Mathematisches Forschungsinstitut Oberwolfach

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# Miniworkshop: Index Theorems and Modularity in Operator Algebras

June 23rd – June 29th, 2002

The meeting was organized by Klaus Fredenhagen (Hamburg), Roberto Longo (Rome) and Karl-Henning Rehren (Göttingen). Most of the 15 participants gave talks on their recent research. Essentially all talks were presented in the morning, and covered recent developments and interplays between Conformal and Algebraic Quantum Field Theory, Subfactors, Tensor Categories. The afternoons were left to scientific discussions.

It had long been realized that Operator Algebra methods are natural and fruitful in Quantum Field Theory, in particular concerning the description of super-selection structure. More recent was the application of Operator Algebras and Endomorphism Analysis to the study of the structure of chiral conformal field theory, for example concerning braid group statistics and modular invariants. In this context several new classification results were presented at the meeting, in particular by listing the discrete series of local conformal nets on $S^1$.

One evening was devoted to a joint round table discussions with the participants of the parallel miniworkshop "Geometry of Operators". The theme concerned possible extensions of the index theorem to Quantum Field Theory. An introduction was presented by R. Longo. The endomorphism of a factor played a rôle analogous to the Fredholm operators, the analytic index became the Jones index and geometric formulas appeared in the context of QFT on black hole spacetimes. A stimulating discussion followed.

The abstract of the talks are given below (in alphabetical order). The excellent working conditions at Oberwolfach created an inspiring atmosphere among the small group of experts and junior scientists that made this Miniworkshop particularly fruitful and successful.

# Abstracts

### Sectors with infinite statistical dimension for the Virasoro algebra

Sebastiano Carpi

The category of (separable) representations of a conformal net of von Neumann algebras can be studied using the DHR theory of superselection sectors. In particular one can define a dimension for these representations called statistical dimension and related to the Jones index for subfactors. It is shown that the positive energy representations of the net associated to the Virasoro algebra with central charge equal to one have infinite statistical dimension if their lowest conformal energy is not the square of a half integer.

### Nuclearity and split property for free Dirac fields on globally hyperbolic spacetimes

Claudio D'Antoni
(joint work with S. Hollands)

We consider the representations associated to quasi-free Hadamard states on free Dirac fields propagating on globally hyperbolic spacetimes and establish local quasi equivalence. As a consequence the local algebras turn out to be (hyperfinite) $III_1$ factors. Nuclearity is proved in the static case. Then split property is obtained in the general globally hyperbolic case. These properties allow to consider the free Dirac field as an example of a local covariant quantum field (to appear in Commun. Math. Phys.).

### Product of representations

Klaus Fredenhagen

Crucial for the DHR analysis of superselection sectors is the notion of a product of representations. Related products occur in the theory of subfactors and in conformal field theory ("fusion"). A new definition of products of representations of nets of algebras was given which reduces in the case of DHR representations to the DHR product.

### Free fields and de Sitter holography

Daniele Guido
(joint work with R. Longo)

As is known, the vacuum state on the Minkowski space can be thermalized on the Rindler wedge, replacing the original time-evolution with the one-parameter group of boosts. This corresponds to the fact that, in presence of a black hole, a freely falling observer feels the Hawking radiation. There is no natural notion of vacuum for quantum fields on the de Sitter space, but there are KMS states corresponding to natural evolutions. We propose a framework for dethermalization, namely the construction of evolutions w.r.t. which the KMS state becomes a ground state. Such evolutions act on local algebras and have only a partial geometric meaning. In the conformal case however, such evolutions correspond to the implementation of suitable conformal symmetries. In the two-dimensional case, we describe the holography maps associated with the event horizons of inertial observers. Such maps give rise to a dS-CFT correspondence only for positive-energy representations of the de Sitter group. In this case the given net is indeed a degenerate chiral conformal net (work in progress).

### $Q$-systems

Masaki Izumi

For a subfactor $M \supset N$, the bimodule ${}_NM_M$ is a fundamental object to consider. "Does ${}_NM_M$ uniquely determine the subfactor $M \supset N$?" The answer of this question turns out to be negative in general, and the obstruction is of cohomological nature. In the type III case, $M \supset N$ corresponds to the canonical endomorphism $\gamma \in \mathrm{End}(N)$, and the above question is equivalent to whether the set of equivalence classes of $Q$-systems for $\gamma$ is a singleton. In the case of the crossed product $M = N \times_\alpha G$ where $G$ is a finite group, there is a one-to-one correspondence between this set and $H^2(G, \mathbb{T})$. There are a few other cases where the number of equivalence classes of $Q$-systems is known.

### Classification of local conformal nets: case $c < 1$

Yasu Kawahigashi
(joint work with R. Longo)

We completely classify diffeomorphism covariant local nets of von Neumann algebras on the circle with central charge $c$ less than 1. The irreducible ones are in bijective correspondence with the pairs of $A$-$D_{2n}$-$E_{6,8}$ Dynkin diagrams such that the difference of their Coxeter numbers is equal to 1, and they consist of the Virasoro nets, their simple current extensions of index 2, and four exceptional ones. Two of the exceptionals were conjectured to exist as cosets and we have shown their existence. The other two exceptionals appear to be completely new.

We first identify the nets generated by irreducible representations of the Virasoro algebra for $c < 1$ with certain coset nets. Then, by using the classification of modular invariants for the minimal models by Cappelli-Itzykson-Zuber and the method of $\alpha$-induction in subfactor theory, we classify all local irreducible extensions of the Virasoro nets for $c < 1$ and infer our main classification result.

### On Longo-Rehren inclusions and Popa's symmetric enveloping algebras

Toshihiko Masuda

In subfactor theory, there are several methods for quantum double construction. Among them, we discuss the relation of two constructions. One is Longo-Rehren construction, and another is Popa's construction of symmetric enveloping algebras. In type $\mathrm{II}_1$ case, Popa gave the definition of symmetric enveloping algebras. However in type III case, we do not know the general methods for construction. If we assume the existence of a state with a large centralizer, then we can define the symmetric enveloping algebra for a type III factor $N \subset M$ as in type $\mathrm{II}_1$ case. In the Longo-Rehren inclusion, we can construct the Jones projection for $N \subset M$ and $N^{\mathrm{opp}} \subset M^{\mathrm{opp}}$, and hence we can relate the Longo-Rehren inclusions and the symmetric enveloping algebras.

### Superselection structure and stability of charges in the ultraviolet

Gerardo Morsella

The description of the confinement phenomenon in QFT is conceptually rather unsatisfactory, since it depends in an essential way on the choice of the unobservable fields used to construct the observables. As remarked by Buchholz, a clarification of this problem can be addressed in the framework of Algebraic Quantum Field Theory, by means of a comparison

between the superselection structures of the theory and of its scaling limit (as constructed by Buchholz and Verch).

In this talk we address the problem of establishing such a comparison, by developing a canonical way to identify (a subset of) charges of the underlying (finite scale) theory, with charges of the scaling limit one. To this end, it is shown that it is possible to establish a one to one correspondence between superselection sectors of the scaling limit theory and suitable sequences in the scaling algebra, called "asymptotic charge transfer chains", which are a generalization of the notion of charge transfer chain of DHR theory. It is then shown that if a charge of the underlying theory complies with a natural notion of ultraviolet stability, then there is a charge in the scaling limit admitting corresponding charge transfer chains which are also, in a weak sense, DHR charge transfer chains creating the considered underlying theory's charge at each finite scale, so that the charge at finite scales and the one in the scaling limit are naturally identified. An intrinsic notion of confinement of charges then is obtained if not all charges of the scaling limit arise in this way from charges of the underlying theory, i.e. if there are charges of the scaling limit which cannot be created by operations at finite scales.

## Conformal orbifold models, crossed G-categories and quasiabelian cohomology

Michael Müger

It is well known that modular categories as defined by Turaev give rise to topological quantum field theories (TQFTs) in 2+1 dimensions. Recently, Kawahigashi, Longo and I proved that the representation category of a 'completely rational' chiral conformal field theories (CFTs) is modular. On the other hand, Turaev defined 'homotopy TQFTs' and in particular invariants for 3-manifolds together with a principle G-bundle starting from a 'modular crossed G-category'. Such a category has a G-grading (on the objects), a compatible G-action and a suitably defined braiding.

We have the following results: 1. A completely rational chiral CFT A with a finite symmetry group $G$ gives rise to a modular crossed $G$-category $G-\mathrm{Rep}A$ with full spectrum. Thus we have an equivariant version of the chain

$$\text{rational CFT} \to \text{modular category} \to d = 2+1 \text{ TQFT}.$$

2. The representation category of the $G$-fixed conformal subtheory $A^G$ (orbifold theory) is given by $(G-\mathrm{Rep}A)^G$. 3. If $\mathrm{Rep}A$ is trivial ('holomorphic case') then $G-\mathrm{Rep}A$ is a group category with one isomorphism class of objects for every g in G. Such categories are classified by $H^3_{qa}(G,T)$, the 'quasiabelian cohomology' recently defined by Ospel. There is a map from $H^3_{ab}(G,T)$ to $H^3(G,T)$, and if $[w]$ is the image under this map of the class $[(w,c)]$ corresponding to $G-\mathrm{Rep}A$, we find that $\mathrm{Rep}A^G$ is equivalent to the representation category of the twisted quantum double $D^w(G)$. The braiding of $\mathrm{Rep}A^G$ is also determined by $[(w,c)]$ in $H^3_{ab}(G,T)$ and will in general differ from that of $D^w(G)-\mathrm{mod}$.

## Some remarks on modular invariants

Michael Müger

After briefly recalling the notions of algebras, Co-algebras, Frobenius algebras and Q-systems in tensor categories and their relations we turn to some observations concerning two approaches to the classification of modular invariants: The quantum field theoretic one due to Moore and Seiberg and improved by Rehren and the subfactor theoretic one due to Ocneanu and extended by Böckenhauer, Evans and Kawahigashi (OBEK). We show that

the modular invariants for a pair $(A_L, A_R)$ of chiral CFTs according to the first approach are classified by pairs $(B_L, B_R, F)$, where $B_L, B_R$ are finite local extensions of $A_L, A_R$, respectively, and $F : \mathrm{Rep}B_L \to \mathrm{Rep}B_R$ is an equivalence of braided tensor categories. (Note that finite local extensions of A are classified by commutative algebras in Rep$A$.) The OBEK approach for $A = A_L = A_R$ is based on not necessarily commutative algebras in RepA. We show that such an algebra gives rise to two commutative algebras $Q_+, Q_-$ and an equivalence of braided tensor categories $F : Q_+ - \mathrm{mod}_0 \to Q_- - \mathrm{mod}_0$. We conjecture that every such triple $(Q_+, Q_-, F)$ arises from an algebra Q as above. This would establish the equivalence of both approaches in the case $A_L = A_R$.

## Examples of module categories

Viktor Ostrik

In this talk I will describe some known results on classification of fusion categories (= semisimple rigid monoidal categories with finitely many simple objects) and module categories over fusion categories. For fusion categories the results are: the classification of categories with two simple objects and the classification of categories whose Frobenius-Perron dimension is a prime number or square of a prime. The module categories are classified over the following fusion categories: representations of a finite group; categories where all simple objects are invertible; Drinfeld doubles of finite groups; fusion category of integrable representations of affine $sl_2$.

## The braid-permutation group in QFT

Karl-Henning Rehren
(joint work with J. T. Neumann)

The simplest conformal quantum field theories (CFT) in $\geq 3$ dimensions exhibit commutation relations which are governed by the "braid-permutation groups" $PB_n$. An open question is, whether this is a general feature of CFT, as suggested by Schroer. In the examples analyzed, $PB_n$ appear as a structure of a $C^*$ category without a monoidal product, unlike the well-known cases of $B_n$ (QFT in $\geq 3$ dimensions) and $S_n$ (QFT in $\leq 2$ dimensions). A general theory (e.g., in terms of endomorphisms) is lacking so far (work in progress).

Presentation of $PB_n$:

Generators: $g_i^r$ $(i = 1 \ldots n, r = 0, \pm 1)$;

Relations: $g_i^r g_i^{-r} = e$, $\quad g_i^r g_{i+1}^s g_i^{-r} = g_{i+1}^{-r} g_i^s g_{i+1}^r$, $\quad g_i^r g_j^s = g_j^s g_i^r$ $(|i-j| \geq 2)$.

*Edited by Roberto Longo*

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