

$$f = \sum_{\nu \in \{0,1\}^r} s_\nu \cdot g_1^{\nu_1} \cdots g_r^{\nu_r}$$

with sums of squares s_ν of degree $\leq N$. A theorem obtained in joint work with V. Powers says that under certain natural algebro–geometric conditions on the set

$$K = \{g_1 \geq 0, \ldots, g_r \geq 0\},$$

the preorder is stable and closed. This implies a large class of non–compact sets K for which the K–moment problem is not finitely solvable. On the other hand, we discuss compact sets K. If $\dim K \geq 3$, P is never stable. The question is considered in dimensions ≤ 2. We illustrate it by applying the following local–global principle: If K is compact (of any dimension) and $f \geq 0$ on K, with only finitely many zeros $M_1, \ldots, M_m$ on K, then $f \in P$ iff $f \in \widehat{P_{M_i}}$ for $i = 1, \ldots, m$, where $\widehat{P_{M_i}}$ is the preorder generated by P in the completed local ring $\widehat{\mathcal{O}_{M_i}} = \widehat{\mathbb{R}[\underline{x}]_{\mathfrak{m}_{M_i}}}$. A variety of concrete examples is discussed, and the question is raised whether P is stable in these cases. After the talk was given, Prestel gave an argument which shows that the answer to this question is negative in many cases.

Positive Polynomials and Moment Problems

Konrad Schmüdgen

In the last decade a close interaction between semi–algebraic geometry and the moment problem emerged. In the first part of the talk the operator–theoretic approach to the moment problem is developed. Let A be the complex unital $*$–algebra generated by k real functions $h_1, \ldots, h_k$ on a set. Let L be a linear functional on A such that $L(f\overline{f}) \geq 0$ for all $f \in A$. The generators $h_1, \ldots, h_k$ act as pairwise commuting symmetric operators on the Hilbert space $\mathcal{H}$ obtained from L by the GNS–construction. If the operators $h_1, \ldots, h_k$ are bounded, then

$$L(p(h)) = \int_{\sigma(h)} p(\lambda) d\langle E(\lambda)1, 1\rangle,$$

$p \in \mathbb{C}[x] \equiv \mathbb{C}[x_1, \ldots, x_k]$, where $E(\lambda)$ is the joint spectral resolution and $\sigma(h)$ is the joint spectrum of the tuple $h = (h_1, \ldots, h_k)$. In the general case there exists a positive Borel measure μ on $\mathbb{R}^k$ such that

$$L(p(h)) = \int p(\lambda) d\mu(\lambda),$$

$p \in \mathbb{C}[x]$, if and only if there is a tuple $H = (H_1, \ldots, H_k)$ of strongly commuting self–adjoint operators on a Hilbert space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$ such that $H_1 \supseteq h_1, \ldots, H_k \supseteq h_k$. It is shown that the latter is true if the operators $h_2, \ldots, h_k$ are bounded.

In the second part of the talk the moment problem and a possible generalization of the strict positivstellensatz for non–compact semi–algebraic sets are discussed. Among others we obtain the following result: Let C be a compact semi–algebraic subset of $\mathbb{R}^d$ and let K be a semi–algebraic set in $\mathbb{R}^{d+1}$ with preorder P. Let L be a linear functional on $\mathbb{C}[x_1, \ldots, x_{d+1}]$ such that $L(P) \geq 0$. If $K \subseteq C \times \mathbb{R}$, then there exists a positive Borel measure μ on $\mathbb{R}^{d+1}$ such that

$$L(p(x)) = \int p(\lambda) d\mu(\lambda) \qquad \text{for all } p \in \mathbb{C}[x_1, \ldots, x_{d+1}].$$

If $K = C \times \mathbb{R}$, then the measure μ can be chosen such that supp $\mu \subseteq K$.

Some interesting recent results by S. Kuhlmann / M. Marshall and by V. Powers / C. Scheiderer are also discussed.

A new approach to Schmüdgen's theorem and complexity

Markus Schweighofer

We prove the following bound for Schmüdgen's Positivstellensatz: Suppose $g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n]$ are polynomials defining a non–empty semialgebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

contained in the open ball around 0 of radius r. Suppose $\varepsilon > 0$. Then there exists $c \in \mathbb{N}$ such that all $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $d \in \mathbb{N}$ strictly positive on S can be written

$$f = \sum_{\delta \in \{0,1\}^m} q_\delta g_1^{\delta_1} \cdots g_m^{\delta_m}$$

where, for all $\delta \in \{0,1\}^m$, q_δ is a sum of squares of polynomials such that the degree of $q_\delta g_1^{\delta_1} \cdots g_m^{\delta_m}$ does not exceed

$$cd^2 \left(\left(d^2 (n+\varepsilon)^d r^d \frac{\|f\|}{\min\{f(x) \mid x \in S\}} \right)^c + 1 \right).$$

Here $\|f\|$ the maximum of the absolute values of the coefficients of f. The proof combines a "tame" version version of the speaker's "algorithmic approach to Schmüdgen's Positivstellensatz" (Journal of Pure and Applied Algebra **166** (2002) 307–319) based on Pólya's theorem on positive forms with a complexity bound for Pólya's theorem as given by Loera and Santos and improved by Powers and Reznick, and a Łojasiewicz inequality. The result can be used to make statements about the duality gap in optimization of polynomials using partial moment sequences and Positivstellensätze.

Applications of Positive Polynomials in Control Theory

Bernd Tibken

In the design of control systems the main issue is to ensure asymptotic stability of the closed loop system, i.e. the state $x(t)$ of the system has to be bounded and $\lim_{t\to\infty} x(t) = 0$ has to hold for all $x(t)$ with initial condition $x(0)$ near the origin 0 of the state space. The basic tool to investigate asymptotic stability and to estimate the region of attraction

$$\Omega = \{x^0 \mid \lim_{t\to\infty} x(t) = 0, x(0) = x^0\}$$

are Lyapunov functions. These functions are assumed to be positive definite near 0 and the time derivative along the flow of the control system has to be negative definite near 0 in order to ensure asymptotic stability. An estimate of the region of attraction is given by

$$S = \{x \mid V(x) < c\} \quad \text{with} \quad c = \min\{V(x) \mid \dot{V}(x) = 0, x \neq 0\}$$

where $V(x)$ is the Lyapunov function used and $\dot{V}(x)$ is the time derivative. For polynomial dynamical systems and polynomial Lyapunov functions this is a polynomial optimization problem. In order to solve the problem globally optimal the representation of positive polynomials on compact semialgebraic sets introduced by Jacobi and Prestel is used. Namely, we have

$$-\dot{V}(x) = q_0(x) + q_1(x)(\tilde{c} - V(x))$$

with q_0 and q_1 sums of squares and $\tilde{c} \leq c$, respectively. This condition is reformulated as an LMI–problem using a simple ansatz of bounded degree for q_1 and solving for q_0 by comparison of coefficients. The gramian matrices of q_0 and q_1 define the LMI constraints and $\tilde{c}$ (which has to be optimized) enters as generalized eigenvalue. Thus, c is computed by a LMI constrained generalized eigenvalue problem. Some examples show the effectiveness of this approach. In principle only lower bounds for c are computed but these lower bounds increase strictly with the degree of the ansatz for q_1. In most of the practical cases degree two or four are sufficient. The method has been tested for several benchmark examples from literature and performed very well.

Functional Analysis Methods in the Study of Positive Polynomials

Florian-Horia Vasilescu

The description of positive polynomials is a subject of interest in both algebraic geometry and functional analysis, involving these two domains in a rather intricate manner. In spite of various difficulties related to the structure of positive polynomials, in some cases one can solve moment problems using results of algebraic geometry. Conversely, solving appropriate moment problems turns out to be an efficient method leading to description of some classes of positive polynomials.

Using functional analysis methods, more precisely methods related to the theory of commuting self–adjoint operators, M. Putinar and myself proved the following result:

Theorem: Let $p, p_1, \ldots, p_m$ be polynomials in n variables, having real coefficients and even degrees. Let also

$$\Theta(t) = (1 + t_1^2 + \cdots + t_n^2)^{-1}, \quad t = (t_1, \ldots, t_n) \in \mathbb{R}^n .$$

We denote by $P, P_1, \dots, P_m$ the homogenizations of $p, p_1, \dots, p_m$ respectively, and assume that $P(x) > 0$ whenever $x \in \bigcap_{k=1}^m P_k^{-1}(\mathbb{R}^+)$, $x \neq 0$. Then there exists an integer $\nu \geq 0$ and a finite collection of real polynomials $\{q_l, q_{kl}\}$, $l \in L$, $k = 1, \dots, m$, such that

$$p(t) = \Theta(t)^{2\nu} \left(\sum_{l \in L} q_l(t)^2 + \sum_{k=1}^m \sum_{l \in L} p_k(t) q_{kl}(t)^2 \right), \quad t \in \mathbb{R}^n.$$

The proof is based on an integral representation formula as well as a separation lemma.

Edited by Markus Schweighofer

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