MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Calculus of Variations

June 30th – July 6th, 2002

The conference was organised by Gero Friesecke (Warwick), Tristan Riviere (Zuerich) and Gianni DalMaso (Trieste).

There were a total of 30 presentations, covering a wide range of topics including gradient flows, mass transportation, geometric analysis, minimal surfaces, Lipschitz maps, singularities, atomistic systems, quantum mechanics and water waves.

The stimulating discussions and the marvellous working conditions provided by the Institute of Oberwolfach created a lively scientific atmosphere.

In particular, the participation of many young researchers and mathematicians from different fields such as applied analysis, geometry and partial differential equations shows that calculus of variations is a growing and active topic with connections to many mathematical disciplines.

The abstracts are listed in the order they have been entered in the book of abstracts.

Abstracts

Surface Energies in Lattice Systems

F. Theil

We study the minimum energy configurations of cubic two-dimensional lattices. The energy is given by the sum over nearest and second nearest neighbour interaction potentials.

$$E = \sum_{x,x' \in \mathcal{L}} V_{x,x'}(|y(x) - y(x')|) \tag{1}$$

$$V_{x,x'} = 0 \quad \text{if} \quad |x - x'| > \sqrt{2}, \tag{2}$$

where $\mathcal{L} \subset \mathbb{Z}^2$. The presence of surfaces leads to the creation of boundary layers and the concentration of energy on the boundary. We show that in a suitable limit (the equilibrium lengths of the springs are roughly comparable) the minimisers of half space systems are asymptotically one-periodic in the tangential direction and the minimum energy is determined by the solution of a finite dimensional nonlinear equation. Since the half space system is infinite-dimensional and the energy is nonconvex, this result is an important step towards a qualitative and quantitative understanding of atomistic systems with surface energies.

Geometric Structure of Null Sets in the Plane (and some applications)

G. Alberti

This talk summarises (part of) a joint research with Marianna Csörnyei and David Preiss (University College London). Although our results are not directly related to the calculus of variations, the purpose of this talk is to highlight an elementary geometric fact that lies behind most of our proofs and that might be useful elsewhere.

Statement: every Borel set $E \subset \mathbb{R}^2$, $|E| = \Sigma$ can be covered by horizontal and vertical stripes ($\simeq$ ϵ-neighbourhood of graphs of 1-Lipschitz functions $x = x(y)$ or $y = y(x)$) so that the sum of the heights is less than $C\sqrt{\Sigma}$ – C a universal constant.

Variations of this lemma have been used to prove the existence of a (weakly defined) tangent field to null sets and singular measures in the plane, constructing Lipschitz maps that take a given set of positive measure onto a disk (following earlier proofs of D. Preiss and J. Hatoušek) and Lipschitz maps that are almost nowhere differentiable with respect to given singular measures in the plane. The lemma is obtained by discretization from a geometric version of classical Erdős-Szekeres Theorem. Unfortunately, it is not known if some equivalent statements holds in higher dimension (in fact, we have some partial counterexample).

Comparison between the Classical and Intrinsic Rectifiability in Carnot Groups

F. S. Cassano

We compare in the setting of the Carnot groups, endowed with its Carnot-Carathéodory metric, the classical and a new intrinsic notion of rectifiability. We prove that the classical definition does not fit the geometry of the Carnot groups while the intrinsic one does. Moreover, we prove that the classical one always implies the intrinsic one and the converse fails.

An optimization problem in mass transportation

G. Buttazzo

Given a connected open regular bounded subset Ω of $\mathbb{R}^n$ and two probability measures f^+ and f^- on $\overline{\Omega}$, for every distance d on Ω we consider the Monge-Kantorovich cost

$$F(d) = \inf\left\{\int\int_{\overline{\Omega}\times\overline{\Omega}} \Psi(d(x,y))d\nu(x,y) : \ \nu \text{ transport plan of } f^+ \text{ onto } f^-\right\}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is a given continuous increasing function with $\Psi(0) = 0$. The distance d is taken in the admissible class of Riemannian distances of the form

$$d_a(x,y) = \inf\left\{\int_0^1 a(\gamma)|\gamma'|dt : \ \gamma(0) = x, \ \gamma(1) = y\right\}$$

where the coefficient $a(x)$ varies in the class

$$\mathcal{A}(\alpha,\beta,m) = \left\{\alpha \le a(x) \le \beta, \ \int_\Omega a(x)dx \le m\right\} \ ,$$

with α, β, m positive constraints. It is shown that the maximization problem

$$\max\left\{F(d_a) : \ a \in \mathcal{A}(\alpha,\beta,m)\right\}$$

admits a solution. The proof is based on density result of isotropic Riemannian metrics in the class of all Finsler metrics.

Fast diffusion to self-similarity: complete spectrum, long time asymptotics, and numerology

R. McCann

(joint work with J. Denzler)

The complete spectrum is determined for the operation

$$H = -m\rho^{m-1}\Delta + x \cdot \nabla$$

on the closure of $C_c(\mathbb{R}^n)$ in the Hilbert space norm

$$\|\Psi\|^2 := \int_{\mathbb{R}^n} |\nabla\Psi|^2 d\rho.$$

Here the Barenblatt profile ρ is the stationary attractor of the rescaled diffusion equations

$$\frac{\partial u}{\partial t} = \Delta(u^m) + \nabla \cdot (xu)$$

in the fast supercritical regime $m \in]\frac{n-2}{n}, 1[$. If $m \geq \frac{n}{n+2}$, the same diffusion dynamics represent steepest descent of an entropy $E(u)$ on probability measures with respect to Wasserstein distance d_2. Formally, $H = \mathrm{Hess}_\rho E$ on the spectral gap $H \geq \alpha = 2 - n(1-m)$ found below suggests the sharp rate of asymptotic convergence:

$$\lim_{t\to\infty} \frac{\log d_2(u(t), \rho)}{t} \leq -\alpha < 0$$

from any centred initial data $0 \leq u(0,x) \in L^1(\mathbb{R}^n)$. Further eigenfunctions – all hypergeometric polynomials – and the presence of continuous spectrum suggest the long time asymptotics of $u(t)$ while yielding insight into the relations between the symmetries of $\mathbb{R}^n$ and the flow. Much of the strange numerology of the spectrum is explained in terms of the number of the moments of ρ.

Winding behaviour of finite-time singularities of the harmonic map flow

P. Topping

The harmonic map flow $u : D^2 \times [0, \infty) \to N \hookrightarrow \mathbb{R}^N$ from the 2-disc to a compact Riemannian manifold is liable to blow up in finite time.

At a singular point $(x, T) \in D \times [0, \infty)$, "bubbling" occurs when we blow up the flow restricted to times $t_n \uparrow T$

We show:

(1) The bubbling is, informally, non unique; it depends on the sequence $t_n \uparrow T$ chosen.
(2) The bubbling can be "winding" in the sense that bubbling convergence may fail when the flow is lifted to the universal cover of N.
(3) The map $u(T) \in W^{1,2}(D, N)$ can be discontinuous.
(4) The polynomial rate of blow-up can, in certain cases, be determined.

Convergence of the Yamabe flow for "large" energies

M. Struwe

(joint work with H. Schwetlick)

In joint work with Hartmut Schwetlick we show sub-convergence of the Yamabe flow on any smooth, compact 3-manifold (M, G_0) without boundary, provided the initial average scalar curvature s_0 and the Yamabe invariant $Y_0 = Y(M, G_0)$ satisfy the condition

$$0 < Y_0 < s_0 \leq (Y_0^{n/2} + S_*^{n/2})^{2/n} ,$$

where $s_* = Y(S^n, G_{S^n})$ and $n = 3$. The proof uses a Kazdan-Warner type estimate to rule out concentration, as in the 2-dimensional case, treated in [1]. If $n = 3$, or if $3 \leq n \leq 6$ and if (M, G_0) is locally conformally flat, this key estimate may be deduced from the positive mass theorem.

[1] M. Struwe. Curvature flows on surfaces. To appear in *Annali. Sc. Norm. Sup. Pisa* (2002).

Deformations and degenerations of Einstein metrics

R. Mazzeo

I describe two projects: the first, with B. Yang, involves the study of Einstein metrics with isolated conical singularities. Specifically, we prove analogues of various theorems due to Hardt, Simon et al. for minimal surfaces in this context. The goals include studying desingularisations of these singularities, and a finiteness theorem for the moduli space of Einstein's on a compact 4-fold (smooth) under certain conditions.

The other project, with F. Pacard, is a "boundary" gluing result for asymptotically hyperbolic Einstein metrics. This produces many new examples and shows (modulo the Poincaré conjecture) that every scalar-positive 3-mfd. bounds an asymptotically hyperbolic 4-mfd.

On the total variation of the Jacobian

P. Marcellini

(joint work with I. Fonseca and N. Fusco)

Ω is an open set of $\mathbb{R}^n$. We denote by $\det Du$ the Jacobian determinant of a map $u : \Omega \to \mathbb{R}^n$ $(n \geq 2)$. We also denote by Det Du, capitalised, the distributional Jacobian determinant, where it exists. Given $u \in L^\infty(\Omega; \mathbb{R}^n) \cap W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p > n-1$ the total variation $TV(\Omega)$ of the Jacobian determinant is defined by

$$TV(u,\Omega) = \inf \left\{ \liminf_{k\to\infty} \int_\Omega |\det Du_k(x)|\, \mathrm{d}x \ : \ u_k \overset{W^{1,p}}{\rightharpoonup} u, u_k \in W^{1,n}(\Omega, \mathbb{R}^n) \right\}.$$

In a work in collaboration with Irene Fonseca (CMU, Pittsburgh) and Nicola Fusco (University of Napoli) we proved some general n-dimensional results and we give some examples. Here are two examples:

- **Example 1:** Let $u : \Omega \setminus \{0\} \to \mathbb{R}^n$ (Ω open, with $0 \in \Omega$) be defined by $u(x) = (w(x) - w(0))/|w(x) - w(0)|$, where w is a Lipschitz-continuous map, classically differentiable at $x = 0$ with $\det Dw(0) \neq 0$. Then $TV(u, \Omega) = |B_1| = w_n$.
- **Example 2 (The "eight" curve):** Let $\gamma = \gamma^+ \cup \gamma^- \subset \mathbb{R}^2$ be the union of two circles of radius 1, with centres respectively at $(+1, 0)$ and $(-1, 0)$. Let $h, \kappa \in \mathbb{Z}$. Let $v : [0, 2\pi] \to \gamma = \gamma^+ \cup \gamma^-$ be the curve whose image turns $|h|$ times in γ^- and $|\kappa|$ times in γ^+ according to the parametric representation $v(\theta) = (-1, 0) + (\cos 2h\theta, \sin 2h\theta)$ if $0 \leq \theta \leq \pi, v(\theta) = (1, 0) = (\cos 2\kappa\theta, \sin 2\kappa\theta)$ if $\pi \leq \theta \leq 2\pi$ let $u(x) = v(x/|x|)$. Then $TV(u, B_1) = (|h| + |\kappa|)\pi$, while $|\text{Det } Du|(B_1) = |h - \kappa|\pi$.

Applications of Scans

R. Hardt

(joint work with T. Rivière)

Smooth maps between Riemannian manifolds M^m, N^n often fail to be dense in $W^{1,p}(M, N)$ in both the strong and weak topologies. The energy drop in a weakly convergent sequence has a local topological point (called bubbling) attributable to $\Pi_{[p]}(N)$.

Following the $W^{1,2}(B^3, S^2)$ case studied by Bethuel and Giaquinta, Modica, Souček, one may, for an individual map, look for a "topological singularity" of dimension $m - [p] - 1$ whose absence or presence determines the strong approximability by smooth maps. For the weak limit of smooth maps one expects this singularity to occur as the "boundary" of a "bubbling chain" of dimension $m - [p]$. Here we describe precisely, for $m - 1 \leq p < n$, the singularity and bubbling caused by $\Pi_{m-1}(N) \otimes \mathbb{Q}$. Unlike in the $W^{1,2}(B^3, S^2)$ case, we find that in general (as in the $W^{1,3}(B^4, S^2)$ case) the bubbling is *not* given by a current. We introduce a generalization, called *scans* to handle such bubbles. We again obtain a scan criteria for strong approximability. Here the bubbled scans correspond to oriented/rectifiable sets of finite lengths with integer multiplicity function θ that is L^α for some $0 < \alpha \leq 1$. We show that the optimal constant α is 1 for $\Pi_n(N^n)$, $\frac{3}{4}$ for $\Pi_3(S^2)$, and is estimable, for *any* rational homotopy invariant, in terms of a diagram derivable from Novikov-Sullivan data.

Remarks on Jesse Douglas

M. Micallef
(joint work with J. Gray)

This was an informal talk given in the evening on Wednesday, July 3, 2002, the 105th anniversary of the birthday of Jesse Douglas. A key problem in minimal surface theory is the determination of a parametrization $\underline{r} : [0, 2\pi] \to \mathbb{R}^n$ of a simple closed curve $\Gamma \subset \mathbb{R}^n$, so that the harmonic extension $\underline{F} : \mathbb{B} \to \mathbb{R}^n$ of $\underline{r}$ is conformal; $\mathbb{B}$ = unit disk in $\mathbb{R}^2$. In this talk, I explained how Douglas formulated this problem as an integral equation for $\underline{r}$. Douglas never published this work, the result is merely stated in abstracts of meetings of the American Mathematical Society during 1926-1929. More important, Douglas eventually realised that this integral equation was the Euler-Lagrange equation for his famous A-functional, which he used to solve the Plateau problem. This formulation of the conformality of $\underline{F}$ as a variational problem for $\underline{r}$ was a major breakthrough in minimal surface theory; its development by Courant and others continues to play an important role in minimal surface theory to the present day. This bears out Carathéodory's citation for the award of the Fields Medal in 1936 to Douglas: Douglas's method for solving the Plateau problem is 'entirely original' and 'of great significance'.

A brief account of the life of Jesse Douglas was also presented. The talk was followed by many interesting and valuable comments from the audience, especially Professor Stefan Hildebrandt.

Time continuous optimal transportation problems

Yann Brenier

The usual Monge-Kantorovich formulation of optimal transportation problems is shown to be equivalent to a "Continuum Mechanics" type formulation where the time variable is explicitly introduced and used. This formulation is far more flexible that the usual one and includes the (simplified) Moser construction for the solution of the Jacobian equation. Many generalizations of the time continuous formulations are possible, including optimal transportation of currents, relaxed geodesics on groups of volume preserving diffeomorphisms, generalized harmonic maps etc.

Elliptic approximation of curvature-sensitive image segmentation energies

A. Braides
(joint work with R. March)

In 1990 Mumford and Nitzberg proposed to improve the celebrated Mumford Shah variational approach to image segmentation by replacing the length of the segmentation set by a term involving also the curvature (the "Elastica" functional). In their formulation a segmentation (U, K) is given by a set K (union of H^2-curves) and a function $u : \Omega \setminus K \to$ minimising

$$\min\left\{\int_{\Omega\setminus K} |\nabla u|^2 \, \mathrm{d}x + \int_K (1+\kappa^2) \, \mathrm{d}\mathcal{H}^1 + \int_{\Omega\setminus K} |u-g|^2 \, \mathrm{d}x\right\}$$

(all constant are set to 1), where K_0 is the set of end points of the curves in K. The problem above can be proved to be approximable by elliptic problem of the type

$$\int_\Omega (v^2|\nabla u|^2 + \tfrac{1}{2}M_\eta(v)\cdot w^2(1+\mathrm{div}(\tfrac{Dv}{|Dv|})^2) + \tfrac{1}{4\pi}M_\eta(w)(\tfrac{1}{\varepsilon}+\varepsilon\mathrm{div}(\tfrac{Dw}{|Dw|})^2)) + \int_{\Omega\setminus K} |u-g|^2 \, \mathrm{d}x$$

where $u \in H^1(\Omega)$, $v, w \in H^2(\Omega)$. $M_\eta(z)$ denotes the "Modica-Mortola" energy density energy density modified by a singular perturbation term:

$$M_\eta(v) = \frac{W(\eta)}{\eta} + 2\eta|\nabla v|^2 + \frac{(v-1)^2}{\sqrt{\eta}}.$$ W is double-well potential with zeros at $0, 1$.

The main issue is the term $\int_\Omega M_\eta(w)(\frac{1}{\varepsilon}+\varepsilon\mathrm{div}(\frac{Dw}{|DW|})^2)\,\mathrm{d}x$ which is shown to behave as $\int_{\partial E}(\frac{1}{\varepsilon}+\varepsilon\kappa^2)\,\mathrm{d}\mathcal{H}^1$, whose minima are circles of radius ε.

Elliptic problems in vortex theory

G. Tarantello

We discuss the role of elliptic problems of Liouville-type in the study of vortices in various gauge field theories, such as Chern-Simons theory, Electroweak theory etc. The feature of such elliptic problems are captured essentially by the following elliptic equations:

$$-\Delta u = \lambda e^u - 4\pi \sum_{j=1}^{N} \delta_{P_j} + f \tag{3}$$

over a 2-manifold M without boundary. The points $\{P_1, \ldots, P_N\}$ are given in M, $f \in L^1(M)$ and $\lambda > 0$ is a given parameter. We discuss a concentration compactness concerning (3), and its consequences towards existence results as λ varies.

Cross-tie patterns and limiting minimization problems in a model for micromagnetism

S. Serfaty
(joint work with F. Alouges and T. Rivière)

We describe a joint work with Francois Alouges and Tristan Rivière, in which we study the family of functionals

$$\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H|^2 + \frac{1}{\varepsilon}\int_\Omega |u|^2$$

where u is a map: $\Omega \to S^2$ (magnetization in physics). H is given by $\nabla\Delta^{-1}\mathrm{div}(u\chi_\Omega)$, the demagnetizing field. Ω is a bounded simply connected domain of $\mathbb{R}^2$. We study the asymptotics $\varepsilon \to 0$. Families of uniformly bounded energy converge (after extraction) to divergence-free vector fields of unit norm, which have line singularities along which the limiting u jumps. We prove a lower bound for the energy in terms of the angle of those jumps; which is optimal. This lower bound can be achieved by a one-dimensional profile for jumps less than $\pi/2$ and by two-dimensional profiles called "cross-tie", which we construct, for jumps between $\pi/2$ and π in angle.

Structure of entropy solutions

F. Otto

(joint work with Camillo De Lellis)

Consider the variational problem:

$$E_\varepsilon(w_\varepsilon) = \varepsilon \int_\Omega |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}\int_\Omega (1-|w_\varepsilon|^2)^2, \quad \Omega \subset \mathbb{R}^2$$

$m_\varepsilon : \Omega \to \mathbb{R}^2$ constraint to $\nabla \cdot m_\varepsilon = 0$. Let m be a strong limit of a sequence $\{m_\varepsilon\}_{\varepsilon>0}$ with bdd $\{E_\varepsilon(m_\varepsilon)\}_{\varepsilon>0}$. The limit satisfies $|m|^2 = 1$ a.e., $\nabla \cdot u = 0$ distributionally. It is expected to have line singularities. But in general $m \notin BV(\Omega)$. We nevertheless prove that m has the structure as if it were in $BV(\Omega)$. The main tool is the control of $\nabla \cdot [\phi(m)]$ as a (signed) Radon measure, where ϕ belongs to a certain class of nonlinear transforms ("entropies").

Two-dimensional parametric variational problems

S. Hilderbrandt

(joint work with H. van der Mosel)

Let $F : \mathbb{R}^n \times \mathbb{R}^N, N = \frac{1}{2}n(n-1)$, be a parametric integrand i.e.

(4) $$F(x, tz) = tF(x, z) \text{ for } t < 0$$

which satisfies

(5) $$m_1|Z| \le F(x,z) \le m_2|z| \text{ with constants } m_1, m_2 > 0, \text{ and}$$

(6) $$F(x,z) \text{ is convex in } z.$$

Then the integral $\mathcal{F}(X) := \int_B F(X, X_u \vee X_v)\, \mathrm{d}u\, \mathrm{d}v$ is well defined on the class $C(\Gamma)$ of surfaces $X : B \to \mathbb{R}^n$, $B = \{(u,v) \in \mathbb{R}^2 \ : \ u^2+v^2 < 1\}$ such that $X \in H^{1,2}(B, \mathbb{R}^N)$ which map ∂B monotonically (with degree 1) onto a closed rectifiable Jordan curve Γ in $\mathbb{R}^n$.

Theorem 1. There is a solution $X \in C(\Gamma)$ of the minimisation problem "$\mathcal{F} \to$ min in $C(\Gamma)$" which is a.e. conformally parametrized and Hölder continuous in B with exponent $\alpha = m_1/m_2$. Moreover, $X \in C^\beta(\overline{B}, \mathbb{R}^n)$ for some $\beta \in (0,1)$ if Γ satisfies a chord-arc condition.

Theorem 2. This minimiser is of class $H^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^n) \cap C^{1,\sigma}(B, \mathbb{R}^n)$ if there exists a "perfect dominance function" $G(x,p)$ of F. Moreover, we also have $X \in H^{2,2}(B, \mathbb{R}^n) \cap C^{1,\sigma}(\overline{B}, \mathbb{R}^n)$ if $\Gamma \in C^3$.

Theorem 3. If F is of the form $F = kA + F^*$, where $A(z) = |z|$ is the area integrand and F^* satisfies (4), (5) (with constants m_1^*, m_2^*) and the standard parametric ellipticity condition

$$|z|F^*_{z^iz^j}(x,z)\zeta^i\zeta^j \geq \lambda^*(|\zeta|^2 - |z|^{-2}\langle z,\zeta\rangle^2)$$

with $\lambda^* > 0$, then F possesses a perfect dominance function, provided that $k < k_0 := 2[m_2^* - \min\{\lambda^*, m_1^*\}]$.

The concept of a dominance function was introduced by C.B. Morrey. The proof of Theorem 3 is based on a construction of dominance functions given by Morrey. Theorem 1 is derived by considering the penalized functionals $\mathcal{F}_\varepsilon := \mathcal{F} + \varepsilon\mathcal{D}$, $\varepsilon > 0$, where $\mathcal{D}$ is the Dirichlet integral $\mathcal{D}(X) = \frac{1}{2}\int_B |\nabla X|^2 \,\mathrm{d}u\,\mathrm{d}v$. Theorem 2 follows by considering the weak Euler equation $\delta\mathcal{G}(X,\phi) = 0$ where $\mathcal{G}(X) = \int_B G(X,\nabla X)\,\mathrm{d}u\,\mathrm{d}v$.

Rigidity in nonlinear elasticity and the derivation of plate theories

S. Müller

(joint work with G. Friesecke and R.D. James)

The energy functional of nonlinear plate theory is a curvature functional for surfaces first proposed on physical grounds by Kirchhoff in 1850. We show that it arises as a Γ-limit of three-dimensional nonlinear elasticity theory as the thickness of a plate goes to 0. A key ingredient in the proof is a sharp rigidity estimate for maps $v \in W^{1,2}(U,\mathbb{R}^n)$, $U \subset \mathbb{R}^n$ a bounded Lipschitz domain. We show that there exists $R \in SO(n)$ such that

$$\int_U |\nabla v - R|^2 \,\mathrm{d}x \leq C(U)\int_U \mathrm{dist}^2(\nabla v, SO(n)).$$

A Γ-convergence approach to generalised Sobolev inequalities

A. Garroni

Well known concentration phenomena arise in problems with lack of compactness due to the critical growth. The most famous example is given by the Sobolev inequality. The same kind of phenomena appear in a more general situation, as has been proved by Flucher and Müller. They study the behaviour of 'almost' maximizers for the functional

$$\int_\Omega \frac{f(\varepsilon u)}{\varepsilon^{2^*}}\mathrm{d}x \text{ for } u = 0 \ \partial\Omega \text{ s.t. } \int_\Omega |\nabla u|^2 \,\mathrm{d}x \leq 1$$

with $\varepsilon > 0$, $0 \leq f(t) \leq c|t|^{\frac{2n}{n-2}}$. They prove concentration by means of a generalised version of the concentration-compactness alternative of P.L. Lions. We approach this problem using the Γ-convergence. This permits us to read easily the concentration directly by the structure of the Γ-limit. The localization of concentration points can be also obtained by the second order expansion in Γ-convergence.

Quasiminimal Partitions and Uniform Rectifiability

Séverine Rigot

A quasiminimal partition is a Caccioppoli partition of $\mathbb{R}^n$ for which one controls the variation of a surface like energy under relatively compact perturbations that preserve the measure of each component. Roughly speaking, one knows that this variation is negligible compared to the initial surface energy.

We prove quantitative and uniform rectifiability properties for the set of interfaces of quasiminimal partitions, namely uniform rectifiability in the terminology of G. David and S. Semmes. To this aim the main issue is to handle properly the volume constraint. Using ideas and constructions inspired by a previous work of Almgren about minimal partitions with prescribed measure, one can get a new quasiminimality condition without volume constraint anywhere and which is much easier to work with. Then the regularity properties follow by fairly standard comparison and covering arguments.

Surface water waves as saddle points of the energy

E. Séré

(joint work with B. Buffoni and J.F. Toland)

By applying the mountain-pass lemma to an energy functional, we establish the existence of two-dimensional water waves on the surface of an infinitely deep ocean in a constant gravity field. The formulation used, which is due to K.I. Babenko, (and later to others, independently), has as its independent variable an amplitude function which gives the surface elevation, its nonlinear term is purely quadratic but nonlocal (it involves the Hilbert transform $\mathcal{C}$). The waves are found as critical points of the functional

$$I(w) = \int_{-\pi}^{\pi} w\mathcal{C}w' - \lambda \int_{-\pi}^{\pi} w^2(1+\mathcal{C}w'), \quad w \in W^{1,2}_{2\pi}.$$

Since this functional is rather degenerate, we have to truncate it, penalize it, and regularize it. To prove the convergence of the critical points, in the limit of vanishing regularization, to a nontrivial wave, we use the Morse index, in the spirit of a work by Amann and Zehnder.

Vortex energy for rotating Bose-Einstein condensates

A. Aftalion

(joint work with T. Rivière and R. Junard)

We find an asymptotic expansion for the energy describing a Bose Einstein condensate in terms of the rotational velocity Ω and a small parameter ε. This simplified energy allows us to understand why in the present experiments the vortex line line is not straight along the axis of rotation but bending.

A relative Morse index for the Dirac-Fock functional

E. Paturel

We prove the existence of infinitely many stationary solutions of the Dirac-Fock model describing atoms and molecules, under the assumption: $N < Z + 1$ and $\max(Z, N) < Z_c$ where N is the electron number, Z the total positive charge, α the electromagnetic coupling constant ($\approx \frac{1}{137}$) and $Z_c = \frac{2}{\alpha(\frac{2}{\pi}+\frac{\pi}{2})}$. This work is an improvement of an article of Esteban and Séré, where the claim was proved under more restrictive assumptions on N. The stress is put on the construction of a relative Morse index for the functional, which allows us to control the energy of the mean field operator.

Douglas condition for Willmore surfaces of prescribed genus

Ernst Kuwert

Let β_p be the infimum of the Willmore functional among oriented, immersed surfaces of genus p in $\mathbb{R}^n$. By a result of L. Simon, for each $p \in \mathbb{N}$ there is a partition $p = p_1 + \ldots + p_r$ with $p_i \geq 1$ such that each of the β_{p_i} is attained and moreover one has the equation

$$e_p = e_{p_1} + \ldots + e_{p_r}, \quad (e_p = \beta_p - 4\pi).$$

By extending the case $r \geq 2$ we obtain the following

Theorem. For any $p \in \mathbb{N}_0$ the infimum β_p is attained.

Simon's work reduces the problem to proving $e_{p_1+p_2} < e_{p_1} + e_{p_2}$.

Removability of point singularities of Willmore surfaces

R. Schätzle

We prove that single, unit-density point singularities can be removed. In particular, this implies that blowup limits of Willmore flows with energy $\leq 8\pi$ are smooth at infinity. As consequences we determine 8π as the optimal energy level such that Willmore flows of spheres below this level exist globally and converge to round spheres, and we obtain compactness results for Gilmore tori.

Det vs det

I. Fonseca

(joint work with P. Marcellini and N Fusco)

It is well known that $u \in W^{1,N}(\Omega : \mathbb{R}^N) \to \int_\Omega \; |\det Du|$ is $W^{1,N}$–sequentially weakly lower semicontinuous, where $\Omega \subset \mathbb{R}^N$ is an open set. However, many variational problems lead us to consider the case where the setting is now in $W^{1,p}(\Omega, \mathbb{R}^N)$ for some $p < N$. Two questions naturally arise:

Q1. What are the "minimal regularity" assumptions on u guaranteeing that

$$\mathrm{Det}Du = \det D \text{ where } \mathrm{Det}Du = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(u^1(\mathrm{adj}Du)_1^j)?$$

Q2. What is the "weakest notion" of convergence under which

$$u_n \in W^{1,N}_{\mathrm{loc}}(\Omega; \mathbb{R}^N), \quad u_n \rightharpoonup u \Rightarrow \mathrm{Det}Du_n = \det Du_n \rightharpoonup \mathrm{Det}Du?$$

The understanding of these questions is relevant to the study of vorticity for Ginzburg-Landau equations, cavitation of nonlinear elastic rubber-like materials, singularities of harmonic mappings with values on the sphere, etc. When addressing Q2 it is tempting to introduce the relaxed functional $TV(u,\Omega)$, the total variation of the Jacobian determinant given by

$$TV(u,\Omega) := \inf\{\underline{\lim} \int_\Omega |\det Du_n| \ : \ u \rightharpoonup u \, W^{1,p}, \ u_n \in W^{1,N}_{\text{loc}}\}.$$

Jointly with N. Fusco and P. Marcellini, it was shown that if $p > N-1$ and $TV(u,\Omega) < +\infty$, then $TV(u,\cdot)$, $\mathrm{Det}Du$ are finite Radon measures, $\det Du \in L^1(\Omega)$,

$$TV(u,\cdot) = |\det \nabla u| \mathcal{L}^n|\Omega + \lambda_s, \quad \mathrm{Det}Du = \det Du \mathcal{L}^n|\Omega + \mu_s,$$

where λ_s, μ_s are finite Radon measures, singular with respect to $\mathcal{L}^n|\Omega$, and $|\mu_s| \le \lambda_s$. The proof of this result is strongly hinged on a theorem obtain in collaboration with G. Leoni and J. Maly, stating that if $u_n \in W^{1,N}$, $u \in BV$, $u_n \to u \ L^1$, $\{u_n\}$ bounded in $W^{1,N-1}$, and if $\det Du_n \overset{*}{\rightharpoonup} \mu$ for some Radon measure μ, then $\frac{d\mu}{d\mathcal{L}^N} = \det Du$.This result is sharp, in that there are examples asserting that one cannot, in general, assume that $\{u_n\}$ is bounded in $W^{1,p}$, $p < N-1$ and unbounded in $W^{1,N-1}$, and also one cannot, in general, assume that $u_n \in W^{1,p} \setminus W^{1,N}$ for some $p < N$.

Three dimensional water waves by variational and dynamical methods

R. Pego

We describe travelling waves in models of wave propagation on water of finite depth, for three models:

(1) the KP - equation
(2) the Benney Luke equation (an isotropic model for long wave of small amplitude)
(3) the exact Euler equations for water waves

When surface tension is strong, the equation for travelling waves is elliptic. Finite energy solitary waves had been found for KP by concentration-compactness methods, and we achieve the same for BL and demonstrate Γ-convergence to KP in the appropriate scaling limit. The problem for the exact Euler equations is open.

When surface tension is zero, looking for fast waves yields problems best addressed through spatial dynamics. For BL and the Euler equations this yields an ill-posed system, but for BL we prove (as for KP) there is an infinite-dimensional family of travelling waves that corresponds to a centre manifold of infinite dimension and codimension. The exact equations admit formally a conserved "energy" for spatial dynamics, but the existence of a centre manifold for nonlinear waves remains open.

Harnack inequalities on scale irregular fractals

Umberto Mosco

Following Barlow-Hambly (1997) we construct a family of homogeneous non self-similar Sierpinski curves in $\mathbb{R}^D$, $D \geq 2$. Each curve $K(\xi)$ of the family depends on an "environment" sequence $\xi = (\xi_1, \xi_2, \ldots)$, where each ξ_i takes its value in a finite set of "scales", A. The main scaling exponents associated with $a \in A$ are

$$\alpha_a > 1,\ N_a \geq 2,\ \rho_a > 1$$

for length, volume (mass), energy, respectively. The asymptotic frequency of $a \in A$ in $\xi \in A^{\mathbb{N}}$ is described by probabilities $0 \leq p_a \leq 1$, $\sum_{a \in A} p_a = 1$, on A:

$$p_a = \lim_{n \to \infty} h_a^{(\xi)}(n),\ h_a^{(\xi)}(n) = \frac{1}{n} \sum_{i=1}^{n} 1_{\xi_i}.$$

Under the assumption

$$|h_a^{(\xi)}(n) - p_a| \leq \frac{g(n)}{n}, \quad n \geq 1, \tag{7}$$

where g is a regular increasing function on the real line, $g(0) = 1$, we are able to carry out an "effective" description of $K^{(\xi)}$ inspired by homogenization theory. We replace the complicated fine structure of $K^{(\xi)}$ by an intrinsic quasi-metric structure within $K^{(\xi)}$ and we estimate the scaling laws for volume and spectral gap on balls B_R of d. The effective quasi-metric d is of the kind $d(x, y) = |x - y|^{\delta}$, where $\delta > 0$ is an index of the ramifications in $K^{(\xi)}$, which is chosen to be

$$\delta = \frac{1}{2} \frac{\sum_a p_a \log(N_a \rho_a)}{\sum_a p_a \log \alpha_a}$$

We then prove the volume estimate $vol(B_R) \approx R^{\nu} e^{G(R)}$ where

$$\nu = 2 \frac{\sum_a p_a \log(N_a)}{\sum_a p_a \log(N_a \rho_a)}$$

and the "universal" spectral gap scaling $\lambda_1(B_R) \approx R^{-2} e(-cG(R))$, where $G(R) \sim g(c \log(1/R))$. By assuming fastest convergence, i.e. $g(s) = O(1)$, we then prove Harnack inequality on balls B_R and Green function estimates on concentric balls $B_r \subset B_R$ of the kind

$$g_{B_R}(x_0, x)|_{x \in \partial B_r} \approx \frac{1}{2 - \nu} (R^{2-\nu} - r^{2-\nu}).$$

Singularities of minimal submanifolds

L. Simon

This talk focussed on a PDE method for finding singular minimal surfaces of codimension 1. One begins with the equation

$$\mathcal{M}u := \sum_{i=1}^{n} D_i \left(\frac{D_i u}{\sqrt{1+|\nabla u|^2}} \right) = \frac{m}{u\sqrt{1+|Du|^2}},$$

where m is an integer ≥ 1 and $n \geq 2$. In the case where we actually use the $(n+1)$-variable version of this equation, so that $u = u(x,y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ a "weak linearization" process was introduced to demonstrate the existence of a rich class of solutions of the form $u(x,y) = \sqrt{m}r + v(x,y)$, where $|v(x,y)| \leq CR^Q$, $r = |x|$, $R = \sqrt{r^2+y^2}$.

Strict convexity and the existence of optimal transports

B. Kirchheim

(joint work with L. Ambrosio and A. Pratteli)

We consider the Monge problem. Given two (absolutely continuous) probabilities μ_1,μ_2 in the n-dimensional space and a norm $\|\cdot\|$ on that space, try to find a mapping $\phi : R^n \to R^n$ that maps the first measure onto the second ($\mu_1(\phi^{-1}(A)) = \mu_2(A)$ for all A) and minimizes the average d-distance the points are moved, i.e.

$$\int \|\phi(x) - x\| d\mu_1(x) \to \ \min.$$

The existence of such an optimal transport map in case of a norm having a sufficiently curved unit ball was established by several authors, including Caffarelli, Evans, Feldmann, Gangbo, McCann, Trudinger and Wang. In joint work with L.Ambrosio and Aldo Pratteli we can prove the existence of an optimal transport also for general norms in the plane and crystaline norms (corresponding to polyhedral unit balls) in any dimension.

Edited by Florian Theil

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