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Arithmetic and Differential Galois Groups

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Recently observations have shown that the Galois theories for number theory, algebraic geometry and differential equations have remarkable similarities. The idea of this conference on Arithmetic and Differential Galois Groups was to bring together researchers working in these different areas in order to discuss the progress in these theories and to exhibit their interactions.

More specifically, the conference was centered around the following topics:

- Inverse problems
- Topological and algebraic fundamental groups
- Absolute and universal Galois groups
- Rigidity for groups and local systems
- Galois action for differential equations
- Differential equations in both characteristic 0 and characteristic p
- Moduli for equations, coverings and differential equations
- Invariant theory and group theory connected with the above

In the following report the abstracts of the lectures presented during the meeting are collected in chronological ordering.

D. Harbater, B. H. Matzat, and M. van der Put

Abstracts

Intrinsic differential Galois groups

Daniel Bertrand

Let (K, ∂) be a differential field with an algebraically closed constant field C of characteristic 0, and let M be a $K[\partial]$-module of finite dimension over K. The intrinsic differential Galois group $G^K(M)$ of M is the stabilizer in $GL(M/K)$ of all the $K[\partial]$-submodules in the tensor algebra generated by M and its dual. It is an inner K-form of the usual differential Galois group $G(M)$ (an algebraic group over C), but as simple computations in rank 2 already show, $G^K(M)$ is seldom isomorphic to the constant form $G(M) \otimes_C K$. More generally, we have on assuming that K *has cohomological dimension* ≤ 1:

Proposition 1: *let G be an algebraic group over C, and let $X = G/G^0$ be its group of connected components. Then $G \otimes_C K$ occurs as intrinsic differential Galois group if and only if the center of G maps onto X, in which case no other form may occur.*

On the other hand, at least in the classical case $K = \mathbb{C}(z)$, we have:

Proposition 2: *let $\mathbf{g}$ be a Lie algebra over $\mathbb{C}$, and let $\mathbf{g}^K$ be any K-form of $\mathbf{g} \otimes_{\mathbb{C}} K$. Then, there exists a $K[\partial]$-module M such that $\mathbf{g}^K \simeq LieG^K(M)$.*

Non linear differential Galois theory

Bernard Malgrange

Let X be a complex analytic manifold, and let Aut$X$ be the groupoid of germs of automorphisms of X. A "Lie groupoid" is roughly speaking, a subgroupoid of Aut$X$ defined by partial differential equations. To any foliation F (with singularities) on X, one associates a Lie groupoid, i.e. the smallest one whose Lie algebra contains F. Several examples are given; for instance, if the foliation comes from a linear differential equation, one obtains essentially the differential Galois group of the equation.

On the inverse problem in differential Galois theory

Julia Hartmann

Differential Galois theory generalizes the usual Galois theory for polynomials to differential equations. There is the notion of a splitting field (Picard-Vessiot extension) of a differential equation, and the differential Galois group is the group of automorphisms of this extension which fix the base field and commute with the derivation. Differential Galois groups are linear algebraic groups over the field of constants of the base field. In analogy to the classical situation, one considers the following inverse problem: *Which linear algebraic groups occur as differential Galois groups over a given differential field?*

The main result of this talk is the following

Theorem. *Let $\mathcal{G}$ be a linear algebraic group defined over the algebraically closed field K of characteristic zero. Then $\mathcal{G}$ occurs as the differential Galois group of some Picard-Vessiot extension of $(K(t), \frac{d}{dt})$.*

Previously, there had been two main approaches towards this result: When $K = \mathbb{C}$, a positive solution follows from the solution of the Riemann-Hilbert problem. In contrast to ordinary Galois theory, this result could not be carried over to arbitrary algebraically closed fields without further assumptions on the group. On the other hand, there was an algebraic solution for connected groups and solvable-by-finite groups.

We use the technique of embedding problems to combine the two approaches.

Galois groups and geometry of modular varieties

Alexander Goncharov

Let $X_N := P^1 - \{0, \infty, \mu_N\} = G_m - \mu_N$ where μ_N is the group of all N-th roots of unity. Let v_0 be the tangent vector at zero dual to dt where t is the canonical coordinate on P^1.

Our goal was to explain a mysterious relationship between the structure of the motivic fundamental group

$$\pi_1^{\mathcal{M}}(X_N, v_0)$$

and geometry of the modular varieties

$$Y_1(m; N) = \Gamma_1(m; N) \backslash GL_m(R)/O_m \cdot R_+^*$$

when $m = 1$ we define $Y_1(m; N)$ as the set of all complex points of the zero dimensional scheme $S_N := \mathrm{Spec}Z[\zeta_N][1/N]$.

One can prove that $\pi_1^{\mathcal{M}}(X_N, v_0)$ is a Lie algebra in the category of mixed Tate motives over S_N. The category itself is canonically equivalent (by a certain canonical fiber functor Ψ) to the category of finite–dimensional modules over a graded Lie algebra $L_\bullet(S_N)$, the motivic Lie algebra of S_N.

Let $\mathcal{G}_\bullet(\mu_N)$ be the image of the motivic Lie algebra $L_\bullet(S_N)$ acting on $\Psi(\pi_1^{\mathcal{M}}(X_N, v_0))$. The fundamental group, and hence $\mathcal{G}_\bullet(\mu_N)$ have a so called depth filtration given by the lower central series of the kernel of the map of the fundamental groups induced by the map $X_N \longrightarrow G_m$. Let $\mathcal{G}_{\bullet,\bullet}(\mu_N)$ be the associated graded for the depth filtration.

We defined a complex of $GL_m(Z)$–modules $M_{(m)}^*$ of length m and prove that the complex (where V_m is the standard representation of GL_m)

$$M_{(m)}^* \otimes S^{w-m} V_m$$

maps surjectively to the depth m, weight w part of the standard cochain complex of the Lie algebra $\mathcal{G}_{\bullet,\bullet}(\mu_N)$. The details can be found at my preprints at xxx.lanl.gov.

Arithmetic monodromy arising from elliptic curves and generalized Dedekind sums

Hiroaki Nakamura

Given a family of elliptic curves E over S, one can associate the outer monodromy representation

$$\varphi : \pi_1(S) \to Out(\Pi), \quad \Pi := \pi_1(E_{\bar{x}} \setminus O).$$

After taking the Belyi lifting and reduction modulo the double commutator subgroup Π'' of Π, one obtains the meta-abelian core of this monodromy

$$\bar{\varphi} : \pi_1(S_\infty) \to Aut(\Pi/\Pi''),$$

where $\pi_1(S_\infty) := ker(\pi_1(S) \to GL(\Pi^{ab}))$ denotes the congruence kernel of $\pi_1(S)$. In this talk, we consider a certain function $\mathbb{E} : \pi_1(S_\infty) \to \hat{\mathbb{Z}}[[\Pi^{ab}]]$ which is equivalent to considering $\bar{\varphi}$. Especially, when $S = M_{1,1}/\bar{\mathbb{Q}}$ (the moduli stack of elliptic curves), the coefficient characters of the pro-l version $\mathbb{E}^{(l)} : \pi_1(S_{l^\infty}) \to \mathbb{Z}_l[[T_1, T_2]]$ can be described by certain l-adic (Eichler-Shimura type) Eisenstein cocycles which involve arithmetic of generalized Dedekind sums.

Computing étale cohomology with Galois action.

BAS EDIXHOVEN

The aim of the talk was to explain a strategy for generalizing Schoof's algorithm (for computing the number of rational points on elliptic curves over finite fields) to more general varieties, or say motives. The explicit example of the modular form Delta was given. The question is then: *can one evaluate the Ramanujan τ function at a prime p in time polynomial in* $\log(p)$*?*

I believe that the answer is yes, but currently I have no proof. The strategy is to compute, for small l, the two-dimensional Galois representation V_l over F_l, and use that $\tau(l)$ is the trace of Frobenius at p acting on V_l. The representation V_l sits in the jacobian of the modular curve $J_1(l)$. One chooses random divisors D, effective, of degree g (genus of $X_1(l)$) on $X_1(l)$, of reasonably small height and defined over a small solvable extension of $\mathbb{Q}$. Let x be a non-zero point of V_l (sitting in $J_1(l)$). Then one approximates numerically (over $\mathbb{C}$) the unique D' (effective, degree g) for which $D' - D$ represents x, and one evaluates a suitable function (j, for example) on D' (e.g., if D' is the sum of points P_i, one considers the sum of the $j(P_i)$). The remaining problem is then to bound the height of $j(D')$. Bounding this height is a problem that one should solve using Arakelov geometry, i.e., with arithmetic Grothendieck Riemann Roch.

Class field theory of arithmetic surfaces

IVAN B. FESENKO

Let B be the spectrum of the ring of integers of a global number field k, or a proper smooth connected curve C over a finite field with function field k. Let S be an arithmetic surface, i.e. an integral normal excellent scheme of dimension two, flat over B, whose generic fibre $S \times_B \mathrm{k}$ is a nonsingular projective curve over k. In addition, assume that S is a regular scheme, proper over B, and with geometrically irreducible generic fibre. Denote by K the function field of S.

For $y \in S_1$ denote by K_y the field of fractions of the completion $\mathcal{O}_y$ of the local ring of S at y; K_y is a complete discrete valuation field with residue field $k(y)$. Fix its local parameter t_y. For every $x \in S_0$ denote by K_x the field of fractions of the completion $\mathcal{O}_x$ of the local ring of S at x. For $x \in S_0$, $y \in S_1$ such that $x \in \overline{\{y\}}$ denote by $y(x) \subset (\operatorname{Spec} \mathcal{O}_x)_1$ the set of branches of $\overline{\{y\}}$ at x and put $K_{x,y} = \prod_{z \in y(x)} K_{x,z}$ where $K_{x,z}$ is the z-adic completion of K_x, so $K_{x,z}$ a two dimensional local field.

For an archimedean place σ of k let k^σ be the completion of k with respect to σ. Denote $S^\sigma = S \times_B \mathrm{k}^\sigma$, $K^\sigma = k(S^\sigma)$. For a $\omega \in S_0^\sigma$ let K_ω^σ be the fraction field of the completion of the local ring of S^σ at ω; so it is a two dimensional local field.

Define certain restricted products

$$J_S = \prod_{y,x} K_2^t(K_{x,y}) \times \prod_{\sigma,\omega} K_2^t(K_\omega^\sigma), \qquad \prod_{y,x} = \prod_y \prod_{x \in \overline{\{y\}}}, \quad \prod_{\sigma,\omega} = \prod_\sigma \prod_{\omega \in S_0^\sigma}.$$

Define

$$P_S = \Delta \prod_y K_2(K_y) + \Delta \prod_x K_2(K_x) + \Delta \prod_\sigma K_2(K^\sigma)$$

where the restricted products are the intersection of the products with $\Delta^{-1}(J_S)$.

Theorem. *Continuous characters of finite order of the Galois group of the function field K of S are in one-to-one correspondence with continuous characters of finite order of K-delic class group $C_S = J_S/P_S$ via the reciprocity map*

$$\Phi_S\colon\ J_S/P_S \to \mathrm{Gal}\,(K^{ab}/K).$$

The reciprocity map Φ_S is the product of local reciprocity maps $\Phi_{x,z}$, and $\prod \Phi_{x,z}(P_S) = 1$ corresponds to two dimensional reciprocity laws.

The proof uses the main results of K. Kato's and Sh. Saito's class field theory.

The locus of curves with prescribed automorphism group

Kay Magaard

(joint work with T. Shaska, S. Shpectorov and H. Völklein)

Let M_g be the moduli space of genus g curves, and $H(g, G, c)$ be the Hurwitz space of G-covers with signature c. For $3 \leq g \leq 48$ we study the locus of curves in M_g with large automorphism group. Recall that for a compact Riemann surface X, $Aut(X)$ is large if its order is greater than $4(g-1)$. We find the loci by analyzing the map $\phi : H(g, G, c) \to M_g$ for all choices of (G, c) from T. Breuers database of groups acting on Riemann surfaces of genus at most 48. We also determine inclusions between loci.

Iterative differential equations; a survey

Marius van der Put

Let K be a field of characteristic $p > 0$ and suppose that $[K : K^p] = p$. Choose an element $z \in K \setminus K^p$. Then K is a differential field with respect to the differentiation $f \mapsto f'$, satisfying $z' = 0$. For differential modules over the differential field K there is a classification. This classification leads to the definition of the differential Galois group. Further one sees by example that there does not exists a suitable Picard-Vessiot theory.

In order to obtain a richer theory, one considers instead of K above a field with a higher derivation. The two fields $C(z)$ and $C((z))$ have natural higher derivations. Over a field with higher derivation one considers more complicated structures, called iterative differential modules. For iterative differential modules, Picard-Vessiot theory works well and there is a well defined differential Galois group. The latter is a reduced linear algebraic group over the field of constants C (supposed to be algebraically closed).

Examples lead to the formulation of a conjecture on the differential Galois groups occurring for the function field K of a smooth projective curve X over C and with singularities in a prescribed finite subset S of X. This conjecture includes Abhyankar's conjecture (proved by Raynaud and Harbater). The conjecture is also very much related with Ramis' theorem for a similar situation for differential equations on a compact Riemann surface.

For connected linear algebraic groups the conjecture is proved. For the non-connected case, the next lecture by H. Matzat provides the information. Finally, it is shown that iterative differential equations arise from p-adic differential equations, Frobenius structures and F-isocrystals.

On the differential Abhyankar conjecture

B. HEINRICH MATZAT

The Differential Abhyankar Conjecture (DAC) is a generalization of the classical Abhyankar conjecture to differential fields. Let $F = K(t)$ be the rational function field over an algebraically closed field of characteristic $p > 0$ with the iterative derivation on t. Then the DAC can be formulated as follows:
A reduced linear algebraic group $G(K)$ can be realized as differential Galois over F with at most one singular point if and only if $G(K)$ is generated by its unipotent subgroups.

In the case of finite groups unipotently generated groups are quasi-p groups. Thus in this case the conjecture coincides with the classical Abhyankar Conjecture proved by Raynaud [1]. In the case of connected groups DAC has been formulated and proved recently in [2]. In this talk a proof for non-connected linear groups has been presented based on the solution of differential embedding problems with connected kernel and finite cokernel (see [3]). In addition the inverse problem of differential Galois theory over F has also been solved:
Every reduced linear algebraic group G(K) over K can be realized as (iterative) differential Galois group over F.

Here again the connected case has already been solved in [2].

[1] M. RAYNAUD, *Revêtements de la droite affine en charactéristique p.* Invent. Math. 11 (1994), 425-462.
[2] B. H. MATZAT, M. VAN DER PUT, *Iterative differential equations and the Abhyankar conjecture.* J. r. a. Math. (to appear).
[3] B. H. MATZAT, *Differential Galois theory in positive characteristic.* IWR-Preprint 2001-35.

Tame class field theory for arithmetic surfaces

ALEXANDER SCHMIDT

In the talk we explained the ingredients and methods of proof of the following theorem.

Theorem. *Let X be an arithmetic surface, i.e. a two-dimensional regular connected scheme, flat and proper over $Spec(\mathbb{Z})$ and let Y be the support of a divisor on X. Then there exists a natural isomorphism of finite abelian groups*

$$rec_{X,Y} : CH_0(X,Y) \xrightarrow{\sim} \tilde{\pi}_1^t(X,Y)^{ab} \, .$$

Here $\tilde{\pi}_1^t(X,Y)^{ab}$ is the abelianized modified tame fundamental group, which classifies finite abelian tale coverings of $U = X - Y$ which are at most tamely ramified along Y and in which every real point splits completely. $\mathrm{CH}_0(X,Y)$ is the relative Chow group of zero-cycles.

Differential modules and skew fields

Mark van Hoeij

Let K and k be differential fields with fields of constants C_K and C_k, characteristic 0, and K/k Galois. Let M be a differential module over K, and suppose that M is isomorphic to its conjugates under $\mathrm{Gal}(K/k)$. Does then M descend to a module defined over k, i.e. is M isomorphic to $N \otimes_k K$ for some differential module N over k? If $K = C_K((x))$ and $k = C_k((x))$ then the answer to this question is yes. However, if $K = C_K(x)$ and $k = C_k(x)$, then there exists a counter example for every nontrivial element of the Brauer group $\mathrm{Br}(C_K/C_k)$. We will show that for each skew field with center C_k and splitting field C_K one can construct counter examples explicitly.

Differential equations for the convolution functor

Michael Dettweiler
(joint work with Stefan Reiter)

In his book "Rigid local systems" (1996), N. Katz showed that every irreducible rigid local system can be obtained from a one-dimensional local system by succesively applying a certain middle convolution functor and tensoring with onedimensional local systems (local systems are interpreted as perverse l-adic sheaves on the affine line in characteristic p). In a previous work (M. Dettweiler, S. Reiter: An Algorithm of Katz and its application to the inverse Galois problem, (2000)) we gave a purely algebraic analogon of the middle convolution functor.

In this talk it was demonstrated that one can actually write down the effect of the convolution functor on the level of differential equations. The main tool is the theory of Okubo sytems. Moreover, I gave a cohomological interpretation of the convolution functor. Finally, it was shown how to construct new differential equations for which Grothendieck's p-currvature conjecture holds (starting from complex representations of finite groups and applying the convolution functor).

Galois theory in dimension 2

David Harbater

I consider the Galois theory of surfaces, especially analogs of questions that have been studied in dimension 1. In dimension 1, say X is an affine curve over an algebraically closed field k. If char $k = 0$, then one knows $\pi_1(X)$ by topology, and therefore one knows which finite groups are the Galois groups of unramified covers of X. In characteristic p, $\pi_1(X)$ is unknown, but we know which finite groups are Galois groups of unramified covers of X, by Abhyankar's Conjecture (proven by Raynaud and the speaker). In dimension 2, we may consider the complement X in $\mathbb{P}^2$ of a normal crossing divisor. The fundamental group of X is known in characteristic 0 (Zariski, Fulton, Deligne), and therefore it is known which finite groups occur. In characteristic p, the tame part of π_1 can be described similarly (Abhyankar, Fulton). But concerning the full π_1 in characteristic p, the "obvious" analog of Abhyankar's Conjecture fails, because there is an extra necessary condition for a group to occur (van der Put, Guralnick, and the speaker).

Similarly, one can consider the corresponding question for the Galois extensions of the function field K of the curve or surface X. For curves, all finite groups are Galois groups, and the absolute Galois group G_K is free (Douady in characteristic 0; Pop and the speaker in characteristic p). This is shown by using that G_K is projective (cohomological dimension $= 1$), and proving that every finite split embedding problem has many proper solutions. (Here "many" means of the same cardinality as k.) In dimension 2, consider the field $k(x, y)$ and its more local analog $k((x, y))$. Then every finite group is a Galois group over these fields K; but the absolute Galois group is *not* free (since the cohomological dimension is 2). Still, it turns out that every finite split embedding problem for K has many proper solutions; so G_K is "as free as possible given that it is not projective".

Some of these results are known to have analogs in differential Galois theory. It would be interesting to know whether the others (particularly in dimension 2) do as well.

Birational anabelian geometry –REVISITED–

Florian Pop

The main tools consists of defining/introducing so called abstract abstract Galois structures and studying properties of such objects. The aim of this talk was to show that there are birational anabelian phenomena over the algebraic closure of finite/global fields. More precisely, we have indicated how one can prove the following result: Let k be the algebraic closure of a finite/global field. For a field extension $K|k$, we denote by $K'|K$ the maximal pro-ℓ extension of K, and set $G'_K = \mathrm{Gal}(K'|K)$.

Theorem. *In the above context, suppose that $K|k$ is a function field with $td(K|k) > 1$ having simply connected smooth models $X \to k$. Then there exists a group theoretic recipe by which we can recover the isomorphy type of $K|k$ from G'_K. This recipe is invariant under pro-finite group isomorphisms. In particular, if $G'_K \cong G'_L$, then $K \cong L$ up to pure inseparable covers in a functorial way.*

The main tools consists in defining/introducing so called abstract Galois structures and studying properties of such objects. Along the same lines one obtains further birational anabelian results, in particular a sharpening of the result of Mochizuki, that the category of all geometrically connected varieties over some given sub-p-adic field k is equivalent via the absolute Galois group functor to a full sub-category of the pro-finite G_k-groups.

Logarithmic good reduction of curves

Jakob Stix

Let S be the spectrum of an excellent henselian (e.g. complete) discrete valuation ring R with perfect (e.g. algebraically closed, finite) residue field $k = R/(\pi)$ of exponential characteristic $p \geq 0$ and field of fractions K. Let η (resp. s) denote its generic (resp. closed) point. We fix a geometric point $\overline{\eta} = \mathrm{Spec}(\overline{K})$ (resp. $\overline{s} = \mathrm{Spec}(\overline{k})$) over η (resp. s). The absolute Galois group G_K of K comes equipped with the (wild) inertia subgroups $P < I < G_K$. Let ℓ be a prime number different from p.

The theory of good reduction asks for a criterion that decides whether smooth objects over η extend to smooth objects over S. For example, an abelian variety A/K has good reduction if and only if I acts trivial on the ℓ-adic Tate module $\mathrm{T}_\ell(A)$. Secondly, a smooth hyperbolic curve X/K has good reduction if and only if the restriction to I of its outer Galois representation $\rho_X^\ell : G_K \to \mathrm{Out}\left(\pi_1(X_{\overline{\eta}})^\ell\right)$ on the pro-ℓ quotient of π_1 is trivial.

Question: What are the geometric implications of trivial action of the wild inertia group? For the case of smooth projective curves we obtain the following theorem that improves a theorem of T. Saito in connecting it with logarithmic geometry.

Theorem. *Let X_K be a smooth projective curve over* $\mathrm{Spec}(K)$ *of genus* $g \geq 2$. *Then the following are equivalent:*

(a) *P acts trivially on $\pi_1(X_{\overline{K}})^\ell$ for some prime number $\ell \neq p$.*

(b) *P acts trivially on $\pi_1(X_{\overline{K}})^{ab,\ell}$ for some prime number $\ell \neq p$.*

(c) *The minimal regular model of X_K over S such that the reduced special fibre is a divisor with normal crossings satisfies the following: any component of the special fibre with multiplicity divisible by p is isomorphic to $\mathbb{P}^1_k$ and meets the rest of the special fibre in exactly two points lying on components of multiplicity prime to p.*

(d) *X_K has log-smooth reduction over S, i.e., there is a model X/S such that $M(\log X_s)$ is a fs-log structure on X that renders it a log-smooth fs-log scheme over S with its standard fs-log structure induced by $\mathbb{N} \to R$, $1 \mapsto \pi$.*

The implications $(d) \implies (a) \iff (b) \iff (c)$ follow from results of Vidal, Kisin/Asada et al., and T. Saito. For the remaining purely geometric step we use M. Artin's contributions to the theory of rational singularities of surfaces.

Counting Galois extensions

Jürgen Klüners

Let k be a number field and $G \leq S_n$. We define:

$$Z(k, G; x) := |\{K/k \mid \mathrm{Gal}(K/k) = G, N_{k/\mathbb{Q}}(d_{K/k}) \leq x\}|.$$

We present the Malle-conjecture which predicts the asymptotic behaviour of this function for $x \to \infty$: $Z(k, G; x) \sim c(k, G) x^{a(G)} \log(x)^{b(k,G)}$, where a is a constant depending on G and b is a constant depending on k and G, which can be explicitly given. The conjecture is known to be true for abelian groups and some small groups. In this talk we prove this conjecture for nilpotent groups in a weak form and give improved results for 2-groups.

Images of modular and geometric 3 and 4-dimensional Galois representations

Nuria Vila, Luis V. Dieulefait

We consider compatible families of non-selfdual three-dimensional Galois representations with coefficients in an imaginary quadratic field K, either those coming from geometry (in particular a family of examples constructed by van Geemen and Top) and those conjecturally attached to certain automorphic representations of $GL(3)$ via the Langlands correspondence (as formulated by Clozel). We give conditions on the representations to ensure that the images of the residual representations be "as large as possible" for almost every prime (all but finitely many), namely:
$SL(3, \ell)$ if ℓ splits in K, and $SU(3, \ell)$ if it is inert.
The conditions for the validity of the theorem are verifyed in one geometric example. We also verify the same for some examples (conjecturally) coming from automorphic forms.

In the second part of the talk, the case is considered of compatible families of symplectic four-dimensional Galois representations attached by Taylor-Laumon-Weissauer to a cuspidal genus 2 Siegel modular form with multiplicity one; only the case of level 1, and weight $k > 3$ is considered. Here it is proved that for such a cusp form f, if f is not of Saito-Kurokawa type and verifies other two conditions (easy to verify in any given example) then the images of the attached Galois representations are "as large as possible" (the maximal possible symplectic group given the field of coefficients and the restrictions

on the determinant) for every prime outside certain density 0 set, and also that assuming Serre's conjecture (for residual 2-dimensional irred. odd representations) the same holds for almost every prime.
Examples are given of Siegel forms f verifying all the conditions of the previous result, of weight 20 and 28. Considering the projectivisation of the residual representations, new groups $PGSp(4, \mathbb{F}_\lambda)$ and $PSp(4, \mathbb{F}_\lambda)$ are realized as Galois groups over $\mathbb{Q}$.

Descent for differential operators over $\overline{\mathbf{C}(x)}[\frac{d}{dx}]$

Jacques-Arthur Weil
(joint work with Elie Compoint)

Let $\mathbf{C}$ denotes the field of complex numbers, $k = \mathbf{C}(x)$ with derivation $\frac{d}{dx}$; let k_0 denote a finite Galois extension of k.
Let $L_1 \in k_0[\frac{d}{dx}]$. We say that L descends to k if there exists $M \in k[\frac{d}{dx}]$ such that L_1 and M are isomorphic over $\overline{k}$.
If k_1 is a finite Galois extension of k_0 such that L_1 is isomorphic over k_1 to M, we say that k_1 is a descent field and that L_1 descends to M over k_1.
In this lecture, we study how to caracterize descent data, descent field, and how to effectively achieve the descent (i.e construct M). Applications of this to absolute factorization and its impact on the differential Galois group are developped. We also discuss the similarity and differences between this approach and the arithmetic situation of the same problem for a non-algebraically closed field studied by van Hoeij and van der Put.

Pullbacks of second order differential equations

Maint Berkenbosch

Consider a normalised second order differential operator $L = d^2 - r$ over $k(x)$, with finite primitive Galois group. We can give a nice proof of Klein's theorem, which states that L is the normalised pullback of the standard equation, under some $F \in \bar{k}$. In particular this gives a description of all possibilities for F. Moreover we can actually calculate F using some semi-invariants of L.

Differential Galois realization of double covers

Zbigniew Hajto
(joint work with Teresa Crespo Vicente)

In the talk we present an effective construction of homogeneous linear differential equations of order 2 with Galois group a double cover $2G$ of a group G equal to one of the alternating groups A_4, A_5 or the symmetric group S_4 over a differential field k of characteristic 0 with algebraically closed field of constants $\mathcal{C}$. It is known that, if $K|k$ is an algebraic extension of the differential field k, then the derivation of k can be extended to K in a unique way and every k-automorphism of K is a differential one. Thus a realization of a finite group G as an algebraic Galois group over k is also a realization of G as a differential Galois group. If such a group G has a faithful irreducible representation of dimension n over $\mathcal{C}$, then G is the Galois group of a homogenous differential equation of order n over k. Given a polynomial $P(X) \in k[X]$ with Galois group G and splitting field K, we give an equivalent condition in terms of a quadratic form over k for the existence of a homogeneous linear differential equation with Galois group $2G$ such that its Picard-Vessiot extension $\widetilde{K}$ is a

solution to the Galois embedding problem associated to the field extension $K|k$ and the double cover $2G$ of G. When this condition is fulfiled, we determine explicitly all such differential equations. Our result has been anounced in:
T. Crespo, Z. Hajto, *Recouvrements doubles comme groupes de Galois différentiels*, C.R. Acad. Sci. Paris, Série I, **333** (2001) 271-274.

Differential Galois realization of covers

Teresa Crespo Vicente
(joint work with Zbigniew Hajto)

In this talk we consider the Galois embedding problem

$$(GEP) : 3A_6 \to A_6 \simeq \mathrm{Gal}(K|k)$$

over a field k of characteristic 0 and containing the roots of unity of order 15. Here A_6 denotes the alternating group on 6 letters and $3A_6$ the Valentiner group, which is a non trivial central cover of A_6. We prove that (GEP) is solvable if and only if a certain quadratic cone Q defined over k, which can be made explicit, has a non-trivial k-rational point. To this end, we use a unimodular irreducible faithful representation $\widetilde{\rho}$ of the group $3A_6$ of dimension 3 and its third symmetric power $\rho = \widetilde{\rho}^{(3)}$, which factorizes through A_6. Our method is based on the explicit determination of all possible copies of ρ inside K. Whenever (GEP) is solvable, we obtain explicitly all possible solutions to it from the non-trivial k-rational points of Q. If k is a differential field of characteristic 0, we obtain all possible differential equations with differential Galois group $3A_6$ such that its Picard-Vessiot extension $\widetilde{K}$ is a solution to (GEP).

Torsion on Abelian varieties over large algebraic fields

Moshe Jarden

Theorem. *Let A be an abelian variety over a number field K. Then K has a finite Galois extension L such that for almost all $\sigma \in \mathrm{Gal}(L)$ there are infinitely many prime numbers l with $A_l(\tilde{K}(\sigma)) \neq 0$.*

Here $\tilde{K}$ denotes the algebraic closure of K and $\tilde{K}(\sigma)$ the fixed field in $\tilde{K}$ of σ. The expression "almost all σ" means "all but a set of σ of Haar measure 0". This theorem, proved jointly with Wulf-Dieter Geyer weakly settles part of a conjecture we made in 1978.

To prove the theorem we construct a finite Galois extension L of K, a number field N, a set Λ of prime numbers, a connected reductive algebraic subgroup H of GL_{2d} over N (with $d = \dim(A)$), a connected linear algebraic group $\hat{H}$, and an isogeny $\theta \colon \hat{H} \to H$ over N which satisfies the following conditions:

1. Λ is contained in the set $\mathrm{Splt}(N)$ of all l that split completely in N.
2. $\sum_{l\in\Lambda'} \frac{1}{l} = \infty$ for each $\Lambda' = \Lambda \cap \mathrm{Splt}(N')$ with N' a finite extension of N.
3. For each $l \in \Lambda$, $\theta(\hat{H}(\mathbb{F}_l)) \leq G_L(l) \leq H(\mathbb{F}_l)$ and $(H(\mathbb{F}_l) : \theta(\hat{H}(\mathbb{F}_l))) \leq |\mathrm{kernel}(\theta)|$. Here $G_L(l)$ is the image in $\mathrm{GL}_{2d}(\mathbb{F}_l)$ of $\mathrm{Gal}(L)$ under the l-ic representation arising from the action on $A_l(\tilde{K})$.
4. The fields $L(A_l)$, $l \in \Lambda$, are linearly disjoint over L.

In proving these conditions we use results of Serre proved during his course at the Collége de France in 1985-1986 and classification theorems of connected semisimple algebraic groups.

Arithmetic on Hurwitz towers

Pierre Debes

Given a projective system of finite groups G_n $(n > 0)$, the aim of the proposed talk is to construct a tower $(\mathcal{H}_n)_{n>0}$ of varieties with the following properties:

- $\mathcal{H}_n$ is geometrically irreducible and defined over k_o (depending on the situation),
- k-rational points on $\mathcal{H}_n$ yield k-regular realizations of G_n, $n > 0$, (for any field $k \supset k_o$),
- for $k = k_o\mathbf{Q}_p$, $k = k_o((x))$ and $k = \mathbf{R}$, there exists projective systems of k-rational points (which yield k-regular realizations of $\varprojlim G_n$).

The construction conjoins patching techniques for algebraic covers (including recent developments for infinite covers) and the Hurwitz space theory (including Fried's modular towers and related work on the so-called Harbater-Mumford components and their boundary).

Reduction of Hurwitz spaces

Irene Bouw

In this talk, I compute the stable reduction of some Galois covers of the projective line branched at three points. These covers are constructed using Hurwitz spaces parameterizing metacyclic covers. The reduction is determined by a hypergeometric differential equation. This generalizes the result of Deligne–Rapoport on the reduction of the modular curve $X(p)$.

Induction and restriction in formal deformation of coverings

Ariane Mezard

Let X/S be a semistable curve with an action of a finite group G and let H be a normal subgroup of G. We present a new condition under which for any base change $T \to S$, $(X/G)\times_S T$ is isomorphic to $(X\times_S T)/G$. This allows us to define induction and restriction morphisms between the G-equivariant deformation functor of X and the G/H-equivariant (resp. H-equivariant) deformation functor of X/H (resp. X).

On the existence of finite Galois stable subgroups of GL_n

Dmitry Malinin

Let E be a finite extension of a number field F with Galois group Γ, and let O_E and O_F be the maximal orders of E and F. Let $F(G)$ be a field obtained via adjoining to F all matrix coefficients of all matrices $g \in G \subset GL_n(E)$.

Theorem 1. *1) For a given number field F and integers n and t, there is only a finite number of normal extensions E/F such that $E = F(G)$ and G is a finite abelian Γ-stable subgroup of $GL_n(O_E)$ of exponent t.*

2) For a given number field F and integers n and $d = [E\colon F]$, there is only a finite number of fields $E = F(G)$ for some finite Γ-stable subgroup G of $GL_n(O_E)$.

Theorem 2. *Let $d > 1, t > 1$ and $n \geq [E(\zeta_t) : E]d$ be given integers where ζ_t is a primitive t-root of 1, and let E/F be a given extensionof degree d. Then there is an abelian Γ- stable subgroup $G \subset GL_n(E)$ of exponent t such that $E = F(G)$.*

The following theorem was proved jointly by H.-J. Bartels and the author using the classification of finite flat group schemes over $\mathbf{Z}$ annihilated by a prime p obtained by V. A. Abrashkin and J.- M. Fontaine:

Theorem 3. *Let $K/\mathbf{Q}$ be a normal extension with Galois group Γ, and let $G \subset GL_n(O_K)$ be a finite Γ-stable subgroup. Then $G \subset GL_n(O_{K_{ab}})$ where K_{ab} is the maximal abelian over $\mathbf{Q}$ subfield of K.*

Non-classical parabolic cohomology and the Manin–Drinfeld principle

Stefan Wewers

Let Γ be a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$, l a prime and $k \geq 0$. Let $U := \mathbb{H}/\Gamma$, $X := \mathbb{H}^*/\Gamma$ and $j : X \to \mathbb{P}^1 \cong \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ the natural map to the j-line. Let $\pi : E \to U$ be the 'universal' elliptic curve over U (with j-invariant j) and set $\mathcal{F} := \mathrm{Sym}^k(R^1\pi_*\mathbb{Q}_l)$. If K is a field of definition of $j : X \to \mathbb{P}^1$ and E, we obtain a short exact sequence of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$–modules

$$0 \to H^1(U, \mathcal{F})^{\mathrm{cusp}} \to H^1(U, \mathcal{F}) \to \bigoplus_{\mathrm{cusps}} \mathbb{Q}_l(-k-1)(0) \to 0$$

The Manin–Drinfeld principle asserts that this sequence splits if Γ is a congruence subgroup.

Generalizing a criterion of T. Scholl (Inv. Math. 124, 1996) and using results of myself on the stable reduction of Galois covers (*Three point covers with bad reduction*, Preprint 2002), I gave an infinite family of counterexamples to the Manin–Drinfeld principle for non-congruence subgroups and all $k > 0$.

Finite monodromy for p-adic Galois covers

Claus Lehr

(joint work with Michel Matignon)

In this talk we report on the reduction of covers of curves over p-adic fields. Let (R, K, k) be a complete mixed characteristic $(0, p)$ DVR and X a smooth, complete K-curve that is a p-cyclic cover of the projective K-line. Denote by B the branch locus of the cover and assume B has equidistant geometry. By this we mean that there is a smooth R-model for $\mathbb{P}^1_K$ such that the points of B specialize to distinct points on the closed fiber. Set $m = |B| - 1$ and assume that m is not of the form $p^k + 1$. Under these assumptions we obtain an explicit criterion to test if X has potentially good reduction. In this case we determine the finite monodromy, i.e. the minimal extension R'/R necessary to obtain a smooth model $X_{R'}$ for X. This extension is known to be Galois and there is an injection from $\mathrm{Gal}(R'/R)$ into the group of k-automorphisms of the special fiber X_k of the stable model. In this context we are lead to study automorphism groups of certain curves in characteristic p, with the goal to bound the finite monodromy. We obtain such bounds improving those known previously.

Edited by Thomas Oberlies

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