Mathematisches Forschungsinstitut Oberwolfach

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Arbeitsgemeinschaft mit aktuellem Thema: A-Infinity Structures and Mirror Symmetry

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The Arbeitsgemeinschaft was organized by Paul Seidel (IAS) and Kenji Fukaya (Kyoto).

The main theme of the meeting was homological mirror symmetry. The first talks introduced the concept of A_∞ – algebras and A_∞ – categories, the main algebraic structure underlying the theory. A_∞ – structures were first introduced by Stasheff, and subsequently applied to homological algebra by many authors. Floer homology groups for Lagrangian intersections carry the structure of A_∞ – category, called Fukaya category. According to homological mirror conjecture, this structure should coincide with the similar structure arising in the category of coherent sheaves on complex manifolds. After discussing several special cases of this conjecture, the last part of the meeting focused on the general homological mirror conjecture.

Inspiring talks at this meeting gave a comprehensive overview of the homological mirror symmetry and stimulated many fruitful discussions.

Abstracts

A_∞ – algebras

M. Markl

The aim of this introductory talk is to give a precise definition of an A_∞ – algebra and of all its “materialisations” – as a co–differential on the tensor co–algebra as well as a degree 1 vector field in a formal non–associative geometry.

We discuss homotopy invariance of A_∞ – algebra and we show how these structures relate to our chain homotopy equivalence. This all follows from the fact that A_∞ – algebras are homotopy invariant concepts in algebra.

A_∞ – categories

Yu. Drozd

The talk covers main definitions: A_∞ – categories and pre - categories, A_∞ – functors, their natural transformations, A_∞ – category of functors. We also give some examples: DG – categories viewed as A_∞ – categories, A_∞ – (pre)category of curves on a Riemann surface etc. Especially, representable functors (to the category of complexes) are considered, as well as an A_∞ – analogue of Yoneda's theorem. We also consider the notions of units (strict unit, homological and homotopy units) of quasi – isomorphisms, especially of equivalence of A_∞ – categories, and their relations.

Morse theory 1

C.G. Liu

In this talk we give a brief introduction to the Morse homology theory and the cup product structure of the Morse homology.

For a closed Riemann manifold M and a Morse function f defined on it, we define the Morse – Witten complex with the boundary ∂. Since $\partial^2 = 0$ we have the Morse homology $H_*(C_*(f), \partial)$. This Morse homology is isomorphic to the usual homology: $H_*(C_*(f), \partial) \cong H_*(M, \mathbf{Z})$.

Morse theory 2

S. Goette

Starting from the non-associative cup product in Morse theory introduced in the previous talk, Fukaya and Oh define an A_∞-pre-category $\mathcal{MS}(M,g)$ on a Riemannian manifold (M,g). The objects of $\mathcal{MS}(M,g)$ are the smooth functions on M. For transversal pairs f_1, f_2, they take $\big(\mathrm{Hom}_{\mathcal{MS}}(f_1,f_2), m_1\big)$ to be the Thom-Smale complex generated by the critical points of $f_2 - f_1$. The higher compositions maps m_2, m_3, $\ldots$ are defined by counting embeddings of ribbon trees into M such that every edge follows the gradient flow of one of the functions $f_j - f_i$. Thus, m_2 is just the cup product of the previous talk. The transversal sequences form a Baire set chosen such that the intersections at the vertices of each tree are transversal.

Kontsevich and Soibelman noticed that the definition above fits nicely together with results of Merkulov and Harvey-Lawson. They start with an A_∞-category with objects as above, but let $\big(\mathrm{Hom}_{\mathcal{MS}}(f_1,f_2), \mu_1\big)$ be the complex of smooth differential forms on M. The composition map is $\mu_2 = \wedge$, $\mu_3 = \mu_4 = \cdots = 0$. Passing to currents, Harvey and Lawson show how to project the de Rham complex homotopically onto the complex generated by δ-distribution along the stable cells of $f_2 - f_1$, if $f_2 - f_1$ is a Morse function and ∇F satisfies Smale's transversality conditions. In such a situation, Merkulov gave inductive formulas for the definition of an A_∞-structure on the image of such a projection. In this special case, one obtains precisely the A_∞-precategory of Fukaya and Oh. The remarkable fact is that the combinatorics of Merkulov's formulas can be described by precisely those ribbon trees that appear geometrically in the construction of Fukaya and Oh.

Obstructions

T. Kuessner

We consider weak A_∞-algebras, i.e., families of maps $\{m_n : V^{\otimes n} \to V\}_{n \geq 0}$ of degree $2 - n$ such that $\hat{d}(v_1 \otimes \ldots \otimes v_n) :=$
$\sum_{k \geq 0} \sum_{l=1}^{n-k+1} (-1)^{|x_1|+\ldots|x_l|+l-1} x_1 \otimes \ldots \otimes x_{l-1} \otimes m_k(x_l \otimes \ldots \otimes x_{l+k-1}) \otimes m_{l+k} \otimes \ldots \otimes x_n$
satisfies $\hat{d}^2 = 0$. In particular, we have $m_1^2(v) = (-1)^{|v|} m_2(v, m_0(1)) - m_2(m_0(1), v)$, that is, m_0 is an obstruction to the existence of $H^*(V, m_1)$.

However, for any b of degree 1 satisfying the Maurer-Cartan equation
$\sum_{k \geq 0} m_k(b \otimes \ldots \otimes b) = 0$ we get that the deformed m_1-operator $m_1^b(v) :=$
$\sum_{k,l \geq 0} m_{k+l+1}(b^{\otimes k} \otimes v \otimes b^{\otimes l})$ satisfies $(m_1^b)^2 = 0$. Thus we get a collection of 'linearised cohomology groups' $H^*(V, m_1^b)$ indexed by solutions of the Maurer-Cartan equation.

We illustrate the use of linearised cohomology groups with the following example. Chekanov constructed invariants of Legendrian knots which are stronger than the classical ones and which are actually a toy example of the Fukaya category. The invariant is a differential algebra, associated to a knot diagram by a combinatorial construction such that the Legendrian Reidemeister moves change the differential algebra only up to stable tame isomorphisms, in particular let its cohomology unaffected. However, to get computable invariants one has to consider a collection of linearised cohomology groups (which are indeed strong enough to distinguish Legendrian knots with the same Maslov- and Thurston-Bennequin-numbers).

Introduction to Floer Theory

H. V. Le

With the name of Floer there are 3 homology theories: instanton Floer homology, Floer homology of periodic Hamiltonian systems and Lagrangian Floer homology. All of three Floer homology theories are generalizations of the Morse theory for the infinite dimensional spaces. Our introduction is devoted to the Lagrangian Floer homology. We consider the Floer homology the the exact case, i.e. $\pi_2(M) = 0$, where the Floer homology of a pair of Hamiltonian deformation equivalent Lagrangian submanifolds can be computed. We also recall the Oh extension of Floer homology for the monotone case, i.e. $\pi_2(M, L) = 0$ and the introduction by Seidel of Floer homology for graded Lagrangian submanifolds. Several applications of Floer homologies are considered.

Fukaya category for exact symplectic manifolds

B. Siebert

The topic of this talk was a discussion of the Fukaya category under the (restrictive) assumption that all involved Lagrangian submanifolds L are exact. This means $\omega = d\theta$ and $\theta|_L = dK$. In this case the technical problems of the general case are absent. In the talk I gave the heuristic definition of the higher products and mentioned the problems and the possible solutions of making it precise in the general case. Then I explained the use of the assumption "exact". Finally I discussed the case of cotangent bundles. In particular, I sketched the proof of Fukaya and Oh, which provides an equivalence with A_∞ pre-category defined by gradient flow of Morse function as introduced in previous talk.

Derived categories

B. Keller

We first define the derived category of an abelian category following Grothendieck-Verdier. Then we give Beilinson's description of the derived category of coherent sheaves on projective n-space following his two-page paper from 1978. Finally, we construct the (perfect) derived category of an A-infinity category using twisted objects following Kontsevich's talk at the 1994 ICM in Zurich.

Introduction to homological mirror symmetry 1

J. Stienstra

The hypergeometric systems of differential equations of Gelfand – Kapranov – Zelevinsky, for appropriate choice of the parameters, are highly relevant in Mirror Symmetry. Solutions to this system in the form of period integrals correspond to looking from that side of the mirror which corresponds to variation of complex structure on Y. Solutions in series form following the classical Frobenius method correspond to the other side of the mirror dealing with variations of symplectic structure on X. The actual solutions on this side correspond to elements in the $\oplus_p H^{p,p}(X)$ part of the cohomology, or rather to elements of Chow ring $\otimes\mathbf{Q}$ or $K_0(X) \otimes \mathbf{Q}$. In the spirit of homological mirror symmetry one should "lift" these elements to objects in the bounded derived category of coherent sheaves on X, and then study the monodromy representation in $Aut(\mathcal{D}^b(cohX))$. For some loops the monodromy action consists of tensoring with a line bundle.

Not mentioned in the talk was the fact that the talk focused in fact just to one "phase" (in physics terminology) and that understanding of the monodromy representations in $Aut(\mathcal{D}^b(cohX))$ requires also understanding phase transitions.

Introduction to homological mirror symmetry 2

K. Fukaya

This is a survey talk on homological mirror symmetry. Following Kontsevich's paper in ICM Zürich, homological mirror symmetry is explained as an isomorphism of two derived categories $\mathcal{D}^b(coh(M_c))$ and $\mathcal{D}^b(F(M_s))$. $\mathcal{D}^b(coh(M_c))$ is a derived category of coherent sheaves on complex manifold M_c and $\mathcal{D}^b(F(M_s))$ is a derived category of A_∞ category $F(M_s)$, whose objects are roughly Lagrangian submanifolds of a symplectic manifold M_s. One of the consequences $Aut(\mathcal{D}^b(coh(M_c))) \cong Aut(\mathcal{D}^b(F(M_s)))$ is discussed, according to the results of P. Seidel, R. Thomas etc. Namely, the symplectic diffeomorphism $M_s \to M_s$ gives a Dehn twist along a Lagrangian sphere $S^n \subset M_s$, which corresponds to a Fourier – Mukai transform $\mathcal{F} \to [Ext(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}]$, where $\mathcal{E}$ is a mirror of S^n, which is called a spherical object.

Floer homology in general case

Y.-G. Oh

In this talk, I will explain Fukaya-Oh-Ohta-Ono's construction of Lagrangian intersection Floer homology in the framework of deformation theory of filtered A-infinity algebras and their bi-modules. First I will explain how to associate a filtered A-infinity algebra (with m_0 term) to each (oriented spin) Lagrangian submanifold which is a quantum deformation of an A-infinity algebra over a countably generated singular cochain complex of the Lagrangian submanifold and then explain an obstruction theory for killing the m_0-term of the filtered A-infinity algebra. This then will be used to define the Floer cohomology for the pairs of un-obstructed Lagrangiain submanifolds.

Virtual fundamental chains and Kuranishi structure

U. Frauenfelder, K. Wehrheim

The moduli spaces of pseudo-holomorphic curves that are used to define the differential and higher products in Floer homology are in general not smooth manifolds. This is due to the occurrence of multiply covered holomorphic discs or spheres. These have a nontrivial isotropy group, and it may not be possible to perturb the equation such that the linearized operators become surjective, and at the same time preserve this symmetry. So in general, the moduli spaces are locally homeomorphic to $s^{-1}(0)/\Gamma$, where Γ is a finite group and s is an equivariant section of a finite dimensional vector bundle $E \to V$. Now a Kuranishi structure on a space X assigns to each point a germ of such a local description with some compatibility conditions. In particular, $\dim V - \operatorname{rank} E$ is constant, but $\operatorname{rank} E$ itself can vary – it is closely related to the cokernel of the linearized operator. For a compact space with a Kuranishi structure, one can define a virtual Euler class on the union (modulo transition maps) of all V/Γ. By a strongly continuous map $f : X \to Y$ (that extends to V locally), this cycle can be pushed forward to a rational homology class on Y of the expected dimension.

As an example and application, we indicate how the space of stable maps to a symplectic manifold M can be equipped with a Kuranishi structure, which leads to the definition of Gromov-Witten invariants with rational coefficients for general symplectic manifolds.

SYZ conjecture

M. Schwarz

This talk gave an introduction to the Strominger – Yau – Zaslow approach to mirror symmetry. The main objects are special Lagrangian submanifolds of a given Calabi – Yau manifold together with their moduli spaces of flat $U(1)$ – connections.

After the definition of special Lagrangian submanifolds as calibrated submanifolds with respect to the calibration by $Re\Omega$, where Ω is a covariantly constant holomorphic $(n, 0)$ – form, it was shown that the deformation space $\mathcal{M}_{SL}$ of a given special Lagrangian submanifold is unobstructed and is a smooth finite dimensional manifold with harmonic 1 – forms $\theta \in \mathcal{H}^1(L)$ corresponding to first order deformations. This is a result due to McLean.

If L is a 3 – torus in a Calabi – Yau 3 – fold, then $dim(\mathcal{M}_{SL}) = 3$ and together with the dual 3 – torus of flat $U(1)$ – connections this forms a 6 – dimensional manifold with canonical Kähler metric.

Tori

B. Kreussler

The homological mirror symmetry conjecture, as formulated in the previous talks, is up to now only partially shown and this only in very special cases. The major achievements are available in the case of tori, especially of real dimension two.

Let E^τ be the real two torus $\mathbb{R}^2/\mathbb{Z}^2$ equipped with complexified Kähler form $\tau \mathrm{d}x \wedge \mathrm{d}y$, where τ is in the upper half plane $\mathcal{H} = \mathbb{R} \times \mathbb{R}_{>0}$. Its mirror complex manifold is the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. There is an explicit description of the pre-A_∞-category $\mathcal{F}(E^\tau)$, Fukaya's category. Its objects are triples (Λ, α, M), where $\Lambda \subset E^\tau$ is the image of a line with rational slope in the universal cover $\mathbb{R}^2$ of E^τ, i.e. a special Lagrangian submanifold with respect to the standard complex structure on E^τ. The grading of Λ is given by a real number α, such that $i\pi\alpha$ is a logarithm of the slope of a line representing Λ. Finally, M denotes a local system on Λ whose monodromy has eigenvalues of modulus one. The morphisms and compositions m_k are defined for transversal collections of special Lagrangians in the usual way.

Passing to cohomology $H^0(\mathcal{F}(E^\tau))$ allows us to construct a category (and not only a pre-category), by using cohomology of local systems in the non-transversal case. By $\mathcal{FK}^0(E^\tau)$ we denote the formal closure under finite direct sums of $H^0(\mathcal{F}(E^\tau))$. The theorem of Polishchuk, Zaslow and the speaker gives an equivalence of categories $\Phi_\tau : \mathcal{D}^b(\mathrm{Coh}(E_\tau)) \to \mathcal{FK}^0(E^\tau)$ which is compatible with shift functors and finite direct sums.

In the higher dimensional case, a similar result is available. Work of C. van Enckevort and K. Fukaya constructs a bijection on the level of objects and morphisms (which is in general not known to be functorial) between subcategories of the relevant categories. These subcategories consist of semi-homogeneous vector bundles on the holomorphic side and linear special Lagrangian submanifolds, being the graph over a horizontal section of the torus fibration on the symplectic side.

Picard – Lefschetz theory

J. Ayoub

This talk deals with Lefschetz fibrations in symplectic geometry. Such a fibration $\pi : X \to S$ can only have finitely many singular points, and all of them are non – degenerate.

We first define the canonical parallel transport in "horisontal" direction, and use it to construct Lagrangian sphere in smooth fibers. These are the vanishing cycles.

We then show how we express the monodromy maps around critical values using the Dehn twist associated to these vanishing cycles.

Finally we associate to a Lefschetz fibration a directed Fukaya category. The derived category of this A_∞ category will be an invariant of the Lefschetz fibration.

Fukaya category in general case

P. Seidel

This talk explained the framework of general Fukaya categories, as developed by Fukaya-Oh-Ohta-Ono. A particularly important point is the rather complicated relation between Lagrangian submanifolds and objects of the category. This is due to the "obstruction" coming from holomorphic discs ("instanton effects"). The main applications are of course to homological mirror symmetry.

Edited by Darko Milinković

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