Mathematisches Forschungsinstitut Oberwolfach

Report No. 47/2002

Arbeitsgemeinschaft mit aktuellem Thema: Einstein Metrics and Geometrization of 3-Manifolds

October 6th – October 12th, 2002

The subject of the conference was an analytic approach towards the proof of the geometrization conjecture of Thurston, which states that any closed 3-manifold can be cut along finitely many spheres and tori into domains, which carry a geometric structure.
This approach, which is essentially due to Michael Anderson, who was one of the organizers of this Arbeitsgemeinschaft relies on a profound understanding of certain curvature functionals and their critical points.
The aim of the lectures was to unfold the main ideas and techniques included and developed in this approach.

Abstracts

Geometrization Conjecture

Bernhard Leeb

We outline the development in 3-dimensional topology leading to Thurston's Geometrization Program, including the Kneser-Milnor prime decomposition, the Loop and Sphere Theorems, the torus splitting, the Seifert Fibre Space Conjecture and the Jaco-Shalen-Johannsen characteristic submanifold theory. Thurston's Geometrization Conjecture states that the pieces resulting from the topological decomposition process admit geometric structures, i.e. complete locally homogeneous metrics on their interiors.
We state the results by Gromov-Lawson on positive scalar curvature metrics on 3-manifolds. A closed 3-manifold admits no metric of $scal > 0$ if it contains an aspherical prime factor. The Hyperbolization Conjecture, one of the two open cases in the Geometrization Program, follows from the geometrization of 3-manifolds with nonpositive Yamabe constant, as proposed by Anderson.

Cheeger-Gromov theory ($L^\infty case$)

Joan Porti

We outline the proof of the following theorems, due to Cheeger and Gromov.
Theorem Let M^3 be an orientable closed 3-manifold, with a sequence of Riemannian metrics g_m such that $vol = 1$ and $|sec| < \Lambda$, for some uniform Λ. Then one of the following occurs (up to subsequence):

- Convergence Case. There exists a metric g_∞ on M of class $C^{1,\alpha}$ and a sequence of diffeomorphisms $\varphi_m : M^3 \to M^3$, such that $\varphi_m^* g_m \to g_\infty$, in the $C^{1,\alpha'}$ topology, $0 < \alpha' < \alpha$.

- Collapse Case. $inj_{g_m} \to 0$ and M has an F structure, so that orbits are arbitrarily short. In particular M is a graph manifold.

- Partial Collapse. There are some regions of M where the injectivity radius goes to zero (and they have an $F-$structure), and some regions where the g_m do not collapse and converge to complete submanifolds for the pointed $C^{1,\alpha'}$-topology.

The second case has topological consequences on M: it is a graph manifold. However the third case has no topology consequences, unless we have some control on the metric!

Cheeger-Gromov theory (L^2-case)

Burkhard Wilking

Following a paper of Anderson we show that in the 3-dimensional case many results of Cheeger and Gromov (c.f. 2^{nd} talk) remain valid if one replaces a priori $L^\infty-$bounds on R by a priori L^2-bounds on R. In this setting the volume radius plays an important role. It is defined by

$$\nu(x) := max\{r | \forall B_y(s) \subset B_x(r), vol(B_y(s)) \geq \frac{s^3}{100}\}$$

One then divides the manifold (M^3, g) into a thick and a thin part

$$\begin{aligned} M^\epsilon &= \{x \in M | \nu(x) > \epsilon\} \\ M_\epsilon &= \{x \in M | \nu(x) \leq \epsilon\} \end{aligned}$$

The main goal of this talk is to prove the following theorem by Anderson.

Theorem Consider a sequence of (unit volume) metrics g_i on a closed 3-manifold M satisfying $\|R^2_{g_i}\|_{L^2} < \Lambda$. Then after passing to a subsequence one can choose $\epsilon_i > 0$ with $\epsilon_i \longrightarrow 0$ such that

$$(M^{\epsilon_i}, g_i) \longrightarrow (\Omega, g_\infty)$$

in the pointed Gromov Hausdorff topology.

There exist $U_i \subset \Omega$ open such that $\cup_{i\in N} U_i = \Omega$ and diffeomorphisms $f : U_i \longrightarrow M^{\epsilon_i}$ such that $(U_i, f^* g_i) \longrightarrow (\Omega, g_\infty)$ in the weak $L^{2,2}-$topology.

Moreover, for i large enough the thin part M_{ϵ_i} is contained in a graph manifold $V \subset M$.

Curvature Integrals I: Einstein-Hilbert action.

Bernhard Hanke

The Einstein-Hilbert action is the simplest, (most natural), curvature integral. Its Euler-Lagrange equation is $z = 0$ and is of 2^{nd} order and lies at the foundation of general relativity.

This lecture discussed Yamabe metrics, the space $\mathcal{C}$ of Yamabe metrics, min-max process and definition of the Yamabe invariant $\sigma(M)$. The linearization L of scalar curvature and its adjoint

$$L^* u = D^2 u - \Delta u - ur.$$

was introduced. The main aspects of the analysis of the scalar curvature functional on $\mathcal{C}$ were discussed; in particular the two splittings of z.

It was shown that Yamabe metrics realizing $\sigma(M)$ are Einstein when $\sigma(M) \leq 0$. The last lecture by Rozoy shows that the same is true when $\sigma(M) > 0$; this is very recent work. That the potential u is global was emphasized; it was demonstrated that u cannot be determined from the local geometry.

Curvature Integrals II: Scalar curvature.

Bernd Ammann

The discussion of this lecture related to integrals involving scalar curvature s which are bounded below, so one can try to seek minima. There are several reasons for preferring such integrals, but one reason is the difficulty of understanding the global nature of the potential u in the case of the Einstein-Hilbert action, (the horizon problem).

Thus integrals of the form $\mathcal{S}^2, \mathcal{S}^2_-, \mathcal{S}^p$, the $L^2(L^p)$ norm of s or of $s^- = \min(s, 0)$ were introduced. The proof that the infimum of such functionals equals $|\sigma(M)|$, when $\sigma(M) \leq 0$ was discussed. Conjectures I-II were restated in terms of minimizing $\mathcal{S}^2_-$.

The Euler-Lagrange equations for $\mathcal{S}^2$ and $\mathcal{S}^2_-$ were stated - they are of 4th order. All critical points of $\mathcal{S}^2_-$ are of constant curvature; this is unknown for $\mathcal{S}^2$ when the scalar curvature changes sign. The relation of these equations with that for Einstein-Hilbert action was discussed. The (now local) potential $-\frac{s^-}{\sigma}$, (or $-\frac{s}{\sigma}$), in the Euler-Lagrange equation corresponds to u from the previous lecture.

In general it is very difficult to minimize scalar curvature integrals - one has very little control on the behaviour of a general minimizing sequence. One needs to find a “controlled” minimizing sequence.

Curvature integral III: Full Curvature

Laurent Bessieres

On M^3 closed oriented manifold, one studies minimizing sequences in $\mathcal{M}_1 = \{g$ Riemannian metric on M, $vol_g M = 1$ $\}$ for the functional $|R_g|_{L^\infty}$ or $|R_g|_{L^2}$.
$\inf |R_g|_{L^\infty}$ is closely related to the minimal volume defined by Gromov: $MinVol(M) = \inf\{vol_g M : |K_g| \leq 1\}$. By the isolation theorem (Cheeger-Gromov, Rong) $\exists \epsilon_0 > 0$ such that $MinVol(M) < \varepsilon_0$ if and only if M is graph manifold (collapse case). In the non collapse case $MinVol(M) > \varepsilon_0$, the Cheeger-Gromov theory provides for any minimizing sequences (M, g_i), convergence in the pointed $C^{1,\alpha}$-topology to a limit (Ω, g), Ω a domain in M, $g \in C^{1,\alpha}$ complete, $\alpha < 1$. The domain Ω has finite number of connected components, $vol_g \Omega \leq MinVol(M)(=)$. For hyperbolic manifolds, $MinVol$ is uniquely realized by the hyperbolic metric. But a minimizing sequence for $MinVol(M\#M)$, M closed hyperbolic, does not realize geometrization because $MinVol(M\#M) > 2MinVol(M)$.
The case of $\mathcal{R}^2(g) = \int_M |R_g|^2 dV_g$ is very similar: there is a collapse / noncollapse dichotomy according to existence of point $x_i \in M$ such that $\nu_i(x_i) > \varepsilon > 0$, ($\nu_i$ = volume radius of g_i), for all g_i of the minimizing sequence. Graph structure in the collapse case. The Cheeger-Gromov L^2-theory (Anderson) provides in the non collapse case (Ω, g), $\Omega \subset M$ with g complete C^∞ (regularization obtained from Euler-Lagrange equation for $\mathcal{R}^2(g)$, elliptic theory) and

$$vol_g = 1, \Omega = \Omega_1 \cup \cdots \cup \Omega_k, k \leq k(M), \int_\Omega |R_g|^2 = \inf_{\mathcal{M}_1} \mathcal{R}^2(g)$$

It is conjectured that hyperbolic metric realizes $\inf_{\mathcal{M}_1} \mathcal{R}^2_g$ if Ω hyperbolic. But on $M\#M$, M closed hyperbolic, this process does not give geometrization: the limit is complete C^∞, but the metrics g_i cannot crush the essential S^2 in the limit, with limit hyperbolic closed metric in each M. Which implies we need to pass to scalar curvature functional, regularized by $\mathcal{R}^2$.

The functionals $I_\varepsilon{}^-$

Uwe Abresch

The purpose of this lecture was to introduce the functionals $I_\varepsilon{}^-(g) = \mathcal{S}^2_-(g)^{1/2} + \varepsilon \cdot \mathcal{Z}^2(g)$ that are used in order to analyze the structure of the 3-manifold M in the cases where $\sigma(M) := \max_{[g]} \mu(g) \leq 0$. Here $\mu(g) := \inf_{\hat{g} \in [g]} \mathcal{S}(\hat{g})$, $\mathcal{S}(g) := vol(g)^{1/3} \int_M sc_g dv$ The basic motivation for setting up such a penalty method is to get some a priori regularity for a minimizing sequence g_ε for $\mathcal{S}^2_-$. We explained that functionals like $\mathcal{R}^2(g)$ that guarantee pointwise bounds on curvature are not appropriate for the geometrization program either; its Euler equation are of fourth order and are too complicated to yield any reasonable information about the structure of its solutions. The example of the connected sum $M_1\#M_2$ of two hyperbolic spaces shows that there exist (likely a lot of) non-trivial solutions. But in order to be helpful with the proof of the geometrization conjecture it will be necessary that

(1) the structure of the minimizers can be understood and related to the homogeneous geometries
(2) the minimizing sequence develops singularities and non-compact limits in order to cope respectively with the prime (or sphere) decomposition and the torus decomposition theorem

The goal is to get this information about M from the variational problem. Strong convergence of minimizing sequences is a must in order to have Euler equations, get regularity, understand collapse, and show that there are no spurious degeneracies.
The Euler Lagrange equations of $I_\varepsilon{}^-$ are $\nabla I_\varepsilon{}^- = \varepsilon \cdot \nabla \mathcal{Z}^2_{g_\varepsilon} + L^*_g(\varepsilon) + \chi \cdot g$. A brief discussion of this equation has been given. We explained why it is advantages to work with the negative part of scalar curvature rather than the full scalar curvature. We stated (without proof) that the principal term $L^*_g(u)$ in the Euler-Lagrange equations gives a direct connection to static solutions of the vacuum Einstein equations.

Proof of Conjectures I-II for Tame 3-Manifolds.

Joseph Ayoub

The definition of tame 3-manifolds. M^3 tame implies that

$$\rho(x) \geq \rho_o,$$

for all $x \in (\Omega_\varepsilon, g_\varepsilon)$ as $\varepsilon \to 0$, where $\rho_o = \rho_o(M)$ depends only on M. The basic idea of the proof of this was sketched.

In brief, one applies the L^2 Cheeger-Gromov theory. As $\varepsilon \to 0$, metrics either converge to compact limit metric on M, form cusps in M, or collapse. The smoothness and convergence to limits were discussed. All limits (compact or cusps) are of constant curvature.

The proof that tori in cusps are incompressible in M was studied in some detail; this is the analogue of the Thurston cusp closing theorem. The geometrization of graph manifolds, via the geometry of the collapse was discussed.

Outline and Overview of the Program

Michael Anderson

This lecture gave an overview of the previous lectures, motivating and bringing together the various themes and issues.
Curvature functionals on the space of metrics on a given 3-manifold can be divided into two basic classes.

(1) Full curvature $\mathcal{R}$: For example $|R_{L^\infty}| = L^\infty$ norm of full or Ricci curvature, $MinVol$ of Gromov. L^2 or L^p norms of R, or of $z = Ric - \frac{s}{3}g$, etc.
(2) Scalar curvature functionals $\mathcal{S}$: For example - Einstein-Hilbert functional, L^2 or L^p norms of s, L^2 or L^p norms of $s^- = \min(0, s)$, etc.

These two classes each have their own advantages/disadvantages which turn out to be essentially complementary.

(1) $\mathcal{R}$-functionals:
Good Side: One can control the geometry of minimizing sequences via the Cheeger-Gromov theory and related compactness theorems.
Bad Side: Limits are typically not of constant curvature, but instead only satisfy very complicated equations. Thus, limits are not related to the Thurston or geometric decomposition of M.

(2) $\mathcal{S}-$functionals:
Good Side: Minimizers, when they exist, are typically of constant curvature and reflect the geometric decomposition of M (sphere / torus decomposition, etc.)
Bad Side: One has no control on geometry of minimizing sequences, and so no possibility of obtaining existence of limits.
These constructions motivate the choice of seemingly complicated functional

$$I_\varepsilon{}^- = \varepsilon(vol)^{1/3}\int_M |z|^2 + (vol^{1/3}\int_M (s^-)^2)^{1/2}$$

combining features of both classes.
We reviewed the existence and geometry of minimizers of $I_\varepsilon{}^-$, namely $(\Omega_\varepsilon, g_\varepsilon)$, and discussed some basic estimates on the geometry as $\varepsilon \to 0$, referring in particular to the results of lectures 7 and 8 (Abresch and Ayoub).

The static vacuum Einstein equation in General Relativity

Bernd Siebert

Under the presence of a hypersurface orthogonal time-like Killing field the vacuum Einstein equations $Ric_N = 0$ (for (N^4, g) Lorentzian) reduce to the system of equations on a 3-manifold (M, g).

$$Ric_g = 0, \Delta u = 0$$

where $u > 0$. These are the static vacuum Einstein equations on M. In Anderson's geometrization program solutions to these equations occur at various places as limits of rescaled metrics and a good understanding in particular of the asymptotics of such solutions is central.
In the talk I discussed the most symmetric solution of this system with one end; the Schwarzschild solution, and its characterization by asymptotic flatness and smoothness up to horizon. There are many more solutions by a construction due to Weyl, that are S^1-symmetric. I mentioned a few special cases showing the necessity of the assumptions in the mentioned characterization at the Schwarzschild metric ("black hole uniqueness".
Another important result is the triviality of complete solutions (generalized Lichnerowicz theorem), which by contradiction with a blow-up construction also gives important a priori estimates for the curvature and the gradient of the potential function u. Similarly one also obtains a priori estimates for (the L^2-curvature radius of) Yamabe metrics.

Introduction to Curvature Blow-up

Fabian Ziltener

Given a sequence of metrics g_i on M^3 that satisfy some non collapse assumption and whose scalar curvature is uniformly bounded in the L^2-norm, but whose Ricci curvature is unbounded, one can find a solution $(\bar{g}, \bar{u})$ on $B_1 \subseteq \mathbb{R}^3$ of the static vacuum Einstein equations

$$\begin{aligned} D^2\bar{u} &= Ric_{\bar{g}}\bar{u}, \\ \Delta_{\bar{g}}\bar{u} &= 0, \end{aligned}$$

$\bar{g}$ is obtained as the weak $L^{2,2}$ limit of a suitably rescaled subsequence of the g_i's (modulo diffeomorphisms) and $\bar{u}$ is constructed from the sequence $u_i \in L^{2,2}(M, \mathbb{R})$, where u_i satisfies

$$L^*_{g_i}u_i + \xi_i = -\frac{s_i}{3}g_i$$

for some $\xi_i \in \ker L_{g_i}$. There are some obvious solutions of the static vacuum Einstein equations. Under some additional hypotheses one can construct non trivial solutions $(\bar{g}, \bar{u})$.

Structure of blow up limits

Thomas Schick

In previous talks, metrics $(\Omega_\varepsilon \subset\subset M, g_\varepsilon)(\varepsilon > 0)$ were constructed, minimizing the functional I^-_ε. We assume that $\sigma(M) \leq 0$ and $\int_{\Omega_\varepsilon} |z|^2 \to_{\varepsilon\to 0} \infty$. The limiting behaviour has to be studied.

We pick appropriate $x_\varepsilon \in \Omega_\varepsilon$ with $\rho(x_\varepsilon) \to 0$ and study the blow ups $(\Omega_\varepsilon, g'_\varepsilon = \rho_\varepsilon(x_\varepsilon)^{-2}g_\varepsilon, x_\varepsilon)$.

Theorem If $\{x_\varepsilon\}$ is chosen correctly,
$(\Omega_\varepsilon, g'_\varepsilon, x_\varepsilon)$ converges in the <u>strong</u> $L^{2,2}-$topology to a complete limit (Ω, g, x).
The limiting metric is locally $L^{3,p}$, and smooth outside a (possibly empty) junction set $\Sigma \subset \Omega$.

It satisfies (weakly) one of three partial differential equations, either:

$$\begin{array}{lll} (1) & \nabla \mathcal{Z}^2 = 0 & \\ (2) & \alpha\nabla^2\mathcal{Z} + L^*(\tau) = 0 & \alpha \neq 0, \tau \leq 0, \tau \not\equiv 0 \\ (3) & L^*(\tau) = 0, & \tau \leq 0 \end{array}$$

with trace equations

$$\begin{array}{ll} (1') & \Delta s = -3|z|^2 \\ (2') & \Delta(\tau + \frac{\alpha}{12}s) = -\frac{1}{4}\alpha|z|^2 \\ (3') & \Delta\tau = 0 \end{array}$$

To get from uniform $L^{2,2}-$bounds (given by the curvature radius $\rho'(x_\varepsilon) = 1$) to uniform $L^{3,2}-$bounds (which give by compactness of $L^{3,2} \hookrightarrow L^{2,2}$, strong convergence), uniform elliptic estimates are used. They work only because $\tau \leq 0$, i.e. because we use the cutoff functional I^-_ε.

Blow-Up-Limits II: Non-Existence Results

Matthias Schwarz

¿From the previous talk No. 11 (Schick) the possible blow-up limits for a sequence of minimizers $(\Omega_\varepsilon, g_\varepsilon)$ for the functional $I_\varepsilon{}^-$, have been classified to be of at most one of the following types:
(N, g_∞) is a complete, non-flat, 3-manifold with potential function $\tau \leq 0$, scalar curvature $s \geq 0$ such that $s \cdot \tau \equiv 0$
satisfying either

$$\begin{array}{lllll} (1) & \mathcal{Z}^2 - eq: & \nabla \mathcal{Z}^2(g) = 0 & s > 0, \tau \equiv 0 \\ & & \Delta s = -3|z|^2, & s \\ (2) & \mathcal{Z}^2_c - eq: & \alpha \nabla \mathcal{Z}^2(g) + L^*(\tau) = 0, & s > 0, \tau \leq 0, \tau \not\equiv 0, s\,\tau \equiv 0 \\ & & \Delta(\tau + \frac{\alpha}{12} s) = -\frac{1}{4}\alpha |z|^2 & \\ (3) & \text{static vacuum}: & L^*(\tau) = 0, & \tau < 0, s \equiv 0 \\ & & \Delta \tau = 0 & \end{array}$$

The subject of this talk was to rule out cases (1) and (3) and any limit metric which carries together with τ a free isometric S^1-action.
Proofs for this were sketched, in case (1) for the analogous case of the $\mathcal{R}^2$ - eq, in case (3) under the additional assumption that $-\lambda \leq \tau < 0$ for some $\lambda > 0$.
Based on this, the Theorem B in Anderson's work (M. Anderson: Scalar curvature, metric degenerations and the static vacuum Einstein equations on 3-manifolds, II, Geom. & Funct. Analysis, 11, (2001), 273-381) was completed, ruling out any collapse which would lead to a free isometric S^1-action.

Asymptotically flat ends I

Janko Latschev

In this talk, we continued the discussion of blow-up limits for sequences of minimizers $(\Omega_\varepsilon, g_\varepsilon)$ of the functionals $I_\varepsilon{}^-$. We introduced the
Sphere Conjecture: There exists a blow-up limit (N, g', γ) with an asymptotically flat end $E \subset N$ i.e. $E \sim \mathbb{R}^3 \setminus B$, and on E the metric g' has the form

$$g'_{ij} = (1 + \frac{2m}{r})\delta_{ij} + h$$

with $m > 0$ and $|h| = O(r^{-2}), |Dh| = O(r^{-3}), |Dh^2| = O(r^{-4})$
In the main part of the talk, we discussed how the sphere conjecture implies the main conjectures I and II. This is based on showing that the natural 2-spheres in the end are essential in M. Assuming this were not the case, we derive a contradiction to the minimizing properties of g_ε by constructing suitable comparison metrics.

Asymptotically Flat Manifolds II

Frank Loose

The topic of this talk was to explain M. Anderson's generalization of the so-called "Black Hole Uniqueness Theorem" (see M. Anderson: On the structure of solutions of the static vacuum Einstein equations, Ann. Henri Poincaré $\underline{1}(2002), 995 - 1042)$, which studies the asymptotic topology and geometry of the static vacuum Einstein equations on a 3-manifold. As it turns out the techniques used there are also applicable to prove the existence of asymptotically flat ends for solutions of the $\mathcal{Z}_c^2$-equations which occur as limit manifolds of rescaled minimizers for the functional I_ε^-, under certain compactness conditions on the level sets of the potential (see M. Anderson: Scalar curvature, metric degenerations, and the static vacuum Einstein equations on a 3-manifold II, Geom. & Funct. Analysis $\underline{11}(2001), 273-381$; in particular §7).

Where Things Stand and What Remains: Sigma -Tame manifolds

Michael Anderson

After discussion of the basic items and results from the previous two Lectures , we focus on the value distribution theory of the potential function $u=-\tau$ from earlier lectures.
The Sphere Conjecture from Lecture 13 (Latschev) is hard to prove, since one has no global mass bound on the potential u. On the other hand, one has an explicit mass bound on the potential u on the "original" unscaled $(\Omega_\varepsilon, g_\varepsilon)$. One then combines this with uniform local lower mass bounds. This leads to the following definition and theorem.

Definition Let $\tilde\sigma(g)=\sigma(g)-|\sigma(M)|, \sigma(g)=\left(\int_M (s^-)^2\right)^{\frac12}$. Then M is $\sigma-$tame if there exists a sequence of unit volume metrics g_i on M such that

$$\tilde\sigma(g_i)\to 0$$

and constants $K_1, K_2<\infty$ such that

$$[\tilde\sigma(g_i)]^{K_1}\cdot \mathcal{Z}^2(g_i)\le K_2$$

Theorem Conjectures I and II are true for $\sigma-$tame 3-manifolds.
It is conjectured that any 3-manifold (closed, oriented) is $\sigma-$tame.

Scalar Curvature and Black-holes

Luc Rozoy

Theorem Let (M,g,f) be such that (M,g) is a compact, analytic Riemannian 3-manifold, $f\in C^\omega(M)$ and $L^*(f)=z(=Ricci-\frac{scal}{3}g)$
Then M is Einstein
Proof Step 1 $\delta L^*(f)=\delta z \Rightarrow (f+1-\frac{2}{n=3})d(Scal)=0$ so $s=R=$ constant
Step 2 Cotton tenor: Set $S=Ricci-\frac{Scal}{2}g$. Then $C(x\wedge y,\zeta)=D_xS(y,\zeta)-D_yS(x,\zeta)$ is a symmetric traceless, divergence free tensor and if $\bar g=e^{2\sigma}g$, then $\bar C=e^{-3\sigma}C$
At $p\in M$ with $f(p)\neq -1$ and $\nabla f(p)\neq 0$ take $(\varepsilon_1,\varepsilon_2,\frac{\nabla f}{|\nabla f|})$ with $(\varepsilon_1,\varepsilon_2)$ unit vectors giving the curvature direction and k_1,k_2 the corresponding principal curvature:

$$C=\frac{1}{(f+1)^2}\begin{pmatrix} 0 & 2w(k_2-k_1) & -D_{\varepsilon_2}W \\ * & 0 & D_{\varepsilon_1}W \\ * & * & 0\end{pmatrix} \text{ with } W=|df|^2$$

Step 3 Conformally flat case:
In a open set where $df\neq 0, ds^2=\frac{df^2}{w}+\frac{2}{2}g_{AB}(x^1,n^2,f)dx^Adx^B$. Then $C=0\Rightarrow W=W(f)$ and $ds^2=dx^2+a^2(x)g_0$; g_0 is of constant curvature u_0.
Write $L^*(f)=z$, $a'f''-a''f=a''+\frac{R}{6}a$, $f''+\frac{2a'}{a}f'+\frac{R}{2}f=0$, $a^2\frac{R}{2}+a^{1,2}+2aa''=\frac{u_0}{2}$
We get **Lafontaine's seesaw** $(a')^2=\frac{c}{a}-\frac{R}{6}a^2 \Longrightarrow L^*(a')=0$
Yamabe's swing $f=-1+\lambda a'+Pr(\frac{a}{(a')^2})a\frac{R}{6}$
In case $c<0$, $f'(n_3)-f'(n_1)\neq 0 \Longleftrightarrow \int_{a_1}^{a_2}\frac{u^{5/2}du}{((u-a,)(a_2-u)(u+a,+a_2))^{3/2}}\neq 0$
and we are done

Step 4 If $f \geq -1$, $L^*(f) = z \Longrightarrow \int_M |z|^2 du_g = \int < L^*(f), z >= \int_M fL(z)dw_g = -\int_M f|z|^2 dw_g$. So $\int_M (1+f)|z|^2 dw_g = 0$ and we are done.
Step 5 If $\{f \leq -1\} \neq \emptyset$ we get $\{f \leq -1\} = (\bigcup_{k=0}^{m_1} S_k) \cup (\bigcup_{k=0}^{m_2} \{P_k\})$ where S_k is an embedded surface on which $(df) \neq 0$ and P_k a Morse singular point of f $\big(df(P_k) = 0,\ \text{Hess }(P_k) = \frac{R}{6} g\big)$. If $\bigcup \{P_k\} \neq \emptyset$ easy (same argument but more simple) if $\bigcup \{P_k\} = \emptyset \rightsquigarrow$
Step 6 $\exists k_0$ such that $\chi(S_k) > 0$
Proof: With $\bar{g} = \frac{g}{(f+1)^2}$, one has $\Delta_g (W + \frac{R}{6} f^2) = 2(f+1)^4 |z|^2$. So maximum of $W + \frac{R}{6} f^2$ on $\{|f+1| > \varepsilon\}$ at $+ M_\varepsilon$ If $\varepsilon \to 0$ at least one $S_k = S_0$ give the max of $W + \frac{R}{6} f^2$ on M. Let $W(S_0) = A$, $W + \frac{R}{6} f^2 \leq A + \frac{R}{6} f^2 \cdots / \cdots \Longrightarrow$ Gauss $(S_0) \geq \frac{R}{3} + \frac{(trh)^2}{2}$, h second fundamental form of f on the f-level.
Step 7 In a neighbourhood of S_0 , $\chi(f - level) > 0$. If the $f-$level are totally umbilical it is easy. If not, expand the field of curvature directions in a field with only finite singular points where index $\neq 0$ But $\sum(index) = \chi(S_0) > 0$. We get a one dimensional strata transverse to the $f-$levels of umbilic with index > 0.
Step 8 index (umbilic) $\neq 0 \Rightarrow$ Cotton=D Cotton=0 (See Caratheodory conjecture + meromorphic Lagrange preparation theorem).
Step 9 $W_0(f) = W$ (umbilic) give $\ddot{W}_0 \frac{\dot{W}_0}{f+1} + \frac{R}{3(f+1)} = \frac{3}{4W_0}(\dot{W}_0 + \frac{R}{3})(\dot{W}_0 + Rf)$ in the conformally flat case. S_0 give the maximum $\Longrightarrow c \leq 0$, $c = 0$ easy. If $c < 0$ solve f not globally with step 3 with $f_c \leq f \leq -1, W_0(f_i) = 0$ and take $W_0(f) = 0$ if $f \leq f_c \bar{\Delta}_{\bar{g}}(W - W_0) = ()^2 + ()\partial_z(W - W_0) - \frac{3C}{a^3}(f+1)(\frac{Rf}{3} + \frac{C}{a^3}(f+1))(W - W_0)$ but $\frac{Rf}{3} + \frac{C}{a^3}(f+1) = 0 \Longrightarrow f = -\frac{1}{3}$.
Step 10 $\mathcal{O}$ connected component of

$$\left\{ \begin{array}{ccc} df & \neq & 0 \\ f < & -1 & \end{array} \right\}$$

such that $\mathcal{C} \subseteq \bar{\mathcal{O}}$. Inside $\mathcal{O}$ maximum principle give Cotton $= 0$!

Edited by Joachim Lohkamp

Participants

Prof. Dr. Uwe Abresch
abresch@math.ruhr-uni-bochum.de
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
D–44801 Bochum

Peter Albers
albers@mathematik.uni-leipzig.de
Fakultät für Mathematik/Informatik
Universität Leipzig
Augustusplatz 10/11
D–04109 Leipzig

Prof. Dr. Bogdan Alexandrov
bogdan@mathematik.hu-berlin.de
Institut für Mathematik
Humboldt-Universität
Rudower Chaussee 25
D–10099 Berlin

Dr. Bernd Ammann
ammann@math.uni-hamburg.de
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
D–20146 Hamburg

Prof. Dr. Michael Anderson
anderson@math.sunysb.edu
Department of Mathematics
State University of New York
at Stony Brook
Stony Brook, NY 11794-3651 - USA

Prof. Dr. Joseph Ayoub
jayoub@clipper.ens.fr
Departement des Mathematiques
Ecole Normale Superieure
45, rue d'Ulm
F-75230 Paris

Dr. Andreas Bernig
bernig@math.unizh.ch
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Dr. Laurent Bessieres
laurent.bessieres@ujf-grenoble.fr
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Maciej Borodzik
mcboro@duch.mimuw.edu.pl
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa - POLAND

Dr. Owen Dearricott
owen.dearricott@math.uni-augsburg.de
Institut für Mathematik der
Universität Augsburg
Universitätsstr. 14
D–86159 Augsburg

Prof. Dr. Gerd Faltings
faltings@mpim-bonn.mpg.de
Max-Planck-Institut für Mathematik
Vivatsgasse 7
D–53111 Bonn

Urs Adrian Frauenfelder
ufrauenf@math.ethz.ch
ETH Zürich
Hofstr.24
CH-8136 Gattikon

Prof. Dr. Kenji Fukaya
fukaya@kusm.kyoto-u.ac.jp
Dept. of Mathematics
Faculty of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502 - JAPAN

Dr. Roland Gunesch
gunesch@mathematik.uni-leipzig.de
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
D–04109 Leipzig

Dr. Bernhard Hanke
Bernhard.Hanke@mathematik.uni-muenchen.de
Mathematisches Institut
Universität München
Theresienstr. 39
D–80333 München

Dr. Michael Heusener
heusener@math.univ-bpclermont.fr
Lab. de Mathematiques Pures
UFR Recherche Scientifique et
Technique
Compl. des Cezeaux Bat. Maths
F-63177 Aubiere Cedex

Daniel John
John@math.ruhr-uni-bochum.de
Am Göpel 26
D–44795 Bochum

Dr. Thilo Kuessner
thilo@whitney.mathematik.uni-tuebingen.de
Mathematisches Institut
Universität München
Theresienstr. 39
D–80333 München

Dipl.-Math. Tobias Lamm
lamm@sunpool.mathematik.uni-freiburg.de
Mathematisches Institut
Universität Freiburg
Eckerstr.1
D–79104 Freiburg

Dr. Janko Latschev
janko@math.unizh.ch
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Bernhard Leeb
leeb@riemann.mathematik.uni-tuebingen.de
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D–72076 Tübingen

Joachim Lohkamp
lohkamp@math.uni-augsburg.de
Institut für Mathematik der
Universität Augsburg
Universitätsstr. 14
D–86159 Augsburg

Dr. Frank Loose
frank.loose@uni-tuebingen.de
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D–72076 Tübingen

Thomas Mautsch
mautsch@math.ethz.ch
Departement Mathematik
ETH-Zentrum
HG G 32.1
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Darko Milinkovic
milinko@poincare.matf.bg.ac.yu
Faculty of Mathematics
University of Belgrade
Studentski Trg. 16, P.B. 550
11001 Beograd - SERBIA

Gregoire Montcouquiol
montcouq@ens.fr
Universite Paul Sabatier
U.F.R. M.I.G.
118, route de Narbonne
F-31062 Toulouse Cedex 4

Marc Nardmann
marc.nardmann@web.de
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
D–04109 Leipzig

Prof. Dr. Joan Porti
porti@mat.uab.es
Departamento de Matematicas
Universitad Autonoma de Barcelona
Campus Universitario
E-08193 Bellaterra Barcelona

Luc Rozoy
luc.rozoy@ujf-grenoble.fr
Institut Fourier
UMR 5582 du CNRS
Universite de Grenoble I
100 rue de Maths
F-38402 Saint-Martin d'Heres

Thomas Schick
schick@uni-math.gwdg.de
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5
D–37073 Göttingen

Prof. Dr. Matthias Schwarz
mschwarz@mathematik.uni-leipzig.de
Fakultät für Mathematik/Informatik
Universität Leipzig
Augustusplatz 10/11
D–04109 Leipzig

Prof. Dr. Bernd Siebert
bernd.siebert@math.uni-freiburg.de
Bernd.Siebert@rz.ruhr-uni-bochum.de
Mathematisches Institut
Universität Freiburg
Eckerstr.1
D–79104 Freiburg

Dr. Miles Simon
msimon@mathematik.uni-freiburg.de
msimon@bingo.mathematik.uni-freiburg.de
Mathematisches Institut
Universität Freiburg
Eckerstr.1
D–79104 Freiburg

Juan Souto
souto@math.uni-bonn.de
Mathematisches Institut
Universität Bonn
Beringstr. 1
D–53115 Bonn

Hartmut Weiss
hartmutw@riemann.mathematik.uni-tuebingen.de
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D–72076 Tübingen

Dr. Burkhard Wilking
wilking@math.uni-muenster.de
Mathematisches Institut
Universität Münster
Einsteinstr. 62
D–48149 Münster

Fabian Ziltener
fabian@math.ethz.ch
Nansenstr. 3
CH-8050 Zürich