Mathematisches Forschungsinstitut Oberwolfach

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Renormalization Group

June 9th – June 15th, 2002

The workshop was organised by D. Brydges (British Columbia), H. Knörrer (ETH) and M. Salmhofer (Leipzig). 46 scientists from 8 countries attended. Amongst these were 10 from departments of theoretical physics. There were 15 lectures given by 13 speakers, 2 organised discussion sessions and several well-attended, spontaneous, disorganised discussion sessions. The workshop concluded with an after-dinner piano concert on Friday June 14.

The Renormalization Group is a set of principles used in the analysis of problems with many degrees of freedom associated with length scales. Such problems arise in many contexts including probability, statistical mechanics, quantum field theory and partial differential equations. The common theme is to classify examples into "universality classes" with canonical representatives obtained by "scaling limits". A beginning example is provided by the Donsker Invariance Principle. Here we start with random walk (without drift). The standard deviation of the step distribution is the smallest length scale, while an entire infinite trajectory has structure on all larger length scales. The *scaling limit* is equivalent to viewing the walk from far away so that only long distance structure is retained. The viewer will see the path of Brownian motion. This means that the random walk is in the universality class whose canonical representative is Brownian motion. In a similar way models in statistical mechanics are in universality classes labelled by quantum field theories.

The implementation of this theme varies according to the core discipline: the context is operators and Hilbert spaces for quantum field theory, microlocal analysis for PDE and combinatoric expansions for statistical mechanics. The workshop allows both ideas and scientists to migrate between these subcommunities. The whole enterprise has some grandeur, but it is hard for the younger scientists to see the ramifications without this type of meeting. The Oberwolfach traditions worked well and the meeting gave us all new optimism and a sense of the gathering power to prove analytic results in the domain of mathematical physics.

Abstracts

Operator–valued Renormalization Group Flow

V. Bach

(joint work with Th. Chen, J. Fröhlich and I.M. Sigal)

We present an improved version of the RG flow built on the Feshbach projection method. The new RG method is based on a generalization of the Feshbach projection method called the "Smooth Feshbach Map". It uses smooth cutoff functions χ rather than projections, and it maps a given Hamiltonian H on a Hilbert space $\mathcal{H}$ to an effective Hamiltonian $F_\chi(H)$ on a (smaller) Hilbert space $\mathrm{Ran}\,\chi \subseteq \mathcal{H}$. The important feature of this map is its isospectrality: H is invertible iff $F_\chi(H)$ is invertible.

As a main application, we construct a convergent RG flow on a subspace $\mathcal{W} \subseteq B(\mathcal{H})$ of operators on $\mathcal{H} \subseteq \mathcal{F}_b\big[L^2(\mathbf{R}^d)\big]$ which represent Hamiltonians in quantum field theory. Thanks to the smoothness of the cutoff function χ, the convergence proof is much simplified, and the norm requirements defining $\mathcal{W}$ (as a Banach space) are weaker (and more natural) than before, using projections.

QED on the 3–torus

J. Dimock

We consider quantum electrodynamics on a three dimensional torus. We start with the lattice gauge theory and attempt to control the singularities as the lattice spacing is taken to zero. This is accomplished by following the flow of the renormalization group transformations. The method is similar to that of T. Balaban for scalar electrodynamics.

Interacting Fermions in 3d at finite temperature

M. Disertori

(joint work with J. Magnen and V. Rivasseau)

It is believed that a system of weakly interacting Fermions in 2 or 3 dimensions, with a rotation invariant Fermi surface, is a Fermi liquid (in the sense of Salmhofer) above the critical temperature $T_c = e^{-\frac{k}{|\lambda|}}$, where k is a constant and λ is the strength of the interaction.

In the 2d case we proved this behaviour (in a work with V. Rivasseau), using a Fermionic expansion and angular analysis in momentum space. Due to the difference between 2d and 3d geometry, this proof cannot be generalized directly to the 3d case.

In a recent work with J. Magnen and V. Rivasseau we completed the first step of the proof, namely the uniform bound on completely convergent contributions. The analysis relies on a direct space decomposition of the propagator, on a bosonic multi-scale cluster expansion and on a Hadamard inequality.

Local Aspects of Renormalization II: Gauge Theories

M. Duetsch

(joint work with K. Fredenhagen and F-M. Boas)

A *local* formulation enables a consistent perturbative treatment of massless Yang-Mills theories. Such a construction requires the validity of BRST-symmetry in a suitable form. A sufficient renormalization condition is the *'Master BRST-Identity'*. To find its precise form we start with classical field theory. We formulate the most general identity which can classically be derived from the field equation: this is the *'Master Ward Identity'*. Then we quantize by the principle that we want to maintain as much as possible of the algebraic structure of the perturbative classical fields. In particular we require the Master Ward Identity as a renormalization condition. Its application to the BRST-current yields the (wanted) Master BRST-Identity.

Construction of a 2–d Fermi Liquid

J. Feldman

(joint work with H. Knörrer and E. Trubowitz)

I discuss the main ideas behind a proof that the temperature zero renormalized perturbation expansions of a class of interacting many–fermion models in two space dimensions have nonzero radius of convergence. The models have "asymmetric" Fermi surfaces and short range interactions. One consequence of the convergence of the perturbation expansions is the existence of a discontinuity in the particle number density at the Fermi surface.

The proof uses a multiscale analysis, discrete renormalization group flow and renormalization of the Fermi surface. Generalized particle–particle and particle–hole ladder diagrams require special treatment. Particle–particle ladders have improved power counting due to the assumed asymmetry of the Fermi surface, suppressing the Cooper channel. A sign cancellation between scales is used to control particle–hole ladders.

Local Aspects of Renormalization I: Renormalization of Quantum Field Theory on curved Spacetime

K. Fredenhagen

(joint work with R. Brunetti, M. Dütsch, R. Verch)

In contrast to classical field theory, the standard formulation of quantum field theory contains many nonlocal elements which have no obvious generalization to curved spacetime. Among them are the use of the Fouriertransform which relies on translation invariance, the concept of a vacuum state and of particles, the choice of a distinguished Hilbert space of states and of a Feynman propagator. Also the Euclidean formulation of quantum field theory makes not much sense on a curved spacetime since the generic spacetime with a Lorentzian metric has no analytic continuation containing a Riemannian space. Nevertheless, the ultraviolet problem of quantum field theory admits, at least at a local level, a satisfactory treatment, in agreement with the equivalence principle. The infrared problem, on the other hand, whose general treatment on curved spacetime seems to be hopeless, can be completely separated. The technique used is the algebraic formulation of field theory combined with methods from microlocal analysis. One first enlarges the algebra of the free field such that it also contains Wick products. One then analyses the Dyson series for a

Lagrangian with a spacetime cutoff described by a test function with compact support. The problem is then reduced to the definition of time ordered products as operator valued distributions. In terms of them the observables of the interacting theory can be defined such that they are, up to unitary equivalence, independent of the spacetime cutoff. The work described was published in Comm. Math. Phys. 2000-2002.

Interacting stochastic systems:
Longtime behaviour and its renormalization analysis

A. Greven

We describe typical phenomena arising in the longtime behaviour of interacting spatial stochastic systems and explain how they can be analyzed using the technique of renormalization by multiple space-time scales. We shall focus on models which arise in population genetics, in particular interacting Feller diffusions and Fisher-Wright diffusions.

The main mathematical point is to give an approximate picture of the spatial stochastic system by passing to a large space-time scale view. This will lead to a simpler stochastic process called the interaction chain. The analysis of this object reduces mainly to the study of the orbit of iterations of a certain nonlinear map in function space. Properties of this orbit can be derived by finding fixed points or fixed shapes of the nonlinear map and by showing convergence properties of general orbits to the special ones generated by fixed points or fixed shapes.

An important point is that this analysis allows to explain the special role of certain specific stochastic models, which correspond to the fixed points and fixed shapes and which characterize a universality class of longtime behaviour in a larger class of models.

Finally we outline the possible applications of the multi-scale analysis in mathematical biology, in particular evolution theory.

Triviality of Hierarchical Ising model in Four Dimensions

T. Hara

(joint work with T. Hattori and H. Watanabe)

We consider the Renormalization Group (RG) transformation for a so-called *Hierarchical Ising model*. This is a version of Ising models with specially arranged hierarchical spin interactions. Thanks to the special fractal structure of the interactions between spins, the Renormalization Group transformation (RGT) $\mathcal{R}$ takes on the following very simple form:

$$(\mathcal{R}h)(x) = \mathcal{N}\, \exp\bigl(\frac{\beta}{2}x^2\bigr) \int_{-\infty}^{\infty} h\bigl(\frac{x}{\sqrt{c}}+y\bigr)\, h\bigl(\frac{x}{\sqrt{c}}-y\bigr)\, dy \tag{1}$$

where $h(x)$ roughly denotes the Gibbs factor (or a single site measure; x corresponds to a spin variable), $\mathcal{N}$ is a normalization constant, and $c \equiv 2^{1-2/d}$ and $\beta \equiv \frac{1}{c} - \frac{1}{2}$ are parameters which depend on d. (d itself is a parameter which mimics lattice dimension.)

It is easy to see that

$$h(x) = e^{-x^2/4} \tag{2}$$

is a fixed point of the above transformation $\mathcal{R}$ (called the *gaussian fixed point*). A natural question would be to investigate local and global structure of the RGT flows (not necessarily in the vicinity of the gaussian fixed point).

To partially answer this question, we studied RGT flows starting from the Ising initial data:

$$h(x) = \delta(x^2 - K^2), \qquad K > 0, \tag{3}$$

for $d = 4$. (In the above, K is roughly proportional to the inverse temperature.)

Our result can be summarized as follows:

Theorem. *For $d = 4$, there is $K_c > 0$, such that the RGT flow starting from the initial condition (3) with $K = K_c$ converges to the gaussian fixed point, (2).*

The above result, supplemented by a more detailed estimate derived in the proof, shows that the continuum limit of this model is gaussian (i.e. triviality).

A word on the proof: The proof uses characteristic functions and correlation inequalities (which are nice), but is partly computer-supported (which is a bit disappointing, but still is rigorous).

Reference: T. Hara, T. Hattori, and H. Watanabe: *Commun. Math. Phys.* **220** (2001) 13–40

Renormalization Group and Ward Identities in $d = 2$ Grassmann Integrals

V. Mastropietro

(joint work with G. Benfatto)

We present a detailed study of the correlation functions of the XYZ Heisenberg spin chain and of models of classical Ising systems in $d = 2$ like the eight vertex model or the Aschkin–Teller model. The correlations can be written in terms of Grassmann integrals which can be evaluated by a multiscale analysis and Ward identities can be implemented in order to prove cancellations. The critical indices are written as a convergent power series and non–universality is found. Convergence is proved by bounding the determinants appearing in the Fermi expectations by the Gram–Hadamard inequality. Such results can be found in Benfatto, Mastropietro RMP (2001) and CMP (2002), and Mastropietro, Preprint (2002)

Flow Equations for Hamiltonians: Applications to dissipative quantum Systems

A. Mielke

The aim of this talk is to show how flow equations can be used to diagonalize dissipative quantum systems. Applying a continuous unitary transformation to the spin-boson model, one obtains flow equations for the Hamiltonian and for observables. Depending on the parameters, different representations of the Hamiltonian are suitable. For the super-Ohmic case the flow equations are solved approximately, yielding very accurate results. The model with an Ohmic bath and a coupling $\alpha = \frac{1}{2}$ can be solved exactly using flow equations. This approach can be used to construct controllable approximation schemes for $\alpha \neq \frac{1}{2}$.

Diffusive Dynamics in pattern forming systems

G. Schneider

We use renormalization theory to prove diffusive behaviour in pattern forming systems. Examples are the nonlinear stability of spatially periodic equilibria, as Taylor vortices of spatially periodic Bénard rolls, the nonlinear stability of modulated fronts connecting stable Taylor vortices with the Couette flow, etc. The proof uses renormalization theory, Bloch wave analysis and a fixed point argument.

Multichannel Nonlinear Scattering

A. Soffer

The Nonlinear Schrödinger equation, which appears in the Hartree Fock approximation and in nonlinear optics, is an example of a dispersive wave equation which has many different asymptotic states depending on the initial data. Such time dependent equations play a central role in many recent scientific advances,such as Bose-Einstein condensates and optical devices. I will discuss the solutions of such equations,including the large time behaviour. Rigorous results have shown, for the first time, the phenomena of ground state selection, asymptotic instability of the excited states and more. These results are obtained by deriving a novel Nonlinear Master equation and multitime scale analysis of its properties. The talk will be general for Physics and Mathematics audience.

Continuous Diagonalization of Hamiltonians

F. Wegner

A method to diagonalize/block-diagonalize Hamiltonians by means of a continuous unitary transformation is presented (F.W., Flow Equations for Hamiltonians, Annalen der Physik (Leipzig) 3 (1994) 77). Applications are given to the problem of the elimination of the electron-phonon interaction (P. Lenz, F.W., Flow Equations for Electron-Phonon Interactions, Nucl. Phys. B482 (1996) 693) and to symmetry breaking in the Hubbard model (I. Grote, E. Krding, F.W., Stability Analysis of the Hubbard Model, J. Low Temp. Phys. 126 (2002) 1385). Problems concerning the asymptotic behaviour are indicated.

Edited by Horst Knörrer

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